A NOTE ON THE SIMPSON CORRESPONDENCE FOR SEMISTABLE HIGGS BUNDLES

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Abstract. In this note we provide an elementary proof of the Simpson correspondence between semistable Higgs bundles with vanishing Chern classes and representation of fundamental groups of Kähler manifolds.

0. Introduction

Recently, J. Cao [Cao16] proved a longstanding conjecture by Demailly-Peternell-Schneider: for any smooth projective manifold whose anticanonical bundle is nef, the Albanese map of X is locally isotrivial. A crucial step of his proof relies on an elegant criteria in [CH17] for the local isotriviality of the fibration, which is based on deep results for the *numerically flat vector bundles* (see Definition 1.4 below) in [DPS94] and the Simpson correspondence in [Sim92].

Theorem 0.1. *Let E be a holomorphic vector bundle over a Kähler manifold X which is* numerically flat*. Then*

(i) [DPS94, Theorem 1.18]. *E admits a filtration*

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_p = E \tag{0.1}$$

by vector subbundles such that the quotients E_k/E_{k-1} are hermitian flat, that is, given by unitary representations $\pi_1(X) \to U(r_k)$. In particular, E is semistable and all the Chern classes of E vanish.

(ii) [Sim92, §3]. *E has a holomorphic structure which is an extension of unitary flat bundles.*

Theorem 0.1.(ii) is indeed a special case (*i.e.* the Higgs fields vanish) of the equivalence between the category of semistable Higgs bundles with vanishing Chern classes and the category of representations of the fundamental groups of Kähler manifolds established by Simpson [Sim92, §3].

Theorem A ([Sim92, Corollary 3.10]). *Let X be a compact Kähler manifold equipped with a smooth vector bundle V. Then the following statements are equivalent*

(1) (V, D) is a flat vector bundle over X, i.e. $D^2 = 0$.

(2) V can be equipped with a Higgs bundle structure $(V, \bar\partial, \theta)$ which is semistable.

Moreover, these equivalences are compatible with extensions in the following sense:

(i) *let*

$$\{0\} = (V_0, D_0) \subsetneq (V_1, D_1) \subsetneq \ldots \subsetneq (V_m, D_m) = (V, D)$$

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be the filtration of flat vector bundles such that $D_i := D_{\upharpoonright V_i}$ *and induced graded terms* $(V_i/V_{i-1}, \nabla_i)$ *correspond to irreducible representations of the fundamental group* $\pi_1(X)$*. Then each* V_i *is* θ *and* $\bar{\partial}$*-invariant, and the induced Higgs bundle structures on the graded terms* $(V_i/V_{i-1}, \bar{\partial}_i, \tau_i)$ *are* stable *with vanishing Chern classes. Moreover,* $(V_i/V_{i-1}, \bar{\partial}_i, \tau_i)$ *is the (unique) Higgs bundle induced by* $(V_i/V_{i-1}, \nabla_i)$ *from the Simpson correspondence.*

(ii) *Let*

$$\{0\} = (V_0, \bar{\partial}_0, \theta_0) \subsetneq (V_1, \bar{\partial}_1, \theta_1) \subsetneq \ldots \subsetneq (V_m, \bar{\partial}_m, \theta_m) = (V, \bar{\partial}, \theta)$$

be the filtration of sub Higgs bundles such that $\bar{\partial}_i := \bar{\partial}_{\upharpoonright V_i}$ *and* $\theta_i := \theta_{\upharpoonright V_i}$*, and each induced graded terms* $(V_i/V_{i-1}, \bar{\partial}_i, \tau_i)$ *is a stable Higgs bundle (the existence of such a filtration is proved by Simpson in [Sim92, Theorem 2] for projective manifolds and by Nie-Zhang [NZ15] for Kähler manifolds). Then each* V_i *is a* D*-invariant subbundle and the induced flat bundle* $(V_i/V_{i-1}, \nabla_i)$ *corresponds to irreducible representation of* $\pi_1(X)$*. Moreover,* $(V_i/V_{i-1}, \nabla_i)$ *is the (unique) flat bundle induced by* $(V_i/V_{i-1}, \bar{\partial}_i, \theta_i)$ *from the Simpson correspondence.*

In [Sim92], Simpson introduced *differential graded category* [Sim92, §3], plus the *formality isomorphism* [Sim92, Lemma 2.2] to reduce the proof of Theorem A to his correspondence between polystable Higgs bundles with vanishing Chern classes and semisimple representations of fundamental groups of Kähler manifolds in [Sim88]. While the correspondence for *an extension* of polystable Higgs bundles (*i.e.* $m = 2$ in Theorem A) was written down explicitly in [Sim92, §3, p. 37], the cases of successive extensions follow from the aforementioned differential graded categories.

The purpose of this note is to provide an elementary proof of Theorems A.(i) and A.(ii). Precisely speaking, we will construct the explicit equivalences in Theorem A. When $m = 2$, Simpson applied the *Hodge decompositions for harmonic bundles* in [Sim92, §2] to build this concrete correspondence. In this note, we applied his $\partial\bar{\partial}$*-lemma for harmonic bundles* in [Sim92, §2] instead to deal with the general cases $m > 2$.

1. Technical Preliminaries

In this section we recall the definition of Higgs bundles, harmonic metrics for flat bundles, and the Simpson correspondence between polystable Higgs bundles and semisimple representations of fundamental groups. We refer the readers to [Cor88, Sim88, Sim92] for further details.

1.1. **Higgs bundles.**

Definition 1.1. *Let* X *be a* n*-dimensional Kähler manifold with a fixed Kähler metric* ω*. A* Higgs bundle *on* X *is a triple* $(V, \bar{\partial}, \theta)$*, where* V *is a smooth vector bundle,* $\bar{\partial}$ *is a* $(0,1)$*-connection satisfying the integrability condition* $\bar{\partial}^2 = 0$*, and* θ *is a map* $\theta : V \to V \otimes \mathscr{A}^{1,0}(X, V)$ *such that*

$$(\bar{\partial} + \theta)^2 = 0. \tag{1.1.2}$$

By the theorem of Koszul-Malgrange, $\bar{\partial}$ gives rise to a holomorphic structure on V, and we denote by E the holomorphic vector bundle $(V, \bar{\partial})$. Thus (1.1.2) is equivalent to that

$$\bar{\partial}(\theta) = 0, \quad \text{and} \quad \theta \wedge \theta = 0.$$

Hence we can write abusively (E, θ) for the definition of Higgs bundle, where E is a holomorphic vector bundle, and $\theta : E \to E \otimes \Omega^1_X$ with $\theta \wedge \theta = 0$.

We say that a Higgs bundle (E,θ) is stable (*resp.* semistable) if for all θ-invariant torsion-free coherent subsheaves $F \subsetneq E$, say *Higgs subsheaves* of (F,θ), we have

$$\mu_\omega(F) := \frac{c_1(\det F)\cdot[\omega]^{n-1}}{\operatorname{rank} F} < (\text{resp.} \leqslant) \frac{c_1(\det E)\cdot[\omega]^{n-1}}{\operatorname{rank} E} =: \mu_\omega(E)$$

where $\det F = (\wedge^{\operatorname{rank} F} F)^{\star\star}$ is the determinant bundle of F, and we say that $\mu_\omega(F)$ is the *slope* of F with respect to ω. A Higgs bundle (E,θ) is *polystable* if it is a direct sum of stable Higgs bundles with the same slope.

For any Higgs bundle (E,θ) over a Kähler manifold X, if h is a metric on E, set $D(h)$ to be its Chern connection with $D(h)^{0,1} = \bar\partial$. Consider furthermore the connection

$$D_h = D(h) + \theta + \theta_h^*,$$

where $\theta_h^\star$ is the adjoint of θ with respect to h, and let $F_h := D_h^2$ denote its curvature. Then the metric h is called *Hermitian-Yang-Mills* if

$$\Lambda F_h = \mu_\omega(E)\cdot \mathbb{1}.$$

1.2. **Higher order Kähler identities for harmonic Bundles.** Let (V,D) is a flat bundle equipped with a metric h. Decompose $D = d' + d''$ into connections of type $(1,0)$ and $(0,1)$ respectively. Let δ' and δ'' be the unique $(1,0)$ and $(0,1)$ connections respectively, such that the connections $\delta' + d''$ and $d' + \delta''$ preserve the metric h. Set

$$\theta = \frac{d'-\delta'}{2}, \quad \bar\partial = \frac{d''+\delta''}{2}, \quad \partial = \frac{d'+\delta'}{2}, \tag{1.2.3}$$

then we can decompose the connection D into

$$D = \bar\partial + \theta + \partial + \theta_h^*,$$

here θ_h^* is the adjoint of θ with respect to h, and it is easy to verify that $\bar\partial + \partial$ is also a metric connection. In general, the triple $(V,\bar\partial,\theta)$ might not be a Higgs bundle.

However, since the hermitian metric h on V can be thought of as a map

$$\Phi_h : X \to GL(n,\mathbb{C})/U(n),$$

by the series of the work of Siu-Sampson-Corlette-Deligne, when Φ_h happens to be a *harmonic map*, $(V,\bar\partial,\theta)$ is a Higgs bundle. Such a metric h on V is called *harmonic* metric, and we say that (V,D,h) is a *harmonic* bundle.

Suppose that (V,D,h) is a harmonic bundle.The *harmonic decomposition* is defined by

$$D = D' + D'', \quad \text{where} \quad D'' = \bar\partial + \theta, \quad D' = \partial + \theta_h^*. \tag{1.2.4}$$

Define the Laplacians

$$\begin{aligned} \Delta &= DD^* + D^*D \\ \Delta'' &= D''(D'')^* + (D'')^*D'' \end{aligned}$$

and similarly Δ'. Then by [Sim92, §2] we have

$$\Delta = 2\Delta' = 2\Delta'', \tag{1.2.5}$$

and thus the spaces of harmonic forms with coefficients in V are all the same, which are denoted by $\mathscr{H}^\bullet(V)$. By the Hodge theory we have the following orthogonal decompositions of V-valued forms with respect to the L^2-inner product:

$$\begin{aligned} A^p(V) &= \mathscr{H}^p(V) \oplus \operatorname{Im}(D'') \oplus \operatorname{Im}\big((D'')^*\big) && (1.2.6) \\ &= \mathscr{H}^p(V) \oplus \operatorname{Im}(D) \oplus \operatorname{Im}(D^*) && (1.2.7) \end{aligned}$$

Consequencely one has the following *$\partial\bar\partial$-lemma for harmonic bundles* in [Sim92, Lemma 2.1].

Lemma 1.2 ($\partial\bar\partial$-lemma)**.** *If (V,D,h) is a harmonic bundle, then*

$$\ker(D')\cap\ker(D'')\cap\big(\mathrm{Im}(D'')+\mathrm{Im}(D')\big)=\mathrm{Im}(D'D''). \tag{1.2.8}$$

We will define the *de Rham cohomology* $H^i_{\mathrm{DR}}(X,V)$ for the flat bundle (V,D). We identify V with the locally constant sheaf of flat sections of V. Consider the sheaves of $\mathscr{C}^\infty$ differential forms with coefficients in V:

$$V\to\big(\mathscr{A}^0(V)\xrightarrow{D}\mathscr{A}^1(V)\xrightarrow{D}\cdots\big),$$

which are fine, and thus the cohomology $H^i_{\mathrm{DR}}(X,V)$ is naturally isomorphic to the cohomology of the complex of global sections

$$\big(A^\bullet(V),D\big)=A^0(V)\xrightarrow{D}A^1(V)\xrightarrow{D}\cdots$$

Let us finish this subsection by recalling the following Corlette-Simpson correspondence.

Theorem 1.3. *Let (X,ω) be a compact Kähler manifold of dimension n.*

(i) [Cor88, Don87] *A flat bundle V has a harmonic metric if and only if it arises from a semisimple representation of $\pi_1(X)$.*
(ii) [Sim88] *A Higgs bundle (E,θ) admits an Hermitian-Yang-Mills metric if and only if it is polystable. Such a metric is harmonic if and only if* $\mathrm{ch}_1(E)\cdot\{\omega\}^{n-1}=\mathrm{ch}_2(E)\cdot\{\omega\}^{n-2}=0$.

1.3. **Numerically Flat Vector Bundle.** Let E be a holomorphic vector bundle of rank r over a compact complex manifold X. We denote by $\mathbb{P}(E)$ the projectivized bundle of hyperplanes of E and by $\mathscr{O}_{\mathbb{P}(E)}(1)$ the tautological line bundle over $\mathbb{P}(E)$. Recall the following definition in [DPS94].

Definition 1.4. *Let X be a compact Kähler manifold.*

(i) *We say that a line bundle L is nef, if for any $\varepsilon>0$, there exists a smooth hermitian metric h_ε on L such that $i\Theta_{h_\varepsilon}(L)\geqslant-\varepsilon\omega$, where ω is a fixed Kähler metric on X.*
(ii) *A holomorphic vector bundle E is said to be nef if $\mathscr{O}_{\mathbb{P}(E)}(1)$ is nef over $\mathbb{P}(E)$.*
(iii) *We say that a holomorphic vector bundle E is* numerically flat *if both E and its dual $E^\star$ is nef.*

2. Proof of Theorem A

Proof Theorem A. (i) Let $\rho:\pi_1(X)\to GL(n,\mathbb{C})$ be the representation of the fundamental group corresponding to the flat vector bundle (V,D). After taking some conjugation, one can put the representation in block upper triangular form

$$\begin{bmatrix}\rho_1 & * & \ldots & *\\ 0 & \rho_2 & \ldots & *\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \rho_m\end{bmatrix}$$

such that for every $i=1,\ldots,r$, $\rho_i:\pi_1(X)\to GL(r_i,\mathbb{C})$ is an irreducible representations. Thus there is a filtration of flat vector bundles

$$\{0\}=V_0\subsetneq V_1\subsetneq\ldots\subsetneq V_m=V$$

such that V_i corresponds to

$$\begin{bmatrix} \rho_1 & * & \dots & * \\ 0 & \rho_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_i \end{bmatrix}$$

In particular,

(i) each V_i is invariant under the flat connection D, that is, $D(V_i) \subset V_i \otimes \mathscr{A}^1(X)$.
(ii) The quotient connection D_i on $Q_i := V_i/V_{i-1}$ induced by D is also flat, which corresponds to the irreducible representation $\rho_i : \pi_1(X) \to GL(r_i, \mathbb{C})$.

By Theorem 1.3, we can find a (unique) harmonic metric h_i such that (Q_i, D_i, h_i) is an harmonic bundle. By (1.2.4) for each $i = 1, \dots, m$, there is a unique harmonic decomposition

$$D_i'' = \bar{\partial}_i + \theta_i, \quad D_i' = \partial_i + \theta_i^*, \tag{2.9}$$

where θ_i^* is the adjoint of θ_i with respect to h_i. Moreover, Q_i can be equipped with a Higgs bundle structure $(Q_i, \bar{\partial}_i, \theta_i)$.

For simplicity we first consider the case that V is an extension of an irreducible representation by another one, that is, $m = 2$ and we have an exact sequence of flat vector bundles over X:

$$0 \to Q_1 \to V \to Q_2 \to 0,$$

and thus there is $\eta \in A^1\big(X, \hom(Q_2, Q_1)\big)$ such that D is given by

$$D = \begin{bmatrix} D_1 & \eta \\ 0 & D_2 \end{bmatrix}.$$

We denote by $D_{2,1}$ the induced flat connection on the bundle $\hom(Q_2, Q_1)$ by D_1 and D_2. By $D^2 = 0$, one has $D_{2,1}(\eta) = 0$, and thus $\{\eta\} \in H^1_{\mathrm{DR}}\big(X, \hom(Q_2, Q_1)\big)$.

Claim 2.1. *The cohomology class $\{\eta\} \in H^1_{\mathrm{DR}}\big(X, \hom(Q_2, Q_1)\big)$ characterizes the isomorphism class of (V, D) among all extensions of Q_1 by Q_2.*

*Proof of Claim 2.1.*For any $\eta' \in A^1\big(X, \hom(Q_2, Q_1)\big)$ such that $\eta' \in \{\eta\}$. Then $\eta' = \eta + D_{2,1}(a)$ for some $a \in A^0\big(X, \hom(Q_2, Q_1)\big)$. We define a *gauge transformation* $g \in \mathrm{Aut}^\infty(V)$ by

$$g = \begin{bmatrix} \mathbb{1} & -a \\ 0 & \mathbb{1} \end{bmatrix}, \tag{2.10}$$

Then

$$g \circ D \circ g^{-1} = \begin{bmatrix} D_1 & \eta' \\ 0 & D_2 \end{bmatrix} =: \tilde{D}. \tag{2.11}$$

Hence (V, D) and $(V, \tilde{D})$ are isomorphic flat bundles. □

Since both (Q_1, D_1, h_1) and (Q_2, D_2, h_2) are both harmonic bundles, so is $\big(\hom(Q_2, Q_1), D_{2,1}, h_1 h_2^*\big)$. Set $D_{2,1}'$ and $D_{2,1}''$ to be the harmonic decomposition of $D_{2,1}$ as (1.2.4), and let $\Delta_{2,1}$ and $\Delta_{2,1}''$ be the Laplacians of $D_{2,1}$ and $D_{2,1}''$ respectively. By Claim 2.1 one can assume that η is the (unique) harmonic representation in its extension class $\{\eta\} \in H^1_{\mathrm{DR}}\big(X, \hom(Q_2, Q_1)\big)$. Then

$$\Delta_{2,1}''(\eta) = \frac{1}{2}\Delta_{2,1}(\eta) = 0.$$

In particular

$$D''_{2,1}(\eta) = 0. \tag{2.12}$$

Let η' and η'' to be the $(1,0)$ and $(0,1)$-parts of η respectively. Set

$$\bar{\partial} := \begin{bmatrix} \bar{\partial}_1 & \eta'' \\ 0 & \bar{\partial}_2 \end{bmatrix} \quad \text{and} \quad \theta := \begin{bmatrix} \theta_1 & \eta' \\ 0 & \theta_2 \end{bmatrix}.$$

Then (2.12) is equivalent to $(\bar{\partial} + \theta)^2 = 0$, and by Definition 1.1 $(V, \bar{\partial}, \theta)$ is a Higgs bundle over X. Moreover, it is compatible with the Higgs bundle structures $(Q_i, \bar{\partial}_i, \theta_i)$. We prove the theorem when $m = 2$.

For general $m \geqslant 2$, we will prove the theorem by inductions. Set $\nabla_j := D_1 \oplus \cdots \oplus D_j$ to be the flat connection on $Q_1 \oplus \cdots \oplus Q_j$, and

$$\nabla'_j := D'_1 \oplus D'_2 \oplus \cdots \oplus D'_j, \quad \nabla''_j := D''_1 \oplus D''_2 \oplus \cdots \oplus D''_j.$$

Then $\nabla_i = \nabla'_i + \nabla''_i$ is the harmonic decomposition defined in (2.9).

Assume that

$$\tilde{\nabla}_j = \begin{bmatrix} D_1 & & B_j \\ & \ddots & \\ \text{\huge 0} & & D_j \end{bmatrix}, \tag{2.13}$$

is the flat connection on $Q_1 \oplus \cdots \oplus Q_j$ defining V_j. Here $B_j \in A^1\big(X, \operatorname{End}(Q_1 \oplus \ldots \oplus Q_j)\big)$ which is strictly upper-triangle such that

$$\nabla_j(B_j) + B_j \wedge B_j = 0 \tag{2.14}$$

by $\tilde{\nabla}_j^2 = 0$. Here we write abusively ∇_j the induced flat connection of $\operatorname{End}(Q_1 \oplus \cdots \oplus Q_j)$ by $(Q_1 \oplus \cdots \oplus Q_j, \nabla_j)$

Claim 2.2. *Assume that we can find $B_{j-1} \in A^1\big(X, \operatorname{End}(Q_1 \oplus \ldots \oplus Q_{j-1})\big)$ which is strictly upper-triangle such that*

(i) *for $\tilde{\nabla}_{j-1}$ defined in (2.13), the pair $(Q_1 \oplus \ldots \oplus Q_{j-1}, \tilde{\nabla}_{j-1})$ defines V_{j-1}.*
(ii) *$\nabla''_{j-1}(B_{j-1}) + B_{j-1} \wedge B_{j-1} = 0$, or equivalently $\nabla'_{j-1}(B_{j-1}) = 0$.*

Then so is true for j.

*Proof of Claim 2.2.*Since V_j is an extension of V_{j-1} by Q_j

$$0 \to V_{j-1} \to V_j \to Q_j \to 0,$$

we denote by $\beta \in H^1_{\mathrm{DR}}\big(X, \hom(Q_j, V_{j-1})\big)$ the extension class. Choose any representative $A \in \beta$, then $(Q_1 \oplus \ldots \oplus Q_j, \tilde{\nabla}_j)$ defining V_j can be written as

$$\tilde{\nabla}_j = \begin{bmatrix} D_1 & & B_{j-1} & a_1 \\ & \ddots & & \vdots \\ \text{\huge 0} & & D_{j-1} & a_{j-1} \\ 0 & \ldots & 0 & D_j \end{bmatrix} \tag{2.15}$$

where $A = a_1 \oplus \cdots \oplus a_{j-1}$ with $a_i \in A^1\big(X, \hom(Q_j, Q_i)\big)$. Then by $\tilde{\nabla}_j^2 = 0$, one has

$$\tilde{\nabla}_{j-1} \circ A + A \circ D_j = 0. \tag{2.16}$$

In particular, $D_{j,j-1}(a_{j-1})=0$, where $D_{j,i}$ the connection on $\hom(Q_j,Q_i)$ induced by D_j and D_i. Since $\big(\hom(Q_j,Q_i),D_{j,i},h_ih_j^*\big)$ is also a harmonic bundle, we set $D'_{j,i}$ and $D''_{j,i}$ to be the harmonic decomposition of $D_{j,i}$ as (1.2.4). By (1.2.7) there exists

$$c_{j-1}\in A^0\big(X,\hom(Q_j,Q_{j-1})\big)\subset A^0\big(X,\hom(Q_j,V_{j-1})\big)$$

such that

$$\Delta_{j,j-1}(a_{j-1}+D_{j,j-1}c_{j-1})=0, \tag{2.17}$$

where $\Delta_{j,i}$ (resp. $\Delta'_{j,i}$) is the Laplacian of $D_{j,i}$ (resp. $D'_{j,i}$). Denote by $\tilde\nabla_{j,j-1}$ the induced flat connection on $\hom(Q_j,V_{j-1})$ by the connections D_j and $\tilde\nabla_{j-1}$, then

$$A_1:=A+\tilde\nabla_{j,j-1}(c_{j-1})=a_1\oplus\cdots\oplus a_{j-1}+(\tilde\nabla_{j-1}\circ c_{j-1}-c_{j-1}\circ D_j)$$

belongs to the same extension class as A_1. If we write $A_1=a'_1\oplus\cdots\oplus a'_{j-1}$ with $a'_i\in A^1\big(X,\hom(Q_j,Q_i)\big)$, then $a'_{j-1}=a_{j-1}+D_{j,j-1}(c_{j-1})$. By (2.17), one has $\Delta'_{j,j-1}(a'_{j-1})=\frac{1}{2}\Delta_{j,j-1}(a'_{j-1})=0$, and thus

$$D'_{j,j-1}(a'_{j-1})=0.$$

This gives us hints that we can use some *ad hoc* methods to find the proper A.

Assume now for some $A=a_1\oplus\cdots\oplus a_{j-1}\in\beta$ such that $D'_{j,i}(a_i)=0$ for all $i=k+1,\dots,j-1$. By (2.16) we have

$$D_{j,k}(a_k)+\sum_{i=k+1}^{j-1}b_{ki}a_i=0,$$

here b_{ki} is the projection of $B_{j-1}\in A^1\big(X,\mathrm{End}(Q_1\oplus\ldots\oplus Q_{j-1})\big)$ to the component $A^1\big(X,\hom(Q_i,Q_k)\big)$. By the assumption that $\nabla'_{j-1}(B_{j-1})=0$, we have $D'_{i,k}(b_{ki})=0$. Hence

$$\begin{aligned}0&=D'_{j,k}D_{j,k}(a_k)+D'_{j,k}(\sum_{i=k+1}^{j-1}b_{ki}a_i)\\&=D'_{j,k}D_{j,k}(a_k)+\sum_{i=k+1}^{j-1}\big(D'_{i,k}(b_{ki})a_i-b_{ki}D'_{j,i}(a_i)\big)\\&=D'_{j,k}D''_{j,k}(a_k)\\&=-D''_{j,k}D'_{j,k}(a_k)\end{aligned}$$

Applying Lemma 1.2 to $D'_{j,k}(a_k)$, there exists $c_k\in A^0\big(X,\hom(Q_j,Q_k)\big)$ such that

$$D'_{j,k}(a_k)=-D'_{j,k}D''_{j,k}(c_k)=-D'_{j,k}D_{j,k}(c_k). \tag{2.18}$$

Set

$$\begin{aligned}\tilde A:&=A+\tilde\nabla_{j,j-1}(c_k)\\&=A+(\tilde\nabla_{j-1}\circ c_k-c_k\circ D_j)\\&=a'_1\oplus\ldots\oplus a'_{k-1}\oplus\big(a_k+D_{j,k}(c_k)\big)\oplus a_{k+1}\oplus\ldots\oplus a_{j-1}.\end{aligned}$$

which belongs to the extension class as A. In other words, the components of $\tilde A$ in $A^1\big(X,\hom(Q_j,Q_i)\big)$ for $i=k+1,\dots,j-1$ are the same as those of A, and the component of $\tilde A$ in $A^1\big(X,\hom(Q_j,Q_k)\big)$ are replaced by $a_k+D_{j,k}(c_k)$, such that $D'_{j,k}\big(a_k+D_{j,k}(c_k)\big)=0$ by (2.18). Thus by the induction on k we can choose $A\in\beta$ properly such that $D'_{j,k}(a_k)=0$ for all $k=1,\dots,j-1$. This is equivalent to $\nabla'_j(B_j)=0$. The claim is thus proved. $\square$

By Claim 2.2 we conclude that there exists $\eta \in A^1\big(X, \mathrm{End}(Q_1 \oplus \ldots \oplus Q_m)\big)$ which is strictly upper-triangle, such that $(Q_1 \oplus \ldots \oplus Q_m, \nabla_m + \eta)$ defines the flat bundle V, and

$$\nabla'_m(\eta) = 0 \quad \Leftrightarrow \quad D'_{j,k}(\eta_{kj}) = 0 \quad \forall\, 1 \leqslant k < j \leqslant m. \tag{2.19}$$

Here we denote by η_{kj} the component of η in $A^1\big(X, \hom(Q_j, Q_k)\big)$. Hence

$$\nabla''_m(\eta) + \eta \wedge \eta = 0. \tag{2.20}$$

Set

$$\bar\partial := \begin{bmatrix} \bar\partial_1 & & \eta'' \\ & \ddots & \\ \text{\Large 0} & & \bar\partial_m \end{bmatrix} \quad \text{and} \quad \theta = \begin{bmatrix} \theta_1 & & \eta' \\ & \ddots & \\ \text{\Large 0} & & \theta_m \end{bmatrix}.$$

Here η' and η'' are the $(1,0)$ and $(0,1)$-parts of η respectively, and $D''_i = \bar\partial_i + \theta_i$ is defined as (2.9). Then (2.20) is equivalent to $(\bar\partial + \theta)^2 = 0$, and thus that $(V, \bar\partial, \theta)$ is a Higgs bundle over X. In this setting, for each $1 \leqslant i \leqslant m$, $(V_i, \bar\partial_{\upharpoonright V_i}, \theta_{\upharpoonright V_i})$ is a Higgs subbundle of E, and the induced Higgs bundle structure on the graded term $Q_i := V_i/V_{i-1}$ coincides with $(Q_i, \bar\partial_i, \theta_i)$.

(ii) The proof of Theorem A.(ii) proceeds along the same lines as Theorem A.(i). We will only sketch a proof for Theorem 0.1.(ii), *i.e.* the case that $\theta = 0$. Let us start with a holomorphic vector bundle $(V, \bar\partial)$. Suppose that it admits a filtration

$$\{0\} = (V_0, \bar\partial_0) \subsetneq (V_1, \bar\partial_1) \subsetneq \cdots \subsetneq (V_m, \bar\partial_m) = (V, \bar\partial)$$

of holomorphic vector bundles such that for each $i = 1, \ldots, m$, the graded term $Q_i := (V_i/V_{i-1}, D''_i)$ is hermitian flat, *i.e.* it can be equipped with a hermitian metric h_i so that the Chern connection D_i is flat. Set $D'_i := D_i - D''_i$, which is a $(1,0)$-connection. Then there exists $\eta \in A^{0,1}\big(X, \mathrm{End}(Q_1 \oplus \ldots \oplus Q_m)\big)$ such that

$$\bar\partial := \begin{bmatrix} D''_1 & & \eta \\ & \ddots & \\ \text{\Large 0} & & D''_m \end{bmatrix}$$

Set $\nabla_j := D_1 \oplus \cdots \oplus D_j$ to be the hermitian flat connection on $Q_1 \oplus \cdots \oplus Q_j$, and

$$\nabla'_j := D'_1 \oplus \cdots \oplus D'_j, \quad \nabla''_j := D''_1 \oplus \cdots \oplus D''_j.$$

A similar proof as Claim 2.1 shows the following result.

Claim 2.3. *One can choose* $\eta \in A^{0,1}\big(X, \mathrm{End}(Q_1 \oplus \ldots \oplus Q_m)\big)$ *property such that*

$$\nabla'_m(\eta) = 0 \quad \Leftrightarrow \quad D'_{j,k}(\eta_{kj}) = 0 \quad \forall\, 1 \leqslant k < j \leqslant m. \tag{2.21}$$

Here we denote by η_{kj} *the component of* η *in* $A^{0,1}\big(X, \hom(Q_j, Q_k)\big)$.

Let us denote $D := \bar\partial + \nabla'_m$, which is flat by (2.21). Since ∇'_m is a $(1,0)$-connection, then the underlying holomorphic structure of (V, D) coincides with $(V, \bar\partial)$. It follows from our construction that, $(V_i, \bar\partial_i)$ is a D-invariant subbundle for each $i = 1, \ldots, m$, with D_i the induced flat bundle structure on the graded terms. $\square$

Remark 2.4. (i) *Note that by the proof of Theorem A.(i), the category of successive of unitary extensions of fundamental groups is larger than that of semistable vector bundles with vanishing Chern classes. In fact, even if all the graded terms are unitary representations of* π_1*, the corresponding semistable Higgs bundles might have non-vanishing (nilpotent) Higgs fields.*

(ii) *The condition of numerical flatness in Theorem 0.1.(ii) is necessary. Indeed, in [BH15] Biswas-Heu constructed an example of an extension of flat vector bundles, which does not admit any holomorphic connection.*

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References

[BH15] Indranil Biswas and Viktoria Heu, *Non-flat extension of flat vector bundles*, Internat. J. Math. **26** (2015), no. 14, 1550114, 6, https://doi.org/10.1142/S0129167X15501141.

[Cao16] Junyan Cao, *Albanese maps of projective manifolds with nef anticanonical bundles*, arXiv:1612.05921, to appear in Ann. Sci. Éc. Norm. Supér. (2016).

[CH17] Junyan Cao and Andreas Höring, *Manifolds with nef anticanonical bundle*, J. Reine Angew. Math. **724** (2017), 203–244, https://doi.org/10.1515/crelle-2014-0073.

[Cor88] Kevin Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), no. 3, 361–382, http://projecteuclid.org/euclid.jdg/1214442469.

[Don87] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 127–131, https://doi.org/10.1112/plms/s3-55.1.127.

[DPS94] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. **3** (1994), no. 2, 295–345.

[NZ15] Yanci Nie and Xi Zhang, *A note on semistable Higgs bundles over compact Kähler manifolds*, Ann. Global Anal. Geom. **48** (2015), no. 4, 345–355, https://doi.org/10.1007/s10455-015-9474-0.

[Sim88] Carlos T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918, https://doi.org/10.2307/1990994.

[Sim92] ———, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 5–95, http://www.numdam.org/item?id=PMIHES_1992__75__5_0.

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