Exponential decay of connection probabilities for subcritical
Voronoi percolation in $\mathbb{R}^d$

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Abstract

We prove that for Voronoi percolation on $\mathbb{R}^d$ ($d \geq 2$), there exists $p_c = p_c(d) \in (0,1)$ such that

• for $p < p_c$, there exists $c_p > 0$ such that $\mathbb{P}_p[0$ connected to distance $n] \leq \exp(-c_p n)$,

• there exists $c > 0$ such that for $p > p_c$, $\mathbb{P}_p[0$ connected to infinity] $\geq c(p - p_c)$.

For dimension 2, this result offers a new way of showing that $p_c(2) = 1/2$. This paper belongs to a series of papers using the theory of algorithms to prove sharpness of the phase transition; see [DRT17a, DRT17b].

1 Introduction

Motivation. Bernoulli percolation was introduced in [BH57] by Broadbent and Hammersley to model the diffusion of a liquid in a porous medium. Originally defined on a lattice, the model was later generalized to a number of other contexts. Of particular interest is the developments of percolation in continuum environment, see [MR08] for a book on the subject.

One of the most fundamental example of continuum models is Voronoi percolation (or Poisson Voronoi percolation), where the Voronoi cells associated to a Poisson point process in $\mathbb{R}^d$ are colored independently black or white with respective probability $p$ and $1 - p$. Voronoi percolation behaves very similarly to Bernoulli percolation, but is harder to study, due to local dependencies (the colors of two disjoint points are always correlated, since two points have always a positive probability to belong to the same cell). Because of these dependencies, several techniques for Bernoulli percolation do not apply, and the analysis of Voronoi percolation requires new and more robust methods to be developed.

In the celebrated work [BR06a], Bollobás and Riordan proved that Voronoi percolation in the plane undergoes a sharp phase transition at the critical parameter $p = 1/2$. More precisely, for $p > 1/2$, the connected component of black cells containing origin is infinite with positive probability, while for $p < 1/2$, the connected component of black cells containing origin has radius larger than $n$ with probability decaying exponentially fast in $n$. Since this result, several other results came to complement the picture on planar Voronoi percolation, including a fine description of the critical behavior [AGMT16, Tas16]. Let us also mention an earlier result obtained by Benjamini and Schramm [BS98] which states asymptotic invariance of the crossing probabilities with respect to a conformal change of metric. We refer the reader to their article for more on the history of the model.

The recent advances in the understanding of Voronoi percolation were mostly restricted to the planar case, and several fundamental questions, including sharpness of the phase transition, remained widely open in higher dimension. This article provides a first proof of sharpness for Voronoi percolation in any dimension $d \geq 2$. As a consequence, it also offers an alternative computation of the critical point in the two-dimensional case.

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Let \( d \geq 2 \) be a positive integer and let \( \mathbb{R}^d \) be the \( d \)-dimensional Euclidean space with \( \| \cdot \| \) denote the \( \ell^2 \) norm. For \( r > 0 \), set \( \mathbb{B}_r := \{ y \in \mathbb{R}^d : \| y \| \leq r \} \) and \( \mathbb{S}_r := \{ y \in \mathbb{R}^d : \| y \| = r \} \) for the ball and sphere of radius \( r \) around the origin.

Let \( \mathbb{P}_p \) denote the Voronoi percolation measure with parameter \( p \) on \( \mathbb{R}^d \), that is \( \mathbb{P}_p \) is the law of two independent point processes \( \eta^b \) and \( \eta^w \) with respective intensities \( p \) and \( 1 - p \) (here, \( \eta^b \) and \( \eta^w \) are two locally finite subsets of \( \mathbb{R}^d \)). Define \( \eta = \eta^b \cup \eta^w \). For a point \( x \in \eta \), define the Voronoi cell of \( x \)

\[
C(x) := \{ y \in \mathbb{R}^d : \| x - y \| = \min_{x' \in \eta} \| x' - y \| \}.
\]

The measure \( \mathbb{P}_p \) induces a coloring \( \omega \) on the points of \( \mathbb{R}^d \) defined as follows. Set \( \omega(y) = 1 \) for every \( y \) belonging to the Voronoi cell of some \( x \in \eta^b \). Set \( \omega(y) = 0 \) for all the other points in \( \mathbb{R}^d \). We say that \( y \) is black if \( \omega(y) = 1 \), and white otherwise.

For \( x, y \in \mathbb{R}^d \), let the event \( x \) connected to \( y \) (denoted by \( \{ x \leftrightarrow y \} \) be the existence of a continuous path of black points connecting \( x \) to \( y \). If \( X, Y \in \mathbb{R}^d \), the event \( \{ X \leftrightarrow Y \} \) denotes existence of \( x \in X \) and \( y \in Y \) such that \( x \) is connected to \( y \). Also, \( \{ 0 \leftrightarrow \infty \} \) is the event that \( 0 \) belongs to an unbounded connected component of black points. For \( p \in [0, 1] \) and \( n \geq 0 \), define \( \theta(p) := \mathbb{P}_p[0 \leftrightarrow \infty] \) and \( \theta_n(p) := \mathbb{P}_p[0 \leftrightarrow S_n] \). Finally, we set \( p_c := \inf\{ p \in [0, 1] : \theta(p) > 0 \} \).

The main result of this paper is the following theorem.

**Theorem 1.** Fix \( d \geq 2 \). For any \( p < p_c \), there exists \( c_p > 0 \) such that for any \( n \geq 1 \),

\[
\theta_n(p) \leq \exp(-c_p n).
\]

Furthermore, there exists \( c > 0 \) such that \( \theta(p) \geq c(p - p_c) \) for any \( p > p_c \).

This result has an immediate corollary, namely the result of Bollobás and Riordan [BR06a] on planar Voronoi percolation.

**Corollary 2.** The critical parameter of Voronoi percolation on \( \mathbb{R}^2 \) is equal to \( 1/2 \). Furthermore, \( \theta(1/2) = 0 \).

Existing proofs of exponential decay for more standard models such as Bernoulli percolation [Men86, AB87, DT10] or the Ising model [ABF87, DT16] do not extend to the context of Voronoi percolation. The reason is a lack of a BK-type inequality. In two dimensions, Bollobás and Riordan use crossing probabilities and introduce tools from Boolean functions [EK96] to bypass this difficulty. This strategy was proved very fruitful in two dimension, since several results were proved for dependent percolation models using similar ideas; see e.g. [BD12, DRT16]. Unfortunately, applying such arguments in higher dimension seemed to be very challenging, so that even Bernoulli-type percolation models remained out of reach of the previous method. Recently, a new technique based on randomized algorithms was introduced to prove sharpness of the phase transition for the random-cluster and Potts models on transitive graphs [DRT17a]. This method, based on an inequality connecting randomized algorithms and influences in a product space first proved in [OSSS05], seems applicable to a variety of continuum models including Voronoi percolation or Boolean percolation [DRT17b].

The strategy consists in proving a family of differential inequalities. More precisely, for every \( \delta \in (0, \frac{1}{2}) \), we will prove that there exists \( c = c(\delta) > 0 \) such that for all \( n \geq 1 \) and \( p \in [\delta, 1 - \delta] \),

\[
\theta_n(p) \geq c \frac{n}{S_n(p)} \theta_n(p),
\]

where \( S_n := \sum_{k=0}^{n-1} \theta_k \).

The proof of Theorem 1 follows easily. Indeed, it is well-known that \( 0 < p_c < 1 \) [BR06a, p270] so that the following lemma can be applied to \( \alpha_0 := \delta < p_c < 1 - \delta =: \alpha_1 \), and \( f_n := \theta_n/c \).

**Lemma 3.** Consider a converging sequence of increasing differentiable functions \( f_n : [\alpha_0, \alpha_1] \to [0, M] \) satisfying

\[
f_n' \geq \frac{n}{\Sigma_n} f_n
\]

for all \( n \geq 1 \), where \( \Sigma_n = \sum_{k=0}^{n-1} f_k \). Then, there exists \( \beta_1 \in [\alpha_0, \alpha_1] \) such that
• For any $\beta < \beta_1$, there exists $c_\beta > 0$ such that for any $n$ large enough, $f_n(\beta) \leq M \exp(-c_\beta n)$.
• For any $\beta > \beta_1$, $f = \lim_{n \to \infty} f_n$ satisfies $f(\beta) \geq \beta - \beta_1$.

This lemma can be found in [DRT17a]. Also, note that Theorem 1 can be proved without the knowledge of $p_c < 1$. Indeed, if $p_c = 1$, Lemma 3 applied to $\alpha_0 = \delta$ and $\alpha_1 = 1 - \delta$ implies exponential decay for every $p < 1 - \delta$. Since $\delta$ is arbitrary, we obtain exponential decay for every $p < 1$. This comment could be relevant when working on spaces different from $\mathbb{R}^d$ for which $p_c < 1$ is not proved yet.

The paper is organized as follows. The next section contains some preliminaries. In Section 3, we prove (2). Section 4 contains the proof of Corollary 2. For completeness, we include the proof of Lemma 3 in Section 5.

2 Preliminaries

2.1 Monotone events and the FKG inequality

An event $A$ is said to be increasing if for every configurations $(\eta^b, \eta^w)$,
\[
(\eta^b, \eta^w) \in A \quad \eta^b \subseteq \eta^b, \eta^w \supseteq \eta^w \implies (\eta^b, \eta^w) \in A.
\]
An event is said to be decreasing if its complement is increasing. The FKG inequality for Voronoi percolation (see e.g. [BR06b, p278]) states that for any increasing events $A$ and $B$,
\[
P_p[A \cap B] \geq P_p[A]P_p[B].
\]

(FKG)

Note that it implies that $P_p[A \cap B] \leq P_p[A]P_p[B]$ whenever $A$ is increasing and $B$ is decreasing.

2.2 A Russo-type formula for Voronoi percolation

For an increasing event $A$, define the set of pivotal points
\[
\text{Piv}_A := \left\{ x \in \eta : 1_A(\eta^b \setminus \{x\}, \eta^w \cup \{x\}) \neq 1_A(\eta^b \cup \{x\}, \eta^w \setminus \{x\}) \right\}.
\]
Call an increasing event $A$ local if there exists $n \geq 0$ such that $A$ is measurable with respect to the $\sigma$-algebra generated by $\{\omega(x)\}_{x \in \mathbb{B}_n}$.

Lemma 4. Consider a local increasing event $A$. Then, $p \mapsto P_p[A]$ is finitely differentiable and
\[
\frac{dP_p[A]}{dp} = E_p[[\text{Piv}_A]].
\]

Proof of Lemma 4. In this proof, $d\eta$ denotes the law of $\eta$ (in particular it does not contain information on colors). Write
\[
P_{p+\delta}[A] - P_p[A] = \int_{\eta} P_{p+\delta}[A | \eta] - P_p[A | \eta] \ d\eta.
\]
Conditioned on $\eta$, the law of $\eta^b$ is Bernoulli percolation with parameter $p$ on points of $\eta$. Since $A$ is measurable with respect to the $\sigma$-algebra generated by $\{\omega(x)\}_{x \in \mathbb{B}_n}$, apply Russo’s formula (for Bernoulli percolation) and Fubini to get
\[
P_{p+\delta}[A] - P_p[A] = \int_{\eta} \int_{p \leq s < p+\delta} E_s[[\text{Piv}_A] | \eta] ds \ d\eta
= \int_{p \leq s < p+\delta} \left( \int_{\eta} E_s[[\text{Piv}_A] | \eta] d\eta \right) ds
= \int_{p \leq s < p+\delta} E_s[[\text{Piv}_A]] ds.
\]
The proof follows by continuity in $s$ of $E_s[|\Pi v_A|]$, which is a direct consequence of the domination $(\mathbb{I})$.

Note that even though the event $A$ may depend only on the colors of the points in $\mathcal{B}_n$, the set $\Pi v_A$ can a priori contain points outside the ball. To check that $|\Pi v_A|$ is integrable, we have

$$|\Pi v_A| \leq |D_n(\eta)|$$

(4)

where $D_n(\eta)$ is the set of points in $\eta$, whose cells intersect the ball $\mathcal{B}_n$. The integrability of $D_n$ follows from standard estimates of the Poisson-Voronoi tessellation. For example, observe that there exists $c > 0$ such that for every $t \geq n$,

$$\mathbb{P}_p[D_n \cap (\mathbb{R}^d \setminus \mathcal{B}_t) \neq \emptyset] \leq \mathbb{P}_p[\eta \cap \mathcal{B}_t = \emptyset] \leq e^{-ct^d}$$

(5)

and

$$\mathbb{P}_p[D_n \cap \mathcal{B}_t \geq t^{d+1}] \leq \mathbb{P}_p[\eta \cap \mathcal{B}_t \geq t^{d+1}] \leq e^{-ct}.$$  

(6)

2.3 The OSSS inequality

Assume $I$ is a countable set, and let $(\Omega^I, \pi^I)$ be a product probability space, and $\mathcal{F} : \Omega^I \rightarrow \{0,1\}$. An algorithm $T$ determining $\mathcal{F}$ takes a configuration $\omega = (\omega_i)_{i \in I} \in \Omega^I$ as an input, and reveals the value of $\omega$ in different coordinates one by one. At each step, which coordinate will be revealed next depends on the values of $\omega$ revealed so far. The algorithm stops as soon as the value of $\mathcal{F}$ is the same no matter the values of $\omega$ on the remaining coordinates. Here, we always assume that $T$ determines $\mathcal{F}$ in finite steps almost surely. We will use the following inequality. For any function $\mathcal{F} : \Omega^I \rightarrow \{0,1\}$, and any algorithm $T$ determining $\mathcal{F}$,

$$\text{Var}(\mathcal{F}) \leq \sum_{i \in I} \delta_i(T) \text{Inf}_i(\mathcal{F}),$$

(OSSS)

where $\delta_i(T)$ and $\text{Inf}_i(\mathcal{F})$ are respectively the revelation and the influence of the $i$-th coordinate defined by

$$\delta_i(T) := \pi^I[T \text{ reveals the value of } \omega_i],$$

$$\text{Inf}_i(\mathcal{F}) := \pi^I[\mathcal{F}(\omega) \neq \mathcal{F}(\tilde{\omega})].$$

Above, $\tilde{\omega}$ denotes the random element in $\Omega^I$ which is the same as $\omega$ in every coordinate except the $i$-th coordinate which is resampled independently.

Remark 5. The (OSSS) inequality is originally stated for the case when the sets $\Omega$ and $I$ are finite. However, the proof of [OSSS05] carries over to the case where $(\Omega, \pi)$ is a general probability space and $I$ is countably infinite without any need for modification. The reader could also consult [DRT17a, Theorem 2.5].

2.4 Tensorization of Voronoi percolation

We will eventually apply (OSSS). In order to do so, we introduce a suitable finite product space to encode the measure of Voronoi percolation.

Fix $\varepsilon > 0$. For $x \in \varepsilon \mathbb{Z}^d$, introduce the box $R^x := x + [0, \varepsilon)^d$ as well as $\eta^b_x = \eta^b \cap R^x$, $\eta^w_x = \eta^w \cap R^x$ and $\eta_x = \eta^w_x \cup \eta^b_x$. Let $(\Omega_x, \pi_x)$ be the measured space associated to the random variable $\eta_x = (\eta^b_x, \eta^w_x)$, and consider the product space $(\prod_{x \in \varepsilon \mathbb{Z}^d} \Omega_x, \otimes_{x \in \varepsilon \mathbb{Z}^d} \pi_x)$. Since the random variables $(\eta^b_x, \eta^w_x)$ are independent for different $x$, this space is in direct correspondence with the original space on which Voronoi percolation was defined.

For $x \in \varepsilon \mathbb{Z}^d$ and an increasing event $A$, define

$$\text{Inf}^x_A := \mathbb{P}_p[1_{A}(\eta) \neq 1_{A}(\tilde{\eta})],$$

(7)

where $\eta = (\eta_x)_{x \in \varepsilon \mathbb{Z}^d}$ has law $\otimes_{x \in \varepsilon \mathbb{Z}^d} \pi_x$ and $\tilde{\eta}$ is equal to $\eta$ except on the $x$-coordinate which is resampled independently. Here and below, we use a slight abuse of notation by denoting the measure on the probability space in which $\eta$ and $\tilde{\eta}$ are defined by $\mathbb{P}_p$.  


Lemma 6. For a local increasing event $A$,

$$
\frac{d\mathbb{P}_p[A]}{dp} \geq \frac{1}{2} \limsup_{\varepsilon \to 0} \sum_{x \in \mathbb{Z}^d} \text{Inf}_x^\varepsilon [A].
$$

(8)

Proof. Assume $A$ depends on the colors in $B_n$ only. Let us start by proving that for any $m \geq 1$,

$$
\frac{d\mathbb{P}_p[A]}{dp} \geq \frac{1}{2} \limsup_{\varepsilon \to 0} \sum_{x \in \mathbb{Z}^d \cap B_m} \text{Inf}_x^\varepsilon [A].
$$

(9)

Fix $x \in \varepsilon \mathbb{Z}^d \cap B_m$ and use the notation for $\eta$ and $\tilde{\eta}$ introduced above. Observe that with probability $1 - O(\varepsilon^{2d})$, the union $\eta_x \cup \tilde{\eta}_x$ contains at most one point. Then, using that $\eta = \tilde{\eta}$ when $\eta_x = \tilde{\eta}_x = \emptyset$ and that $\eta_x$ and $\tilde{\eta}_x$ play symmetric roles, we obtain that

$$
\text{Inf}_x^\varepsilon [A] = 2\mathbb{P}_p[1_A(\eta) \neq 1_A(\tilde{\eta}), |\eta_x| = 1, |\tilde{\eta}_x| = 0] + O(\varepsilon^{2d}).
$$

Under the condition that $|\eta_x| = 1$ and $|\tilde{\eta}_x| = 0$, the configuration $\tilde{\eta}$ is simply obtained from $\eta$ by removing the only point $x$ of $\eta$ in $R_x^c$. Furthermore, by monotonicity, this point $x$ must be pivotal in $\eta$ when $1_A(\eta) \neq 1_A(\tilde{\eta})$. Hence, writing $\text{Piv}_A$ for the pivotal set corresponding to $\eta$, the equation above implies

$$
\text{Inf}_x^\varepsilon [A] \leq 2\mathbb{P}_p[|\text{Piv}_A \cap R_x^c| \geq 1, |\eta_x| = 1, |\tilde{\eta}_x| = 0] + O(\varepsilon^{2d})
$$

$$
\leq 2\mathbb{P}_p[|\text{Piv}_A \cap R_x^c|] + O(\varepsilon^{2d}).
$$

Summing this equation over the points $x \in \varepsilon \mathbb{Z}^d \cap B_m$ gives

$$
\sum_{x \in \varepsilon \mathbb{Z}^d \cap B_m} \text{Inf}_x^\varepsilon [A] \leq 2\mathbb{P}_p[|\text{Piv}_A|] + O(\varepsilon^d).
$$

Eq. (9) follows by taking the limsup and using the derivative formula of Lemma 4.

Obtaining (8) from (9) follows readily from the existence of $c > 0$ such that

$$
\text{Inf}_x^\varepsilon [A] \leq 2\varepsilon^d \exp(-c|x|^d)
$$

(10)

uniformly in $\varepsilon$ and $x \in \varepsilon \mathbb{Z}^d$ with $|x| \geq 4n$. To see this, assume that the value of $1_A$ is changed when $\eta$ is replaced by $\tilde{\eta}$. Then, $\eta \cup \tilde{\eta}$ must have at least one point in $R_x^c$ (which occurs with probability smaller than $2\varepsilon^d$), and the cell of one of these points must intersect $B_{\|x\|/4}$, and therefore $B_{\|x\|/4}$ cannot contain a point of $\eta$. Applying (5) concludes the proof. \hfill \Box

3 Proof of Theorem 1

As mentioned in the introduction, we only need to prove (2). For this, we fix $\delta > 0$. Fix $n > 0$ and $p \in [\delta, 1 - \delta]$. Below, constants $c_i$ ($i \leq 4$) are positive and depend on $\delta$ and $d$ only. In particular, these constants are independent of $n$ and $p$.

For $\varepsilon \in (0,1)$, consider the product space $(\prod_{x \in \mathbb{Z}^d} \Omega_x, \otimes_{x \in \mathbb{Z}^d} \pi_x)$ introduced in Section 2.4. Applying (OSSS) to $f = 1_{0 \rightarrow S_n}$ and an algorithm $T_k$ determining $f$ gives that

$$
\theta_n(p)(1 - \theta_n(p)) \leq \sum_{x \in \mathbb{Z}^d} \delta_x(T_k)\text{Inf}_x^\varepsilon [0 \leftrightarrow S_n].
$$

(11)

The algorithm $T_k$ will be provided by the following lemma, whose proof is postponed to the end of this section.

Lemma 7. There exists $c_0 > 0$ such that for any $k \in [0, n]$, there exists an algorithm $T_k$ determining $1_{0 \rightarrow S_n}$ with the property that

$$
\delta_x(T_k) \leq c_0 \mathbb{P}_p[x \leftrightarrow S_k].
$$

(12)
A standard estimate on Poisson Point Processes in $\mathbb{R}^d$ implies that

$$P_p[R_z \leftrightarrow S_k][\text{FKG}] \leq \sum_{z \in \mathbb{Z}^d} P_p[R_z \leftrightarrow S_k] \cdot P_p[E_z].$$

A simple geometric observation using the invariance under translation of Voronoi percolation implies that

$$\sum_{k=1}^{n} P_p[x \leftrightarrow S_k] \leq \sum_{k=1}^{n} \theta_d(x,S_z)(p) \leq 2 \sum_{k=0}^{n-1} \theta_k = 2S_n(p)$$

(above, $d(x,S_z)$ denotes the distance between $x$ and $S_z$) so that

$$\theta_n(p) \leq 2c_0c_1 \frac{S_n(p)}{n} \sum_{x \in \mathbb{Z}^d} \text{Inf}_x^c[0 \leftrightarrow S_n].$$

Lemma 6 implies (2) by letting $\varepsilon$ tend to 0. Overall, the proof of the theorem boils down to the proof of Lemma 7.

Proof of Lemma 7. Fix $0 \leq k \leq n$. We start by defining the algorithm.

For each $y \in \varepsilon \mathbb{Z}^d$, define an auxiliary algorithm \textit{Discover}(y) revealing the random variables $\eta_x$ around the point $y$ until the color of each point in $R^c_y$ is determined. We define this algorithm more formally inductively. Set a parameter $s = 0$. When $s = t$ (for some integer $t$), if the color of all the points inside the box $R^c_y$ are determined by all the revealed coordinates so far, the algorithm stops and returns the colors of points as the output. If not, the algorithm reveals the value of $\eta_x$ for $x \in \varepsilon \mathbb{Z}^d$ satisfying $|x - y| \leq t$ and sets $s = t + 1$. We write $x \in D(y)$ if $x$ is revealed by \textit{Discover}(y). We are now in a position to define the algorithm $T_k$.

Definition 8. The algorithm $T_k$ runs as follows: Initialize the algorithm by setting $X_0 = \emptyset$ and $Z_0 = S_k$. At every step $t$, assume that $X_t \subset \varepsilon \mathbb{Z}^d$ and $Z_t \subset \mathbb{R}^d$ have been constructed. If there is no $y \in \varepsilon \mathbb{Z}^d \setminus X_t$ with $R^c_y \cap Z_t \neq \emptyset$, the algorithm stops. If such a $y$ exists (if more than one exists, pick the smallest for an ordering of $\varepsilon \mathbb{Z}^d$ fixed before running the algorithm), then the algorithm does the following:

- run \textit{Discover}(y).
- Set $X_{t+1} = X_t \cup \{y\}$.
- Set $Z_{t+1} = Z_t \cup \{all \ the \ black \ points \ in \ \omega \cap R^c_y\}$.

Note that this algorithm discovers the union of all the black connected components of $S_k$ in $\omega \cap B_n$. In particular, it clearly determines $1_{0 \leadsto S_n}$. We now bound the revealment of $T_k$.

When $x \in \varepsilon \mathbb{Z}^d$ is revealed, there exist $y \in \varepsilon \mathbb{Z}^d$ and $y' \in R^c_y$ such that $x \in D(y)$ and $y' \leftrightarrow S_k$. Take $z \in \mathbb{Z}^d$ such that $y' \in R^1_z = z + [0,1)^d$. Define $E_z$ to be the decreasing event that $\eta_y$ does not intersect the Euclidean ball of radius $|x - z| - 3\sqrt{d}$ around $z$. The fact that $x \in D(y)$ implies that $E_z$ occurs, since otherwise the colors of the points in $R^1_z$ are independent of the colors of the points in $R^c_y$. We deduce that

$$\delta_x(T_k) \leq \sum_{z \in \mathbb{Z}^d} P_p[R_z^1 \leftrightarrow S_k \text{ and } E_z] \cdot P_p[R_z^1 \leftrightarrow S_k] \cdot P_p[E_z].$$

A standard estimate on Poisson Point Processes in $\mathbb{R}^d$ implies that

$$P_p[E_z] \leq \frac{1}{c_2} \exp(-c_2 |z - x|^d).$$

(14)
Furthermore, when \( z \in B_m \), by choosing a path \( y_0, \ldots, y_k = z \) in \( \mathbb{Z}^d \) with \( x \in R^1_{y_0} \) and \( k \leq c_3 |z-x| \), we deduce that
\[
\mathbb{P}_p[x \leftrightarrow S_k | R^1_z \leftrightarrow S_k] \geq \mathbb{P}_p[x \leftrightarrow R^1_z \text{ and } R^1_z \text{ all black} | R^1_z \leftrightarrow S_k] \geq \mathbb{P}_p[x \leftrightarrow R^1_z \text{ and } R^1_z \text{ all black}] \geq \prod_{i=1}^{k} \mathbb{P}_p[R^1_{y_i} \text{ all black}] \geq \exp(-c_4 |z-x|).
\] (In the last inequality we used that \( p \geq \delta \).) The bounds (14) and (15) imply that
\[
\delta_x(T_k) \leq \mathbb{P}_p[x \leftrightarrow S_k] \sum_{x \in \mathbb{Z}^d} \exp(c_4 |z-x|) \cdot \frac{1}{c_2} \exp(-c_2 |z-x|^d) \leq c_0 \mathbb{P}_p[x \leftrightarrow S_k],
\]
which concludes the proof (note the competition between a term growing exponentially fast in the distance and a term decaying exponential fast in the distance to the power \( d \)).

4 Proof of Corollary 2

Let \( A_n \) be the event that \( \Lambda_n = [-n, n]^2 \) is crossed by a continuous path of black points going from left to right. Since the complement of \( A_n \) is the event that there is a continuous path of white vertices from top to bottom, which has the same probability, we deduce that
\[
\mathbb{P}_{1/2}[A_n] = 1/2.
\] (16)

In particular, (16) implies that \( \mathbb{P}_{1/2}[B_1 \leftrightarrow S_n] \geq \frac{1}{2^n} \) so that
\[
\mathbb{P}_{1/2}[0 \leftrightarrow S_n] \geq \mathbb{P}_{1/2}[B_1 \leftrightarrow S_n] \mathbb{P}_{1/2}[B_1 \text{ all black}] \geq \frac{1}{n} \mathbb{P}_{1/2}[B_1 \text{ all black}].
\]
Since this quantity does not decay exponentially fast, we deduce that \( p_c \leq 1/2 \).

The square-root trick (using the FKG inequality) implies that for any \( n \geq k \geq 1 \),
\[
\mathbb{P}_{1/2}[B_k \text{ is connected in } \Lambda_n \text{ to the top of } \Lambda_n] \geq 1 - \mathbb{P}_{1/2}[B_k \leftrightarrow \infty]^{1/4}
\]
so that
\[
\mathbb{P}_{1/2}[B_k \text{ is connected in } \Lambda_n \text{ to the top and bottom of } \Lambda_n] \geq 1 - 2\mathbb{P}_{1/2}[B_k \leftrightarrow \infty]^{1/4}.
\]
Now, the uniqueness of the infinite connected component \[BR06b, p278\] when it exists implies that
\[
\liminf_{n \to \infty} \mathbb{P}_{1/2}[A_n] \geq 1 - 2\mathbb{P}_{1/2}[B_k \leftrightarrow \infty]^{1/4}.
\]
Assume for a moment that \( \theta(1/2) > 0 \). Letting \( k \) tend to infinity, we would deduce that \( \mathbb{P}_{1/2}[A_n] \) tends to 1 which would contradict (16). This implies \( \theta(1/2) = 0 \) and \( p_c \geq 1/2 \).

**Remark 9.** The argument that \( \theta(1/2) > 0 \) implies \( \mathbb{P}_{1/2}[A_n] \) tends to 1 could have been replaced by Zhang’s argument which is adopted to the setting of Voronoi percolation in \[Zva96\].

**Remark 10.** Since the trace on \( \mathbb{R}^2 \) of Voronoi percolation on \( \mathbb{R}^d \) also has the property of crossing squares with probability \( 1/2 \) when the parameter \( p \) is equal to \( 1/2 \), the previous reasoning readily implies that \( p_c(d) \leq 1/2 \).

5 Proof of Lemma 3

Define \( \beta_1 := \inf \{ \beta : \limsup_{n \to \infty} \frac{\log \Sigma_n(\beta)}{\log n} \geq 1 \} \).
Assume $\beta < \beta_1$. Fix $\delta > 0$ and set $\beta' = \beta - \delta$ and $\beta'' = \beta - 2\delta$. We will prove that there is exponential decay at $\beta''$ in two steps.

First, there exists an integer $N$ and $\alpha > 0$ such that $\Sigma_n(\beta) \leq n^{1-\alpha}$ for all $n \geq N$. For such an integer $n$, integrating $f_n' \geq n^\alpha f_n$ between $\beta'$ and $\beta$ – this differential inequality follows from (3), the monotonicity of the functions $f_n$ (and therefore $\Sigma_n$) and the previous bound on $\Sigma_n(\beta)$ – implies that

$$f_n(\beta') \leq M \exp(-\delta n^\alpha), \quad \forall n \geq N.$$ 

Second, this implies that there exists $\Sigma < \infty$ such that $\Sigma_n(\beta') \leq \Sigma$ for all $n$. Integrating $f_n' \geq n^\alpha f_n$ for all $n$ between $\beta''$ and $\beta'$ – this differential inequality is again due to (3), the monotonicity of $\Sigma_n$, and the bound on $\Sigma_n(\beta')$ – leads to

$$f_n(\beta'') \leq M \exp(-\delta \Sigma n), \quad \forall n \geq 0.$$ 

Assume $\beta > \beta_1$. For $n \geq 1$, define the function $T_n := \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i}{\Sigma_i}$. Differentiating $T_n$ with respect to $\beta$ and using (3), we obtain

$$T'_n = \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i'}{\Sigma_i} \geq \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i}{\Sigma_i} \geq \frac{\log \Sigma_{n+1} - \log \Sigma_1}{\log n},$$

where in the last inequality we used that for every $i \geq 1$,

$$\frac{f_i}{\Sigma_i} \geq \int_{\Sigma_i}^{\Sigma_{i+1}} \frac{dt}{t} = \log \Sigma_{i+1} - \log \Sigma_i.$$

For $\beta' \in (\beta_1, \beta)$, using that $\Sigma_{n+1}$ is increasing in $\beta$ and integrating the previous differential inequality between $\beta'$ and $\beta$ gives

$$T_n(\beta) - T_n(\beta') \geq (\beta - \beta') \frac{\log \Sigma_n(\beta') - \log M}{\log n}.$$

Hence, the fact that $T_n(\beta)$ converges to $f(\beta)$ as $n$ tends to infinity implies

$$f(\beta) - f(\beta') \geq (\beta - \beta') \left( \limsup_{n \to \infty} \frac{\log \Sigma_n(\beta')}{\log n} \right) \geq \beta - \beta'.$$

Letting $\beta'$ tend to $\beta_1$ from above, we obtain $f(\beta) \geq \beta - \beta_1$.

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