On the double random current nesting field

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Abstract

We relate the planar random current representation introduced by Griffiths, Hurst and Sherman to the dimer model. More precisely, we provide a measure-preserving map between double random currents (obtained as the sum of two independent random currents) on a planar graph and dimers on an associated bipartite graph. We also construct a nesting field for the double random current, which, under this map, corresponds to the height function of the dimer model. As applications, we provide an alternative derivation of some of the bozonization rules obtained recently by Dubédat, and show that the spontaneous magnetization of the Ising model on a planar biperiodic graph vanishes at criticality.

1 Introduction

1.1 Random currents and dimers

The Ising model is a random configuration of ± 1 spins. In this article, we think of the spins as living on the faces of a planar graph $G = (V, E)$ with vertex set V and edge set E. In [21], Peierls used the so-called *low-temperature representation* of the model to show the existence of an order/disorder phase transition in the Ising model on $\mathbb{Z}^2$. In this representation, configurations of spins $+1$ or -1 assigned to the faces of G are seen as contour configurations on G. More precisely, for any $B \subset V$, write $\mathcal{E}^B$ for the collection of sets of edges $\omega \subset E$ such that the graph (V, ω) has odd degrees at B and even degrees everywhere else. A connected component of $\omega \in \mathcal{E}^B$ is called a *contour*, and ω itself is called a *contour configuration*. Now, each spin configuration on faces of G is naturally associated with the collection ω of edges $e \in E$ bordering two faces with different spins. Automatically, ω belongs to $\mathcal{E}^\emptyset$. Conversely, every element ω of $\mathcal{E}^\emptyset$ is associated with exactly two spin configurations, one with spin $+1$ on the unbounded face, and one with spin -1.

The low-temperature representation is only one among many classical representations of the Ising model. A few years after Peierls, van der Waerden [27] introduced the *high-temperature expansion*, which was also fruitfully used to study the Ising model on arbitrary graphs. In [13], Griffiths, Hurst and Sherman proposed to expand the partition function (or more complicated weighted sums) of the Ising model into a power series in the inverse

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temperature and expressed it in terms of integer-valued functions on the edges of G. This new method, called the *random current representation*, is particularly useful when studying truncated spin configurations and has since then been a central tool in the study of the Ising model.

More formally, a *current* on G with *sources* B is a set of edges $\omega \subseteq E$ partitioned into two distinguished subsets $\omega_{\text{odd}} \subseteq \omega$ and $\omega_{\text{even}} \subseteq \omega$, called *odd* and *even edges* respectively, such that $\omega_{\text{odd}} \in \mathcal{E}^B$ and $\omega_{\text{even}} = \omega \setminus \omega_{\text{odd}}$. The set of all currents with sources B will be denoted by Ω^B. Let us also introduce the following probability measure on currents with sources B. For each $e \in E$, fix $x_e \in [0,1]$ and set $p_e = 1 - \sqrt{1 - x_e^2}$. The random current model with sources B is a probability measure on Ω^B given by

$$\mathbf{P}^B_{\text{curr}}(\omega) = \frac{1}{Z^B_{\text{curr}}} \prod_{e \in \omega_{\text{odd}}} x_e \prod_{e \in \omega_{\text{even}}} p_e \prod_{e \in E \setminus \omega} (1 - p_e), \qquad \text{for all } \omega \in \Omega^B, \tag{1.1}$$

where Z^B_{curr} is a normalizing constant, called the partition function, which makes the above measure into a probability measure.

Remark 1. Our definition of random currents is derived directly from the original one of Griffiths, Hurst and Sherman [13], where a current is a function assigning to each edge a natural number. It is left to the reader to check that our representation is obtained by forgetting the numerical value of the current but keeping the information about its parity and whether it is zero or not. More precisely, ω_{odd} is the set of edges with odd current, ω_{even} with strictly positive even current, and $E \setminus \omega$ with zero current.

The random current model has been successful in several ways. In the original article [13], it was used to derive correlation inequalities. In 1982, it was used by Aizenman [1] to prove triviality of the Ising model in dimension $d \geq 5$ and a few years later, Aizenman, Barsky and Fernandez proved that the phase transition is sharp [2] (see also [10] for an alternative proof). In recent years, the representation has been the object of a revived interest. It was used to study the continuity of the phase transition (see below) and it was also related to other objects. For instance, a new distributional relation between random currents, Bernoulli percolation and the FK-Ising model was discovered by Lupu and Werner [20]. We refer to [9] for more references.

In most applications, one considers pairs of independent current configurations. The reason comes from the combinatorial properties that this "double current" model enjoys. For two currents ω and ω', define the sum $\omega + \omega'$ to be the current with odd edges $\omega_{\text{odd}} \triangle \omega'_{\text{odd}}$ and even edges $(\omega \cup \omega') \setminus (\omega_{\text{odd}} \triangle \omega'_{\text{odd}})$, where $\triangle$ is the symmetric difference. This simply corresponds to addition mod 2 together with keeping track of whether the current is zero or not. Note that if $\omega \in \Omega^B$ and $\omega' \in \Omega^\emptyset$, then $\omega + \omega' \in \Omega^B$. Define the *double random current model* with sources B to be the probability measure on Ω^B induced by the sum of two independent random currents with sources B and $\emptyset$:

$$\mathbf{P}^B_{\text{d-curr}}(\omega) = \mathbf{P}^B_{\text{curr}} \otimes \mathbf{P}^\emptyset_{\text{curr}}(\{(\omega', \omega'') \in \Omega^B \times \Omega^\emptyset : \omega' + \omega'' = \omega\}), \quad \text{for all } \omega \in \Omega^B.$$

In [19], the double random current model was represented in terms of so-called alternating flows studied by Talaska [26] in relation to the totally positive Grassmannian [22]. In this paper, inspired by the connection of another classical model of statistical physics, namely the dimer model, and the totally positive Grassmannian [17,18,23], we relate the double random

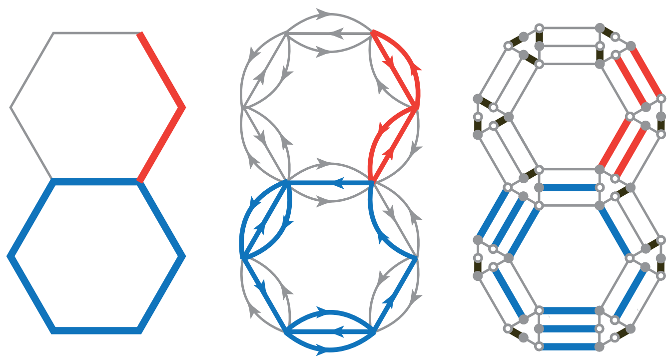

Figure 1: A double random current configuration, a corresponding alternating flow F, and one of the $2^{|V^c(F)|} = 4$ dimer covers associated with the alternating flow, where $|V^c(F)|$ is the number of isolated vertices of F.

current model model to the dimer model. Formally, the *dimer model* is a probability measure on *dimer covers* (also called *perfect matchings*) of a graph, i.e. sets of edges such that each vertex is incident on exactly one edge. We will now define a weighted graph G^d on which the dimer model will be in a correspondence with double random currents. To this end, we proceed in two steps. We first construct a directed graph $\vec{G}$, and then deduce G^d from it.

Let $\vec{G}$ be a directed graph with the same vertex set V as G, and with edge set $\vec{E}$ defined as follows: each $e \in E$ is replaced by three parallel directed edges with the same endpoints as e, and such that that the middle edge $\vec{e}_m$ has the opposite orientation to the two side edges $\vec{e}_{s1}$ and $\vec{e}_{s2}$, see Fig. 2. The middle edge can be oriented arbitrarily, and it is assigned weight $x_{\vec{e}_m} = 2x_e/(1-x_e^2)$, whereas the side edges get weights $x_{\vec{e}_{s1}} = x_{\vec{e}_{s2}} = x_e$.

The graph G^d is constructed from $\vec{G}$ as follows (Fig. 2). For a vertex z, let $r(z)$ be the number of pairs of consecutive edges in $\vec{E}$ around z with the same orientation, and let $\deg(z)$ be the degree of z. Replace each z with a cycle of $3\deg(z) - r(z)$ edges, called *short edges*. By construction, the length of the cycle is even, and hence its vertices can be colored black and white in an alternating way. Now, add *long edges* corresponding to the edges of $\vec{G}$. We do it in such a way that if (z, w) is a directed edge of $\vec{G}$, then the corresponding edge in G^d connects a white vertex in the cycle of z with a black vertex in the cycle of w, and moreover, the cyclic order of edges around each cycle in G^d matches the one in $\vec{G}$. The resulting graph G^d is therefore *bipartite*. We finish the construction by assigning weights. The long edges inherit their weights from their counterparts in $\vec{G}$, and short edges get weight 1.

Let $\mathcal{M}^\emptyset$ be the set of dimer covers of G^d. Define the dimer model probability measure with $\emptyset$ boundary conditions by

$$\mathbf{P}^\emptyset_{\mathrm{dim}}(M) = \frac{1}{Z^\emptyset_{\mathrm{dim}}} \prod_{e \in \mathcal{M}^\emptyset} x_e, \qquad \text{for all } M \in \mathcal{M}^\emptyset. \tag{1.2}$$

Let us now describe a mapping π from the dimer model to current configurations. Consider a dimer model M, and set $\pi(M)$ to be the current configuration $\omega \in \Omega^\emptyset$ defined as follows: an edge e of G will be in ω_{odd} (resp. ω_{even} and $E \setminus \omega$) if there is 1 or 3 dimers (resp. 2

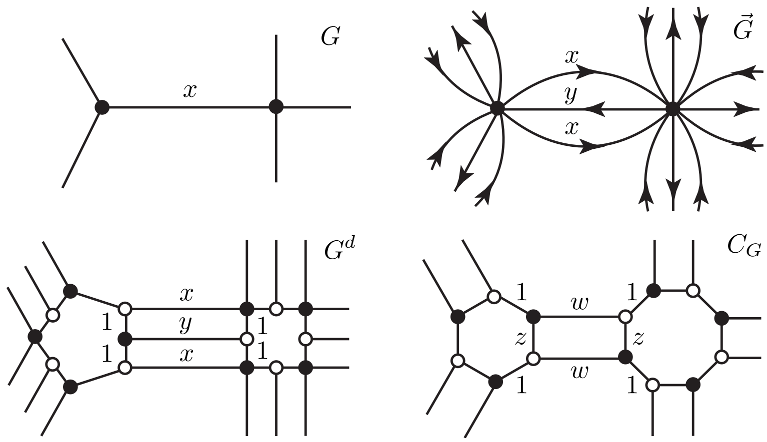

Figure 2: An example of the local structure of G, $\vec{G}$, G^d and C_G. The weights satisfy, $y = 2x/(1-x^2)$, $w = 2x/(1+x^2)$, $z = (1-x^2)/(1+x^2)$.

and 0) covering the three edges of G^d associated with e. Let $\pi_* \mathbf{P}^{\emptyset}_{\text{dim}}$ be the pushforward measure on $\Omega^{\emptyset}$. The main result of this paper is the following.

Theorem 1.1. *For any finite simple planar graph G, we have $\pi_* \mathbf{P}^{\emptyset}_{\text{dim}} = \mathbf{P}^{\emptyset}_{\text{d-curr}}$.*

Remark 2. The theorem can be extended to graphs that are properly embedded in an orientable surface.

1.2 The nesting field of a double random current

One of the main application of Theorem 1.1 is the study of the so-called *nesting field*. The idea behind introducing the nesting field is the interpretation of the contours of a current as level lines of a random surface whose discretization is an integer-valued function defined on the faces of G. The change in height of the discretized surface when crossing a contour is either $+1$ or -1, and for two contours belonging to different clusters, the respective height changes are independent.

For a current ω, a connected component of the graph (V, ω) will be called a *cluster*. In particular, each contour C of ω_{odd} (also called a contour of ω) is contained in a unique cluster of ω, and each cluster $\mathscr{C}$ of ω gives rise to a contour configuration $\mathscr{C} \cap \omega_{\text{odd}}$. Call a cluster $\mathscr{C}$ *odd* around a face u if the spin configuration associated via the low-temperature expansion with the contour configuration $\mathscr{C} \cap \omega_{\text{odd}}$ assigns spin -1 to u if the exterior face has spin $+1$.

Let $(\xi_{\mathscr{C}})$ be a family (indexed by clusters of ω) of iid random variables equal to $+1$ or -1 with probability 1/2. The *nesting field* at u is defined by

$$\mathcal{S}_u = \sum_{\mathscr{C} \text{ odd around } u} \xi_{\mathscr{C}},$$

where the sum is taken over all clusters that are odd around u.

One of the main features of Theorem 1.1 is that it enables to connect the nesting field of a random current ω drawn from the double random current measure to the height function associated with dimer covers of G^d. While the latter notion is classical, we still take a moment to recall it here. In the whole article, a *path* is a sequence of neighboring faces.

To each dimer cover M on G^d, we associate a 1-*form* f_M (i.e. a function defined on directed edges which is antisymmetric under changing orientation) satisfying $f_M((z,w)) = -f_M((w,z)) = 1$ if $\{z,w\} \in M$ and z is white, and $f_M((z,w)) = 0$ otherwise. From now on, we fix a *reference 1-form* f_0 given by $f_0((z,w)) = -f_0((w,z)) = 1/2$ if $\{z,w\}$ is a short edge and z is white, and $f_M((z,w)) = 0$ otherwise.

The *height function* $h = h_M$ of a perfect matching M is defined on the faces of G^d and is given by

(i) $h(u_0) = 0$ for the unbounded face u_0,

(ii) for every other face u, choose a path γ connecting u_0 and u, and define $h(u)$ to be the total flux of $f_M - f_0$ through γ, i.e., the sum of values of $f_M - f_0$ over the edges crossing γ from left to right.

Note that this is independent of the choice of γ since $f_M - f_0$ is a divergence-free flow.

Note that both the faces and vertices of G are embedded naturally in the faces of G^d.

Theorem 1.2. *The law of h under $\mathbf{P}^{\emptyset}_{\text{dim}}$ restricted to the faces of G is the same as the law of the nesting field $\mathcal{S}$ under $\mathbf{P}^{\emptyset}_{\text{d-curr}}$.*

Remark 3. Again, the theorem can be extended to graphs G that are properly embedded in the torus. In this case, the total increment of the nesting field on G between two faces u and v, as it is the case for the dimer height function on G^d, is defined only up to homotopy of the path γ connecting u and v along which the divergence free flows is summed up. We denote these increments by $\mathcal{S}_\gamma$ and h_γ respectively, and conclude that $\mathcal{S}_\gamma$ drawn according to $\mathbf{P}^{\emptyset}_{\text{d-curr},G}$ has the same distribution as h_γ drawn according to $\mathbf{P}^{\emptyset}_{\text{dim},G^d}$. Also, after fixing γ, the increment $\mathcal{S}_\gamma$ is equal to the sum of the ± 1 variables $\xi_{\mathscr{C}}$ for the clusters $\mathscr{C}$ that are *odd with respect to* γ, meaning that the contour configuration $\mathscr{C} \cap \omega_{\text{odd}}$ crosses an odd number of edges of γ.

Consider an infinite *biperiodic* (i.e. invariant under the action of a $\mathbb{Z}^2$-isomorphic lattice) planar graph $\mathbb{G}$. The graph $\mathbb{G}$ is assumed to be *nondegenerate*, in the sense that the complement of the edges is the union of topological disks (in other words, the faces are topological disks). Then, the dimer graph $\mathbb{G}^d$ constructed as in the finite case, is biperiodic and bipartite. The height function of dimers on biperiodic bipartite graphs has been studied in detail, for instance in [16]. Kenyon, Okounkov and Sheffield identified three possible behaviors depending on the phase: gaseous, liquid or frozen, in which the associated dimer model lies. In particular, the height function of dimers in the liquid phase, which is specified by the property that the characteristic polynomial has zeroes on the torus $\mathbb{T}^2$, has unbounded fluctuations. Let $\mathbb{G}_n = \mathbb{G}/(n\mathbb{Z} \oplus n\mathbb{Z})$. The relation between the nesting field and the height function of dimers can be hence combined with Theorem 4.5 of [16] to give the following.

Corollary 1.3. *Assume that the characteristic polynomial of the dimer model on $\mathbb{G}^d$ has a real zero on the torus $\mathbb{T}^2$, then*

$$\lim_{n\to\infty} \mathbf{E}^{\emptyset}_{\text{d-curr},\mathbb{G}_n}[\mathcal{S}_\gamma^2] = \tfrac{1}{\pi} \log[|\phi(u) - \phi(v)|] + o(\log[|\phi(u) - \phi(v)|]),$$

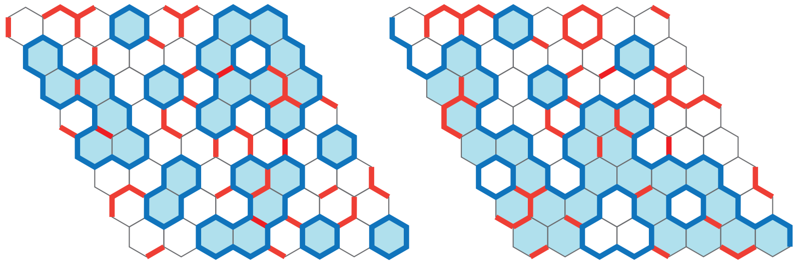

Figure 3: Two random current configurations on a piece of the hexagonal lattice with $\emptyset$, and $\{a,b\}$-boundary conditions from left to right respectively, where a is the top-leftmost and b the bottom-rightmost vertex. The odd edges are drawn in blue and the even edges in red. The colored faces are assigned spin -1 in the contour (low-temperature) expansion of Ising model with $+$ and Dobrushin boundary conditions respectively.

where the limit is taken for a fixed path γ connecting u and v, $\mathbf{E}^{\emptyset}_{\text{d-curr},\mathbb{G}_n}$ is the expectation with respect to $\mathbf{P}^{\emptyset}_{\text{d-curr}}$ on $\mathbb{G}_n$, and ϕ is a linear bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$.

We will not use the specific form of ϕ, but let us say that it is expressed in terms of the characteristic polynomial.

1.3 Application 1: Bozonization rules for the Ising model

For a finite planar graph $G=(V,E)$, define the set Σ_G of configurations σ assigning to each vertex $u\in V$ a spin σ_u, equal to $+1$ or -1. The distribution of the Ising model with free boundary conditions on G at inverse temperature β and with coupling constants $(J_e)_{e\in E}$ is defined on Σ_G by

$$\mu^{\mathrm{f}}_{G,\beta}(\sigma)=\frac{1}{Z_{\text{Ising}}}\exp\Big(-\beta\mathbf{H}_G(\sigma)\Big)\qquad\text{for all }\sigma\in\Sigma_G,$$

where $\mathbf{H}_G=-\sum_{\{u,v\}\in E}J_{\{u,v\}}\,\sigma_u\sigma_v$ is the Hamiltonian of the model.

By construction, the Ising model is related to the double random current on G with parameters $x_e=\tanh(\beta J_e)$ and hence, Theorem 1.1 gives a connection between the Ising model and dimers on a bipartite graph. It is known since [11] that the Ising model on a graph G is related to a dimer model on a modified graph, called the Fisher graph of G. This connection enables to express the partition function of the former model in terms of the partition function of the later, which is more amenable to computations. The Fisher graph of G is not bipartite, a fact which renders the study of the dimer model on it more difficult.

Recently, Dubédat [7] (see also [5]) proved that the Ising model can be related to a dimer model on a bipartite graph C_G where each edge of G is replaced by a quadrilateral and each vertex of degree d by a $2d$-gon face, see Fig. 2. The dimer model defined in this article on G^d can in fact be mapped to the dimer model on C_G with weights as in Fig. 2 via an explicit sequence of vertex splittings and urban renewals (operations which partially preserve the distribution of dimers, and in particular, the height function, see Remark 4). This means that Dubédat's mapping and our mapping are two facets of the same relation.

In [7], Dubédat derived powerful bozonization rules expressing the square of averages of order and disorder variables in terms of averages of certain observables of the height function of a dimer model. Here, we provide an alternative proof of some of these relations (Lemma 3 of [7]). Before stating the result, we define the notion of a disorder variable. A *disorder line* ℓ is a continuous curve drawn in the plane in such a way that it avoids V and crosses E finitely many times. The *disorder variable* μ_ℓ associated with ℓ corresponds to the change of the Hamiltonian flipping the coupling J_e to $-J_e$ for edges $e \in E$ which are traversed an odd number of times by ℓ. Correspondingly, the correlation function involving a collection of disorder variables $(\mu_{\ell_j})_{1\le j\le n}$ and a function $F : \Sigma_G \to \mathbb{C}$ is defined by

$$\mu^{\mathrm{f}}_{G,\beta}\Big[F\prod_{j=1}^{n}\mu_{\ell_j}\Big] := \mu^{\mathrm{f}}_{G,\beta}\Big[F\exp(-\beta\sum_{e\in E_{\mathrm{odd}}}2J_e\sigma_x\sigma_y)\Big], \tag{1.3}$$

where E_{odd} is the set of edges $e \in E$ crossed an odd number of times by $\cup_{j=1}^{n}\ell_j$. Recall that the faces and vertices of G are embedded naturally in the faces of G^d, and hence, with a slight abuse of notation, we can speak of the height function evaluated at a vertex or a face of G.

Theorem 1.4. *Consider a finite planar graph G, and the dimer model on G^d with the associated weights. For any vertices $x_1, \dots, x_k$ and any disordered lines $\ell_1, \dots, \ell_n$ starting from the unbounded face u_0 and ending in the faces $u_1, \dots, u_n$ respectively, we have that*

$$\mu^{\mathrm{f}}_{G,\beta}\Big[\prod_{i=1}^{k}\sigma_{x_i}\times\prod_{j=1}^{n}\mu_{\ell_j}\Big]^2 = \mathbf{E}^{\emptyset}_{\mathrm{dim}}\Big[\prod_{i=1}^{k}\sin(\pi h_{x_i})\times\prod_{j=1}^{n}\cos(\pi h_{u_j})\Big]. \tag{1.4}$$

Note that for a vertex x and a face u, $\sin(\pi h_x) = (-1)^{h_x-1/2}$ and $\cos(\pi h_u) = (-1)^{h_u}$. Also, the fact that the disorder lines are starting on the unbounded face u_0 is a convenient convention to state the result elegantly in terms of the notation introduced in the previous section. The theorem can be extended to graphs G properly embedded in the torus with appropriate modifications.

1.4 Application 2: Continuity of the phase transition for the Ising model on biperiodic planar graphs

For a finite subgraph $G = (V, E)$ of a nondegenerate biperiodic graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, define the set Σ^+_G of configurations σ assigning to each vertex of $\mathbb{G}$ a spin σ_u, equal to $+1$ or -1, with the additional constraint that any vertex of $\mathbb{V} \setminus V$ receives a spin $+1$. The distribution of the Ising model with $+$ boundary conditions on G at inverse-temperature β and with coupling constants $(J_e)_{e\in E}$ is defined on Σ^+_G by

$$\mu^{+}_{G,\beta}(\sigma) = \frac{1}{Z_{\mathrm{Ising}}}\exp\Big(-\beta\mathbf{H}^+_G(\sigma)\Big) \qquad \text{for all } \sigma \in \Sigma_G,$$

with $\mathbf{H}^+_G := -\sum_{\{u,v\}} J_{\{u,v\}}\,\sigma_u\sigma_v$, where the sum is over edges $\{u, v\}$ intersecting V. A measure $\mu^+_{\mathbb{G},\beta}$ can be defined on $\mathbb{G}$ by taking the weak limit of the measures $\mu^+_{G,\beta}$. The model undergoes an order/disorder phase transition on $\mathbb{G}$ at a critical inverse-temperature

$\beta_c = \beta_c(\mathbb{G})$ characterized by the property that $\mu^+_{\mathbb{G},\beta}[\sigma_u] = 0$ if $\beta < \beta_c$ and $\mu^+_{\mathbb{G},\beta}[\sigma_u] > 0$ if $\beta > \beta_c$, where u is an arbitrary vertex of $\mathbb{G}$.

In [6], the critical parameter β_c of the Ising model was proved to correspond to the only value of β for which the dimer model introduced in [7] on $C_{\mathbb{G}}$ (and therefore the one defined here on $\mathbb{G}^d$) is in the liquid phase. Here, we combined this result with the information above to prove the following statement.

Theorem 1.5. *Let $\mathbb{G}$ be a nondegenerate infinite biperiodic planar graph, then*

$$\mu^+_{\mathbb{G},\beta_c}[\sigma_u] = 0.$$

For the square lattice, the result goes back to the exact computation of Yang [28]. In higher dimension, the fact that $\mu^+_{\mathbb{G},\beta_c}[\sigma_u] = 0$ is known for the nearest neighbor Ising model on $\mathbb{G} = \mathbb{Z}^d$ [3, 4]. On trees, the result was proved in [14]. Recently, Raoufi [25] showed that amenable groups with exponential growth undergo a continuous phase transition. To the best of our knowledge, a proof which is valid for any infinite biperiodic planar graph was not available until now.

A byproduct of the proof is the following result about non-percolation of spins.

Corollary 1.6. *Let $\mathbb{G}$ be a nondegenerate infinite biperiodic planar graph, then the $\mu^+_{\mathbb{G},\beta_c}$-probability that there exists an infinite cluster of pluses or minuses is zero.*

1.5 Application 3: a scaling limit result

Lastly, we provide one example to illustrate a general principle. In recent years, several observables of the dimer and double dimer model have been computed in the scaling limit (see e.g. the work of Kenyon [15]). In some cases, this translates into scaling limit results for certain observables of the double random current. For instance, in Section 5, we present an exact computation of the scaling limit of a natural observable of the double random current on a cylinder.

1.6 Extension to Dobrushin boundary conditions

Much of what has been described above can be extended to cover the case of the Ising model with Dobrushin boundary conditions. Consider two vertices a and b on the exterior face of G. Configurations in $\mathcal{E}^{\{a,b\}}$ correspond to (the so-called Dobrushin) spin configurations where the external face is split into two faces of opposite spins by adding an additional edge joining a and b. In particular, this construction implies that $\omega \in \mathcal{E}^{\{a,b\}}$ necessarily contains a contour connecting a and b.

The definition of the nesting field for a current with $\{a, b\}$-boundary conditions is almost the same with the exception that the variable $\xi_{\mathscr{C}_0}$ corresponding to the cluster $\mathscr{C}_0$ connecting a and b is set to 1. Moreover, the cluster $\mathscr{C}_0$ is called odd around u if its contours assign spin -1 to u in the model with Dobrushin boundary conditions with $+1$ spin on the (external) face adjacent to the clockwise boundary arc from a to b, and -1 spin on the face adjacent to the arc from b to a.

Consider an augmented graph $\vec{G}_{(a,b)}$ where an additional edge $e_{(a,b)}$ directed from b to a is added in the external face of $\vec{G}$ in such a way that the clockwise boundary arc of $\vec{G}$ from

a to b is bordering the unbounded face of $\vec{G}_{(a,b)}$. We define the graph $G^d_{(a,b)}$ out of $\vec{G}_{(a,b)}$ exactly as we defined G^d out of $\vec{G}$. Let $\mathcal{M}_{(a,b)}$ be the set of dimer covers of $G^d_{(a,b)}$ containing the edge (b,a). Also, introduce the height function h of M in $\mathcal{M}_{(a,b)}$ by choosing a reference 1-form corresponding to a matching that represents a current composed only of a path of odd edges that form the clockwise arc from b to a on the boundary of G.

Then, we have the following extension of Theorems 1.1 and 1.2.

Theorem 1.7. *For any finite, simple planar graph G,*

(i) $\pi_* \mathbf{P}^{\{a,b\}}_{\text{dim}} = \mathbf{P}^{\{a,b\}}_{\text{d-curr}}$,

(ii) the law of h under $\mathbf{P}^{(a,b)}_{\text{dim}}$ restricted to the faces of G is the same as the law of the nesting field $\mathcal{S}$ under $\mathbf{P}^{\{a,b\}}_{\text{d-curr}}$.

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2 Proofs of Theorems 1.1, 1.2 and 1.7

There will be no difference in working with $B = \emptyset$ or $B = \{a, b\}$. For this reason, we simply refer to B as being the set of sources. The proofs rely on the notion of *alternating flows* and their *height function*. For this reason, we define a probability measure on flows which will be later naturally related to the double random current measure and its nesting field. We should mention that the proofs of the theorems can be obtained by hand, meaning without using alternating flows. Nonetheless, we believe that alternative flows offer an elegant way of deriving the connection between dimers and double random currents.

A *sourceless alternating flow* F is a set of directed edges of $\vec{G}$ such that for each vertex v, the edges in F around v alternate between being oriented towards and away from v when going around v. In particular, the same number of edges enters and leaves v. For two vertices a and b on the outer face of $\vec{G}$, an *alternating flow with source a and sink b* is a sourceless alternating flow on $\vec{G}_{(a,b)}$ containing $e_{(a,b)}$ (note that, here, (a,b) is an oriented edge and should not be confused with $\{a,b\}$). Denote the set of sourceless alternating flows on $\vec{G}$ by $\mathcal{F}^{\emptyset}$, and the set of alternating flows with source a and sink b by $\mathcal{F}^{(a,b)}$.

Define a probability measure on alternating flows with $B = \emptyset$ or $B = (a,b)$ by

$$\mathbf{P}^B_{\text{a-flow}}(F) = \frac{1}{Z^B_{\text{a-flow}}} 2^{|V^c(F)|} \prod_{\vec{e} \in F} x_{\vec{e}}, \qquad \text{for all } F \in \mathcal{F}^B, \tag{2.1}$$

where $V^c(F)$ is the set of isolated vertices in the graph (V,F).

Define a map $\theta : \mathcal{F}^B \to \Omega^B$ as follows. For every $F \in \mathcal{F}^B$ and every $e \in E$, consider the number of corresponding directed edges $\vec{e}_m$, $\vec{e}_{s1}$, $\vec{e}_{s2}$ present in F. Let $\omega_{\text{odd}} \subset E$ be the set with one or three such present edges, and $\omega_{\text{even}} \subset E$ the set with exactly two such edges. Then, set $\theta(F) = \omega$. It follows from the definition of alternating flows that $\omega = \omega_{\text{odd}} \cup \omega_{\text{even}}$ is a current with sources B. Denote by $\theta_* \mathbf{P}^B_{\text{a-flow}}$ the pushforward measure on Ω^B. The following result was previously obtained in [19].

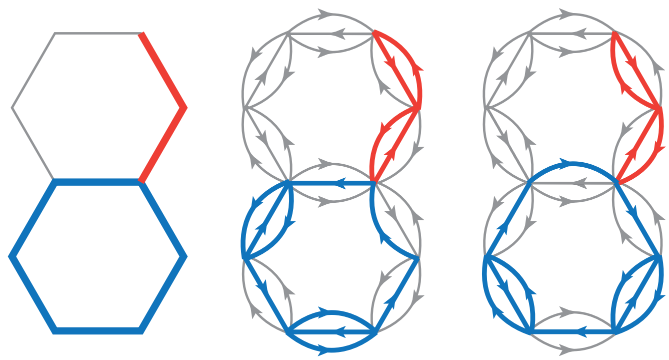

Figure 4: A double random current configuration with odd edges marked blue and even edges marked red, and two out of 2×27 corresponding alternating flows. Note the opposite orientations of the outer boundaries of the flows which account for the factor 2. Every second odd edge of the cycle can be represented in exactly 3 ways which yields the factor 27.

Theorem 2.1 ([19])**.** *For any finite, simple planar graph G, we have that $\theta_* \mathbf{P}^B_{\text{a-flow}} = \mathbf{P}^B_{\text{d-curr}}$.*

Proof. Since the theorem is a special case of [19, Thm 4.1], we only outline the proof here for completeness.

Let $|\omega|$ be the number of edges in ω and $k(\omega)$ be the total number of clusters of the graph (V, ω) (note that isolated vertices count as a cluster). Using that the number of even subgraphs of the graph (V, ω) is equal to $2^{|\omega|-|V|+k(\omega)}$, it can be checked that the double random current measure takes the following form (see [19, Thm 3.2] for a detailed proof):

$$\mathbf{P}^B_{\text{d-curr}}(\omega) = \frac{1}{Z^B_{\text{d-curr}}} 2^{|\omega|+k(\omega)} \prod_{e \in \omega_{\text{odd}}} x_e \prod_{e \in \omega_{\text{even}}} x_e^2 \prod_{e \in E \setminus \omega} (1 - x_e^2). \tag{2.2}$$

Now, fix ω and observe that $\theta^{-1}(\omega)$ is simple to understand (see Fig. 4). Once given the orientations of the boundaries of each one of the non-trivial (meaning not reduced to an isolated vertex) clusters in F, not much freedom remains for the edges. More precisely, the even edges of F necessarily contain the edge e_m, and the second edge is determined by the alternating condition. An odd edge e can be of two types: either F contains only e_m, or it is of a second type, where it contains either e_{s1} only, e_{s2} only, or the three edges e_{s1}, e_m and e_{s2}. Again, which type it is is determined by the alternating condition.

Observe that the sum over all configurations in $\theta^{-1}(\omega)$ with prescribed orientations of the boundaries of the non-trivial clusters is equal to

$$2^{|\omega|} \prod_{e \in \omega_{\text{odd}}} \frac{x_e}{1 - x_e^2} \prod_{e \in \omega_{\text{even}}} \frac{x_e^2}{1 - x_e^2}.$$

Indeed, each even edge contributes the multiplicative weight $x_{e_m} x_{e_{si}} = 2\frac{x_e^2}{1-x_e^2}$ (with i equal to 1 or 2), each odd edge of the first type $x_{e_m} = 2x_e/(1 - x_e^2)$, each odd edge of the second type $x_{e_{s1}} + x_{e_{s2}} + x_{e_{s1}} x_{e_m} x_{e_{s2}} = 2x_e/(1 - x_e)^2$ (we take into account that there are three possibilities for the alternating flow at this edge). Finally, each edge not in ω does not contribute any multiplicative weight.

The result follows from the fact that the outer boundary of each non-trivial cluster can be oriented in two possible ways, hence the weight $2^{k(\omega)-|V^c(F)|}$. □

We now describe a straightforward measure preserving mapping from the dimer model to alternating flows. To each matching $M \in \mathcal{M}^B$, we associate a flow $\eta(M) \in \mathcal{F}^B$ by replacing each long edge in M by the corresponding directed edge in $\vec{G}$. One can see that this always produces an alternating flow. Indeed, assuming otherwise, there would be two consecutive edges in $\eta(M)$ of the same orientation, and therefore, the path of odd length connecting them in a cycle would have a dimer cover, which is a contradiction. Let $\eta_* \mathbf{P}^B_{\text{dim}}$ be the pushforward measure on $\mathcal{F}^B$ under the map η.

Theorem 2.2. *For any finite, simple planar graph G, we have that $\eta_* \mathbf{P}^B_{\text{dim}} = \mathbf{P}^B_{\text{a-flow}}$.*

Proof. Comparing (2.1) with (1.2), and knowing that the long edges of G^d have the same weights as in $\vec{G}$, we only need to account for the factor $2^{|V^c(F)|}$ from the definition of the alternating flow measure. To this end, note that the only freedom in the dimer covers in $\eta^{-1}(F)$ is the way they match the short edges in the cycles corresponding to the isolated vertices of (V, F). Each such cycle has two matchings, and the matchings of different cycles are independent. This completes the proof. □

Proof of Theorems 1.1 and 1.7 (i). We define $\pi = \theta \circ \eta : \mathcal{M}^B \to \Omega^B$ to be the many-to-one map projecting dimer covers to currents (note that it is the mapping defined in the introduction). Let $\pi_* \mathbf{P}^B_{\text{dim}}$ be the pushforward measure on Ω^B. Combining the two previous theorems yields the corresponding statements of the introduction. □

We now turn to height functions. Let $h = h_F$ be the *height function* of a flow F defined on the faces of $\vec{G}$ (or $\vec{G}_{(a,b)}$ if we consider (a, b)-boundary conditions) in the following way:

(i) $h(u_0) = 0$ for the unbounded face u_0,

(ii) for every other face u, choose a path γ connecting u_0 and u, and define $h(u)$ to be total flux of F through γ, i.e., the number of edges in F crossing γ from left to right minus the number of edges crossing γ from right to left.

The obtained value is independent of the choice of γ, since at each $v \in V$, the same number of edges of h enters and leaves v (and so the total flux of F through any closed path of faces is zero).

Proof of Theorem 1.2 and 1.7(ii). It is clear that h_F is equal to the height function of the dimer cover $M = \eta(F)$. We therefore relate h_F to $\mathcal{S}(\omega)$, where $\omega = \theta(F)$.

Recall from the proof of Theorem 2.1 that for each cluster of a double random current, there are two opposite orientations of the boundary of the corresponding connected component of the associated alternating flows in $\theta^{-1}(\omega)$. Set $\xi_{\mathcal{C}}(F) = +1$ if F is oriented counterclockwise around the boundary of the cluster $\mathcal{C}$ of ω, and $\xi_{\mathcal{C}}(F) = -1$ otherwise. By the proof of Theorem 2.1, $(\omega, \xi(F))$ is in direct correspondence with F. Furthermore, by construction, h_F is equal to the nesting field $\mathcal{S}(\omega)$ obtained from the $\xi(F)$. The fact that for each cluster $\mathcal{C}$, the two opposite orientations carry the same weight implies that under the law of alternating flows, conditionally on ω, $\xi(F)$ is a iid family of random variables which are equal to $+1$ or -1 with probability $1/2$. This concludes the proof. $\square$

3 Proof of Theorem 1.4

The proof is based on classical properties of the double random current model combined with the properties of the mapping π. First, observing that changing J_e to $-J_e$ amounts to changing x_e to $-x_e$, and not changing p_e, the classical representation of spin-spin correlations in terms of the random current gives

$$\mu^{\mathrm{f}}_{G,\beta}\Big[\prod_{i=1}^{k}\sigma_{x_i}\times\prod_{j=1}^{n}\mu_{\ell_j}\Big]=\frac{1}{Z^{\emptyset}_{\mathrm{curr}}}\sum_{\omega\in\Omega^X}w(\omega)(-1)^{|\omega_{\mathrm{odd}}\cap E_{\mathrm{odd}}|},$$

where $X=\{x_1,\dots,x_k\}$, E_{odd} is the set of edges crossed an odd number of times by $\cup_{j=1}^{n}\ell_j$, and where $w(\omega)$ is the weight associated with a current ω through (1.1). The switching lemma for double currents [1, 8, 9] implies that

$$\mu^{\mathrm{f}}_{G,\beta}\Big[\prod_{i=1}^{k}\sigma_{x_i}\times\prod_{j=1}^{n}\mu_{\ell_j}\Big]^2=\mathbf{E}^{\emptyset}_{\text{d-curr}}[(-1)^{|\omega_{\mathrm{odd}}\cap E_{\mathrm{odd}}|}\mathbb{I}_{\omega\in\mathcal{F}_X}],$$

where $\mathcal{F}_X$ is the event that every cluster of ω intersects X an even number of times (the points in X are in general counted with multiplicity). To conclude the proof of the theorem, we therefore need to show that

$$\mathbf{E}^{\emptyset}_{\text{d-curr}}[(-1)^{|\omega_{\mathrm{odd}}\cap E_{\mathrm{odd}}|}\mathbb{I}_{\omega\in\mathcal{F}_X}]=\mathbf{E}^{\emptyset}_{\mathrm{dim}}\Big[\prod_{i=1}^{k}\sin(\pi h_{x_i})\times\prod_{j=1}^{n}\cos(\pi h_{u_j})\Big].\tag{3.1}$$

In order to see this, fix a current ω and denote its nesting field by $\mathcal{S}$. Observe first that for every dimer configuration $M\in\pi^{-1}(\omega)$, $h_u=\mathcal{S}_u$ on every face u of the graph and therefore, since all disorder lines start on the unbounded face,

$$\prod_{j=1}^{n}\cos(\pi h_{u_j})=\prod_{j=1}^{n}\cos(\pi\mathcal{S}_{u_j})=\prod_{j=1}^{n}(-1)^{|\omega_{\mathrm{odd}}\cap E_{\mathrm{odd}}|}.$$

(In particular it does not depend on M.)

Let us now treat the case of the product of sines in (3.1). The definition of the reference 1-form f_0 together with the structure of the graph G^d imply that h is constant on the vertices of every cluster of ω. In particular, if $|X\cap\mathcal{C}|$ is even for every cluster $\mathcal{C}$ of ω, the product of sines is equal to 1. To treat the case where $|X\cap\mathcal{C}|$ is odd for some $\mathcal{C}$, observe that while the height function of M is not determined by $\omega=\pi(M)$, it is determined by the alternating flow $F=\eta(M)$, except on isolated vertices, where it is obtained by adding $\pm\frac{1}{2}$ to the height function at neighboring faces, independently for each isolated vertex. Since the orientations of the clusters of F are chosen uniformly at random in the coupling introduced in the previous section (they are given by the $\xi_{\mathcal{C}}$), we conclude that

$$\mathbf{E}^{\emptyset}_{\mathrm{dim}}\Big[\prod_{i=1}^{k}\sin(\pi h_{x_i})\,\Big|\,\pi(M)=\omega\Big]=\mathbb{I}_{\omega\in\mathcal{F}_X}.$$

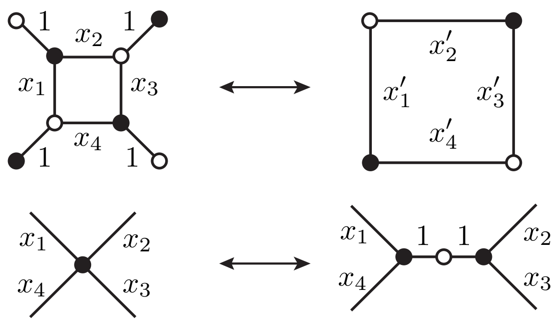

Figure 5: Urban renewal and vertex splitting are transformations of weighted graphs preserving the distribution of dimers and the height function outside the modified region. The weights in urban renewal satisfy $x'_1 = x_3/(x_1x_3 + x_2x_4)$, $x'_2 = x_4/(x_1x_3 + x_2x_4)$, $x'_3 = x_1/(x_1x_3 + x_2x_4)$, $x'_4 = x_2/(x_1x_3 + x_2x_4)$.

By combining the two displayed equations with Theorems 1.1 and 1.2, we deduce (3.1).

Remark 4. The relations obtained in Theorem 1.4 are the same as in Lemma 3 of [7]. Indeed, the dimer model on G^d is associated with the dimer model of [5, 7] as follows. Given an edge of G, select a quadrilateral face in G^d corresponding to the edge and (if necessary) split each vertex that the chosen quadrilateral shares with a quadrilateral corresponding to a different edge of G. In this way we find ourselves in the situation from the upper left panel in Fig. 5. After performing urban renewal (i.e. the transformation from Fig. 5) and collapsing the doubled edge, we are left with one quadrilateral as desired. One can check that the weights that we obtain match those from Fig. 2. We then repeat the procedure for every edge of G and the resulting graph is C_G.

Note that the height function on faces is not modified by vertex splitting and urban renewal. Nonetheless, there is indeed loss of information between the dimer model on G^d and the one on C_G, and we the former is more suitable for understanding double random currents.

4 Proof of Theorem 1.5

We will in fact work with the Ising model on the dual graph $\mathbb{G}^*$ obtained by putting a vertex in each face of $\mathbb{G}$, and edges between vertices corresponding to neighboring faces. As such, the Ising model below will be seen as a random assignment of spins to the faces of $\mathbb{G}$. While we use the notation $\mathbb{G}$ as in the introduction, the outcome of the proof will be Theorem 1.5 for $\mathbb{G}^*$. Since the dual graph of a nondegenerated biperiodic graph is itself non-degenerated and biperiodic, this is sufficient. The reason for working with the Ising model on $\mathbb{G}^*$ is that we will use the connection with the dimer on $\mathbb{G}$, and that this makes the study more coherent with other sections of the article.

Below, we restrict our attention to the Ising model on $\mathbb{G}^*$ at $\beta = \beta_c(\mathbb{G}^*)$ and drop β from the notation. Let $\mu^+_{\mathbb{G}^*}$ (resp. $\mu^-_{\mathbb{G}^*}$) be the infinite volume Ising measure on $\mathbb{G}^*$ with + (resp. −) boundary conditions, and for a face u of $\mathbb{G}$, let $\mathbf{C}_u(\sigma)$ be the minimum number of spin changes in σ over infinite self-avoiding paths starting from u. The architecture of the proof is the following:

Step 0 We introduce relevant auxiliary infinite volume measures.

Step 1 We show that $\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma)] = +\infty$.

Step 2 We prove that $\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma) = 0] = 0$.

Step 3 We deduce that $\mu^+_{\mathbb{G}^*}[\sigma_u] = 0$.

Remark 5. Note that Step 2 can be restated as follows: there is no infinite cluster of pluses or minuses $\mu^+_{\mathbb{G}^*}$-almost surely. As a byproduct, we obtain Corollary 1.6.

Step 1 is the major novelty of the proof. It relies on Theorem 1.2 and Corollary 1.3. Step 3 is directly extracted from [8, Prop. 4.1]. We refer to [12] for classical facts on the Ising model.

Let $\Lambda \approx \mathbb{Z} \oplus \mathbb{Z}$ be a group acting transitively on $\mathbb{G}$. Let $\mathbb{G}_n = \mathbb{G}/(n\mathbb{Z} \oplus n\mathbb{Z})$ be the toroidal graph of size $n \in \mathbb{N}$, and let $\mathbb{G}^d_n = \mathbb{G}^d/(n\mathbb{Z} \oplus n\mathbb{Z})$ be the bipartite toroidal dimer graph corresponding to $\mathbb{G}_n$. Below, we consider the random current, double random current and dimer models on $\mathbb{G}_n$ and $\mathbb{G}^d_n$ with n tending to infinity, and where the weights x_e on $\mathbb{G}_n$ are defined as follows: if e is the edge between the faces u and v, then $x_e := \exp[-2\beta(\mathbb{G}^*)J_{\{u,v\}}]$. In what follows, we add subscripts to the already introduced notation to mark the dependency of the probability measures on the underlying graph.

Step 0. Note that for topological reasons, some current configurations on $\mathbb{G}_n$ do not correspond to spin configurations on the faces of $\mathbb{G}_n$. To overcome this obstacle, we will resort to the construction of infinite volume measures for the different models, where planarity is recovered in the limit as n tends to infinity. There are several ways to proceed and we simply explain here the shortest one (this is not the most self-contained one).

By [16], $\mathbf{P}^{\emptyset}_{\text{dim},\mathbb{G}^d_n}$ converges weakly to a Λ-invariant measure $\mathbf{P}^{\emptyset}_{\text{dim},\mathbb{G}^d}$ on dimer covers of $\mathbb{G}^d$. Since the sourceless double random current on $\mathbb{G}_n$ is a local function of the dimer model on $\mathbb{G}^d_n$, we get that $\mathbf{P}^{\emptyset}_{\text{d-curr},\mathbb{G}_n}$ converges weakly to an infinite volume measure $\mathbf{P}^{\emptyset}_{\text{d-curr},\mathbb{G}}$ on sourceless currents on $\mathbb{G}$.

The measures $\mathbf{P}^{\emptyset}_{\text{curr},\mathbb{G}_n}$ also converge weakly to a measure $\mathbf{P}^{\emptyset}_{\text{curr},\mathbb{G}}$ on sourceless currents on $\mathbb{G}$. In order to see this, we go back to the original definition of single and double currents in terms of integer-valued functions. Since the integer value of the double current at an edge is obtained from the parity independently for any edge, the integer-valued double random current also converges. With this definition, the integer-valued double random current is simply the sum of two iid integer-valued single random currents, and therefore for any finite set D of edges, the characteristic function of the latter when restricted to D is the square-root of the characteristic function of the former. In particular, it converges for any fixed D. This implies the convergence of the single random current.

We now define a probability measure $\mu_{\mathbb{G}^*}$ on the space of ± 1 spin configurations on the faces of $\mathbb{G}$ by tossing a symmetric coin to decide the spin at a fixed face, and then using the odd part of a current ω drawn according to $\mathbf{P}^{\emptyset}_{\text{curr},\mathbb{G}}$ to define the interfaces between $+1$ and -1 spins. This is well defined since $\mathbb{G}$ is planar and the degree of ω_{odd} at every vertex of $\mathbb{G}$ is even almost surely. Note that $\mu_{\mathbb{G}^*}$ is Λ-invariant since the infinite-volume version of the single random currents inherits the invariance under the action of Λ from the dimer measure.

Using the domain Markov property of ω_{odd} under $\mathbf{P}^{\emptyset}_{\text{curr},\mathbb{G}_n}$, and the fact that a spin configuration under $\mu_{\mathbb{G}^*}$ carries the same information (up to a spin flip) as ω_{odd}, one can check that $\mu_{\mathbb{G}^*}$ satisfies the Dobrushin–Lanford–Ruelle conditions for an infinite volume Gibbs state of the Ising model with parameters β and $(J_e)_{e\in E}$.

A result of Raoufi [24] classifying Λ-invariant Gibbs measures for the Ising model, and the ± 1 symmetry of $\mu_{\mathbb{G}^*}$ readily yield

$$\mu_{\mathbb{G}^*} = \tfrac{1}{2}(\mu^+_{\mathbb{G}^*} + \mu^-_{\mathbb{G}^*}). \tag{4.1}$$

(Note that the result in [24] is stated for vertex transitive graphs, and it can be generalized to the quasi-transitive case which includes biperiodic graphs).

Step 1. Fix two faces u and v of $\mathbb{G}_n$, and a self-avoiding path γ connecting u and v. Recall that a cluster $\mathscr{C}$ of ω is odd with respect to γ if $\mathscr{C}\cap\omega_{\text{odd}}$ crosses an odd number of edges of γ. For $k = 1, 2, \ldots, \infty$, let $\mathbf{N}^k_\gamma(\omega)$ be the number of clusters of ω that are odd with respect to γ in the current configuration obtained by restricting ω to the set of edges at distance at most k from γ. The quantities $\mathbf{N}^k_\gamma$ are subadditive, i.e.

$$\mathbf{N}^k_\gamma(\omega+\omega') \le \mathbf{N}^k_\gamma(\omega) + \mathbf{N}^k_\gamma(\omega'). \tag{4.2}$$

We give a proof of this inequality that is independent of k so we may assume that $k = \infty$. Indeed, note that if $\mathscr{C}$ is a cluster of $\omega+\omega'$, then the parity of the number of edges in $\mathscr{C}\cap(\omega+\omega')_{\text{odd}}$ crossing γ is equal to the parity of the sum of the numbers of edges in $\mathscr{C}\cap\omega_{\text{odd}}$ and $\mathscr{C}\cap\omega'_{\text{odd}}$ crossing γ. Hence, if the former number is odd, exactly one of the latter numbers is odd, which means that either ω or ω' contain at least one cluster that is odd with respect to γ, and (4.2) is proved.

Note moreover that $\mathbf{N}^k_\gamma$ is decreasing in k since adding connections to the current cannot result in a larger number of odd clusters. By (4.2) and Remark 3, we can therefore write for all k and n,

$$\mathbf{E}^{\emptyset}_{\text{curr},\mathbb{G}_n}[N^k_\gamma] \ge \tfrac{1}{2}\mathbf{E}^{\emptyset}_{\text{d-curr},\mathbb{G}_n}[N^k_\gamma] \ge \tfrac{1}{2}\mathbf{E}^{\emptyset}_{\text{d-curr},\mathbb{G}_n}[N^\infty_\gamma] = \tfrac{1}{2}\mathbf{E}^{\emptyset}_{\text{d-curr},\mathbb{G}_n}[\mathcal{S}^2_\gamma] = \tfrac{1}{2}\mathbf{E}^{\emptyset}_{\text{dim},\mathbb{G}^d_n}[h^2_\gamma], \tag{4.3}$$

where $\mathcal{S}_\gamma$ and h_γ are the increments along γ of the nesting field and the dimer height function respectively. The first equality follows from the fact that conditionally on ω, $\mathcal{S}_\gamma$ is the sum of $\mathbf{N}_\gamma(\omega)$ iid centered random variables of variance 1. As both N^k_γ and h_γ are local functions, taking first the weak limit in n and then the decreasing limit in k on both sides of (4.3), we get

$$\mathbf{E}^{\emptyset}_{\text{curr},\mathbb{G}}[N^\infty_\gamma] \ge \tfrac{1}{2}\mathbf{E}^{\emptyset}_{\text{dim},\mathbb{G}^d}[h^2_\gamma] = \tfrac{1}{2\pi}\log[|\phi(u)-\phi(v)|] + o(\log[|\phi(u)-\phi(v)|]), \tag{4.4}$$

where in the equality, we used Corollary 1.3 together with the fact that at $\beta = \beta_c$, the relation between dimers on $C_{\mathbb{G}}$ and $\mathbb{G}^d$ implies by [6] that the characteristic polynomial of the dimer model on $\mathbb{G}^d$ has a real zero on the torus $\mathbb{T}^2$. (Recall that ϕ is a linear transformation.)

For a spin configuration σ on the faces of $\mathbb{G}$, let $\mathbf{C}_{uv}(\sigma)$ be the minimal number of sign changes in σ along all self-avoiding paths from u to v. It follows that every such path γ should contain at least one spin change per odd cluster, and therefore $\mathbf{C}_{uv}(\sigma) \ge \mathbf{N}^\infty_\gamma(\omega)$, where σ and ω are related by the low-temperature expansion (hence the choice of x_e at the beginning of the proof). We deduce that

$$\mu_{\mathbb{G}^*}[\mathbf{C}_{uv}(\sigma)] \geq \mathbf{E}^{\emptyset}_{\text{curr},\mathbb{G}}[\mathbf{N}^{\infty}_{\gamma}(\omega)], \tag{4.5}$$

and together with (4.4) this gives us

$$\mu_{\mathbb{G}^*}[\mathbf{C}_u(\sigma)] + \mu_{\mathbb{G}^*}[\mathbf{C}_v(\sigma)] \geq \mu_{\mathbb{G}^*}[\mathbf{C}_{uv}(\sigma)] \geq \tfrac{1}{2\pi}\log[|\phi(u)-\phi(v)|] + o(\log[|\phi(u)-\phi(v)|]).$$

Letting $|u-v|$ tend to infinity, and using the invariance of $\mu_{\mathbb{G}^*}$ under the action of Λ, we find that $\mu_{\mathbb{G}^*}[\mathbf{C}_u(\sigma)] = +\infty$ for every face u. To complete this step, it only remains to transfer this estimate to $\mu^+_{\mathbb{G}^*}$ instead of $\mu_{\mathbb{G}^*}$. But since $\mathbf{C}_u(\sigma) = \mathbf{C}_u(-\sigma)$, by (4.1) we deduce that $\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma)] = \mu^-_{\mathbb{G}^*}[\mathbf{C}_u(\sigma)] = \mu_{\mathbb{G}^*}[\mathbf{C}_u(\sigma)]$.

Below, we will use the following notation. For a set of faces F, ∂F denotes the set of faces u such that there exists a neighboring face v which is not in F.

Step 2. We proceed by contradiction. Assume that $\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma) = 0] = p > 0$. We wish to prove that for every $k \geq 0$,

$$\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma) \geq k+2 \mid \mathbf{C}_u(\sigma) \geq k] \leq 1-p. \tag{4.6}$$

This immediately implies that $\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma)] < \infty$, which contradicts the first step.

Note that it suffices to show that for every $k \geq 0$,

$$\begin{aligned} &\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma) \geq 2k+1 \mid \mathbf{C}_u(\sigma) \geq 2k \text{ and } \sigma_u = +] \leq 1-p, \\ &\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma) \geq 2k+2 \mid \mathbf{C}_u(\sigma) \geq 2k+1 \text{ and } \sigma_u = -] \leq 1-p. \end{aligned} \tag{4.7}$$

We prove the first inequality, the second follows similarly.

For $k = 0$, the result is a direct consequence of the definition of p. For $k \geq 1$, let $\mathcal{F}$ be the set of faces v of $\mathbb{G}$ for which every path from u to v contains at least $2k$ changes of signs. Fix a set of faces F. For $\sigma \in A_F := \Sigma_{\mathbb{G}^*} \cap \{\mathcal{F} = F\} \cap \{\mathbf{C}_u(\sigma) \geq 2k\} \cap \{\sigma_u = +1\}$, faces on ∂F have spin $+1$. Therefore, we deduce that

$$\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma) \geq 2k+1 | A_F] \leq 1 - \mu^+_{\mathbb{G}^*}[S_F \mid \sigma_v = +1 \text{ for all } v \in \partial F],$$

where S_F denotes the event that there is an infinite self-avoiding path of pluses starting from ∂F. Note that $\mathbf{C}_u(\sigma) = 0$ is included in S_F. The FKG inequality for the Ising model implies immediately that

$$\mu^+_{\mathbb{G}^*}[S_F | \sigma_v = +1 \text{ for all } v \in \partial F] \geq \mu^+_{\mathbb{G}^*}[S_F] \geq \mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma) = 0] = p,$$

so that

$$\mu^+_{\mathbb{G}^*}[\mathbf{C}_u(\sigma) \geq 2k+1 | A_F] \leq 1-p.$$

Since the events A_F partition $\Sigma_{\mathbb{G}^*} \cap \{\mathbf{C}_u(\sigma) \geq 2k\} \cap \{\sigma_u = +1\}$, summing on all possible F gives (4.7).

Step 3. It suffices to show that $\mu^+_{\mathbb{G}^*}[\sigma_u] \le 0$ since we already know by the first Griffiths inequality that $\mu^+_{\mathbb{G}^*}[\sigma_u] \ge 0$.

Fix a finite subgraph H of $\mathbb{G}^*$ and note that

$$\mu^+_H[\sigma_u] \le \mu^+_H[\mathbf{C}_u(\sigma) = 0] + \mu^+_H[\sigma_u \mathbf{1}_{\mathbf{C}_u(\sigma)\ge 1}]. \tag{4.8}$$

Now, condition on the set $\mathcal{F}$ of faces of $\mathbb{G}$ which are not connected by a path of pluses to the exterior of H. By definition, conditioned on $\mathcal{F} = F$, the configuration outside F is made of pluses, and the configuration inside of F is an Ising model conditioned on faces of ∂F to have spin -1. Furthermore, $\mathbf{C}_u(\sigma) \ge 1$ implies that $u \in F$. Thus, the Gibbs property implies that

$$\mu^+_H[\sigma_u \mathbf{1}_{\mathbf{C}_u(\sigma)\ge 1}] = \sum_{F \ni u} \mu^+_H[\sigma_u | \sigma_v = -1, \forall v \in \partial F] \times \mu^+_H[\mathcal{F} = F \text{ and } \mathbf{C}_u(\sigma) \ge 1] \le 0, \tag{4.9}$$

where the inequality follows from the fact that the FKG inequality implies that

$$\mu^+_H[\sigma_u | \sigma_v = -1, \forall v \in \partial H] \le 0.$$

Plugging this in (4.8) gives that $\mu^+_H[\sigma_u] \le \mu^+_H[\mathbf{C}_u(\sigma) = 0]$. Step 2 implies that $\mu^+_{\mathbb{G}^*}[\sigma_u] \le 0$ by letting H tend to $\mathbb{G}^*$.

5 A computation on a cylindrical annulus

Let $G_{n,m}$ be the cylindrical annulus graph obtained from $[0,n] \times [1,m] \subset \mathbb{Z}^2$ by identifying each vertex $(0,k)$ with (n,k), and embedding it in such a way that one of the n-gon faces is unbounded. Consider the homogenous critical (all edge weights equal to $x = \sqrt{2} - 1$) double random current model on $G_{n,m}$ with no sources. Let $N_{n,m}$ be the number of odd clusters around the interior circumference of the cylinder. The probability generating function of $N_{n,m}$ is defined by

$$g_{n,m}(s) = \sum_{\omega} \mathbf{P}^{\emptyset}_{\text{d-curr}}(\omega) s^{N_{n,m}(\omega)},$$

where the sum is taken over all currents on $G_{n,m}$ with $\emptyset$-boundary conditions We will compute the scaling limit of $g_{n,m}$ as $n \to \infty$ and fixed $\frac{n}{m} = \tau$ using a result of Kenyon [15].

To this end, let $f_{n,m}$ be the probability generating function of the nesting field of the double random current model evaluated at the bounded n-gon face of the cylinder (with 0 boundary conditions on the unbounded face). Then, by the definition of the nesting field, we have that $g_{n,m}(s) = f_{n,m}(\lambda)$ for $s = (\lambda + \lambda^{-1})/2$. By Theorem 1.2, the nesting field on $G_{n,m}$ is distributed as the height function of the dimer model on $G^d_{n,m}$, which by previous remarks, is distributed as the height function on $C_{G_{n,m}}$. The graph $C_{G_{n,m}}$ is a piece of the square-octagon lattice wrapped on a cylinder. If we color the squares of $C_{G_{n,m}}$ black and white in such a way that no two black squares are connected by an edge, we can perform further urban renewals on the black squares so that $C_{G_{n,m}}$ is transformed to the cylinder graph $G_{2n,2m}$. Using criticality of the weights, one can see that the dimer model on $G_{2n,2m}$ is uniform. Hence, the height function of the uniform dimer model on $G_{2n,2m}$ evaluated at the bounded 2n-gon is distributed like the nesting field of the critical double random current model on $G_{n,m}$ evaluated at the bounded n-gon.

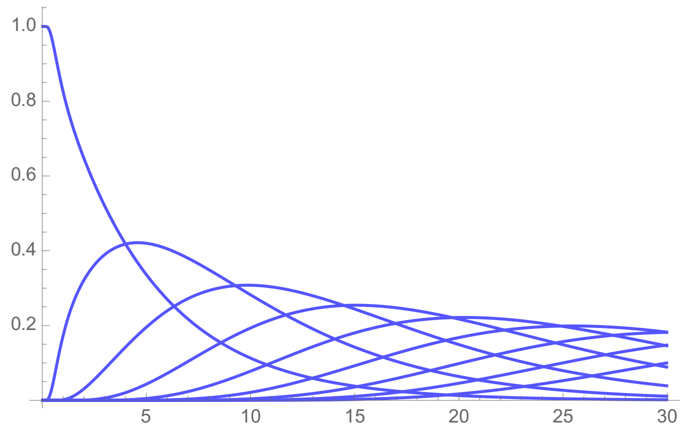

Figure 6: Probability of having $N = 0, \dots, 8$ odd clusters around the cylinder in the scaling limit as a function of $1/\tau$

The setup of the result of Kenyon is the double dimer model: It is the superimposition of two iid dimer configurations which results in a configuration composed of loops (cycles of edges contained in exactly one dimer cover) and doubled edges. To make an analogy with our setup, one can apply the definition of the nesting field to this case by taking the loops to be the odd part of a current, and the doubled edges to be even. Let $\tilde{g}_{n,m}(s)$ (resp. $\tilde{f}_{n,m}$) be the probability generating function of the number of noncontractible loops (resp. the nesting field) on $G_{2n,2m}$ in the uniform double dimer model. Then, as before, $\tilde{g}_{n,m}(s) = \tilde{f}_{n,m}(\lambda)$ for $s = (\lambda + \lambda^{-1})/2$. In [15], Kenyon computed the scaling limit of $\tilde{g}_{n,m}(s)$:

$$\lim_{n\to\infty, \frac{n}{m}=\tau} \tilde{g}_{n,m}(s) = \prod_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \frac{(1+q^j s + q^{2j})^2}{(1+q^j+g^{2j})^2},$$

where $q = e^{-\frac{\tau\pi}{2}}$.

Using the definition of the height function for the dimer model, one can see that the nesting field of the double dimer model is distributed like the difference of two independent height functions of a single dimer model on $G_{2n,2m}$. Hence, $\tilde{f}_{n,m}(\lambda) = f^2_{n,m}(\lambda)$, and therefore $\tilde{g}_{n,m}(s) = g_{n,m}(s)^2$ which means that

$$\lim_{n\to\infty, \frac{n}{m}=\tau} g_{n,m}(s) = \prod_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \frac{1+q^j s + q^{2j}}{1+q^j+q^{2j}},$$

where $q = e^{-\frac{\tau\pi}{2}}$.

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