Seven-dimensional forest fires

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Abstract

We show that in high dimensional Bernoulli percolation, removing from a thin infinite cluster a much thinner infinite cluster leaves an infinite component. This observation has implications for the van den Berg-Brouwer forest fire process, also known as self-destructive percolation, for dimension high enough.

1 Introduction

Think about the open vertices of supercritical percolation as if they were trees, and about the infinite cluster as a forest. Suddenly a fire breaks out and the entire forest is cleared. New trees then start growing randomly. When can one expect a new infinite cluster to appear? The surprising conjecture in [vdBB04] is that in the two-dimensional case, even if the original forest were extremely thin, still a considerable amount of trees must be added to create a new infinite cluster. Heuristically, the conjecture claims that the infinite cluster might occupy a very low proportion of vertices but they sit in a way that separates the remaining finite clusters by gaps that cannot be easily bridged. This conjecture is still open. See [vdBB04, vdBBV08, vdBdL09] for connections to other models of forest fires and more.

Let us define the model formally, in three steps. The model was originally introduced as a site percolation model, but we will define it for bonds, as some of the auxiliary results we need have only been proved for bond percolation. We are given a graph G, a probability $p \in [0,1]$ ("the original density") and a probability $\varepsilon \in [0,1]$ ("the recovered density"). Let $\mathbb{P}_p$ be the Bernoulli bond percolation measure on G with parameter p.

1. Assign independent uniformly distributed values from $[0,1]$ to the edges of G. Let $\omega_p \in \{0,1\}^{E(G)}$ denote the set of edges with value at most p. The configuration ω_p is distributed as $\mathbb{P}_p$, and a *cluster* refers

to a maximal connected component of edges. It will be of importance below that as p ranges over $[0,1]$, we obtain a simultaneous coupling of Bernoulli configurations on G such that $\omega_{p_1} \subset \omega_{p_2}$ when $p_1 \le p_2$.

2. Let $\tilde{\mathbb{P}}_p$ be the law of the configuration $\tilde{\omega}_p$ constructed as follows: for any edge e,

$$\tilde{\omega}_p(e) = \begin{cases} \omega_p(e) & \text{if } e \text{ is in a finite cluster of } \omega_p, \\ 0 & \text{otherwise.} \end{cases}$$

3. Let $\tilde{\mathbb{P}}_{p,\varepsilon}$ be the law of $\tilde{\omega}_{p,\varepsilon}$ where $\tilde{\omega}_{p,\varepsilon}$ is defined as follows: for any edge e, $\tilde{\omega}_{p,\varepsilon}(e) = \max\{\tilde{\omega}_p(e), \omega'_\varepsilon(e)\}$, where ω'_ε is a percolation configuration with edge-weight ε which is independent of ω_p.

We can now define our property of interest.

Definition We say that the graph G *recovers from fires* if for every $\varepsilon > 0$, there exists $p > p_c(G)$ such that $\tilde{\mathbb{P}}_{p,\varepsilon}$ has an infinite connected component, with probability 1. We say that G *site-recovers from fires* if the analogous definitions for site percolation hold.

In [vdBB04] the authors showed that a binary tree site-recovers from fires and conjectured that $\mathbb{Z}^2$ lattice does *not* site-recover from fires. The binary tree is an example of a non-amenable graph, that is, a graph in which the boundary of a (finite) set of vertices is comparable in size to the set itself. Recovery from fires, both in edge and site sense, was proven in [AST13] for a large class of non-amenable transitive graphs. Our result concerns hyper-cubic lattices.

Theorem 1. *For d sufficiently large, $\mathbb{Z}^d$ recovers from fires.*

Here and below, $\mathbb{Z}^d$ refers to the $\mathbb{Z}^d$ nearest neighbour lattice. The main property of $\mathbb{Z}^d$ that we will use is that $\mathbb{P}_{p_c}(0 \longleftrightarrow \partial B(0,r)) \le Cr^{-2}$ (see below for a discussion on this condition, and also for the notations). This was proved in [KN11] based on results of Hara, van der Hofstad & Slade [HvdHS03, Har08]. These establish the necessary estimate for d sufficiently large (19 seems to be enough, though this can be improved) and also for *stretched-out* lattices in $d > 6$. The number 6 is actually meaningful and is the limit of the technique involved, lace expansion. Our proof easily extends to stretched-out 7-dimensional lattice (hence the title of the article), but for simplicity we will prove the theorem only for nearest-neighbour percolation in d sufficiently high. In fact, our proof provides further information in the supercritical percolation regime. Recall the common notation $\mathscr{C}_\infty(\omega_p)$ for the infinite cluster of edges present in ω_p.

Theorem 2. *For every $\varepsilon > 0$ and d sufficiently large, there exists $p > p_c$ such that $\omega_{p_c+\varepsilon} \setminus \mathscr{C}_\infty(\omega_p)$ contains an infinite cluster almost surely.*

Theorem 1 is clearly a corollary of Theorem 2. Another consequence is that for every $\varepsilon > 0$, the critical probability for percolation on the random graph obtained from $\mathbb{Z}^d$ by removing a sufficiently 'thin' supercritical percolation cluster is almost surely at most $p_c + \varepsilon$. Theorem 2 and the last statement cannot possibly hold for (site) percolation on $\mathbb{Z}^2$, since an infinite cluster cuts space up into finite pieces.

Proof sketch We will show that for every $\varepsilon > 0$, there exists some $p > p_c$ such that when removing the infinite cluster of p-percolation from $(p_c + \varepsilon)$-percolation, the remainder still percolates. The proof proceeds by a renormalization procedure.

1. We first choose $\ell \in \mathbb{N}$ sufficiently large such that for any $L \geq \ell$, connectivity properties of boxes of size $L^2 \times \ell^{d-2}$ in $(p_c + \varepsilon)$-percolation behave like $(1-\eta)$-percolation on a coarse grain lattice for some small η. This is a standard application of Grimmett-Marstrand [GM90] and renormalization theory.

2. We then use the fact that the one-arm exponent in high dimensions is 2 to note that for any L, only a small number M of vertices in a box of size $L^2 \times \ell^{d-2}$ can connect to distance L in *critical percolation.*

3. Picking L sufficiently large, one can argue that these M points do not alter the connectivity properties of boxes of size $L^2 \times \ell^{d-2}$ for $(p_c+\varepsilon)$-percolation. In particular, the coarse grain percolation still behaves like $(1 - \eta)$-percolation even after removing that small number of vertices.

4. We now pick p sufficiently close to p_c that the behaviour (for ω_p) at scale L is not altered by moving from p_c to p. Since there are less sites in $\mathscr{C}_\infty(\omega_p)$ than sites connected to distance L in ω_p, this p gives the result.

Examining this a little shows that what the proof really needs is that the one-arm exponent is bigger than 1 i.e. that

$$\mathbb{P}_{p_c}(0 \longleftrightarrow \partial B(0,r)) \leq r^{-1-c} \qquad c > 0.$$

(the number of points removed in the second renormalization step will no longer be bounded independently of L, but would still be too small to block the cluster of the boxes at scale ℓ). This is interesting as it is conjectured to hold also below 6 dimensions. While nothing is proved, simulations hint that it might hold for $\mathbb{Z}^5$ [AS94, §2.7]. On the other hand, let us note that in $\mathbb{Z}^3$ this probability is larger than cr^{-1} (this is well-known but we are not aware of a precise reference – compare to [vdBK85, (3.15)] and [Kes82, Theorem 5.1]). Hence, the approach used here has no hope of working in $\mathbb{Z}^3$ (though, of course, this does not preclude the possibility that $\mathbb{Z}^3$ does recover from fires). We remarks that a similar renormalization technique was recently used in [GHK13], also under the assumption that the one-arm exponent is bigger than 1.

Notations Identify $\mathbb{Z}^2$ with the subgraph of $\mathbb{Z}^d$ of points with the $d-2$ last coordinates equal to 0. Let $S_\ell = \{x \in \mathbb{Z}^d : |x_i| \le \ell \ \forall i \ge 3\}$ be the two-dimensional slab of height $2\ell + 1$. We will also use the following standard notations: For a subgraph G of $\mathbb{Z}^d$, we say that x is connected to y in G if they are in the same connected component of G. We denote it by $x \overset{G}{\longleftrightarrow} y$ or simply $x \longleftrightarrow y$ when the context is clear. We also use the notation $x \longleftrightarrow \infty$ to denote the fact that x is contained in an infinite connected component.

Let $||\cdot||_\infty$ be the infinity norm on $\mathbb{R}^d$ defined by

$$||x||_\infty = \max\{|x_i| : i = 1, \dots, d\}.$$

We consider the hypercubic lattice $\mathbb{Z}^d$ for some large but fixed d. For $\ell, L > 0$, define the ball $B_x(L) = \{y \in \mathbb{Z}^d : ||y - x||_\infty \le L\}$ and let $\partial B_x(L)$ be its inner vertex boundary.

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2 Proof

From now on, d is fixed and is large enough. For $x \in \mathbb{Z}^2$, let $\mathscr{A}(x, \ell, L, M)$ be the event that there are less than M sites y in the $(6L+1) \times (6L+1) \times (2\ell+1)^{d-2}$ box $S_\ell \cap B_x(3L)$ that are connected to a site at distance L from themselves. Note that we do not assume that this connection is contained in the slab S_ℓ, the connection may be anywhere in $B_y(L)$.

Lemma 3. *Let $\eta > 0$ and $\ell > 0$. There exists $M > 0$ such that for any integer L, there exists $p > p_c$ such that*

$$\mathbb{P}_p(\mathscr{A}(x, \ell, L, M)) \ge 1 - \eta.$$

Proof. By [KN11], there exists $C > 0$ such that (for large enough d)

$$\mathbb{P}_{p_c}(0 \longleftrightarrow \partial B_0(n)) \le \frac{C}{n^2}. \tag{1}$$

Choose M in such a way that $\frac{49(2\ell+1)^{d-2}C}{M} < \eta$. For any integer L, Markov's inequality implies

$$\mathbb{P}_{p_c}\big[|\{y \in S_\ell \cap B_x(3L) : y \leftrightarrow \partial B_y(L)\}| \ge M\big] \le \frac{1}{M} \sum_{y \in S_\ell \cap B_x(3L)} \mathbb{P}_{p_c}(y \leftrightarrow \partial B_y(L)).$$

By (1) and the choice of M, the right-hand side is thus strictly smaller than η. By choosing p close enough to p_c, we obtain that

$$\mathbb{P}_p\big[|\{y \in S_\ell \cap B_x(3L) : y \longleftrightarrow \partial B_y(L)\}| \ge M\big] \le \eta. \qquad \square$$

For a set $S \subset \mathbb{Z}^d$, let ω^S be the configuration obtained from ω by closing the edges adjacent to a site in S. Let $\mathscr{B}(x, \ell, L, M)$ be the event that for any set S of M sites contained in $B_x(3L)$, ω^S contains

- a cluster crossing from $\partial B_x(L)$ to $\partial B_x(3L)$ contained in the slab S_ℓ,
- a unique cluster in the box $S_\ell \cap B_x(3L)$ of radius larger than L.

Lemma 4. *Let $\eta > 0$ and $\varepsilon > 0$. There exists $\ell > 0$ such that for any $M > 0$, there is $L > 0$ so that*

$$\mathbb{P}_{p_c+\varepsilon}(\mathscr{B}(x, \ell, L, M)) \ge 1 - \eta.$$

Proof. For a given ℓ and L denote by $E = E(x, \ell, L)$ the event that:

1. There is a crossing from $\partial B_x(L)$ to $\partial B_x(3L)$ in S_ℓ.
2. There is exactly one cluster in $S_\ell \cap B_x(3L)$ of radius larger than L.

Shortly, the event E is just $\mathscr{B}$ without the set S, or if you want $\mathscr{B}$ is the event that E occurred in ω^S for all S with $|S| \le M$.

We claim that for ℓ sufficiently large, $\mathbb{P}_{p_c+\varepsilon}(\neg E) \le \exp(-cL)$ for some $c = c(\varepsilon, \ell) > 0$ independent of L. Finding such an ℓ is a standard exercise in renormalization theory, but let us give a few details nonetheless. Call a box of side-length $2\ell+1$ *good* if it contains crossings between opposite faces in all directions, and if all clusters of diameter at least $\frac{1}{4}\ell$ connect inside the box. By choosing ℓ large, we can require that a box is good with arbitrarily high probability (see e.g. the appendix of [BBHK08]). Considering such boxes centered around the sites in $\ell\mathbb{Z}^2$. The events that these boxes are good are 2-dependent (in the sense of [LSS97] i.e. any box is independent of all boxes not neighbouring it), and hence by [LSS97], if the probability that a box is good is sufficiently large, then the good boxes stochastically dominate two-dimensional percolation at density, say, $\frac{9}{10}$. Now, a cluster of good boxes contains a cluster in the underlying percolation, since the crossings of adjacent boxes must intersect. This means that if either of the conditions in the definition of E fail, then there is a cluster of bad boxes with at least L/ℓ boxes. But the probability for that, from Peierls' argument, is at most $(4/10)^{L/\ell} \cdot (6L/\ell)^2$. This shows the claim.

Fix $M > 0$. Let F_M be the set of configurations in $B_x(3L)$ for which there exists $S \subset B_x(3L)$ with $|S| = M$ and $\omega^S \notin E$. We have

$$\begin{aligned}\mathbb{P}_{p_c+\varepsilon}(F_M) &\le \sum_{S\subset B_x(3L):|S|=M} \mathbb{P}_{p_c+\varepsilon}(\omega^S \notin E)\\ &\le \sum_{S\subset B_x(3L):|S|=M} (1-p_c-\varepsilon)^{-2dM}\, \mathbb{P}_{p_c+\varepsilon}(\neg E)\\ &\le (1-p_c-\varepsilon)^{-2dM}(6L+1)^{dM}\, \mathbb{P}_{p_c+\varepsilon}(\neg E)\\ &\le (1-p_c-\varepsilon)^{-2dM}(6L+1)^{dM}\exp(-cL).\end{aligned}$$

For L large enough, this quantity is smaller than η. The lemma follows from the fact that if $\omega \notin \mathscr{B}(x,\ell,L,M)$, then there exists $S \subset B_x(3L)$ with $|S| = M$ and $\omega^S \notin E$, i.e. $\omega \in F_M$. □

In order to prove Theorem 1 and 2, we will use Lemma 4 to construct an infinite cluster at density $p_c + \varepsilon$, and Lemma 3 to make sure that the infinite cluster present at the lower density p does not interfere too much with this construction.

Proof of Theorems 1 and 2. Recall the notations ω_p, $\tilde{\omega}_p$ and ω'_ε from page 1. We need to show that for any $\varepsilon > 0$, there exists $p > p_c$ such that $\tilde{\omega}_{p,\varepsilon}$ has an infinite component. Note that $(\omega_{p_c} \cup \omega'_\varepsilon) \smallsetminus \mathscr{C}_\infty(\omega_p)$ is stochastically dominated by $\tilde{\omega}_{p,\varepsilon}$. Thus, it suffices to show that for every $\varepsilon > 0$, there is $p > p_c$ such that $\omega_{p_c+\varepsilon} \smallsetminus \mathscr{C}_\infty(\omega_p)$ contains an infinite component. That is, Theorem 1 follows from Theorem 2, and it suffices to prove the latter.

Let therefore $\varepsilon > 0$. Fix $\eta > 0$ such that $1-2\eta$ exceeds the critical parameter for any 8-dependent percolation on vertices of $\mathbb{Z}^2$. Define successively ℓ, M, L and p_0 as follows. Fix $\ell = \ell(\varepsilon,\eta) > 0$ as defined in Lemma 4. Pick $M = M(\eta,\ell) > 0$ as defined in Lemma 3. This defines $L = L(\eta,\varepsilon,\ell,M) > 0$ by Lemma 4, and then $p = p(\eta,\ell,M,L) > p_c$ by Lemma 3.

Let $\mathbf{P}$ denote the joint law of $(\omega_p, \omega_{p_c+\varepsilon})$ under the increasing coupling described above. A site $x \in L\mathbb{Z}^2$ is said to be *good* if $\omega_p \in \mathscr{A}(x,\ell,L,M)$ and $\omega_{p_c+\varepsilon} \in \mathscr{B}(x,\ell,L,M)$. By definition,

$$\mathbf{P}\big[\mathscr{A}(x,\ell,L,M) \cap \mathscr{B}(x,\ell,L,M)\big] \ge 1-2\eta.$$

Since these events depend on edges in $B_x(4L)$ only, the site percolation (on $L\mathbb{Z}^2$) thus obtained is 8-dependent. As a consequence, there exists an infinite cluster of good sites on the coarse grained lattice $L\mathbb{Z}^2$.

On the event that there exists an infinite cluster of good sites on the coarse grained lattice, there exists an infinite path in $\omega_{p_c+\varepsilon} \setminus \mathscr{C}_\infty(\omega_p)$. Indeed, by induction, consider a path of adjacent good sites $x_1, \ldots, x_n$. Consider C_i to be a cluster in

$$[\omega_{p_c+\varepsilon} \setminus \mathscr{C}_\infty(\omega_p)] \cap [B_{x_i}(3L) \setminus B_{x_i}(L)]$$

of radius larger than L. By the definition of $\mathscr{A}$ there are at most M sites in the box $S_l \cap B_{x_i}(3L)$ connected to distance L in ω_p. Hence the same box also contains no more than M sites in $\mathscr{C}_\infty(\omega_p)$ since any site connected to infinity must be connected to distance L. Using the definition of $\mathscr{B}$ with S being exactly $\mathscr{C}_\infty(\omega_p) \cap S_l \cap B_{x_i}(3L)$ we see that $\omega_{p_c+\varepsilon} \setminus \mathscr{C}_\infty(\omega_p)$ contains a crossing cluster for the box $S_l \cap B_{x_i}(3L)$ with all the properties listed before Lemma 4. In particular, the uniqueness property ensures two such crossing clusters in two neighbouring boxes must intersect. The result follows readily. □

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