Finite volume Bootstrap Percolation with balanced threshold rules on $\mathbb{Z}^2$

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Abstract

We prove that there exists a sharp metastability transition for two-dimensional finite Bootstrap Percolation with threshold growth rules. This result extends theorems obtained by Holroyd [Hol03] for the simple two-dimensional Bootstrap Percolation. The method emphasizes physical phenomena involved in the growth of the threshold rule and offers an expandable frame that may be used to prove sharp metastability transition results for a wide class of cellular automata – see [GG97] for an introduction to this class of models. This article represents a further step towards an understanding of universality of two dimensional bootstrap models.

1 Introduction

1.1 Motivation and historical introduction

A threshold bootstrap percolation model is a simple cellular automaton that provides a model for several phenomena such as metastability, dynamics of glasses or crack formation. A famous example of a threshold model is the so-called *simple bootstrap percolation* originally introduced by Chalupa, Leath and Reich [CLR79]. In this model, sites of the square lattice $\mathbb{Z}^2$ are occupied or empty. At time 0, sites are occupied with probability p independently of each others. At each time step, a site becomes occupied if two or more of its nearest neighbors are occupied.

In 1987, A. Van Enter [vE87] established the first rigorous result on simple bootstrap percolation by proving that every site of $\mathbb{Z}^2$ becomes occupied almost surely whenever $p > 0$. This motivates the study of the (random) first time T at which 0 becomes occupied as p goes to 0. In [AL88], Aizenman and Lebowitz proved that there exist two constants $c, C \in (0, \infty)$ such that

$$c \leq \liminf_{p\to 0} p \log T \leq \limsup_{p\to 0} p \log T \leq C \quad \text{almost surely.}$$

We refer to this article for an enlightening exposition of the metastability effects in the model. The question of whether c and C could be chosen arbitrary close to each other was left open for a long time. Finally, a sharp metastability transition was shown to occur in [Hol03]: for every $\varepsilon > 0$, the probability of $|p \log T - \frac{\pi^2}{18}| \geq \varepsilon$ goes to 0 as $p \to 0$. More precise estimates for T were derived later in [GHM10]. Several authors investigated more general growth rules and found the right order of magnitude for T [vEH07, GG97, Mou93, Sch90] – hence generalizing the result of Aizenman and Lebowitz – under a mild condition called the voracious property. The question of sharp metastability transition, though, remained open. The goal of this paper is to prove sharp metastability for a wide class of models.

1.2 Formal definition of threshold rules

Let $\mathbb{Z}^2 = \{x = (x_1, x_2) : x_1, x_2 \in \mathbb{Z}\}$ be the set of all 2-vectors of integers. Elements of $\mathbb{Z}^2$ are called *sites*. A set $K \subseteq \mathbb{Z}^2$ is a *star* set if for any $x \in K$, $[0, x] \cap \mathbb{Z}^2 \subseteq K$. It is *symmetric* if $x \in K$ implies $-x \in K$.

Let $\mathbb{N} = \{0, 1, \dots\}$. Fix $\theta \in \mathbb{N}$ and a symmetric star set $\mathcal{N} \subseteq \mathbb{Z}^2$ containing the origin $(0, 0)$. A map $\mathcal{T}$ from subsets $K \subseteq \mathbb{Z}^2$ to itself is called a symmetric *threshold rule with parameters* $(\theta, \mathcal{N})$ if

$$x \in \mathcal{T}(K) \quad \text{if and only if} \quad x \in K \text{ or } |K \cap (x + \mathcal{N})| \geq \theta,$$

where $|E|$ denotes the cardinality of the set E. The parameter θ is called the *threshold*. The set $\mathcal{N}$ (resp. $x + \mathcal{N}$) is called the *neighborhood* of $(0, 0)$ (resp. of x). The set *spanned by* K can be defined as

$$\langle K \rangle = \lim_{t \to \infty} \mathcal{T}^t(K)$$

where $\mathcal{T}^t = \overbrace{\mathcal{T} \circ \dots \circ \mathcal{T}}^{t \text{ iterates}}$. The variable t is interpreted as a time. A set K is said to be *stable* if $\langle K \rangle = K$. Recall the following classification from [GG97] and [GG96]. A threshold rule is

- *supercritical* if there exists a finite set K such that $\langle K \rangle$ is infinite,
- *subcritical* if there exists a finite set K such that $\mathbb{Z}^2 \setminus K$ is stable,
- *critical* in the remaining cases. For critical threshold rules, any finite set spans a finite set and no finite set can be the complement of a stable set.

For a symmetric star set containing $(0, 0)$, we define $\iota(\mathcal{N})$ to be the maximum cardinality of sets of the form $\ell \cap \mathcal{N}$ where ℓ is a line containing $(0, 0)$. A threshold rule is *balanced* if there exist two distinct lines ℓ and ℓ' containing $(0, 0)$ such that $\ell \cap \mathcal{N}$ and $\ell' \cap \mathcal{N}$ have cardinality $\iota(\mathcal{N})$. The threshold rule is said to be *unbalanced* otherwise.

Examples of critical threshold rules include among many others the following

$$\begin{array}{lll}
\text{ex 1:} & \theta = 2 & \mathcal{N} = \{(0,0), (0,1), (1,0), (0,-1), (-1,0)\}, \\
\text{ex 2:} & \theta = 2k & \mathcal{N} = \{(0,0), (0,1), \dots, (0,k), (1,0), \dots, (k,0), \\
& & \qquad (0,-1), \dots, (0,-k), (-1,0), \dots, (-k,0)\}, \\
\text{ex 3:} & \theta = 2 & \mathcal{N} = \{(0,0), (0,1), (1,1), (-1,0), (-1,-1)\}, \\
\text{ex 4:} & \theta = 4 & \mathcal{N} = \{(0,0), (0,1), (1,1), (1,0), (1,-1), (0,-1), (-1,-1), (-1,0), (-1,1)\}.
\end{array}$$

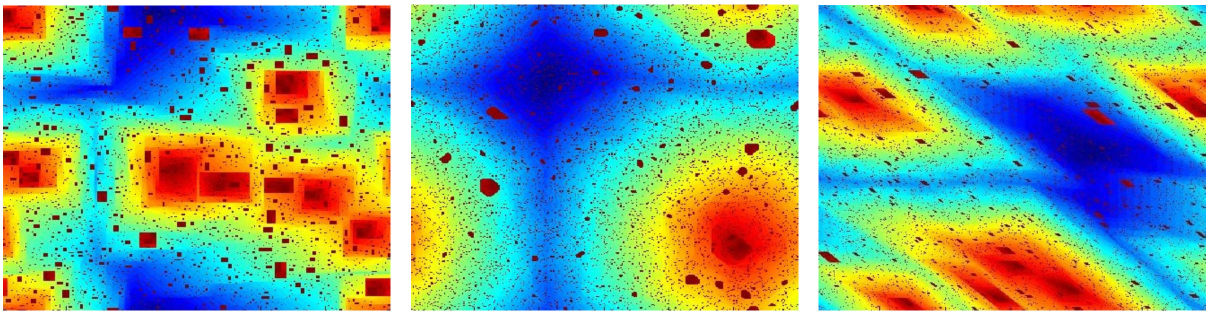

Figure 1: The figure on the left (resp. center, right) corresponds to ex 1 (resp. 4, 3). Sites in red are the sites in $\mathcal{T}^t(X)$ for small t, sites in blue are sites belonging to $\mathcal{T}^t(X)$ one for large t.

1.3 Statement of the theorem

The notion of threshold rule is purely deterministic. We introduce randomness as follows. A *threshold model* is a random model given by a threshold rule and a probability measure on subsets of $\mathbb{Z}^2$. In this article, the probability measure will always be the product measure with parameter p on the product σ-algebra of $\{0,1\}^{\mathbb{Z}^2}$. More formally, define X as the random variable $X(\omega) = \{x \in \mathbb{Z}^2 : \omega(x) = 1\}$. A site x is said to be *occupied* if $x \in X$. Define the random variable

$$T = T(\omega) = \inf\{t \in \mathbb{N} : (0,0) \in \mathcal{T}^t(X(\omega))\},$$

i.e. the smallest t such that the origin belongs to the set $\mathcal{T}^t(X(\omega))$.

Theorem 1.1. *For any balanced critical threshold rule with parameters $(\theta, \mathcal{N})$, there exists $\lambda \in (0,\infty)$ such that for every $\varepsilon > 0$*

$$\mathbb{P}_p\big[|p^\gamma \log T - \lambda| > \varepsilon\big] \to 0 \qquad \text{as } p \to 0$$

where $\gamma = \theta - (|\mathcal{N}| - \iota(\mathcal{N}))/2$.

The constant λ can be computed in any specific model. It is the solution of a variational problem, see Definition 4.1.

The families of supercritical and subcritical models have been studied in [GG97, GG96]. This is the reason why we restrict ourselves to the critical case. Assuming that the threshold rule is balanced and symmetric is rather natural, since non-symmetric or unbalanced threshold rules exhibit very different behavior, see [Dua89, vEH07, Mou93, Sch90, DCvE11].

Our theorem shows a kind of universality – meaning that the global behavior is independent of the microscopic definition of the model. The strategy of the proof extends to many other models, notably the voracious balanced symmetric models, see [GG97, GG96].

1.4 Probabilistic tools

An event $A \subseteq \{0,1\}^{\mathbb{Z}^2}$ is *increasing* if $\omega \in A$ and $\omega \leq \omega'$ imply $\omega' \in A$. Two important correlation inequalities related to increasing events will be used in the article.

The first inequality is the so-called *Harris inequality*. It is a particular case of the FKG inequality [FKG71]. For two increasing events A and B,

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

The second inequality is the *BK inequality*. For A and B two increasing events, the event $A \circ B$ is defined as follows. A configuration $\omega \in \{0,1\}^{\mathbb{Z}^2}$ belongs to $A \circ B$ if there exists a set $F = F(\omega)$ so that $\mathbb{1}_F\omega \in A$ and $\mathbb{1}_{F^c}\omega \in B$. In this case, A and B are said to occur *disjointly*. For k increasing events $A_1, \ldots, A_k$, one can define the disjoint occurrence by

$$A_1 \circ \cdots \circ A_k = A_1 \circ (A_2 \circ \cdots (A_{k-1} \circ A_k)).$$

Then, for any increasing events $A_1, \ldots, A_k$ depending on a finite number of sites,

$$\mathbb{P}_p(A_1 \circ \cdots \circ A_k) \leq \mathbb{P}_p(A_1) \cdots \mathbb{P}_p(A_k).$$

We refer the reader to the book [Gri99] for proofs of these two inequalities.

1.5 Organization of the paper

The next section introduces the notion of *directions of the model*, of *nucleation sets* and *voracious nucleation sets*. This section is reminiscent of Section 3 in [Hol03]. Section 3 introduces the notion of *droplets*. It then studies the probability that they are spanned. Section 4 deals with the proof of a lower bound on the probability that a droplet of size B/p^γ is internally spanned. Section 5 deals with the proof of the corresponding upper bound. Section 6 contains the proof of Theorem 1.1. It is based on all the previous sections.

2 Half-planes and directions of the threshold rule

2.1 Half-planes and nucleation sets

Gravner and Griffeath introduced the notion of half-plane with zero velocity [GG97]. For the sake of completeness we remind the reader of the definitions introduced in [GG97]. We restrict our attention to a threshold rule with a threshold θ and a symmetric star neighborhood $\mathcal{N}$. Let us start with a definition that will be useful in the following.

Definition 2.1. *Two sites x, y are* neighbors *if there exists $z \in \mathbb{Z}^2$ such that x and y belong to $z + \mathcal{N}$. A set S is* connected *if for any $x, y \in S$, there exists $n \in \mathbb{N}$ and $x = x_1, \ldots, x_n = y$ such that x_i and x_{i+1} are neighbors for every $i \in \{1, \ldots, n-1\}$.*

For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, define $\langle x|y \rangle = x_1y_1 + x_2y_2$.

Definition 2.2. *A* direction u *of the threshold rule is an element of* $\mathbb{R}^2 \setminus \{(0,0)\}$ *such that*

- *there exists* $x \in \mathcal{N} \setminus \{(0,0)\}$ *satisfying* $\langle x|v\rangle = 0$ *and*
- $\{\langle x|u\rangle : x \in \mathbb{Z}^2\} = \mathbb{Z}$.

The second direction allows to normalize u in a unique way. The direction u is therefore a vector which is not necessarily of unit norm. The set of directions of the threshold rule is denoted $\mathbb{U}$. Sometimes, we index $\mathbb{U} = \{u_1, \dots, u_{|\mathbb{U}|}\}$ by increasing complex argument when taken in $[0, 2\pi)$. In words, u_1 is the direction of the model with smallest complex argument, u_2 with second smallest complex argument, etc.

The previous definition is justified by the following fact. For any $v \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that there exists $x \in \mathcal{N} \setminus \{(0,0)\}$ satisfying $\langle x|v\rangle = 0$, the set $\{\langle x|v\rangle : x \in \mathbb{Z}^2\}$ is a discrete subgroup of $(\mathbb{R}, +)$.

The *half-plane* $\mathbb{H}_u$ is defined by $\mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x|u\rangle < 0\}$. Note that the boundary $\{x \in \mathbb{Z}^2 : \langle x|u\rangle = 0\}$ of $\mathbb{H}_u$ is not included in $\mathbb{H}_u$.

Definition 2.3. *A set* $S \subseteq \mathbb{Z}^2$ *is a* nucleation set of direction u *if* $\langle \mathbb{H}_u \cup S\rangle \setminus \mathbb{H}_u$ *is infinite. A nucleation set* S *of direction* u *is called* voracious *if* $\{x \in \mathbb{Z}^2 : \langle x|u\rangle = 0\} \subseteq \langle \mathbb{H}_u \cup S\rangle$.

For $u \in \mathbb{U}$, the set of nucleation sets S of direction u with minimal cardinality such that $(0,0) \in \mathcal{T}(S)$ is denoted by $\mathcal{S}^u$.

For a symmetric star set $\mathcal{N}$ and a direction u, define $\iota_u = \iota_u(u, \mathcal{N})$ to be the cardinality of $\{x \in \mathcal{N} : \langle x|u\rangle = 0\}$. Define ι to be the largest element of $\{\iota_u : u \in \mathbb{U}\}$. Set $\mathbb{U}_1 = \{u \in \mathbb{U} : \iota_u = \iota\}$. A threshold rule is said to be *balanced* if $\mathbb{U}_1$ contains two non-colinear vectors. It is said to be *unbalanced* otherwise. In this article, we focus on balanced dynamics only.

Propositions 1.3, 2.4, 2.5 and 2.6 of [GG96] determine the nature (subcritical, critical and supercritical) of the threshold rule in terms of θ and $\{\iota_u, u \in \mathbb{U}\}$. For completeness, we include a restatement of these propositions here. Note that the authors of [GG96] treat the case of balanced or unbalanced rules, and do not require $\mathcal{N}$ to be a star set.

Proposition 2.4. *Consider a balanced threshold rule with a symmetric star neighborhood* $\mathcal{N}$ *and a threshold* θ*. We have the following equivalences:*

- *the threshold rule is critical if and only if* $|\mathcal{N}| - \iota < 2\theta \leq |\mathcal{N}| - 1$.
- *The threshold rule is subcritical if and only if* $|\mathcal{N}| - 1 < 2\theta$.
- *The threshold rule is supercritical in the other cases.*

Instead of working directly with ι_u, it is more convenient to introduce the parameter

$$\gamma_u = \max\left(0, \theta - \frac{|\mathcal{N}| - \iota_u}{2}\right).$$

Set

$$\gamma = \max\left(0, \theta - \frac{|\mathcal{N}| - \iota}{2}\right).$$

From now on, we consider balanced threshold rules. Furthermore, we assume that the threshold rule is critical, which means that $\gamma > 0$.

In [GG96], the existence of voracious nucleation sets was discussed by the authors. Instead of proving their existence, they introduced the additional hypothesis that the model is voracious, meaning that any nucleation set is voracious. The question of wether most models are in fact voracious is also asked in the same article (Conjecture 1.4). We will not deal with this question. We avoided this difficulty by proving that there exists at least one voracious nucleation set among nucleation sets with minimum cardinality. The existence of this voracious nucleation set will be a sufficiently strong property in order to implement our program.

For $u = (u_1, u_2) \in \mathbb{U}$, the set $\{\langle x|(u_2, -u_1)\rangle : x \in \mathbb{Z}^2\}$ is also a discrete subgroup of $(\mathbb{R}, +)$. In particular, there exists $\alpha > 0$ such that $\{\langle x|\alpha(u_2, -u_1)\rangle : x \in \mathbb{Z}^2\} = \mathbb{Z}$. From now on, fix $u^\perp = \alpha(u_2, -u_1)$.

Lemma 2.5. *Consider a balanced critical threshold rule with a symmetric star neighborhood $\mathcal{N}$ and a threshold θ. Then, for every $u \in \mathbb{U}$ with $\gamma_u > 0$, the set*

$$S = \{nu^\perp : 1 \leq n \leq \gamma_u\}$$

is a voracious nucleation set of direction u in $\mathcal{S}^u$.

Proof. The set S is contained in $\mathcal{N}$. Indeed,

$$\gamma_u = \theta - \frac{|\mathcal{N}| - \iota_u}{2} \leq \frac{\iota_u - 1}{2}$$

since $2\theta \leq |\mathcal{N}| - 1$. Now,

$$\gamma_u + \frac{|\mathcal{N}| - \iota_u}{2} = \theta.$$

Therefore, $(0, 0)$ belongs to $\mathcal{T}(\mathbb{H}_u \cup S)$ and thus $-u^\perp + S \subseteq \mathcal{T}(\mathbb{H}_u \cup S)$. By induction, $-nu^\perp$ is in $\mathcal{T}^{n+1}(\mathbb{H}_u \cup S)$ and we deduce that $\mathbb{Z}u^\perp$ is included in $\langle \mathbb{H}_u \cup S\rangle$, thus proving that S is a voracious nucleation set.

In order to prove that $S \in \mathcal{S}^u$, note that $\langle \mathbb{H}_u \cup S'\rangle = \mathbb{H}_u \cup S'$ for any S' with cardinality smaller than γ_u. Indeed, if $x \in \mathbb{Z}^2 \setminus \mathcal{N}$, the set $(x + \mathcal{N}) \cap (\mathbb{H}_u \cup S')$ has cardinality strictly smaller than $\gamma_u + (|\mathcal{N}| - \iota_u)/2 \leq \theta$. Therefore, the minimal cardinality for a nucleation set is larger or equal to γ_u. □

The previous proof shows that sets in $\mathcal{S}^u$ have cardinality γ_u. Therefore, they must all be included in $\mathcal{N}$.

2.2 Occupied lines

For $n \in \mathbb{Z}$, define the n-th *line orthogonal to u* as $\ell^u(n) = \{x \in \mathbb{Z}^2 : \langle x|u\rangle = n\}$.

Definition 2.6. *A line ℓ^u orthogonal to u is* occupied *in K if there exist $x \in \ell^u$ and $S \in \mathcal{S}^u$ such that $x + S \in K$.*

This definition is an extension of the definition of occupied lines and rows for the simple bootstrap percolation, see [Hol03].

For $m, n \in \mathbb{N}$, define the set

$$R^u(m,n) = \big\{x \in \mathbb{Z}^2 : 0 \le \langle x|u\rangle \le m \text{ and } 0 \le \langle x|u^\perp\rangle \le n\big\}.$$

Such a set is called a *rectangle in direction u*. By extension, any translate of $R^u(m,n)$ is also called a rectangle. Define the event

$$\mathcal{A}^u(m,n) = \bigcap_{j=0}^{n} \big\{\omega : \ell^u(j) \text{ is occupied in } X(\omega) \cap R^u(m, n+|\mathcal{N}|)\big\}.$$

Note that we consider the configuration in the rectangle $R^u(m, n+|\mathcal{N}|)$.

The following proposition studies the behavior of $\mathbb{P}_p[\mathcal{A}^u(m,n)]$. In particular, we prove that this probability can be expressed conveniently in terms of a family of functions h_p^u.

Proposition 2.7. *Let $u \in \mathbb{U}$. There exists a family of decreasing functions $(h_p^u)_{p\in(0,1)}$ such that*

(1) **(Expression in terms of h_p^u)** *For any $n > 0$,*

$$\exp\big[-h_p^u[p^{\gamma_u}m](n+|\mathcal{N}|)\big] \le \mathbb{P}_p[\mathcal{A}^u(m,n)] \le \exp\big[-h_p^u[p^{\gamma_u}m]n\big]. \tag{2.1}$$

(2) **(Behavior near 0 and ∞)** *There exist $p_0, c > 0$ such that for every $p < p_0$ and $x > \gamma_u p^{\gamma_u}$ and*

$$-c\log[x/c] \le h_p^u[x] \le -\log[1-e^{-cx}]. \tag{2.2}$$

(3) **(Uniform convergence)** *There exists an integrable function $h^u : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that h_p^u/h^u converges to 1 uniformly on (A,B) for every $A, B > 0$.*

In simple cases, the functions h^u could be computed explicitly. The limit h^u corresponds to the function g in [Hol03] and functions g_k in [HLR04]. Observe that all these functions are much less explicit in the general case. Also note that if m and n are of order $p^{-\gamma_u}$, then $-p^{\gamma_u}\log\mathbb{P}_p[\mathcal{A}^u(m,n)]$ remains of order 1 when p goes to 0. This is a reason why the scale $p^{-\gamma_u}$ is the right scale to consider.

Proof of (1). The main ingredient to construct h_p^u is the sub and super-multiplicativity. Fix $m > 0$. Define $v_{p,m}(n) = \mathbb{P}_p[\mathcal{A}^u(m,n)]$. The FKG inequality and the independence imply

$$v_{p,m}(n)v_{p,m}(n') \le v_{p,m}(n+n') \le v_{p,m}(n-|\mathcal{N}|)v_{p,m}(n').$$

Note that we use the fact that a set in $\mathcal{S}^u$ is necessarily contained in $\mathcal{N}$, as observed at the end of the previous subsection. The sub-additivity lemma implies that there exists $\mu = \mu(u,p,m) \in (0,1)$ such that $\mu^{n+|\mathcal{N}|} \le v_{p,m}(n) \le \mu^n$. For any $m \in \mathbb{N}$, set $h_p^u(p^{\gamma_u}m) = -\log\mu$. Extend h_m^u to all $(0,\infty)$ in a piecewise linear way. Note that h_p^u is decreasing since $\mathcal{A}^u(m,n) \subseteq \mathcal{A}^u(m+1,n)$ for every $m \ge 0$. $\square$

Proof of (2). For $j, k \in \mathbb{Z}$, define the set

$$S_{j,k} = \{x \in \ell^u(j) : k\gamma_u \leq \langle x | u^\perp \rangle \leq k\gamma_u + (\gamma_u - 1)\}.$$

This set is a translate of a voracious nucleation set as proved in Proposition 2.5. Consider the event

$$\mathcal{E} = \bigcap_{j=0}^{n} \bigcup_{k=-\infty}^{\infty} \{S_{j,k} \in X(\omega) \cap R^u(m,n)\}.$$

In words, the event corresponds to the fact that certain translates of a voracious nucleation set are occupied in $X(\omega) \cap R^u(m,n)$. These translates are chosen in such a way that they do not intersect. This allows for the use of independence, which gives

$$\mathbb{P}_p[\mathcal{A}^u(m,n)] \geq \mathbb{P}_p(\mathcal{E}) \geq \left[1 - (1 - p^{\gamma_u})^{\lfloor m/\gamma_u \rfloor}\right]^n \geq \exp\left[-\log\left(1 - e^{cmp^{\gamma_u}}\right) n\right]$$

for any $m \geq \gamma_u$, where c is a constant chosen small enough. The right inequality of (2.2) follows readily for $x = p^{\gamma_u} m \geq \gamma_u p^{\gamma_u}$.

If $\mathcal{A}^u(m,n)$ occurs, every rectangle of the form $k|\mathcal{N}| + R^u(m, |\mathcal{N}|)$ must contain one nucleation set. Since this nucleation set is a connected set of cardinality at least γ_u, we obtain

$$\mathbb{P}_p[\mathcal{A}^u(m,n)] \leq \prod_{j=0}^{\lfloor n/|\mathcal{N}| \rfloor} \left(m|\mathcal{N}| \cdot |\mathcal{N}|^{\gamma_u} \cdot p_u^{\gamma}\right)$$

where $m|\mathcal{N}|$ bounds the cardinality of each rectangle and $|\mathcal{N}|^{\gamma_u}$ the number of connected sets of cardinality γ_u containing a prescribed site. The left inequality of (2.2) follows by taking the logarithm. □

Proof of (3). Fix $A < B$. Let us prove that h_p^u converges to some function h^u as $p \to 0$. The proof of (1) implies

$$\frac{\log \mathbb{P}_p[\mathcal{A}^u(xp^{-\gamma_u}, n)]}{n + |\mathcal{N}|} \leq h_p^u(x) \leq \frac{\log \mathbb{P}_p[\mathcal{A}^u(xp^{-\gamma_u}, n)]}{n}.$$

It is therefore sufficient to prove that for each fixed $n > 0$, $x \mapsto \mathbb{P}_p[\mathcal{A}^u(xp^{-\gamma_u}, n)]$ converges uniformly on $[A, B]$ as $p \to 0$. For any $E \subseteq \{0, \ldots, n\}$, define $\mathcal{A}^u(m, n, E)$ to be the event that lines of index in E are not occupied. Via an inclusion-exclusion principle, it is sufficient to show that $x \mapsto \mathbb{P}_p[\mathcal{A}^u(xp^{-\gamma_u}, n, E)]$ converges uniformly on $[A, B]$ for any fixed E.

Fix k large enough and let us make the assumption that $xp^{-\gamma_u}$ is divided by k. The convergence for multiple of k is sufficient to imply the general result by monotonicity. Divide $R^u(xp^{-\gamma_u}, n)$ into translates $R_1, \ldots, R_{xp^{-\gamma_u}/k}$ of $R^u(k, n)$. It is easy to see that the event $\mathcal{A}^u(xp^{-\gamma_u}, n, E, k)$ that each line with index E is not occupied in any of the rectangles R_i for $i \in \{1, \ldots, m/k\}$ satisfies

$$\left|\mathbb{P}_p[\mathcal{A}^u(xp^{-\gamma_u}, n, E)] - \mathbb{P}_p[\mathcal{A}^u(xp^{-\gamma_u}, n, E, k)]\right| \leq \varepsilon(k)$$

for every $x \leq B$, where $\varepsilon(k)$ goes to 0 as k goes to infinity. Indeed, $\mathcal{A}^u(xp^{-\gamma_u}, n, E) \setminus \mathcal{A}^u(xp^{-\gamma_u}, n, E, k)$ is included in the set of configurations such that one connected set of cardinality γ_u intersects two rectangles. The probability of this set is of order $C_{B,n}p^{\gamma_u}xp^{-\gamma_u}/k \leq C_{B,n}B/k$. In conclusion, we can restrict our attention to $\mathcal{A}^u(xp^{-\gamma_u}, n, E, k)$.

The event $\mathcal{A}^u(xp^{-\gamma_u}, n, E, k)$ occurs if and only if no line of index in E is traversable in any of the R_i. Since the configurations in two different R_i are independent, we obtain

$$\mathbb{P}_p[\mathcal{A}^u(xp^{-\gamma_u}, n, E, k)] = \mathbb{P}_p[\mathcal{A}^u(k, n, E)^c]^{xp^{-\gamma_u}/k}.$$

Now, $\mathbb{P}_p[\mathcal{A}^u(k, n, E)] = 1 - \mathbb{P}_p[\mathcal{A}^u(k, n, E)] = 1 - \mathbb{E}_p[N] - o(1)$ where N is the number of nucleation sets on lines with index E. Note that we replaced the probability of the existence of such nucleation sets by the expectation of their number. This is justified by the fact that the event has probability going to 0. We find, $\mathbb{E}_p[N] = C_{k,n}p^{\gamma_u}$ where $C_{k,n}$ counts the number of possible nucleation sets. When p goes to 0, we obtain the uniform convergence on (A, B)

$$\mathbb{P}_p[\mathcal{A}^u(xp^{-\gamma_u}, n, E, k)] \longrightarrow \exp(-C_{k,n}x/k).$$

□

3 Growth of droplets

3.1 Definition of droplets

Definition 3.1. *A* droplet D *is a set of the form* $D = D[\mathbf{a}] = \bigcap_{u \in \mathbb{U}} \{x \in \mathbb{Z}^2 : \langle x|u \rangle < a_u\}$ *where* $\mathbf{a} \in \mathbb{N}^{\mathbb{U}}$.

For $u \in \mathbb{U}$, define the *edge* $E_u(D)$ of $D = D[\mathbf{a}]$ to be the set of $x \in \mathbb{Z}^2$ such that $\langle x|u \rangle = a_u$ and $\langle x|v \rangle \leq \beta_v a_v$ for $v \neq u$. Note that the edges do not belong to D.

The *dimension* $\mathbf{m} \in \mathbb{N}^{\mathbb{U}}$ of $D[\mathbf{a}]$ is given by $m_u = |E_u(D)|$ for every $u \in \mathbb{U}$. The *perimeter* $\Phi(D)$ of $D[\mathbf{a}]$ is defined as

$$\Phi(D) = \sum_{u \in \mathbb{U}} m_u.$$

We will require a notion of "nice droplet". For $k \in \mathbb{N}$, let $D[k]$ be a droplet with dimensions $(k, \ldots, k)$. The existence of $D[k]$ is fairly elementary. Set $x_1 = 0$ and $x_{i+1} = x_i - ku_i^{\perp}$. Since the neighborhood is symmetric, we obtain $x_{|\mathbb{U}|+1} = x_0$ and $D[k]$ is constructed.

The *location* of $D_1[\mathbf{a}] \subseteq D_2[\mathbf{b}]$ is given by $\mathbf{s} \in \mathbb{N}^{\mathbb{U}}$ where $s_u = |a_u - b_u|$ for every $u \in \mathbb{U}$. The *total location* $\Psi(D_1, D_2)$ of $D_1 \subseteq D_2$ is defined by

$$\Psi(D_1, D_2) = \sum_{u \in \mathbb{U}} s_u.$$

Note that $\Psi(D_1, D_2)$ does not depend on the position of D_1 inside D_2.

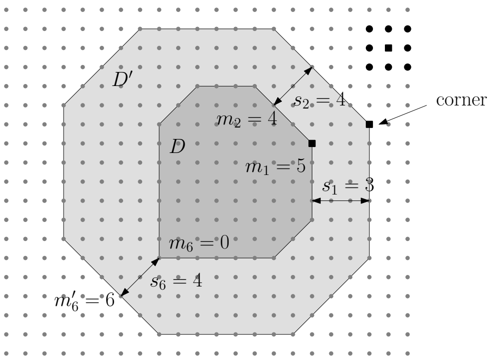

Figure 2: An example of two droplets $D \subseteq D'$ in the case of eight closest neighbors

We are now in position to define a functional depending on two droplets. Recall that $\mathbb{U}_1 = \{u \in \mathbb{U} : \iota_u = \iota\} = \{u \in \mathbb{U} : \gamma_u = \gamma\}$.

Definition 3.2. *For two droplets $D_1 \subseteq D_2$, let*

$$W_p(D_1, D_2) = p^\gamma \sum_{u \in \mathbb{U}_1} h_p^u[p^\gamma m_u] s_u.$$

This functional allows to control some events depending on two droplets, as we will see in the next subsection.

Not every $\mathbf{m} \in \mathbb{N}^{2\kappa}$ necessarily corresponds to the dimension of a droplet. Yet it is easy to verify that the condition is additive in the following way: if $\mathbf{m}$ and $\mathbf{m}'$ are the dimensions of two droplets D and D', then there exists a droplet with dimensions $\mathbf{m} + \mathbf{m}'$.

Definition 3.3 (Sum of droplets)**.** *The* sum *of droplets $D_1, \ldots, D_k$ is the droplet, denoted by $D_1 \oplus \cdots \oplus D_k$, defined as follows:*

- *$D_1 \oplus \cdots \oplus D_k$ has dimension $\mathbf{m^{(1)}} + \cdots + \mathbf{m^{(k)}}$, where $\mathbf{m^{(j)}}$ is the dimension of D_j for every $j \in \{1, \ldots, k\}$.*
- *$E_{u_1}(D_1) \subseteq E_{u_1}(D_1 \oplus \cdots \oplus D_k)$ and $E_{u_2}(D_1) \subseteq E_{u_2}(D_1 \oplus \cdots \oplus D_k)$.*

Note that the first condition determines the droplet up to translations of $\mathbb{Z}^2$. The second condition fixes the position of the droplet. It corresponds to the fact that the "corner" corresponding to directions u_1 and u_2 is the same for D_1 and $D_1 \oplus \cdots \oplus D_k$.

An important property will be used in many places.

Lemma 3.4. *Let $D_1 \subseteq D_2$ and D three droplets. The locations of $D_1 \oplus D \subseteq D_2 \oplus D$ and $D \oplus D_1 \subseteq D \oplus D_2$ are equal to the one of $D_1 \subseteq D_2$.*

Proof. This comes from the fact that s_u is the absolute value of the difference of two linear applications from the space of droplets into $\mathbb{R}$. $\square$

For $p, Z > 0$ and any droplet D, we set $D^Z = D^Z(p) = D \oplus D(Zp^{-\gamma})$.

3.2 Estimates using the functional

Fix $D_1 \subset D_2$. Define $\mathcal{I}(D_1, D_2) = \{\omega : \langle (X(\omega) \cap D_2) \cup D_1 \rangle = D_2\}$. Define $\mathcal{E}(D_1, D_2)$ to be the event that there exists a connected set $K \subseteq \langle (X(\omega) \cap D_2) \cup D_1 \rangle$ such that $K \cup E_u(D_2)$ is connected for every $u \in \mathbb{U}_1$. In words, the event $\mathcal{I}$ corresponds to the fact that D_2 is "spanned" by the configuration inside it knowing that D_1 is already full. The event $\mathcal{E}$ is the corresponding event, when the condition of being spanned is replaced by the existence of a connected set touching each edge of direction $u \in \mathbb{U}_1$.

Remark 3.5. *This second definition is interesting because one can associate to every stable connected set the smallest droplet containing it. By construction, this droplet is crossed. therefore, we can restrain our attention to stable sets that are droplets. Also note that the dynamics has a tendency to create fat clusters. Therefore, the intuition predicts that $\mathcal{I}(D_1, D_2)$ and $\mathcal{E}(D_1, D_2)$ have comparable probabilities.*

We now bound the probabilities of $\mathcal{I}(D_1, D_2)$ and $\mathcal{E}(D_1, D_2)$ using the functional W_p. Note that in order to bound the probability of $\mathcal{E}(D_1, D_2)$, we need to consider the functional of slightly bigger droplets.

Proposition 3.6. *Fix $\varepsilon, Z, B > 0$. There exist $Q, T, p_0 > 0$ such that for any $p < p_0$ and every droplets $D_1 \subseteq D_2 \subseteq D(Bp^{-\gamma})$ satisfying $\Psi(D_1, D_2) \leq Tp^{-\gamma}$, we have*

$$\mathbb{P}_p\left[\mathcal{I}\left(D_1^Z, D_2^Z\right)\right] \geq Q^{-1}\exp\left[-(1+\varepsilon)\frac{W_p\left(D_1^Z, D_2^Z\right)}{p^\gamma} - \frac{\varepsilon}{B}\Psi(D_1^Z, D_2^Z)\right], \tag{3.1}$$

$$\mathbb{P}_p\big[\mathcal{E}(D_1, D_2)\big] \leq Q\exp\left[-(1-\varepsilon)\frac{W_p(D_1^Z, D_2^Z)}{p^\gamma}\right]. \tag{3.2}$$

Proof of (3.1). Let $\varepsilon, Z, B > 0$. Let $T, p_0 > 0$ to be fixed later. Consider two droplets $D_1[\mathbf{a}] \subseteq D_2[\mathbf{b}]$ satisfying $\Psi(D_1, D_2) \leq Tp^{-\gamma}$ and $p < p_0$.

Let us start with some notations. We recommend to look at Fig. 3. Fix $u \in \mathbb{U}$. Let $\mathbb{H} = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < a_u\}$. Let $\tilde{m}_u$ be the largest integer such that a translate of $R^u(\tilde{m}_u + 2Tp^{-\gamma}, s_u)$ is contained in the set

$$T^u := \{x \in D_2^Z : \langle x|u \rangle \geq a_u \text{ and } \langle x|u_{k-1} \rangle \leq a_{u_{k-1}} \text{ and } \langle x|u_{k+1} \rangle \leq a_{u_{k+1}}\},$$

where $u = u_k$ and the indexes are considered in $\mathbb{Z}/|\mathbb{U}|\mathbb{Z}$. Define R^u to be a translate of $R^u(\tilde{m}_u + 2Tp^{-\gamma}, s_u)$ contained in T^u. Further partition R^u into three rectangles R_1^u, R_2^u and R_3^u of width $Tp^{-\gamma}$, $\tilde{m}_u$ and $Tp^{-\gamma}$ respectively, in such a way that R_2^u lies between R_1^u and R_3^u.

Note that

$$\tilde{m}_u \geq m_u + Zp^{-\gamma} - 2(1 + |\mathcal{N}|)Tp^{-\gamma}.$$

Indeed, T^u has the property that for any $x \in \ell^u(j+1) \cap T^u$, we have $(x + \mathcal{N}) \cap \ell^u(i) \subseteq T^u$ for $i \leq j$. This can easily be seen from the fact that $y \in (x + \mathcal{N}) \cap \ell^u(i)$ satisfies $\langle y|u_{k+1} \rangle \leq \langle x|u_{k+1} \rangle$, and the same for u_{k-1}. Therefore, we deduce that the cardinality of $\ell^u(j) \cap T^u$ can decrease by at most $2|\mathcal{N}|$ when going from j to $j+1$.

From now on, set $T \le Z/(4+4|\mathcal{N}|)$ small enough that $\tilde{m}_u \ge m_u + Zp^{-\gamma}/2$. Let $\tilde{\mathcal{A}}(R_2^u)$ be the event

$$\bigcap_{j=a_u}^{a_u+s_u} \{\omega : \ell^u(j) \text{ is occupied in } X(\omega) \cap R_2^u\}.$$

Note that this event is almost a translate of $\mathcal{A}(R^u(\tilde{m}_u, s_u))$ except that lines must be occupied in a translate of $R^u(\tilde{m}_u, s_u)$ instead of $R^u(\tilde{m}_u, s_u + |\mathcal{N}|)$. Property (1) of Proposition 2.7 and the argument used in the proof of Property (2) of this same proposition imply

$$\mathbb{P}_p[\tilde{\mathcal{A}}^u(R_2^u)] \ge (1-(1-p^{\gamma_u})^{c\tilde{m}_u})^R\, \mathbb{P}_p[\mathcal{A}^u(\tilde{m}_u, s_u)] \ge e^{-c} \exp\big[-p^{\gamma_u} h_p^u[\tilde{m}_u] s_u\big], \qquad (3.3)$$

where the constant c can be set to be equal to $|\mathcal{N}|(\log(1-e^{-Z/2}) + \log Z)$.

Let $\mathcal{B}(R^u)$ be the event that each line of R^u contains a voracious nucleation set of the form $z + \{nu^\perp : n = 1, \ldots, \gamma_u\}$ in $\langle (X \cap R^u) \cup D_1^Z \rangle$. If $\omega \in \mathcal{B}(R^u)$ for every $u \in \mathbb{U}$, then $D_2^Z = \langle (X(\omega) \cap D_2^Z) \cup D_1^Z \rangle$. Indeed, fix a direction $u \in \mathbb{U}$. The voracious nucleation set in $\ell^u(a_u) \cap R^u$ spans $\ell^u(a_u) \cap T^u$ since for every site $x \in \ell^u(a_u) \cap T^u$,

$$(x+\mathcal{N}) \cap D_1^Z = (x+\mathcal{N}) \cap \mathbb{H}.$$

By induction on the lines $\ell^u(a_u), \ldots, \ell^u(a_u+s_u)$, we find that T^u belongs to $\langle (X(\omega) \cap D_2^Z) \cup D_1^Z \rangle$. We can apply iteratively this reasoning for each u. Using the Harris inequality, we obtain

$$\mathbb{P}_p\big[\mathcal{I}(D_1^Z, D_2^Z)\big] \ge \prod_{u\in\mathbb{U}} \mathbb{P}_p(\mathcal{B}(R^u)) \ge \prod_{u\in\mathbb{U}} \mathbb{P}_p\big(\tilde{\mathcal{A}}^u(R_2^u)\big) \prod_{u\in\mathbb{U}} \mathbb{P}_p\big(\mathcal{B}(R^u)\big|\tilde{\mathcal{A}}^u(R_2^u)\big) \qquad (3.4)$$

Let us now prove that

$$\mathbb{P}_p\big(\mathcal{B}(R^u)\,\big|\,\tilde{\mathcal{A}}(R_2^u)\big) \ge \exp\Big(-\frac{\varepsilon}{2B} s_u\Big) \qquad (3.5)$$

provided p is chosen small enough. In order to do so, proceed as follows. Since $\tilde{\mathcal{A}}(R_2^u)$ holds, there exists $x \in \ell^u(a_u) \cap R_2^u$ and $S \in \mathcal{S}^u$ such that $x + S \in X(\omega) \cap R_2^u$. In particular, $\langle (x+S) \cup \mathbb{H} \rangle$ is infinite.

Observe the two following easy facts. First, the set $\langle \mathbb{H} \cup (x+S) \rangle \setminus \mathbb{H}$ is connected. Second, $\langle \mathbb{H} \cup (x+S) \rangle$ is included in $\mathbb{H} \cup (x+S) \cup \{z \in \mathbb{Z}^2 : (z+\mathcal{N}) \cap \mathbb{H} \neq \emptyset\}$. Indeed, every set S' with $|S'| \le \gamma_u - 1$ satisfies $\langle S' \cup \mathbb{H} \rangle = S' \cup \mathbb{H}$ and the cardinality of $S \cap \{z \in \mathbb{Z}^2 : (z+\mathcal{N}) \cap \mathbb{H} \neq \emptyset\}$ is smaller or equal to $\gamma_u - 1$.

This implies that the number of sites of $\ell^u(a_u) \cap (R_1^u \cup R_3^u)$ which neighborhood intersects $\langle (x+S) \cup D_1^Z \rangle$ is larger than $T/(p^\gamma |\mathcal{N}|)$. Therefore, $T/(p^{\gamma_u}|\mathcal{N}|^2)$ of these sites have disjoint neighborhood. Fix one of these sites z and assume that $y \in (z+\mathcal{N}) \cap \langle (x+S) \cup D_1^Z \rangle$. If the set $z + \{nu^\perp : n = 1, \ldots, \gamma - 1\}$ does not contain y and is in $X(\omega) \cap (R_1^u \cup R_3^u)$, then z becomes occupied eventually. But this means that the voracious nuclear set $z + \{nu^\perp : n = 0, \ldots, \gamma - 1\}$ belongs to the final configuration. If $y \in z + \{nu^\perp : n = 1, \ldots, \gamma - 1\}$, simply consider the set $z - \{nu^\perp : n = 1, \ldots, \gamma - 1\}$. Conditionally on $\langle X(\omega) \cap R_2^u \rangle$, the probability that $\langle X(\omega) \cap R^u \rangle$ contains a set of the $z + \{nu^\perp : n = 0, \dots, \gamma - 1\}$ is therefore larger than

$$1 - (1 - p^{\gamma-1})^{T/(p^\gamma |\mathcal{N}|^2)} \geq \exp[-\varepsilon/(2B)]$$

provided p_0 small enough. Furthermore, the previous depends only on the configuration $\ell^u(a_u) \cap (R_1^u \cup R_3^u)$. One can iterate this reasoning on every line to obtain the result.

Putting (3.3) and (3.5) into (3.4), we find

$$\mathbb{P}_p\big[\mathcal{I}(D_1^Z, D_2^Z)\big] \geq \exp\left[-\left(\sum_{u \in \mathbb{U}} h_p^u[\tilde{m}_u] s_u\right) - |\mathbb{U}|c - \frac{\varepsilon}{2B}\Psi(D_1^Z, D_2^Z)\right],$$

By choosing T small enough, we find for every $u \in \mathbb{U}_1$,

$$h_p^u[\tilde{m}_u] \leq (1 + \varepsilon) h_p^u[p^\gamma m_u + Z]$$

via the uniform convergence of h_p^u to h^u on $[Z, B + Z]$. For $u \notin \mathbb{U}_1$,

$$h_p^u(\tilde{m}_u) s_u \leq -\log[1 - e^{-cp^{\gamma_u}\tilde{m}_u}] s_u \leq \frac{\varepsilon}{2B} s_u$$

provided that p is chosen small enough. Putting everything together, we obtain the claim with $Q = e^{-|\mathbb{U}|c}$. □

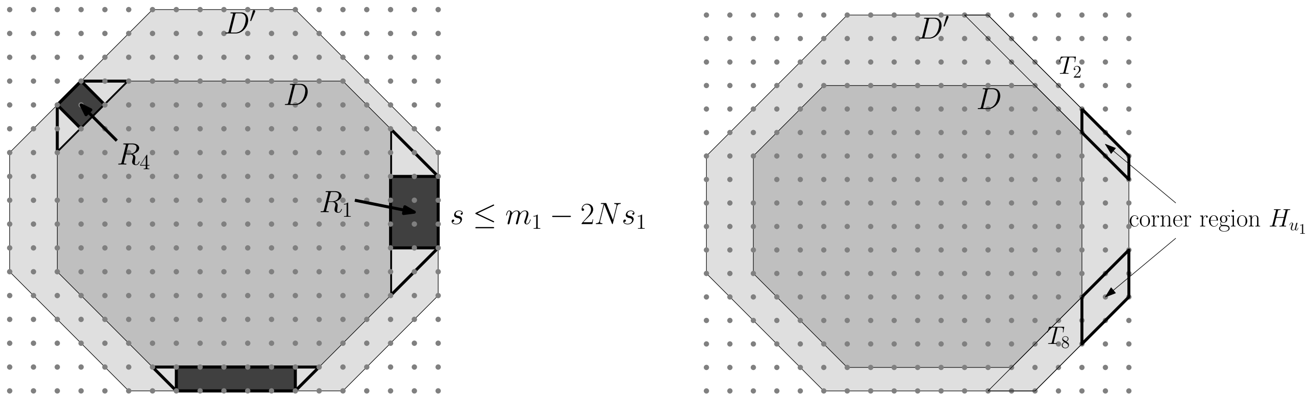

Figure 3: *Left.* Rectangles R_1^u, R_2^u and R_3^u used in the lower bound as well as the set T^u. *Right.* Corner region H^u.

Before turning to the proof of (3.2), let us discuss a lemma first. For any $m, n \in \mathbb{N}$, define the *strip*

$$S^u(n) = \{x \in \mathbb{Z}^2 : 0 \leq \langle x|u \rangle \leq n\}$$

and the events

$$\mathcal{C}^u(m, n, E) = \big\{\ell^u(0) \text{ and } \ell^u(n) \text{ connected in } \big\langle (X(\omega) \cap R^u(m, n)) \cup (\mathbb{Z}^2 \setminus S^u(n)) \cup E \big\rangle \big\},$$

where $E \subseteq \mathbb{Z}^2 \setminus R^u(m, n)$. For such a set E, define s_E to be the number of lines orthogonal to u that intersect a connected set of cardinality γ_u in E.

Lemma 3.7. *Let $u \in \mathbb{U}$. Fix $B, Z, \varepsilon > 0$. There exist $N, p_0 > 0$ so that for $p < p_0$,*

$$\mathbb{P}_p\big[\mathcal{C}^u(m,n,E)\big] \le \exp\big[-(1-\varepsilon)h_p^u(p^{\gamma_u}m)(n - Ns_E)\big] \tag{3.6}$$

uniformly in $Zp^{-\gamma_u} \le m \le Bp^{-\gamma_u}$ and $E \subseteq \mathbb{Z}^2 \setminus R^u(m,n)$.

Proof. Fix $B, Z, \varepsilon > 0$ and $u \in \mathbb{U}$. We prove the result by slicing the rectangle into rectangles of fixed (but large) height ℓ. Let us first prove that there exists $k > 0$ so that for any $p < p_0$ and any E with $s_E = 0$,

$$\mathbb{P}_p\big[\mathcal{C}^u(m,k,E)\big] \le \exp[-(1-2\varepsilon)h_p^u(p^{\gamma_u}m)k]. \tag{3.7}$$

Fix

$$K = \max\{\langle S \cup \mathbb{H}_u\rangle \setminus \mathbb{H}_u : S \text{ non-nucleation connected set and } |S| \le \gamma_u\}.$$

Let $\mathcal{E}^u(m,k)$ be the event that $X(\omega) \cap R^u(m,k)$ contains $\gamma_u + 1$ sites in the same ball of radius $2K$. The number of sets of cardinality $\gamma_u + 1$ in a ball of radius $2K$ included in $R^u(m,k)$ is bounded by Mkm for some universal constant $M = M(K) > 0$. Therefore, (2.2) implies for p small enough

$$\mathbb{P}_p\big[\mathcal{E}^u(m,k)\big] \le Mkmp^{\gamma_u+1} \le MBp \le \exp[-h_p^u(Z)k] \le \exp[-h_p^u(p^{\gamma_u}m)k] \tag{3.8}$$

since $B \ge p^{\gamma_u}m \ge Z$.

Let $k > 2K$. Let us assume in the following that $\mathcal{E}^u(m,k)$ does not occur. Make the further assumption that neither $\ell^u(0)$ nor $\ell^{-u}(-k+1)$ is occupied. We used the notation $\ell^{-u}(i)$ in order to specify that the line must be occupied in direction $-u$. We know that E does not connect $\ell^u(0)$ to $\ell^u(k)$ since $s_E = 0$. Since a non-nucleation connected set S with $|S| \le \gamma_u$ is such that $\text{diam}(\langle S \cup S^u(k)\rangle \setminus S^u(k)) \le K$ and no ball of radius $2K$ contains $\gamma_u + 1$ occupied sites, the configuration inside $R^u(m,k)$ cannot connect $\ell^u(0)$ to $\ell^u(k)$. In conclusion, if $\mathcal{E}^u(m,k)$ does not occur, $\ell^u(0)$ or $\ell^u(k)$ must be occupied.

By induction, we deduce that there exists k' between 0 and k such that $\ell^u(1), \dots, \ell^u(k' - K)$ are occupied and $\ell^{-u}(-k'+K), \dots, \ell^{-u}(-k)$ are occupied. Set $\mathbb{P}_p[\mathcal{A}^u(m,k)] = 1$ for $k < 0$. We find

$$\begin{aligned}\mathbb{P}_p\big[\mathcal{C}^u(m,k,E)\big] &\le \mathbb{P}_p\big[\mathcal{E}^u(m,k)\big] + \sum_{k'=0}^{k} \mathbb{P}_p\big[\mathcal{A}^u(m,k'-K)\big]\mathbb{P}_p\big[\mathcal{A}^{-u}(m,k-k'-K)\big] \\ &\le \mathbb{P}_p\big[\mathcal{E}^u(m,k)\big] + k\exp\big[-h_p^u(p^{\gamma_u}m)(k-2K)\big]\end{aligned}$$

where we used the fact that lines at distance greater than $2K$ are independently occupied for the first inequality, and Proposition 2.7 for the second one. Using the fact that $h_p^u(p^{\gamma_u}m) \le h_p^u(B)$, we choose k such that $k + 1 \le \exp[\varepsilon h_p^u(B)(k-2K)]$ and $k - 2K \ge (1-\varepsilon)k$. We deduce

$$\mathbb{P}_p\big[\mathcal{C}^u(m,k,E)\big] \le \exp\big[-(1-2\varepsilon)h_p^u(p^{\gamma_u}m)k\big].$$

Now, the rectangle $R^u(m,n)$ can be divided into $\lfloor n/k \rfloor$ translations of $R^u(m,k)$. Note that a connected set of cardinality γ_u has radius smaller than $\gamma_u|\mathcal{N}|$. Provided $k \geq \gamma_u|\mathcal{N}|$, each of these connected sets intersects at most two rectangles. Therefore, at least $\lfloor n/k \rfloor - 2s_E$ of these translated rectangles satisfy the condition of (3.7). We thus obtain

$$\begin{aligned}\mathbb{P}_p\big[\mathcal{C}^u(m,n,E)\big] &\leq \big(\mathbb{P}_p\big[\mathcal{C}^u(m,n,E)\big]\big)^{\lfloor n/k \rfloor - 2s_E} \\ &\leq \exp\big[-(1-2\varepsilon)h_p^u(p^{\gamma_u}m)(k\lfloor n/k \rfloor - 2ks_E)\big] \\ &\leq \exp\big[-(1-3\varepsilon)h_p^u(p^{\gamma_u}m)(n - Ns_E)\big]\end{aligned}$$

for n large enough and $N = 2k+2$. This concludes the proof. □

We are now in a position to prove (3.2).

Proof of (3.2). Let $\varepsilon, Z, B > 0$. Let $T, p_0 > 0$ to be fixed later. Consider two droplets $D_1[\mathbf{a}] \subseteq D_2[\mathbf{b}]$ satisfying $\Psi(D_1, D_2) \leq Tp^{-\gamma}$ and $p < p_0$.

For $u \in \mathbb{U}$, define the *corner region* H_u to be the set

$$\{x \in D_2 : \langle x|u\rangle \geq a_u \text{ and } \langle x|u_{k-1}\rangle \geq a_{u_{k-1}}\} \cup \{x \in D_2 : \langle x|u\rangle \geq a_u \text{ and } \langle x|u_{k+1}\rangle \geq a_{u_{k+1}}\},$$

where $u = u_k$ and the indexes are considered in $\mathbb{Z}/|\mathbb{U}|\mathbb{Z}$; see Fig. 3. Let $Y_u = s_{X(\omega)\cap H_u}$ be the number of connected sets of cardinality γ_u in the corner region. Let N be such that Lemma 3.7 holds. The full conditioning formula implies

$$\mathbb{P}_p\big[\mathcal{E}(D_1,D_2)\big] \leq \sum_{V\subseteq \mathbb{U}_1} \mathbb{P}_p\left[\mathcal{E}(D_1,D_2)\,\Big|\, Y_u \leq \frac{\varepsilon s_u}{N} \text{ for } u \in V\right] \cdot \mathbb{P}_p\left[Y_u \geq \frac{\varepsilon s_u}{N} \text{ for } u \in \mathbb{U}_1 \setminus V\right].$$

Fix $V \subseteq \mathbb{U}_1$. Let us prove the following two inequalities

$$\mathbb{P}_p\left[Y_u \geq \frac{\varepsilon s_u}{N} \text{ for } u \in \mathbb{U}_1 \setminus V\right] \leq \exp\left[-(1-\varepsilon)\left(\sum_{u\in\mathbb{U}_1\setminus V} h_p^u[m_u^Z]s_u\right)\right], \tag{3.9}$$

$$\mathbb{P}_p\left[\mathcal{E}(D_1,D_2)\,\Big|\, Y_u \leq \frac{\varepsilon s_u}{N} \text{ for } u \in V\right] \leq \exp\left[-(1-\varepsilon)^2\left(\sum_{u\in V} h_p^u[m_u^Z]s_u\right)\right] \tag{3.10}$$

for p small enough, where $\mathbf{m}^Z$ is the dimension of D_1^Z. The proof follows with $Q = 2^{|\mathbb{U}_1|}$ by plugging (3.9) and (3.10) together.

Let us start with (3.9). Fix $u \in \mathbb{U}_1 \setminus V$ such that s_u is maximal. Since $u \notin V$, there exist $Y_u \geq \varepsilon s_u/N$ lines among $\ell^u(a_u), \ldots, \ell^u(b_u)$ intersecting a connected set of γ occupied sites. The fact that $\Psi(D_1,D_2) \leq Tp^{-\gamma}$ implies that a line orthogonal to u has an intersection with H_u of cardinality smaller than $2Tp^{-\gamma}$. In particular, the probability that it intersects a connected set of γ occupied sites is smaller than $cTp^{-\gamma} \cdot p^\gamma = cT$ for some universal constant $c > 0$. The probability that among the s_u lines $\ell^u(a_u), \ldots, \ell^u(b_u)$, at least $\varepsilon s_u/N$ of them intersect a connected set of γ occupied sites is therefore bounded by the probability that the sum of s_u iid Bernoulli variables of parameter cT exceeds $\varepsilon s_u/N$. An application of Chernoff inequality yields

$$\mathbb{P}_p\left[Y_u \geq \frac{\varepsilon s_u}{N}\right] \leq \exp\big[-\beta(T)s_u\big] \leq \exp\left[-\beta(T)\cdot\frac{1}{|\mathbb{U}|}\left(\sum_{u\notin V} s_u\right)\right]$$

where $\beta(T)$ goes to infinity as T goes to 0. In the second inequality, we used the fact that s_u is maximal among directions $u \in \mathbb{U}_1 \setminus V$. Providing $T = T(\varepsilon, N, Z) \ll Z$ small enough, one can find $\beta(T) \geq |\mathbb{U}|h_p^u(Z) > |\mathbb{U}|h_p^u[m_u^Z]$ for every u and $p < p_0$ small enough so that (3.9) holds.

Let us now turn to (3.10). Fix $u \in V$. If $\mathcal{E}(D_1, D_2)$ holds, then $\tilde{T}^u = \{x \in D_2 : \langle x|u\rangle \geq a_u\}$ must be crossed in direction u. We obtain that

$$\mathbb{P}_p\left(\tilde{T}^u \text{ is crossed in } \langle (X\cap D_2)\cup D_1\rangle \;\middle|\; X\cap H_u\right) \leq \exp\big[-(1-\varepsilon)h_p^u[m_u^Z](s_u - NY_u)\big] \tag{3.11}$$

by applying Lemma 3.7 to $\tilde{T}^u$ and $E = X\cap H_u$. In fact, we do not exactly use Lemma 3.7 since $\tilde{T}^u$ is not a rectangle. Nevertheless, the proof works mutatis mutandis, the important feature of $\tilde{T}^u$ being that $|\ell^u(i)\cap(\tilde{T}^u\setminus H_u)| \geq m_u^Z$. We made this slightly abusive approximation in order to make Lemma 3.7 more appealing.

Conditionally on sets $X(\omega)\cap H_u$, sets $\tilde{T}^u$ are crossed independently of each others. We deduce

$$\begin{aligned}
\text{LHS of (3.10)} &\leq \mathbb{P}_p\left(\mathbb{E}_p\left(\mathcal{E}(D_1,D_2)\,|X\cap(\cup_{u\in\mathbb{U}}H_u)\right) \;\middle|\; Y_u \leq \frac{\varepsilon s_u}{N} \text{ for } u\in V\right)\\
&\leq \mathbb{E}_p\left(\prod_{u\in V}\exp\big[-(1-\varepsilon)h_p^u[m_u^Z](s_u - NY_u)\big] \;\middle|\; Y_u \leq \frac{\varepsilon s_u}{N} \text{ for } u\in V\right)\\
&\leq \prod_{u\in V}\exp\left[-(1-\varepsilon)h_p^u[m_u^Z]\left(1 - N\frac{\varepsilon}{N}\right)s_u\right].
\end{aligned}$$

In the first line, we decoupled events by conditioning on the configuration inside the corner regions and in the second, we harnessed (3.11). This concludes the proof. □

4 Lower bound on the probability of creating a critical droplet

A droplet D is said to be *internally spanned* if $\langle X\cap D\rangle = D$. As emphasized in the introduction, we can prove (see Section 6) that the estimate on the time at which 0 becomes occupied boils down to estimating the probability of being internally spanned for droplets of size $Bp^{-\gamma}$, where $B \gg 1$. A droplet of such size will be called *critical.* This denomination comes from the metastability interpretation of the growth. In this section, this probability is shown to be of order $e^{-2\lambda/p^\gamma}$, where λ is defined below.

4.1 Definition of the constant λ

Considering polygons in $\mathbb{R}^2$ of the form

$$D = D[\mathbf{t}] = \bigcap_{u \in \mathbb{U}} \{x \in \mathbb{Z}^2 : \langle x|u\rangle < t_u\},$$

for $\mathbf{t} \in (\mathbb{R}_+^*)^{\mathbb{U}}$. Define $E_u(D)$ in a similar way to the discrete case. We consider only non-degenerated polygons, meaning that edges $E_u(D)$ are not singletons. We call such a polygon a *continuous droplet.*

Let $\|\cdot\|$ be the Euclidean length. The *continuous dimension* of $D[\mathbf{t}]$ is the uplet $\mathbf{m} \in \mathbb{R}^{|U|}$ where $m_u = \|E_u(D)\|/\|u^\perp\|$ for every $u \in \mathbb{U}$. For two droplets $D[\mathbf{t}] \subseteq D'[\mathbf{t}']$, the *continuous location* is given by $\mathbf{s}$ where $s_u = \|t_u - t'_u\|/\|u\|$. We also define $\Psi(D, D') = \sum_{u\in\mathbb{U}} s_u$.

Definition 4.1. *Define*

$$W(D, D') = \sum_{u\in\mathbb{U}_1} h^u[m_u]s_u$$

where h^u is defined in Proposition 2.7. Let

$$\lambda = \frac{1}{2} \inf \left\{ \sum_{n=-\infty}^{\infty} W(D_n, D_{n+1}) : (D_n)_{n\in\mathbb{Z}} \in \mathfrak{D} \right\}$$

where $\mathfrak{D}$ is the set of bi-infinite increasing sequences of continuous droplets $(D_n)_{n\in\mathbb{Z}}$ such that D_n converges to $\mathbb{R}^2$ (resp. to a singleton) as $n \to \infty$ (resp. as $n \to -\infty$).

The definition of λ as the minimizer of some energy is reminiscent of a metastability phenomenon. Since the creation of a droplet of critical size is very unlikely, the procedure to create it tends to minimize the energy. Here, the energy takes the special form of a work along a certain sequence of droplets. The sequence along which the work is minimized is therefore related to the typical shape of a critical droplet.

We now prove the following easy statement.

Proposition 4.2. *The constant λ belongs to $(0, \infty)$.*

Proof. First, λ is strictly positive. Indeed, h^u being decreasing, the functional along any path of $\mathfrak{D}$ is greater than $h^u(B)B$ for any $B > 0$, where $u \in \mathbb{U}_1$ is fixed.

Let $D(r)$ be the continuous droplet with dimensions $\mathbf{m} = (r, \dots, r)$. The symmetric sequence $D(\varepsilon) \subseteq D(2\varepsilon) \subseteq \dots \subseteq D(n\varepsilon) \subseteq \dots$ has energy bounded from above by

$$\varepsilon \sum_{i=1}^{\infty} h^u(\varepsilon i) \le \int_0^\infty h^u(x)dx.$$

This integral is finite thanks to Proposition 2.7. We deduce that λ is finite. □

4.2 Lower bound for creating a critical droplet

Proposition 4.3. *Fix $\varepsilon, B > 0$. There exists $p_0 > 0$ such that for any $p < p_0$, there exists a droplet $D_p \supseteq D(Bp^{-\gamma})$ with*

$$\mathbb{P}_p(D_p \text{ is internally spanned}) \geq \exp\left[-(2\lambda + \varepsilon)/p^\gamma\right].$$

We start by proving that a small droplet is created with fairly good probability.

Lemma 4.4. *Fix $\varepsilon > 0$. There exist $A, p_0 > 0$ such that for any $p < p_0$,*

$$\mathbb{P}_p(D(Ap^{-\gamma}) \text{ is internally spanned}) \geq \exp[-\varepsilon/p^\gamma].$$

Proof. First, there exists $C > 0$ such that for any $k \geq C$ and p small enough,

$$\mathbb{P}_p\big[\mathcal{I}(D(k), D(k+1))\big] \geq \exp\left(-C \log(1 - e^{-p^\gamma k/C})\right). \tag{4.1}$$

Indeed, define $x_0 = 0$ and $x_{i+1} = x_i - ku_i^\perp$. Similarly, define $y_0 = 0$ and $y_{i+1} = y_1 - (k+1)u_i^\perp$. These points are the "corners" of two translates of $D(k)$ and $D(k+1)$. Then, $|\langle y_i - x_i | u\rangle| \leq \sum_{v\in\mathbb{U}} |\langle u_i^\perp | u\rangle|$. We deduce that $\Psi(D(k), D(k+1)) \leq C$ for some universal $C > 0$. If $k \geq C|\mathcal{N}|$, one can consider a translate of $R^u(k - C|\mathcal{N}|, s_u)$ included in T^u, where T^u is defined as in the proof of (3.1). If $\mathcal{B}(R^u)$ denotes the fact that each line of this rectangle contains a translate of $z + \{nu^\perp : n = 1, \ldots, \gamma\}$, we obtain (4.1) by using a lower bound on $\mathbb{P}_p[\mathcal{B}(R^u)]$ similar to the one obtained in the proof of Property (2) of Proposition 2.7.

Now, let $\mathcal{E}$ be the event that every site in $D(C)$ is occupied. The Harris inequality implies

$$\begin{aligned}
\mathbb{P}_p\big[D(Ap^{-\gamma}) \text{ is internally spanned}\big] &\geq \mathbb{P}_p(\mathcal{E}) \prod_{k=C}^{Ap^{-\gamma}} \mathbb{P}_p\big[\mathcal{I}(D(k), D(k+1))\big] \\
&\geq p^{|D(C)|} \prod_{k=0}^{Ap^{-\gamma}} \exp\left[-C\log(1 - e^{-p^\gamma k/C})\right] \\
&\geq p^{|D(C)|} \exp\left[-\frac{\int_0^A C \log(1 - e^{-x/C})dx}{p^\gamma}\right]
\end{aligned}$$

for p small enough. The claim follows by choosing A small enough. □

Proof of Proposition 4.3. Let $\varepsilon, B > 0$. Extend the notation $D^Z(p) = D \oplus D(Zp^{-\gamma})$ to continuous droplets by setting $D^Z = D \oplus D(Z)$. Choose A such that Lemma 4.4 holds.

Let $0 < T \ll Z \ll A$ such that for every $p < p_0$, there exists a sequence of continuous droplets $(D_n)_{n\leq N}$ such that:

- D_0^Z is included in a translate of $D(A)$,
- $D(B) \subseteq D_N^Z \subseteq D(B')$ where $B' > B$,

- $\Psi(D_n^Z, D_{n+1}^Z) \le T$ for every $0 \le n \le N-1$,
- $\sum_{n=0}^{N} W(D_n^Z, D_{n+1}^Z) \le 2\lambda + \varepsilon$.

In order to justify the existence of $(D_n)_{n\ge N}$, we used the fact that h^u is continuous and integrable. Let $D_n^Z = D_n^Z(\mathbf{a}_n)$.

Fix p small enough. Construct the discrete droplets $(D_n^Z)_p = (D_n^Z)_p(\lfloor \mathbf{a}_n p^{-\gamma}\rfloor)$ where $\lfloor \mathbf{a}_n p^{-\gamma}\rfloor = (\lfloor (a_n)_u p^{-\gamma}\rfloor)_{u\in\mathbb{U}}$. We obtain for p small enough

$$\begin{aligned}
\mathbb{P}_p\big[(D_N^Z)_p \text{ int. spanned}\big] &\ge \mathbb{P}_p\big[D(Ap^{-\gamma}) \text{ int. spanned}\big] \cdot \prod_{n=0}^{N-1} \mathbb{P}_p\big[\mathcal{I}((D_n^Z)_p, (D_{n+1}^Z)_p)\big] \\
&\ge \exp[-\varepsilon/p^\gamma] \prod_{n=0}^{N-1} Q^{-1} \exp\left[-(1+\varepsilon)\frac{W_p((D_n^Z)_p, (D_{n+1}^Z)_p)}{p^\gamma} - \frac{\varepsilon}{B}\Psi((D_n^Z)_p, (D_{n+1}^Z)_p)\right] \\
&\ge \exp[-\varepsilon/p^\gamma] \prod_{n=0}^{N-1} Q^{-1} \exp\left[-(1+\varepsilon)^2\frac{W(D_n^Z, D_{n+1}^Z)}{p^\gamma} - \frac{\varepsilon}{B}\Psi(D_n^Z, D_{n+1}^Z)\right] \\
&\ge \exp\left[-\frac{\varepsilon + \varepsilon + (1+\varepsilon)^2(2\lambda+\varepsilon) + \varepsilon c'}{p^\gamma}\right],
\end{aligned}$$

where c' is some universal constant. In the first inequality, we used the Harris inequality and the fact that $(D_0^Z)_p$ can be included in a translate of $D(Ap^{-\gamma})$. In the second, we used Lemma 4.4 and Proposition 3.1. In the third, we used that h_p^u converges uniformly to h^u. In the last, we harnessed the fourth property of the sequence (D_n) together with the fact that $\Psi(D_0^Z, D_N^Z) \le c'Bp^{-\gamma}$ for some universal constant $c' > 0$. We also used that Q^{-N} is constant. The claim follows by choosing ε small enough. □

Let us mention the following corollary, which asserts that once the droplet $D(Bp^{-\gamma})$ has been created, it grows will small cost.

Corollary 4.5. *Fix $\varepsilon > 0$. There exists $p_0 > 0$ such that for any $p < p_0$,*

$$\mathbb{P}_p(D(p^{-3\gamma}) \text{ is internally spanned}) \ge \exp\left[-(2\lambda+\varepsilon)/p^\gamma\right].$$

Proof. Fix $\varepsilon > 0$. Let $B > 0$ such that $\int_B^\infty C\log(1-e^{-x/C})dx \le \varepsilon$. Proposition 4.3 implies that there exists a droplet $D_p \supseteq D(Bp^{-\gamma})$ and

$$\mathbb{P}_p(D_p \text{ is internally spanned}) \ge \exp[-(\lambda+\varepsilon)/p^\gamma]. \tag{4.2}$$

The bound (4.1) applied to $\mathcal{I}(D(k), D(k+1))$ for $Bp^{-\gamma} \le k \le p^{-3\gamma}$ implies

$$\mathbb{P}_p(D(p^{-3\gamma}) \text{ internally spanned}) \ge \exp[-\varepsilon/p^\gamma]\mathbb{P}_p(D_p \text{ is internally spanned}). \tag{4.3}$$

We used the fact that $D(Bp^{-\gamma}) \subseteq D_p$. Equations (4.2) and (4.3) lead to

$$\mathbb{P}_p(D(p^{-3\gamma}) \text{ internally spanned}) \ge \exp[-(\lambda+2\varepsilon)/p^\gamma]$$

. □

5 Upper bound on the probability of crossing a droplet

In this section, we deduce an upper bound on the probability that a droplet of size $Bp^{-\gamma}$ is crossed. We proceed by studying all the possible ways of crossing a droplet.

Proposition 5.1. *Let $\epsilon > 0$. There exist $B, p_0 > 0$ such that*

$$\mathbb{P}_p(D \text{ is crossed}) \leq \exp[-(2\lambda - \varepsilon)/p^\gamma]$$

for any $p < p_0$ and any droplet D with a perimeter between $Bp^{-\gamma}$ and $1 + |\mathcal{N}|^2 Bp^{-\gamma}$.

5.1 Definition of a hierarchy

The notion of hierarchy and the specific vocabulary associated to it is introduced in this section. This notion generalizes the notion of hierarchy pioneered in [Hol03] for simple bootstrap percolation.

Recall that for a tree, the offsprings of a vertex are its neighbors further from the root. A tree is called a d-ary tree if all the vertices have not more than d offsprings.

Definition 5.2 (Hierarchy)**.** *A* hierarchy $\mathcal{H}$ *is a $|\mathcal{N}|$-ary tree with vertex labeled by non-empty droplets such that the label of a vertex contains the labels of its offsprings.*

The label of a vertex v is denoted by D_v. A vertex of $\mathcal{H}$ is called a *seed* (resp. a *normal vertex*, resp. a *splitter*) if it has no offspring (resp. one offspring, resp. strictly more than one).

Definition 5.3 (Precision of a hierarchy)**.** *A hierarchy* of precision (s, t) *(with $t \geq 2s \geq 1$) is a hierarchy such that:*

(1) if u is a seed, then $\Phi(D_u) < t$. If u is a normal vertex or a splitter, then $\Phi(D_u) \geq t$.

(2) If u is a normal vertex with offspring v, then $\Psi(D_v, D_u) \leq s$.

(3) If u is a normal vertex with a seed or a normal vertex v as its only offspring, then $\Psi(D_v, D_u) > s/2$.

(4) If u is a splitter and v is one of its offspring, then $\Psi(D_v, D_u) > s/2$.

The following lemma tells us that there are not too many hierarchies of precision $(Tp^{-\gamma}, Zp^{-\gamma})$. This property is important. It assures that we will not loose too much in the upper bound because of the entropy due to the number of hierarchies.

Lemma 5.4 (number of hierarchies)**.** *Let $0 < 2T < Z < B$ and $p > 0$. There exists $C > 0$ such that the number $N_{T,Z,B}(p)$ of hierarchies of precision $(Tp^{-\gamma}, Zp^{-\gamma})$ for a droplet D with $\Phi(D) \leq Bp^{-\gamma}$ satisfies*

$$N_{T,Z,B}(p) \leq Cp^{-C}.$$

Proof. Let $0 < 2s < t$. The definition of the hierarchy of precision (s,t) implies that every two steps further from the root, the absolute location of droplets decreases of at least $s/2$. Therefore, a hierarchy with root label D is a labeled $|\mathcal{N}|$-ary tree of height $2\Phi(D)/s$ at the most. In particular, the hierarchy has less than $|\mathcal{N}|^{2\Phi(D)/s}$ vertices. There are less than $|\mathcal{N}|^{|\mathcal{N}|^{2\Phi(D)/s}}$ possible $|\mathcal{N}|$-ary trees with less than $|\mathcal{N}|^{2\Phi(D)/s}$ vertices. Moreover, each label is a droplet included in D. Thus, the number of possible labels for a prescribed vertex is smaller than $\Phi(D)^{|\mathbb{U}|} \cdot c\Phi(D)^2$ since one needs to choose its dimension and mark one of its site. We harnessed the fact that the volume of a droplet D is bounded by $c\Phi(D)^2$ where c is a sufficiently large universal constant. We obtain

$$N_{T,Z,B}(p) \leq |\mathcal{N}|^{|\mathcal{N}|^{2\Phi(D)/s}} \left(c(Bp^{-\gamma})^{|\mathbb{U}|+2}\right)^{|\mathcal{N}|^{2\Phi(D)/s}} \leq |\mathcal{N}|^{|\mathcal{N}|^{2B/T}} \left(c(Bp^{-\gamma})^{|\mathbb{U}|+2}\right)^{|\mathcal{N}|^{2B/T}},$$

which is the claim. □

5.2 Encoding of a growth via occurring hierarchies

We now relate the concept of hierarchy to our study.

Definition 5.5 (Occurrence of a hierarchy). *A hierarchy* occurs *if all of the following events occur disjointly:*

(1) if u is a seed, D_u is crossed;

(2) if u is a normal vertex with offspring v, then $\mathcal{E}(D_v, D_u)$ occurs;

(3) if u is a splitter with offsprings $v_1, \dots, v_i$, then $\left\langle D_{v_1} \cup \dots \cup D_{v_j} \right\rangle$ is a connected set and D_u is the smallest droplet containing it.

This section is devoted to the proof of the following deterministic proposition:

Proposition 5.6. *Let $t \geq 2s \geq 1$ and D be a non-empty droplet. If D is crossed, then there exists a hierarchy of precision (s,t) with root-label $D_r = D$ that occurs.*

This proposition illustrates the interest for hierarchies. They encode how a droplet is crossed by decomposing this complicated event into events that are easier to control. The following lemma is a technical step.

Lemma 5.7. *Let $\mathcal{S}$ be an internally spanned and connected set of cardinality greater than $|\mathcal{N}|$, then there exist i non-empty connected sets $\mathcal{S}_1$, ..., $\mathcal{S}_i$ with $i \in \{2, \dots, |\mathcal{N}|\}$ such that*

(1) the strict inclusion $\mathcal{S}_j \subset \mathcal{S}$ holds for every $j \in \{1, \dots, i\}$;

(2) $\langle \mathcal{S}_1 \cup \dots \cup \mathcal{S}_i \rangle = \mathcal{S}$;

(3) $\{\mathcal{S}_1 \text{ is internally spanned}\} \circ \dots \circ \{\mathcal{S}_i \text{ is internally spanned}\}$ occurs.

Proof. Let K be finite. The set $\langle K\rangle$ may be constructed via the following algorithm. For each time step $t \in \{0,\dots,\tau\}$, we find a collection of m_t connected sets $\mathcal{S}_1^t,\dots,\mathcal{S}_{m_t}^t$, and corresponding sets of sites $K_1^t,\dots,K_{m_t}^t$ with the following properties:

(i) $K_1^t,\dots,K_{m_t}^t$ are pairwise disjoint

(ii) $K_i^t \subseteq K$ for every $i \le m_t$,

(iii) $\mathcal{S}_i^t = \langle K_i^t\rangle$ is connected for every $i \le m_t$,

(iv) if $i \neq j$ then $\mathcal{S}_i^t \nsubseteq \mathcal{S}_j^t$,

(v) $K \subseteq \mathcal{S}^t \subseteq \langle K\rangle$ where $\mathcal{S}^t = \bigcup_{i=1}^{m_t} \mathcal{S}_i^t$,

(vi) $\mathcal{S}^\tau = \langle K\rangle$.

We construct the sets by induction. Let K be enumerated as $K = \{x_1,\dots,x_r\}$. Set $m_0 = r$ and $\mathcal{S}_i^0 = K_i^0 = \{x_i\}$. Suppose that we have already constructed the sets $\mathcal{S}_1^t,\dots,\mathcal{S}_{m_t}^t$ and $K_1^t,\dots,K_{m_t}^t$, then

(a) if there exist sets $\mathcal{S}_{i_1}^t,\dots,\ \mathcal{S}_{i_j}^t$ with $2 \le j \le |\mathcal{N}|$ such that $\langle \mathcal{S}_{i_1}^t \cup \dots \cup \mathcal{S}_{i_j}^t\rangle$ is connected, set $K' = K_{i_1}^t \cup \dots \cup K_{i_j}^t$ and $S' = \langle \mathcal{S}_{i_1}^t \cup \dots \cup \mathcal{S}_{i_j}^t\rangle$. We construct the list $(\mathcal{S}_1^{t+1}, K_1^{t+1}),\dots,(\mathcal{S}_{m_{t+1}}^{t+1}, K_{m_{t+1}}^{t+1})$ as follows. From the list $(\mathcal{S}_1^t, K_1^t),\dots,(\mathcal{S}_{m_t}^t, K_{m_t}^t)$, delete every pair $(\mathcal{S}_\ell^t, K_\ell^t)$ for which $\mathcal{S}_\ell^t \subseteq \mathcal{S}'$. Then add $(\mathcal{S}', K')$ to the list.

(b) else stop the algorithm and set $t = \tau$.

Properties (i) to (v) are obviously preserved during the induction and m_t is strictly decreasing with t, so the construction must stop. We aim to prove that property (vi) holds. Since $K \subseteq \mathcal{S}^\tau \subseteq \langle K\rangle$, we have $\langle \mathcal{S}^\tau\rangle = \langle K\rangle$. We now proceed by contradiction. Assume $\langle K\rangle \neq \mathcal{S}^\tau$. There exists a site in $\mathcal{T}(\mathcal{S}^\tau)\setminus\mathcal{S}^\tau$. This site y is such that $y \in \langle K\rangle\setminus\mathcal{S}^\tau$ and $y+\mathcal{N}$ contains θ occupied sites in $\mathcal{S}^\tau$. Furthermore, the occupied sites must lie in sets $\mathcal{S}_{i_1}^\tau,\dots,\mathcal{S}_{i_j}^\tau$ where j must be larger or equal to 2 since y is not in $\mathcal{S}^\tau$. Sets $\mathcal{S}_{i_1}^\tau,\dots,\mathcal{S}_{i_j}^\tau$ constitute an uplet which corresponds to the case (a) of the algorithm (since $y+\mathcal{N}$ connect all these sets). Therefore the algorithm should not have stopped at time τ. We have reached a contradiction.

Let us now construct $\mathcal{S}_1,\dots,\mathcal{S}_i$. Let K such that $\langle K\rangle = \mathcal{S}$ and apply the algorithm described above. There must be at least one time step (i.e. $\tau \ge 1$) since the cardinality of $\mathcal{S}$ is greater than $|\mathcal{N}|$. Consider the last time step of the algorithm (from time $\tau-1$ to time τ). The sets involved in the creation of $\mathcal{S}' = \mathcal{S}^\tau$ fulfill all of the required properties. □

In this proof, the notion of connectedness is used in a crucial way. The following corollary will be primordial in the proof of the theorem.

Corollary 5.8. *For any crossed droplet D and $k \le \Phi(D)$, there exists a crossed droplet D' with $\frac{k}{|\mathcal{N}|^2} - 1 \le \phi(D') \le k$.*

Proof. We may construct the droplet D' as follows. Assume that D is a crossed droplet. There must exist a connected and internally spanned set $\mathcal{S}$ that intersects every edge of D. Lemma 5.7 applied to $\mathcal{S}$ shows the existence of internally spanned sets $\mathcal{S}_1, \ldots, \mathcal{S}_i$ such that $\langle \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_i \rangle = \mathcal{S}$. Now, consider the smallest droplets $D_1, \ldots, D_i$ containing $\mathcal{S}_1, \ldots, \mathcal{S}_i$.

The perimeter of D is smaller than the sum of the perimeters of droplets D_i plus $|\mathcal{N}|^2$. This last term is due to the fact that droplets do not necessarily touch each other. There must exist a crossed droplet with an absolute location between $(\Phi(D) - |\mathcal{N}|^2)/|\mathcal{N}|^2$ and $\Phi(D)$. Reiterating this operation, we can consider a sequence of crossed droplets with decreasing perimeter. We stop the procedure the first time we obtain a droplet with perimeter between $\frac{k}{|\mathcal{N}|^2} - 1$ and k. □

Proof of Proposition 5.6. The proof is an induction on the perimeter of the droplet.

Let D be a crossed droplet. If $\Phi(D) < t$, then the hierarchy with only one vertex r and $D_r = D$ occurs. Consider a droplet D with $\Phi(D) \geq t$. Assume that the proposition holds for any droplet with perimeter less than $\Phi(D)$.

We construct a sequence of droplets iteratively. Fix $D_1^0 = D$ and $\mathcal{S}_1^0 = \langle K \cap D \rangle$. Assume that $D_1^t, \ldots, D_{m_t}^t$ are defined. Using Lemma 5.7, there exist disjoint connected sets $\mathcal{S}_1^{t+1}, \ldots, \mathcal{S}_{j_1}^{t+1}$ spanning $\langle K \cap D_1^t \rangle$. There also exists $\mathcal{S}_{j_1+1}^{t+1}, \ldots, \mathcal{S}_{j_2}^{t+1}$ spanning $\langle K \cap D_2^t \rangle$, etc. We can construct a sequence $\mathcal{S}_1^{t+1}, \ldots, \mathcal{S}_{m_{t+1}}^{t+1}$ of internally spanned sets such that the j_1 first droplets span $\mathcal{S}_1^t$, the j_2 next ones span $\mathcal{S}_2^t$, etc. Let $D_1^{t+1}, \ldots, D_{m_{t+1}}^{t+1}$ be the smallest droplets containing $\mathcal{S}_1^{t+1}, \ldots, \mathcal{S}_{m_{t+1}}^{t+1}$. Stop at the first time step T for which the droplet $D' \in \{D_j^t : j \leq m_t\}$ with largest total location with respect to D satisfies $\Psi(D', D) \geq s/2$. Note that D' exists since

$$\Psi(\{x\}, D) = \sum_{u \in \mathbb{U}} |a_u - a_{-u}| \geq \sum_{u \in \mathbb{U}} \left(\frac{1}{2} \sum_{v \neq u} m_v \right) \geq \Phi(D) \geq t > 2s$$

for any $x \in D$. We have used the fact that $|a_u - a_{-u}|$ denotes the number of line of direction u intersecting D to show that $|a_u - a_{-u}| \geq \frac{1}{2} \sum_{v \neq u} m_v$. We are now facing three cases:

(1) If $\Psi(D', D) \leq s$. Since D' is crossed, the induction hypothesis implies that there exists a hierarchy $\mathcal{H}'$ of precision (s,t) with root r' and root-label $D_{r'} = D'$. By construction, the event $\mathcal{E}(D', D)$ occurs. Let r be a vertex with label $D_r = D$. Construct a hierarchy by adding the edge (r, r') to the hierarchy $\mathcal{H}'$. This hierarchy is indeed a hierarchy of precision (s,t). The event $\mathcal{E}(D', D)$ is disjoint from the event that $\mathcal{H}'$ occurs since $\mathcal{E}(D', D)$ depends on sites outside of D'. This hierarchy thus occurs.

(2) If $\Psi(D', D) > s$ and $T = 1$. By the induction hypothesis, there exist hierarchies $\mathcal{H}_1, \ldots, \mathcal{H}_{m_1}$ occurring for each D_i^1. Note that these hierarchies are occurring disjointly since $\mathcal{S}_1^1, \ldots, \mathcal{S}_{m_1}^1$ occur disjointly by Lemma 5.7. Consider a vertex r with label $D_r = D$. Let $\mathcal{H}$ be the hierarchy created by adding the edges (r, r_j) for $j \leq m_1$, where r_j is the root of $\mathcal{H}_j$. The hierarchy $\mathcal{H}$ is of precision (s,t) since $\Psi(D_i^1, D) \geq s/2$

for every $i \leq m_1$. Moreover, the hierarchy occurs since D is the smallest droplet containing $\langle D_1^1 \cup \ldots \cup D_{m_1}^1 \rangle$ and hierarchies $\mathcal{H}_1, \ldots, \mathcal{H}_{m_1}$ occur disjointly.

(3) if $\Psi(D', D) > s$ and $T \geq 2$. Consider the droplet D'' from which D' has been created. Denote the offsprings of D'' by $D_1 = D', \ldots, D_m$. There exist hierarchies $\mathcal{H}_1, \ldots, \mathcal{H}_m$ associated to $D_1, \ldots, D_m$ that occur disjointly. Consider a root r with label $D_r = D$ and a second vertex y with label $D_y = D''$. Construct a hierarchy through the process of adding $m + 1$ additional edges (r, y) and (y, r_j) for $j = 1, \ldots, m$ where r_j is the root of $\mathcal{H}_j$. This hierarchy occurs since D'' is spanned disjointly by $D_1, \ldots, D_m$. It is therefore sufficient to check that it is a hierarchy of precision (s, t). In order to do so, note that $\Psi(D'', D) \leq s/2$ and $\Psi(D_i, D'') \geq s/2$. This last inequality follows from $\Psi(D'', D) > s/2$ and $\Psi(D_i, D) > s$.

□

5.3 Occurrence probability for hierarchies

Let us introduce the following quantity.

Definition 5.9. *The* pod *of a hierarchy $\mathcal{H}$, denoted by* $\text{Pod}(\mathcal{H})$*, is defined by*

$$\text{Pod}(\mathcal{H}) = \sum_{w \text{ seed of } \mathcal{H}} \Phi(D_w).$$

Our goal is to bound the probability that a hierarchy $\mathcal{H}$ with precision $(Tp^{-\gamma}, Zp^{-\gamma})$ occurs. If $\mathcal{H}$ occurs, two events hold:

(1) the event $\mathcal{S}(\mathcal{H})$ that every seeds of $\mathcal{H}$ are crossed disjointly;

(2) the event $\mathcal{E}(\mathcal{H})$ that $\mathcal{E}(D_v, D_u)$ occurs disjointly for every normal vertex $u \in \mathcal{H}$ with offspring v.

The strategy to bound the probability of an occurring hierarchy is the following. In Lemma 5.10, the probability of $\mathcal{S}(\mathcal{H})$ is proved to be very small whenever $\text{Pod}(\mathcal{H})$ is large. Therefore, we can restrict our attention to hierarchies with small pod. For such a hierarchy $\mathcal{H}$, one can find a hierarchy $\mathcal{H}'$ without splitter such that $\mathcal{E}(\mathcal{H})$ is less likely than $\mathcal{E}(\mathcal{H}')$. This implies that the maximum probability of occurrence over all possible hierarchies is achieved for hierarchies without splitter.

Lemma 5.10. *Let $\mu, A > 0$. There exist $Z, p_0 > 0$ such that for $p \leq p_0$*

$$\mathbb{P}_p(\mathcal{S}(\mathcal{H})) \leq \exp(-\mu/p^{\gamma})$$

for any hierarchy $\mathcal{H}$ of precision $(Tp^{-\gamma}, Zp^{-\gamma})$ satisfying $\text{Pod}(\mathcal{H}) \geq Ap^{-\gamma}$.

This lemma comes from the fact that, proportionally to their size, seeds have very small probability to be crossed.

Proof. Fix $\mu, A > 0$. For any seed $D_w = D_w(\mathbf{a})$ of $\mathcal{H}$, there exists a direction $u \in \mathbb{U}_1$ such that $|a_u - a_{-u}| \geq \phi(D_w)/|\mathbb{U}|$. Therefore, D_w is included in a translate of $R^u(Zp^{-\gamma}, \phi(D_w)/|\mathbb{U}|)$. The probability that D_w is crossed is thus smaller than the probability for $R^u(Zp^{-\gamma}, \phi(D_w)/|\mathbb{U}|)$ to be crossed in direction u. Using Lemma 3.7 with $E = \emptyset$, we find for p small enough

$$\mathbb{P}_p\big[D_w \text{ is crossed}\big] \leq \exp\big[-(1-\varepsilon)h_p^u(Z)\Phi(D_w)/|\mathbb{U}|\big].$$

The BK inequality applied to the disjointly occurring events $\{D_w \text{ is crossed}\}$ gives us

$$\begin{aligned}\mathbb{P}_p(\mathcal{S}(\mathcal{H})) &\leq \prod_{w \text{ seed}} \mathbb{P}_p(D_w \text{ is crossed}) \\ &\leq \prod_{w \text{ seed}} \exp\big[-(1-\varepsilon)h_p^u(Z)\Phi(D_w)/|\mathbb{U}|\big] \\ &= \exp[-(1-\varepsilon)h_p^u(Z)\mathrm{Pod}(\mathcal{H})/|\mathbb{U}|] \leq \exp\left[-\frac{(1-\varepsilon)Ah_p^u(Z)/|\mathbb{U}|}{p^\gamma}\right].\end{aligned}$$

Thanks to Proposition 2.7, $h_p^u \to \infty$ as $p \to 0$ so that $Z > 0$ and $p_0 > 0$ can be chosen in such a way that $A(1-\varepsilon)h_p^u(Z)/|\mathbb{U}| > \mu$ for every $p \leq p_0$. □

From now on, we restrict ourselves to hierarchies with small pod. We aim to construct $\mathcal{H}'$ with no splitter and larger probability of occurrence than $\mathcal{H}$. Lemma 5.11 is a technical statement allowing us to create $\mathcal{H}'$. Lemma 5.13 bounds the probability in terms of the functional W_p. Finally, Proposition 5.1 plugs all these lemmas together.

Lemma 5.11. *Let $\mathcal{H}$ be a hierarchy and s be a splitter of $\mathcal{H}$ with label D_s. If $D_1,\dots,D_I$ are the labels of the offsprings of s, then there exists a translate of $D_1 \oplus \mathcal{N} \oplus \dots \oplus D_I$ that contains D_s.*

Proof. Since s is a splitter, $D_1, \dots, D_I$ form a connected set. Adding neighborhoods $\mathcal{N}$ inbetween droplets (we only need $I-1$ such sets), we can consider only droplets that are connected in the canonical sense of $\mathbb{R}^2$. Also note that it is sufficient to consider the case of two droplets D and D'.

Let $D = D(\mathbf{a})$ and $D' = D'(\mathbf{a}')$. Since $D \cap D' \neq \emptyset$, we have $\max(a_u, a'_u) - \min(a_{-u}, a'_{-u}) \leq |a_u + a'_u - a_{-u} - a'_{-u}|$. The result follows readily from the fact that $D \oplus D'$ is a translate of $D''(\mathbf{a} + \mathbf{a}')$. □

Lemma 5.12. *Let $D_1 \subseteq D_2$ and $D'_1 \subseteq D'_2$ be two pairs of droplets. We have*

$$W_p(D_1, D_2) + W_p(D'_1, D'_2) \geq W_p(D_1 \oplus D'_1, D_1 \oplus D'_2) + W_p(D_1 \oplus D'_2, D_2 \oplus D'_2).$$

This lemma is the main reason why the occupied sites form droplets. It is always more interesting for the occupied sites to appear near existing occupied sites. Hence, the dynamics has a tendency to create large droplets.

Proof. This follows from the facts that h_p^u is increasing and Lemma 3.4. Indeed, it implies $W_p(D_1 \oplus D, D_2 \oplus D) \leq W_p(D_1, D_2)$ and $W_p(D \oplus D_1, D \oplus D_2) \leq W_p(D_1, D_2)$ for any droplets $D_1 \subset D_2$ and D. □

Proposition 5.13. *Let $\varepsilon, Z, B > 0$. There exist $0 < 2T < Z$ and $p_0 > 0$ such that for any hierarchy $\mathcal{H}$ of precision $(Tp^{-\gamma}, Zp^{-\gamma})$ with root label $D \subseteq D(Bp^{-\gamma})$ and M vertices, there exists an increasing sequence of droplets $D_1 \subseteq \cdots \subseteq D_M$ satisfying:*

- $\Phi(D_1) \leq |\mathcal{N}|^2 M + \mathrm{Pod}(\mathcal{H})$,
- $D \subseteq D_M$ *and*
- $\mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq Q^M \exp\left[-(1-\varepsilon)\sum_{n=1}^{M-1} W_p(D_n^Z, D_{n+1}^Z)/p^\gamma\right]$ *for* $p < p_0$.

Proof. Let $\varepsilon, Z, B > 0$ and consider $Q, T, p_0 > 0$ such that Proposition 3.2 holds. We proceed by induction on hierarchies. Let D_r be the label of the root of $\mathcal{H}$.

(1) Assume the root r is a seed. Then $\mathcal{H}$ is a singleton and it is sufficient to set $D_1 = D_M = D_r$.

(2) Assume the root r is a normal vertex. Set D_u to be the label of its unique offspring u. By induction on the hierarchy with root u, there exists a sequence $(D_n)_{n\leq M-1}$ satisfying the conditions of the induction. In particular, $D_{M-1} \subseteq D_u$. Setting D_M to be the smallest droplet containing $\langle D_r \cup D_{M-1}\rangle$, we obtain

$$\mathbb{P}_p(\mathcal{E}(D_u, D_r)) \leq \mathbb{P}_p(\mathcal{E}(D_{M-1}, D_M)) \leq Q \exp\big[-(1-\varepsilon)W_p(D_{M-1}^Z, D_M^Z)/p^\gamma\big].$$

In the first inequality, we used the fact that the event $\mathcal{E}(D_u, D_r)$ is included in $\mathcal{E}(D_{M-1}, D_M)$ and in the second, (3.2). The induction hypothesis is then verified with $(D_n)_{n\leq M}$.

(3) Assume the root is a splitter. Denote by $\mathcal{H}_i$ the associated hierarchies of root r_i where $r_1, \ldots, r_I$ are the offsprings of r. By induction, there exist I sequences $(D_n^i)_{n\leq N_i}$ satisfying for every i

$$\mathbb{P}_p(\mathcal{E}(\mathcal{H}_i)) \leq Q^{M_i} \exp\left[-(1-\varepsilon)\sum_{n=1}^{M_i} W_p\big((D_n^i)^Z, (D_{n+1}^i)^Z\big)/p^\gamma\right],$$

where M_i is the number of vertices in $\mathcal{H}_i$. Considering Lemma 5.12, it is possible to define a sequence $(D_n)_{n\leq M-1}$ such that

$$D_1 = D_1^1 \oplus \mathcal{N} \oplus \ldots \oplus D_1^I$$
$$D_{M-1} = D_{M_1}^1 \oplus \mathcal{N} \oplus \ldots \oplus D_{M_I}^I$$

and

$$\sum_{n=1}^{M-1} W_p\big(D_n^Z, D_{n+1}^Z\big) \leq \sum_{i=1}^{I}\sum_{n=1}^{M_i} W_p\big((D_n^i)^Z, (D_{n+1}^i)^Z\big).$$

We used that $\sum_{i\leq I} M_i = M - 1$. We deduce

$$\mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq Q^{M-1} \exp\left[-(1-\varepsilon)\sum_{n=1}^{M-1} W_p\big(D_n^Z, D_{n+1}^Z\big)/p^\gamma\right].$$

Lemma 5.11 implies that one can choose a translate of the sequence such that D_r is included in D_{M-1}. Moreover,

$$\Phi(D_1) \leq N^2 + \sum_{i=1}^{I} (\text{Pod}(\mathcal{H}_i) + N^2 M_i) = \text{Pod}(\mathcal{H}) + N^2 M$$

so that the induction hypothesis follows if we fulfill the last requirement on the length of (D_n) by adding $D_M = D_{M-1}$.

□

5.4 Proof of Proposition 5.1

Let $\varepsilon > 0$ small enough. By the definition of λ, one can choose $A, B, p_0 > 0$ such that for any $D_0 \subseteq \cdots \subseteq D_N$ such that $\Phi(D_0^Z) \leq 2Ap^{-\gamma}$ and $D(Bp^{-\gamma}) \subseteq D_N^Z$,

$$\sum_{n=0}^{N} W_p(D_n^Z, D_{n+1}^Z) \geq \frac{2\lambda - 3\varepsilon}{(1-\varepsilon^2)^2}$$

for $p \leq p_0$. Let $Z, p_0 > 0$ such that Lemma 5.10 holds with A and $\mu = 2\lambda - 2\varepsilon$. Note that we could be lead to tune p_0 again. Finally, choose $T, p_0 > 0$ so that Lemma 5.13 holds with ε^2.

Let D be a droplet with perimeter between $Bp^{-\gamma}$ and $1 + |\mathcal{N}|^2 Bp^{-\gamma}$. If D is crossed, Proposition 5.6 implies the existence of an occurring hierarchy of precision $(Tp^{-\gamma}, Zp^{-\gamma})$. Using Lemma 5.4, we obtain for p small enough,

$$\mathbb{P}_p(D \text{ is crossed}) \leq N_{B,Z,T,p} \cdot \max\{\mathbb{P}_p(\mathcal{H} \text{ occur})\} \leq \exp[\varepsilon/p^\gamma] \max\{\mathbb{P}_p(\mathcal{H} \text{ occur})\}$$

where the maximum is taken over hierarchies of precision $(Tp^{-\gamma}, Zp^{-\gamma})$ with root-label D. We have used Lemma 5.4 to bound the number of possible hierarchies by a polynomial factor. It is thus sufficient to prove that

$$\mathbb{P}_p(\mathcal{H} \text{ occurs}) \leq \exp[-(2\lambda - 2\varepsilon)/p^\gamma]$$

for any hierarchy $\mathcal{H}$ of precision $(Tp^{-\gamma}, Zp^{-\gamma})$ with root label D satisfying $Bp^{-\gamma} \leq \phi(D) \leq 1 + |\mathcal{N}|^2 Bp^{-\gamma}$. If $\text{Pod}(\mathcal{H}) \geq Ap^{-\gamma}$, Lemma 5.10 implies that

$$\mathbb{P}_p(\mathcal{H} \text{ occurs}) \leq \mathbb{P}_p(\mathcal{S}(\mathcal{H})) \leq \exp[-(2\lambda - 2\varepsilon)/p^\gamma].$$

If $\text{Pod}(\mathcal{H}) \leq Ap^{-\gamma}$, Lemma 5.13 yields the existence of a sequence $D_0, \ldots, D_N$ satisfying:

$$\mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq Q^M \exp\left[-(1-\varepsilon)\sum_{n=0}^{N-1} W_p(D_n^Z, D_{n+1}^Z)/p^\gamma\right],$$

where M is the number of vertices in the hierarchy. Furthermore, $D(Bp^{-\gamma}) \subseteq D_N^Z$ and

$$\Phi(D_0^Z) \leq |\mathcal{N}|^2 M + \text{Pod}(\mathcal{H}) \leq 2Ap^{-\gamma}$$

for p small enough. Therefore,

$$\mathbb{P}_p(\mathcal{H} \text{ occurs}) \leq \mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq Q^M \exp[-(2\lambda - 3\varepsilon)/p^\gamma].$$

Since $M = M(Z, T, B)$ does not depend on p, the proof follows readily by letting p go to 0.

6 Proof of Theorem 1.1

The proof is divided into two parts: the upper and the lower bound on T. These two bounds are very much related to the probability that a droplet of size B/p^γ is internally spanned/crossed at some distance from the origin.

Proof of the upper bound in Theorem 1.1. Let $\varepsilon, B, p_0 > 0$ to be fixed later. For $p < p_0$, fix $L = \exp[(\lambda + \varepsilon)/p^\gamma]$.

Consider $\mathcal{E}$ to be the event that any translate of rectangles $R^u(p^{-3}, 0)$ (for $u \in \mathbb{U}$) included in $D(L)$ contains a translate of $\{nu^\perp : n = 1, \dots, \gamma\}$ in $X(\omega)$. The probability of this event can be bound from below as follows

$$\mathbb{P}_p(\mathcal{E}) \geq \left(\exp\left[-\log\left(1 - e^{-cp^{-3\gamma}}\right)\right]\right)^{|\mathbb{U}|L^2} \longrightarrow 1.$$

Once again, we used Property (2) of Proposition 2.7.

Denote by $\mathcal{F}$ the event that there exists a critical droplet of size $p^{-3\gamma}$ included in $D(L)$ which is internally spanned. Applying (6.1) and dividing $D(L)$ into $c'(Lp^{3\gamma})^2$ disjoint translates of $D(p^{-3\gamma})$, where c' is a universal constant, one easily acquires

$$\mathbb{P}_p(\mathcal{F}) \geq 1 - \left(1 - \exp[-(2\lambda + \varepsilon)/p^\gamma]\right)^{c'(Lp^{3\gamma})^2} \sim 1 - \exp[-c'(Lp^{3\gamma})^2 e^{-(2\lambda+\varepsilon)/p^\gamma}] \longrightarrow 1$$

by harnessing the following bound from Corollary 4.5:

$$\mathbb{P}_p(D(p^{-3\gamma}) \text{ internally spanned}) \geq \exp[-(\lambda + 2\varepsilon)/p^\gamma]. \tag{6.1}$$

Moreover, the occurrence of $\mathcal{E}$ and $\mathcal{F}$ implies that $p^\gamma \log T \leq \lambda + 2\varepsilon$ for p small enough. Indeed, each site in $D(p^{-3\gamma})$ becomes occupied in time shorter than $p^{-6\gamma}$ since $\mathcal{T}$ adds at least one occupied site at each time-step. After the creation of the critical droplet, it only takes a time of order $p^{-3\gamma}L$ to progress and reach 0, thanks to the event $\mathcal{E}$. The Harris inequality yields

$$\mathbb{P}_p(p^\gamma \log T \leq \lambda + 2\varepsilon) \geq \mathbb{P}_p(\mathcal{E} \cap \mathcal{F}) \geq \mathbb{P}_p(\mathcal{E})\mathbb{P}_p(\mathcal{F}) \to 1$$

which concludes the proof of the upper bound. □

Proof of the lower bound in Theorem 1.1. Let $p > 0$. Fix $L = \exp(\lambda - \varepsilon)/p^\gamma$. Let $B, p_0 > 0$ be such that

$$\mathbb{P}_p(D \text{ is crossed}) \leq \exp[-(2\lambda - \varepsilon)/p^\gamma]$$

for any droplet D with perimeter between $Bp^{-\gamma}$ and $1+|\mathcal{N}|^2Bp^{-\gamma}$, and any $p<p_0$.

Let $\mathcal{E}$ be the event that the origin is spanned by the configuration in the box of size $Bp^{-\gamma}$, the probability of this event goes to 0. Indeed, if the origin is not in X, there must exist a connected set of cardinality $\theta > 2\gamma$ at distance less than $Bp^{-\gamma}$ that is full. There are less than $|\mathcal{N}|^\theta \cdot (Bp^{-\gamma})^2$ choices for this set so that the probability is bounded by c_1p for some c_1 depending only on B and the model.

Let $\mathcal{F}$ be the event that no droplet in $D(L)$ of perimeter between $Bp^{-\gamma}$ and $1+|\mathcal{N}|^2Bp^{-\gamma}$ is crossed In this case, Corollary 5.8 implies that no droplet in $D(L)$ of perimeter larger than $Bp^{-\gamma}$ is crossed. In particular, since $0\notin\langle X\cap D(Bp^{-\gamma})\rangle$, then $0\notin\langle X\cap D(L)\rangle$.

It is easy to see that if $\mathcal{F}$ and $\mathcal{E}$ hold, we must have $p^\gamma\log T\geq\lambda-\varepsilon$. Indeed, since $(0,0)\in\langle X\rangle\setminus\langle X\cap D(L)\rangle$, there must exists a sequence of sites $x_0,\dots,x_\ell=(0,0)$ such that $x_k\in\mathcal{T}^k(X)\setminus\mathcal{T}^k(X\cap D(L))$ and x_k and x_{k+1} are neighbors for every k. Since $x_0\notin D(L)$, we obtain that $\ell\geq L/|\mathcal{N}|$ and therefore $T\geq L/|\mathcal{N}|$. We deduce

$$\begin{aligned}\mathbb{P}_p(p^\gamma\log T\leq\lambda-\varepsilon)&\leq\mathbb{P}_p(\mathcal{E})+\mathbb{P}_p(\mathcal{F})\\&\leq c_1p+(|\mathcal{N}|^2Bp^{-\gamma})^{|\mathbb{U}|}L^2\exp[-(2\lambda-\varepsilon)/p^\gamma]\longrightarrow 0\end{aligned}$$

where $(|\mathcal{N}|Bp^{-\gamma})^{|\mathbb{U}|}$ bounds the number of possible dimensions for the internally spanned droplet D and L^2 the number of possible position for D. When p goes to 0, this probability converges to 0 and the lower bound follows. □

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