THE CRITICAL FUGACITY FOR SURFACE ADSORPTION OF SELF-AVOIDING WALKS ON THE HONEYCOMB LATTICE IS $1 + \sqrt{2}$

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Abstract. In 2010, Duminil-Copin and Smirnov proved a long-standing conjecture of Nienhuis, made in 1982, that the growth constant of self-avoiding walks on the hexagonal (a.k.a honeycomb) lattice is $\mu = \sqrt{2+\sqrt{2}}$. A key identity used in that proof was later generalised by Smirnov so as to apply to a general $O(n)$ loop model with $n \in [-2, 2]$ (the case $n = 0$ corresponding to self-avoiding walks).

We modify this model by restricting to a half-plane and introducing a fugacity associated with boundary sites (also called *surface* sites) and obtain a generalisation of Smirnov's identity. The value of the *critical* surface fugacity was conjectured by Batchelor and Yung in 1995. This value also plays a crucial role in our identity, which thus provides an independent prediction for it.

For the case $n = 0$, corresponding to self-avoiding walks interacting with a surface, we prove the conjectured value of the critical surface fugacity. A critical part of this proof involves demonstrating that the generating function of self-avoiding bridges of height T, taken at its critical point $1/\mu$, tends to 0 as T increases, as predicted from SLE theory.

1. Introduction

The n-vector model, also called $O(n)$ model, introduced by Stanley in 1968 [23] is described by the Hamiltonian

$$\mathcal{H}(d, n) = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j,$$

where d denotes the dimensionality of the lattice, i and j are adjacent sites, and $\mathbf{s}_i$ is an n-dimensional vector of magnitude $\sqrt{n}$. When $n = 1$ this Hamiltonian describes the Ising model, and when $n = 2$ it describes the classical XY model. Two other interesting limits, which leave a lot to be desired from a pure mathematical perspective, are the limit $n \to 0$, in which case one recovers the self-avoiding walk (SAW) model, as first pointed out by de Gennes [7], and the limit $n \to -2$, corresponding to random walks, or more generally a free-field Gaussian model, as shown by Balian and Toulouse [2].

Self-avoiding walks will be central in the second half of this paper. They have been considered as models of long-chain polymers in solution since the middle of the last century — see for example articles by Orr [20] and Flory [12]. Since that time they have been studied and extended by polymer chemists as models of polymers; by mathematicians as combinatorial models of pristine simplicity in their description, yet malevolent difficulty in their solution; by computer scientists interested in computational complexity; and by biologists using them to model properties of DNA and other biological polymers of interest.

Of particular importance to this article is the fact that the n-vector model lattice has been shown [8] to be equivalent to a loop model with a weight n attached to closed loops. The partition function of this loop model can be written as

$$Z(x) = \sum_{\gamma} x^{|\gamma|} n^{\ell(\gamma)},$$

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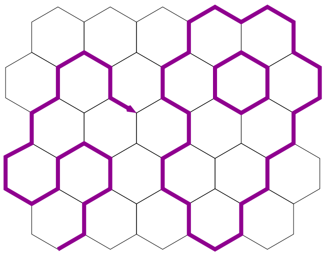

FIGURE 1. A configuration of the loop model on the honeycomb lattice.

where γ is a configuration of non-intersecting loops, $|\gamma|$ is the number of edges and $\ell(\gamma)$ is the number of loops. In the following we consider an $O(n)$ loop model with a defect, *i.e.* a model of closed loops with one self-avoiding walk component[1]. A typical configuration is shown in Fig. 1.

In 1982 Nienhuis [19] showed that, for $n \in [-2, 2]$, the loop model on the honeycomb lattice could be mapped onto a solid-on-solid model, from which he was able to derive the critical points and critical exponents, subject to some plausible assumptions. These results agreed with the known exponents and critical point for the Ising model, and they predicted exact values for those models corresponding to other values of the spin dimensionality n. In particular, for $n = 0$ the critical point for the honeycomb lattice SAW model was predicted to be $x_c = 1/\sqrt{2+\sqrt{2}}$, a result finally proved 28 years later by Duminil-Copin and Smirnov [10]. The starting point of their proof is a *local* identity for a "parafermionic" observable, valid at every vertex of the lattice. Then they obtain a *global* identity linking several walk generating functions by summing over all vertices of a domain[2]. Smirnov [22] then extended the local identity to the general honeycomb $O(n)$ model with $n \in [-2, 2]$. This extension provides an alternative way of predicting the value of the critical point $x_c(n) = 1/\sqrt{2+\sqrt{2-n}}$ as conjectured by Nienhuis.

Nienhuis's results were concerned with bulk systems. Interesting surface phenomena can also be studied if one considers the n-vector model in a half-space, with vertices in the surface (the boundary of the half-space) having an associated fugacity. Clearly, if this fugacity is made repulsive, adsorption onto the surface will be energetically unfavourable; if the fugacity is made attractive, adsorption becomes increasingly favoured. The adsorption transition is an example of a *special* surface transition [6].

In 1995 Batchelor and Yung [3] extended Nienhuis's work to the adsorption problem described above, and making similar assumptions to Nienhuis conjectured the value of the critical surface fugacity for the honeycomb lattice n-vector model, using the integrability of an underlying lattice model.

Conjecture 1 (Batchelor and Yung)**.** *For the $O(n)$ loop model on the semi-infinite hexagonal lattice with $n \in [-2, 2]$, associate a fugacity $x_c(n) = 1/\sqrt{2+\sqrt{2-n}}$ with occupied vertices and an additional fugacity y with occupied vertices on the boundary. Then the model undergoes a special surface transition at*

$$y = y_c(n) = 1 + \frac{2}{\sqrt{2-n}}.$$

In this paper we first show that the local identity proved by Smirnov [22] for the $O(n)$ model with $n \in [-2, 2]$ can be generalised to a half-plane system with a surface fugacity (Lemma 3). We use this to prove a generalisation of the global identity of Duminil-Copin and Smirnov to include a surface fugacity (Proposition 4). The contribution of one of these generating functions vanishes at $y = y_c(n)$, which lends support to the above conjecture.

[1]Defects correspond to correlation functions of the underlying spin model. It follows that the critical point remains the same.

[2]A more formal presentation of their proof has recently been provided by Klazar [16].

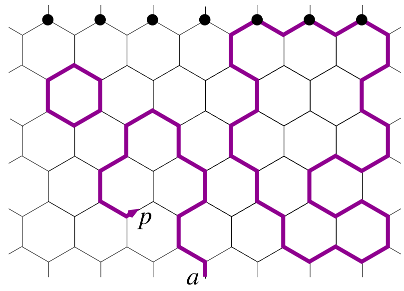

FIGURE 2. A configuration γ on a finite domain, with the weighted vertices on the top boundary indicated. The contribution of γ to $F(z)$ is $\mathrm{e}^{-5i\sigma\pi/3}x^{51}y^3n^2$.

We then focus on the case $n = 0$, corresponding to self-avoiding walks interacting with an impenetrable surface, and prove Conjecture 1 for this value of n: a self-avoiding walk with step fugacity $x_\mathrm{c} = 1/\sqrt{2+\sqrt{2}}$ is adsorbed if $y > 1+\sqrt{2}$ and desorbed if $y < 1+\sqrt{2}$.

Theorem 2. *The critical surface fugacity for self-avoiding walks on the honeycomb lattice is*

$$y_\mathrm{c} = 1+\sqrt{2}.$$

The proof of Theorem 2 relies of course on our global identity, but also requires earlier results dealing with SAW confined to a half-plane or a strip: notably existence of critical values of the fugacity, enumeration of SAW in a strip and the behaviour as the size of the strip increases, among others. Most of these results have been proved for the square (and hypercubic) lattice, but we need to adapt these proofs to the honeycomb case, which we do in Section 3. Section 4 combines these results and the global identity to prove Theorem 2. A third key ingredient, of independent interest, is that the generating function of *bridges* of height T, taken at x_c, tends to 0 as T increases. The proof is probabilistic in nature, and is given in the appendix.

To conclude this introduction, let us mention that we do not even have conjectures for the values of the critical fugacities on other lattices; instead, numerical estimates using series analysis and Monte Carlo methods are the best current results. New methods of estimating the growth constants and critical surface fugacities of the square and triangular lattices, inspired by results presented in [10] and this paper, are explored in [4] and [5].

2. Identity in the presence of a boundary

We consider the honeycomb lattice, embedded in the complex plane $\mathbb{C}$ in such a way that the edges have unit length. This allows us to consider vertices of the lattice as complex numbers. It is also convenient to start and end self-avoiding walks at a mid-edge of the lattice. We restrict the lattice to a half-plane, bounded by a horizontal surface consisting of *weighted sites* (Fig. 2). We further consider a domain D of this half-lattice, consisting of a finite connected collection of half-edges such that for every vertex v incident to at least one half-edge of D, all three half-edges incident to v actually belong to D. We denote by $V(D)$ the set of vertices incident to half-edges of D. Those mid-edges of D which are adjacent to only one vertex in $V(D)$ form the *boundary* ∂D. A *configuration* γ consists of a (single) self-avoiding walk w and a (finite) collection of closed loops, which are self-avoiding and do not meet one another nor w. We denote by $|\gamma|$ the number of vertices occupied by γ (also called the length), by $c(\gamma)$ the number of *contacts* with the surface (i.e. vertices of the surface occupied by γ), and by $\ell(\gamma)$ the number of loops. See Fig. 2 for an example.

Define the following *generating function*, or *observable*: for $a \in \partial D$ and $p \in D$, set

$$F(D, a, p; x, y, n, \sigma) \equiv F(p) := \sum_{\gamma: a \rightsquigarrow p} x^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \mathrm{e}^{-i\sigma W(w)}, \tag{1}$$

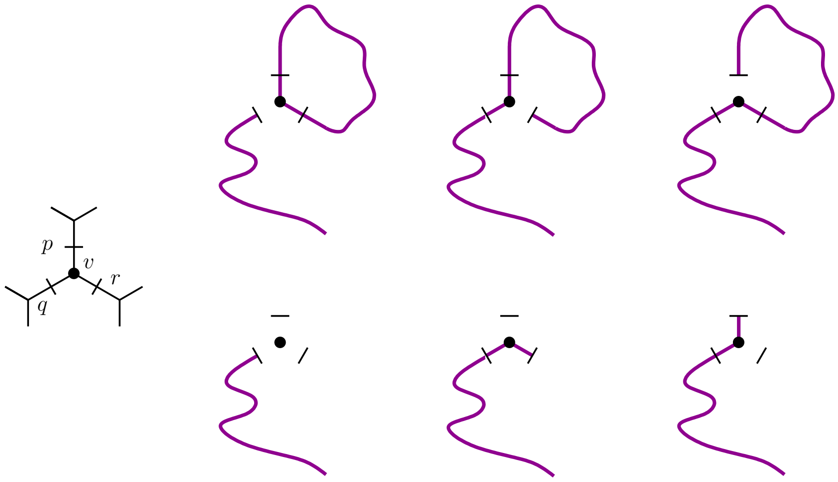

FIGURE 3. The two ways of grouping the configurations which end at mid-edges p, q, r adjacent to vertex v. Above, configurations which visit all three mid-edges; below, configurations which visit one or two of the mid-edges.

where the sum is over all configurations γ in D for which the SAW component w runs from the mid-edge a to the mid-edge p and $W(w)$ is the *winding angle* of that self-avoiding walk, that is, $\pi/3$ times the difference between the number of left turns and the number of right turns.

The case $y = 1$ of the following lemma is due to Smirnov [22].

Lemma 3 (The local identity). *For $n \in [-2, 2]$, set $n = 2\cos\theta$ with $\theta \in [0, \pi]$. Let*

$$\sigma = \frac{\pi - 3\theta}{4\pi}, \qquad x_c^{-1} = 2\cos\left(\frac{\pi+\theta}{4}\right) = \sqrt{2-\sqrt{2-n}}, \qquad \text{or} \tag{2}$$

$$\sigma = \frac{\pi + 3\theta}{4\pi}, \qquad x_c^{-1} = 2\cos\left(\frac{\pi-\theta}{4}\right) = \sqrt{2+\sqrt{2-n}}. \tag{3}$$

Then for a vertex $v \in V(D)$ not belonging to the weighted surface, the observable F defined by (1) *satisfies*

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0, \tag{4}$$

where p, q, r are the mid-edges of the three edges adjacent to v.

If $v \in V(D)$ lies on the weighted surface,

$$\begin{aligned} &(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = \\ &\qquad (q-v)(1-y)(x_c y\lambda)^{-1} \sum_{\gamma: a \rightsquigarrow q,p} x_c^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \mathrm{e}^{-i\sigma W(w)} \\ &\qquad\qquad + (r-v)(1-y)(x_c y\bar{\lambda})^{-1} \sum_{\gamma: a \rightsquigarrow r,p} x_c^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \mathrm{e}^{-i\sigma W(w)}, \end{aligned} \tag{5}$$

where $\lambda = \mathrm{e}^{-i\sigma\pi/3}$ is the weight accrued by a walk for each left turn, p, q, r are the three mid-edges adjacent to v, taken in counterclockwise order, with p just above v, and the first (resp. second) sum runs over configurations γ that go from a to p via q (resp. via r).

Equation (2) corresponds to the larger of the two critical values of the step weight x and hence to the dense regime critical point. Equation (3) corresponds to the line of critical points separating the dense and dilute phases, as predicted by Nienhuis [19]. In what follows, we refer to (2) and (3) as the dense and dilute regimes, respectively.

Proof. If v does not belong to the surface, the proof is completely analogous to the proof of Lemma 4 in [22]: One observes that the left-hand side of (4) counts (weighted) configurations

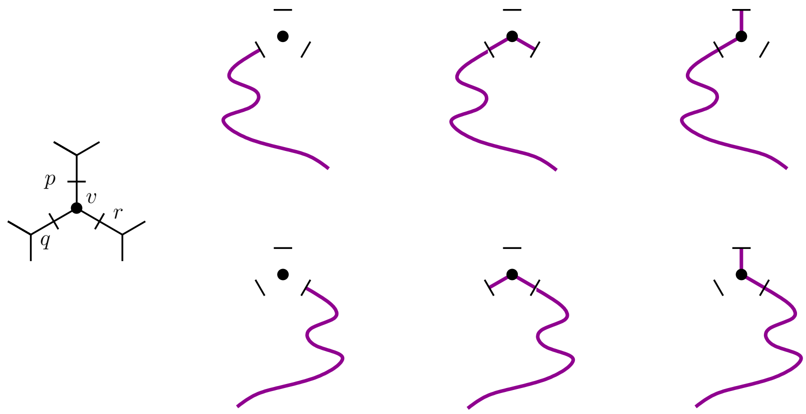

FIGURE 4. The two groupings of walks ending at a mid-edge adjacent to a surface vertex. The top three lead to (6), and the bottom three lead to (7).

ending at a mid-edge adjacent to v, and organizes these configurations by groups of three, as shown in Fig. 3 (which, up to rotations, includes all possible cases). It is then easy to check that, for the given values of σ and x_c, the contribution of each group vanishes. The fact that $y \neq 1$ in our paper makes no difference, because the number of weighted vertices is the same for all walks in a group.

This is not true if v belongs to the surface. Still, let us determine the contribution of a group. We first note that groups of the first type (for which the three mid-edges p, q, and r are visited) cannot exist when v is on the surface. For groups of the second type, we distinguish two cases, depending on whether the walk approaches v via q or via r (Fig. 4). If the leftmost configuration in each group of Fig. 4 is denoted γ_1, and the rightmost one γ, with associated SAW components w_1 and w, then the contribution in the first case is

$$(q-v)x_c^{|\gamma_1|}y^{c(\gamma_1)}n^{\ell(\gamma_1)}e^{-i\sigma W(w_1)}(1+x_c y\bar{\lambda}j+x_c y\lambda\bar{j}) \qquad (6)$$

with $j=e^{2\imath\pi/3}$. But we know that this vanishes when $y=1$, so the last term in parentheses must be $(1-y)$. Moreover,

$$|\gamma_1|=|\gamma|-1, \quad c(\gamma_1)=c(\gamma)-1, \quad \ell(\gamma_1)=\ell(\gamma), \quad W(w_1)=W(\gamma)-\pi/3,$$

and one concludes that groups of walks visiting q give the first sum in (5). Similarly, for a group of walks visiting r, the contribution is

$$(r-v)x^{|\gamma_1|}y^{c(\gamma_1)}n^{\ell(\gamma_1)}e^{-i\sigma W(w_1)}(1+x_c y\bar{j}\lambda+x_c yj\bar{\lambda}) = (r-v)(1-y)x_c^{|\gamma|-1}y^{c(\gamma)-1}n^{\ell(\gamma)}e^{-i\sigma(W(w)+\pi/3)},$$

which gives the second sum in (5).

■

In [10], Duminil-Copin and Smirnov use Lemma 3 to prove that the growth constant of the self-avoiding walk ($n=0$ in the dilute regime (3)) is $x_c^{-1}=2\cos(\pi/8)=\sqrt{2+\sqrt{2}}$. They do so by considering a special trapezoidal domain $D_{L,T}$ as shown[3] in Fig. 5, and deriving from the local identity a global identity that relates several generating functions counting walks in this domain. Here we generalise this identity to a general $O(n)$ model including a boundary weight.

We partition the boundary $\partial D_{T,L}$ into four subsets $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{E}}$ and $\mathcal{E}$ as illustrated in Fig. 5. We also define four generating functions, counting configurations in $D_{T,L}$ starting from a and

[3]The convention on T is chosen in such a way a walk of minimal length going from the bottom to the top of the domain contains exactly T vertical edges, one of them being split into two half-edges.

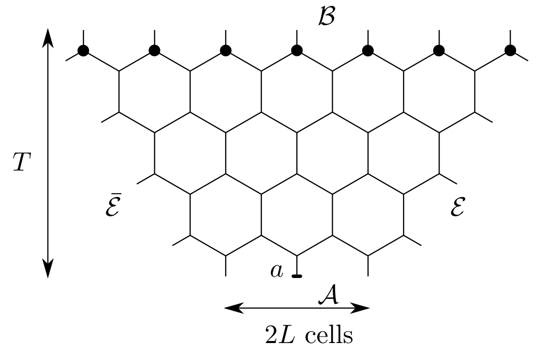

FIGURE 5. Finite patch $D_{4,1}$ of the half hexagonal lattice. The SAW components of configurations start on the central mid-edge of the bottom boundary (shown as a). The weighted vertices, belonging to the surface, are marked with a black disc.

ending in $\partial D_{T,L}$. First,

$$A_{T,L}(x,y) := \sum_{\gamma : a \rightsquigarrow \mathcal{A}\setminus\{a\}} x^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)}, \tag{7}$$

where the sum is over all configurations in $D_{T,L}$ whose SAW component goes from the mid-edge a to a mid-edge of $\mathcal{A}\setminus\{a\}$. We similarly define the generating functions $A^{\circ}_{T,L}(x,y)$, $B_{T,L}(x,y)$ and $E_{T,L}(x,y)$ for configurations ending in $\{a\}$, $\mathcal{B}$, and $\bar{\mathcal{E}} \cup \mathcal{E}$ respectively. Note that configurations counted by A° comprise *only* closed loops inside $D_{T,L}$; that is, their self-avoiding walk component is the empty walk $a \rightsquigarrow a$.

Proposition 4. *For $n = 2\cos\theta$ and $x_{\rm c}^{-1} = 2\cos((\pi \pm \theta)/4)$, the above defined generating functions satisfy*

$$A^{\circ}_{T,L}(x_{\rm c},y) = \cos\left(\frac{3(\pi\pm\theta)}{4}\right) A_{T,L}(x_{\rm c},y) + \cos\left(\frac{\pi\pm\theta}{2}\right) E_{T,L}(x_{\rm c},y) + \frac{y^*-y}{y(y^*-1)} B_{T,L}(x_{\rm c},y), \tag{8}$$

where

$$y^* = \frac{1}{1-2x_{\rm c}^2} = 1 \mp \frac{2}{\sqrt{2-n}}.$$

Observe that in the dilute case, $x_{\rm c}^{-1} = 2\cos((\pi-\theta)/4)$, the value of y^* coincides with the predicted value of $y_{\rm c}(n)$ given in Conjecture 1. In Section 4, we use the above identity to prove Conjecture 1 in the case $n = 0$. In this case the left-hand side of (8) reduces to 1, all coefficients are positive as long as $y < y^*$, so that the polynomials $A_{T,L}$, $B_{T,L}$ and $E_{T,L}$ are uniformly bounded, independently of T, L. Just as in the proof of Duminil-Copin and Smirnov for the growth constant of SAW, the bound on $B_{T,L}$ is an important ingredient of our proof. The identity (8) allows $B_{T,L}(x_{\rm c},y)$ to diverge for $y \geq y^*$ which signals the surface transition at the $\mathcal{B}$ boundary.

Proof. Let p_v, q_v, r_v be the mid-edges adjacent to a vertex v. We compute the sum

$$S := \sum_{v\in V(D_{T,L})} \big((p_v - v)F(p_v) + (q_v - v)F(q_v) + (r_v - v)F(r_v)\big) \tag{9}$$

in two ways.

Firstly, all summands of (9) associated with a non-weighted vertex v are 0 by the first part of Lemma 3. We are left with the contribution of vertices lying on the surface, given in the second part of the lemma. Since $W(w) = 0$ for all walks occurring in (5),

$$2S = e^{-5i\pi/6}(1-y)(x_{\rm c}y\lambda)^{-1} \sum_{p\in\mathcal{B},\gamma:a\rightsquigarrow q,p} x_{\rm c}^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} + e^{-i\pi/6}(1-y)(x_{\rm c}y\bar{\lambda})^{-1} \sum_{p\in\mathcal{B},\gamma:a\rightsquigarrow r,p} x_{\rm c}^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)},$$

where q (resp. r) stands for the SW (resp. SE) mid-edge adjacent to v. The factor 2 accounts for the fact that edges have length 1, so that terms like $(p-v)$ have modulus 1/2. Now reflecting a configuration γ that reaches a mid-edge $p \in \mathcal{B}$ from the SW gives a configuration γ' that reaches a mid-edge $p' \in \mathcal{B}$ from the SE. Moreover, $|\gamma| = |\gamma'|$, $c(\gamma) = c(\gamma')$ and $\ell(\gamma) = \ell(\gamma')$. Hence

$$\begin{aligned}
2S &= (1-y)(x_c y)^{-1} \sum_{p\in\mathcal{B},\gamma:a\rightsquigarrow q,p} x_c^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \left(e^{-5i\pi/6}\bar{\lambda} + e^{-i\pi/6}\lambda\right) \\
&= -2i(1-y)(x_c y)^{-1} \cos\left(\frac{\pi \pm \theta}{4}\right) \sum_{p\in\mathcal{B},\gamma:a\rightsquigarrow q,p} x_c^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \\
&= -i(1-y)(x_c y)^{-1} \cos\left(\frac{\pi \pm \theta}{4}\right) B_{T,L}(x_c, y) \qquad \text{by symmetry} \\
&= -\frac{i}{2}(1-y)(x_c^2 y)^{-1} B_{T,L}(x_c, y).
\end{aligned} \tag{10}$$

To obtain another expression for S, starting from (9), note that any mid-edge p not belonging to $\partial D_{T,L}$ contributes to two terms in the sum, for vertices v_1 and v_2, and these two terms cancel because $(p-v_1) = -(p-v_2)$. Thus we are left with precisely the contributions of those mid-edges in $\partial D_{T,L}$:

$$2S = -i\sum_{p\in\mathcal{A}} F(p) + e^{-5i\pi/6}\sum_{p\in\bar{\mathcal{E}}} F(p) + e^{-i\pi/6}\sum_{p\in\mathcal{E}} F(p) + i\sum_{p\in\mathcal{B}} F(p). \tag{11}$$

We again use symmetry arguments to rewrite this sum. First, denoting $\mathcal{A} = \{a\} \cup \mathcal{A}^- \cup \mathcal{A}^+$ (with $\mathcal{A}^-$ to the left of a), we have

$$\begin{aligned}
\sum_{p\in\mathcal{A}} F(p) &= A^{\circ}_{T,L}(x_c, y) + \sum_{\gamma:a\rightsquigarrow\mathcal{A}^-} x_c^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \left(\lambda^3 + \bar{\lambda}^3\right) \\
&= A^{\circ}_{T,L}(x_c, y) - \cos\left(\frac{3(\pi \pm \theta)}{4}\right) A_{T,L}(x_c, y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
e^{-i\pi/3}\sum_{p\in\bar{\mathcal{E}}} F(p) + e^{i\pi/3}\sum_{p\in\mathcal{E}} F(p) &= \sum_{\gamma:a\rightsquigarrow\bar{\mathcal{E}}} x_c^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \left(e^{-i\pi/3}\lambda^2 + e^{i\pi/3}\bar{\lambda}^2\right) \\
&= -\cos\left(\frac{\pi \pm \theta}{2}\right) E_{T,L}(x_c, y).
\end{aligned}$$

Finally,

$$\sum_{p\in\mathcal{B}} F(p) = B_{T,L}(x_c, y).$$

Equating (10) and (11) gives the proposition. ∎

3. Confined self-avoiding walks

In the remainder of this paper we specialise to $n = 0$, corresponding to self-avoiding walks. In this case, additional results and a proof for the critical surface fugacity can be established. In this section we first review some basic but important background, and then adapt to the honeycomb lattice some known results about *square lattice* self-avoiding walks confined to a half-plane or a strip.

Again, we consider self-avoiding walks on the honeycomb lattice, starting and ending at a mid-edge. The simplest model associates a fugacity x with each visited vertex (or *step*, or *monomer*). One then studies the generating function

$$C(x) = \sum_{n\geq 0} c_n x^n,$$

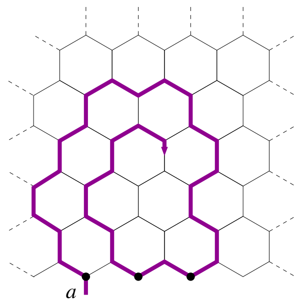

FIGURE 6. A self-avoiding walk in a half-plane, with weights attached to the vertices of the surface (indicated by black discs).

where c_n is the number of SAW of n monomers, considered equivalent up to a translation. Simple concatenation arguments and a classical lemma on sub-multiplicative sequences suffice to prove that the *growth constant*

$$\mu := \lim_{n\to\infty} (c_n)^{1/n}$$

exists and is finite [18, Chap. 1]. Of course, $1/\mu$ is the radius of convergence of the series $C(x)$. Duminil-Copin and Smirnov [10] proved Nienhuis's conjecture [19] that, for the honeycomb lattice, $\mu = \sqrt{2+\sqrt{2}}$.

3.1. Self-avoiding walks in a half-plane

We now consider SAW in a half-plane, originating at a mid-edge a just below the surface (Fig. 6). It is known that the growth constant for such walks is the same as for the bulk case (see [25] or [18, Chap. 3]). We also add a fugacity y to vertices in the surface. In physics terms, $y = \mathrm{e}^{-\epsilon/k_B T}$ where ϵ is the energy associated with a surface vertex, T is the absolute temperature and k_B is Boltzmann's constant.

Let $c_n^+(i)$ be the number of half-plane walks of n-steps, with i monomers in the surface, and define the partition function (or generating function) as

$$C_n^+(y) = \sum_{i=0}^{n} c_n^+(i) y^i.$$

If y is large, the polymer adsorbs onto the surface, while if y is small, the walk is repelled by the surface.

Proposition 5. *For $y > 0$,*

$$\mu(y) := \lim_{n\to\infty} C_n^+(y)^{1/n}$$

exists and is finite. It is a log-convex, non-decreasing function of $\log y$*, and therefore continuous and almost everywhere differentiable.*

For $0 < y \le 1$,

$$\mu(y) = \mu(1) \equiv \mu.$$

Moreover, for any $y > 0$,

$$\mu(y) \ge \max(\mu, \sqrt{y}).$$

This behaviour implies the existence of a critical value y_c, with $1 \le y_c \le \mu^2$, which delineates the transition from the desorbed *phase to the* adsorbed *phase:*

$$\mu(y) \begin{cases} = \mu & \text{if } y \le y_c, \\ > \mu & \text{if } y > y_c. \end{cases}$$

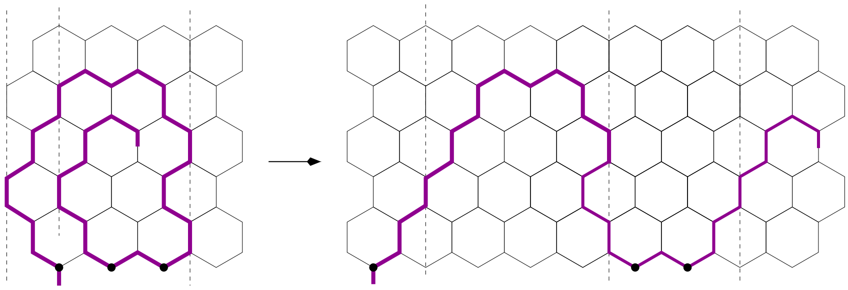

FIGURE 7. Unfolding a half-plane self-avoiding walk on the honeycomb lattice.

Proof. The existence of $\mu(y)$ has been proved by Hammersley, Torrie and Whittington [13] in the case of the d-dimensional hypercubic lattice. Their discussion and proof, which use concatenation and *unfolding* of walks, apply, *mutatis mutandis* to the honeycomb lattice. Unfolding consists of reflecting parts of the walk in lines parallel to the y-axis passing through those vertices of the walk with maximal and minimal x-coordinates (Fig. 7). This unfolding is repeated until the origin and end-point have minimal and maximal x-coordinates respectively. The main advantage of such unfolded walks is that they can be concatenated without creating self-intersections (this may require the addition of a few steps between the walks).

The other results are elementary, and adapted from an earlier paper of Whittington [25]. In particular, the lower bound $\mu(y) \geq \sqrt{y}$ is obtained by counting zig-zag walks sticking to the surface. ■

The situation as $y \to \infty$ has only recently been rigorously established by Rychlewski and Whittington [21], who proved that, on the square lattice, $\mu(y)$ is asymptotic to y. This translates into $\mu(y) \sim \sqrt{y}$ in our honeycomb setting.

Various other quantities exhibit singular behaviour at y_c. For example, the mean density of vertices in the surface is given by

$$\frac{1}{n}\frac{\sum_i i c_n^+(i) y^i}{\sum_i c_n^+(i) y^i} = \frac{y}{n}\frac{\partial \log C_n^+(y)}{\partial y}.$$

In the limit of infinitely long walks this density tends to[4]

$$y\frac{\partial \log \mu(y)}{\partial y}.$$

From the behaviour of $\mu(y)$ given in Proposition 5, it can be seen that the density of vertices on the surface is 0 for $y < y_c$ and is positive for $y > y_c$.

3.2. Self-avoiding walks in a strip

As discussed in the previous subsection, the usual model of surface-interacting walks considers walks originating in a surface and interacting with monomers (or edges) in that surface. One way to study such systems is to consider interacting walks in a strip, and then to take the limit as the strip width becomes infinite. Clearly, if one studies walks in a strip, it is possible to consider interactions with both the top and bottom surface.

Consider a strip of height T on the honeycomb lattice, as shown in Fig. 8. We consider self-avoiding walks that originate at a mid-edge a just below the bottom of the strip. Such walks are said to be *arches* if they end at the bottom of the strip, and *bridges* if they end at the top (Fig. 8). Let $c_{T,n}(i,j)$ be the number of n-step walks in a strip of height T with i vertices in the

[4]The exchange of the limit and the derivative is possible thanks to the convexity of $\log \mu(y)$, see for instance [24, Thm. B7, p. 345].

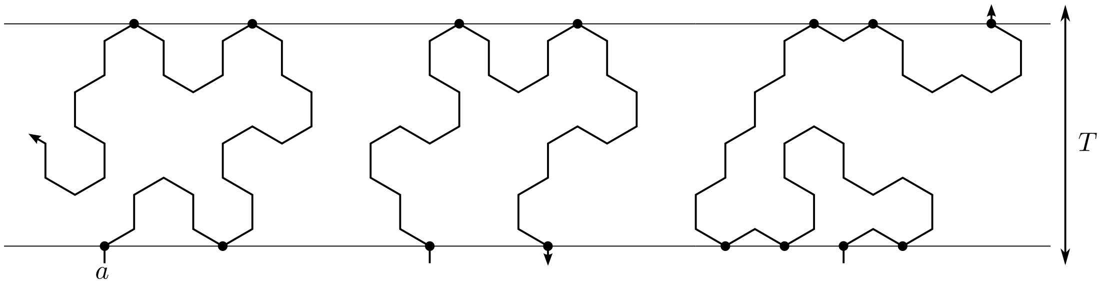

FIGURE 8. Walks confined to a strip of height $T = 5$ with weights attached to vertices along the top and bottom of the strip: a general walk, an arch, and a bridge.

bottom line and j vertices in the top line. Similarly, define the numbers $a_{T,n}(i, j)$ and $b_{T,n}(i, j)$ counting arches and bridges. The associated partition function is given by

$$C_{T,n}(y, z) = \sum_{i,j} c_{T,n}(i, j) y^i z^j,$$

where y and z are the weights associated with visits to vertices at the bottom and top of the strip respectively. We define similar partition functions $A_{T,n}(y, z)$ and $B_{T,n}(y, z)$ for arches and bridges, hoping no confusion will arise with the series $A_{T,L}(x, y)$ and $B_{T,L}(x, y)$ of Section 2.

Proposition 6. *For $y, z > 0$, one has*

$$\lim_{n\to\infty} A_{T,n}(y, z)^{1/n} = \lim_{n\to\infty} B_{T,n}(y, z)^{1/n} = \lim_{n\to\infty} C_{T,n}(y, z)^{1/n} := \mu_T(y, z),$$

where $\mu_T(y, z)$ is finite, and non-decreasing in y and z. By the symmetry of bridges,

$$\mu_T(y, z) = \mu_T(z, y),$$

and so, in particular, $\mu_T(y, 1) = \mu_T(1, y)$. Finally, $\mu_T(1, y)$ is a log-convex and thus continuous function of $\log(y)$.

Proof. Again, the existence of the limits follows from concatenation and unfolding arguments as given in Section 4 of [14]. The log-convexity result is easily adapted from [14, Thm. 6.3]. ∎

Therefore the critical fugacity for self-avoiding walks in a strip is independent of which wall the interacting monomers are situated on. As per our discussion in Section 2, it turns out to be convenient to put the interacting monomers on the top, rather than at the bottom.

The next proposition describes how the growth constant $\mu_T(1, y)$ changes as T grows.

Proposition 7. *For $y > 0$, we have*

$$\mu_T(1, y) < \mu_{T+1}(1, y).$$

Moreover, as $T \to \infty$,

$$\mu_T(1, y) \to \mu(y),$$

the growth constant of self-avoiding walks interacting with a surface (Proposition 5*).*

Again, the proof is an adaptation to the honeycomb lattice of results proved by van Rensburg, Orlandini and Whittington for the hypercubic lattices [14] (similar arguments are also covered in Chapter 8 of [18], but without interactions). Before we describe how to adapt these proofs, let us derive a corollary that will be essential in the next section. It deals with the properties of $\rho_T(y) := 1/\mu_T(1, y)$, which is the radius of convergence of the series

$$C_T(x, y) := \sum_{n\geq 0} C_{T,n}(1, y) x^n$$

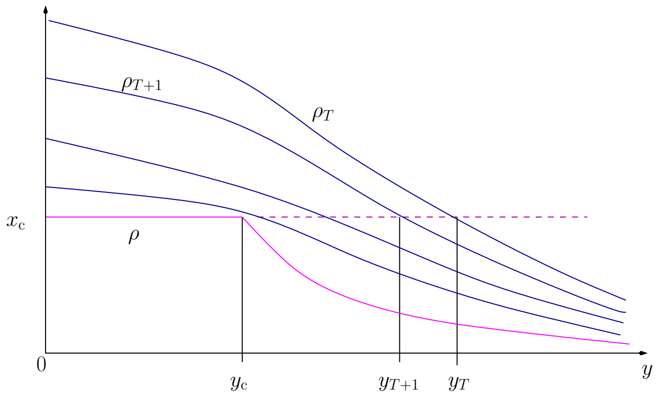

FIGURE 9. An illustration of Corollary 8.

counting walks in a strip that interact with the top boundary, and of the analogous series $A_T(x, y)$ and $B_T(x, y)$ that count arches and bridges. See Fig. 9 for an illustration.

Corollary 8. *Let $y > 0$. The generating functions $A_T(x, y)$, $B_T(x, y)$ and $C_T(x, y)$ all have the same radius of convergence,*

$$\rho_T(y) = 1/\mu_T(1, y).$$

Moreover, $\rho_T(y)$ decreases to $\rho(y) := 1/\mu(y)$ as T goes to infinity. In particular, $\rho_T(y)$ decreases to $\rho := 1/\mu$ for $y \leq y_c$.

There exists a unique $y_T > 0$ such that $\rho_T(y_T) = x_c := 1/\mu$. The series (in y) $A_T(x_c, y)$, $B_T(x_c, y)$ and $C_T(x_c, y)$ have radius of convergence y_T, and y_T decreases to the critical fugacity y_c as T goes to infinity.

Proof. The first part of the lemma is an obvious translation of Propositions 6 and 7.

The existence of y_T follows from the intermediate value theorem: ρ_T is continuous, $\rho_T(1) > x_c$ and $\rho_T(y) \to 0$ as $y \to \infty$ (because $\rho_T(y) \leq 1/\sqrt{y}$ as can be seen by counting zig-zag paths, as in the proof of Proposition 5).

The uniqueness of y_T follows from the log-convexity of $\mu_T(y)$ in $\log y$, which precludes having $\rho_T(y) = \rho_T(y') \leq x_c$. This also means that

$$\rho_T(y) < \rho_T(y_T) \iff y > y_T \qquad \text{and} \qquad \rho_T(y) > \rho_T(y_T) \iff y < y_T. \tag{12}$$

Let us now prove that y_T is the radius of $A_T(x_c, y)$, $B_T(x_c, y)$ and $C_T(x_c, y)$. The argument is the same for the three series, so let us work for instance with C_T. By definition of ρ_T, the series $C_T(x_c, y)$ converges if $x_c < \rho_T(y)$, and diverges if $x_c > \rho_T(y)$. But $x_c = \rho_T(y_T)$, so by (12), this means that $C_T(x_c, y)$ converges if $y < y_T$ and diverges if $y > y_T$, which means that y_T is the radius of $C_T(x_c, y)$.

Let us finally prove that y_T decreases towards y_c. First, since $\rho_T(y_T) = x_c$ and $\rho_{T+1}(y) < \rho_T(y)$ (Proposition 7), we have $\rho_{T+1}(y_T) < x_c$ and thus $y_{T+1} < y_T$. Hence the sequence $(y_T)_{T\geq 1}$ decreases. Let $\bar{y}$ be its limit. For $y \leq y_c$, we have $\rho_T(y) > \rho(y) = x_c$, and thus $y_T > y_c$ for all T. Hence $\bar{y} \geq y_c$. Since $\bar{y} < y_T$, we have $\rho_T(\bar{y}) > \rho_T(y_T) = x_c$, and thus $\rho(\bar{y}) \geq x_c$ (Proposition 7). Since $\rho(y) < x_c$ for $y > y_c$ (Proposition 6), it follows that $\bar{y} \leq y_c$. We have thus proved that y_T decreases to y_c. ■

Proof of Proposition 7. The proof uses arguments similar to Sections 5 and 6 of [14], but is, we believe, somewhat shorter[5].

[5]In particular, working in two dimensions gives a simple argument proving the divergence at their radius of convergence of generating functions that count self-avoiding walks in a strip and moreover, we do not need the full strength of a pattern theorem.

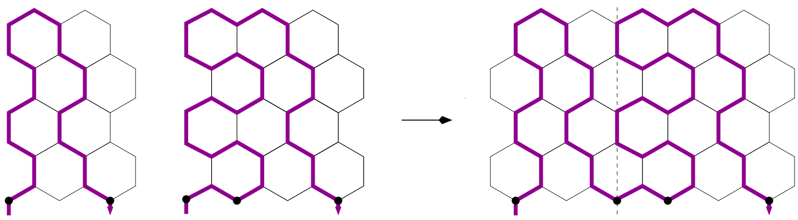

FIGURE 10. Concatenation of two unfolded arches in a strip of height $T = 5$.

First, since $\mu_T(1, y) = \mu_T(y, 1)$, we choose to work with arches in a strip of height T, interacting with the bottom line of the strip. Let us say that an arch going from mid-edge a to mid-edge b is *unfolded* if the abscissa $x(v)$ of every non-final vertex v of the walk satisfies $x(a) \le x(v) < x(b)$. Two unfolded arches can be concatenated (after deleting the last step of the first arch, see Fig. 10) to form a new unfolded arch. We say an unfolded arch is *prime* if it is not the concatenation of two (or more) unfolded arches. The first two arches of Fig. 10 are prime, the third one, by construction, is not.

Let us fix $y > 0$. The arguments of [14, Section 4] show that the generating function $\vec{A}_T(x, y)$ that counts unfolded arches (by the size and the number of contacts with the bottom line of the strip) has the same radius of convergence as the generating function $A_T(x, y)$ that counts all arches. By Proposition 6, this radius is $\rho_T(y) := 1/\mu_T(y)$. Moreover, the above definition of prime arches shows that

$$\vec{A}_T(x, y) = \frac{P_T(x, y)}{1 - P_T(x, y)/(xy)},$$

where $P_T(x, y)$ counts prime unfolded arches.

It follows from the transfer matrix method that the series $\vec{A}_T(x, y)$ (and, in fact, all series counting walks in a strip that occur in this section) is a rational function of x and y (see [11, p. 364], or [1]). Hence $\vec{A}_T(x, y)$ diverges at its radius $\rho_T(y)$, and it follows that $P_T(\rho_T(y), y)/(y\rho_T(y)) = 1$.

Now consider the prime unfolded arch w that consists of a (wavy) column with $2(T-1)$ vertical edges (like the first arch of Fig. 10). This walk contributes a term $x^{4T-1}y^2$ in the series $P_T(x, y)$. Let $\tilde{P}_T(x, y) := P_T(x, y) - x^{4T-1}y^2$. The generating function of unfolded arches that do not contain w as a factor is

$$\frac{\tilde{P}_T(x, y)}{1 - \tilde{P}_T(x, y)/(xy)},$$

and thus its radius is larger than the radius of $\vec{A}_T(x, y)$. The above series counts (among others) walks that do not touch the top line of the strip. Their generating function is $\vec{A}_{T-1}(x, y)$, which has radius $\rho_{T-1}(y)$. Hence $\rho_{T-1}(y) > \rho_T(y)$, or equivalently $\mu_{T-1}(y) < \mu_T(y)$.

The proof that $\mu_T(y)$ tends to $\mu(y)$ is analogous to the proof of Theorem 6.5 in [14].

■

4. THE CRITICAL SURFACE FUGACITY OF SAWS IS $1 + \sqrt{2}$

4.1. THE GLOBAL IDENTITY

Let us write the identity (8) at $n = 0$, that is, at $\theta = \pi/2$. Then no loops are allowed. In particular, the polynomial $A^{\circ}_{T,L}$ reduces to 1. The identity (8) thus reads (for dense and dilute regimes respectively, the bottom sign corresponding to the dilute phase),

$$1 = \cos\left(\frac{3\pi(2\pm1)}{8}\right) A_{T,L}(x_c,y) + \cos\left(\frac{\pi(2\pm1)}{4}\right) E_{T,L}(x_c,y) + \frac{y^*-y}{y(y^*-1)} B_{T,L}(x_c,y)$$
$$= \mp\frac{\sqrt{2\pm\sqrt{2}}}{2} A_{T,L}(x_c,y) \mp \frac{1}{\sqrt{2}} E_{T,L}(x_c,y) + \frac{y^*-y}{y(y^*-1)} B_{T,L}(x_c,y), \tag{13}$$

where

$$x_c^{-1} = 2\cos\left(\frac{\pi(2\pm1)}{8}\right) = \sqrt{2\mp\sqrt{2}} \qquad \text{and} \qquad y^* = 1\mp\sqrt{2}.$$

As exemplified in [10], the identity (13) (there specialized to $y=1$), provides easy bounds, existence of limits, etc. if all coefficients are positive, so we now consider only the dilute regime. We denote

$$\alpha = \cos\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}, \qquad \varepsilon = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \qquad \beta(y) = \frac{y^*-y}{y(y^*-1)} = \frac{1+\sqrt{2}-y}{\sqrt{2}\,y},$$

so that the identity of interest is

$$1 = \alpha A_{T,L}(x_c,y) + \varepsilon E_{T,L}(x_c,y) + \beta(y) B_{T,L}(x_c,y). \tag{14}$$

4.2. A lower bound on y_c

As L increases, the polynomials $A_{T,L}(x_c,y)$ and $B_{T,L}(x_c,y)$ count more and more walks. Hence for any $0 \le y$, their values increase with L. If, in addition, $y < y^*$, the coefficients α, ε and $\beta(y)$ are positive, and (14) shows that the values of $A_{T,L}(x_c,y)$ and $B_{T,L}(x_c,y)$ remain bounded as L increases. Hence, for $0 \le y < y^*$, the limits

$$\lim_L A_{T,L}(x_c,y) \quad \text{and} \quad \lim_L B_{T,L}(x_c,y)$$

exist and are finite. Clearly, these limits are $A_T(x_c,y)$ and $B_T(x_c,y)$, where $A_T(x,y)$ and $B_T(x,y)$ are respectively the generating functions of arches and bridges in a strip of height T (called a T-strip below), defined just above Corollary 8. Thanks to this corollary, we obtain $y^* \le y_T$ and hence

$$y^* \le y_c. \tag{15}$$

4.3. A limit identity

Proposition 9. *For $0 \le y < y_T$ (the radius of convergence of $A_T(x_c,\cdot)$ and $B_T(x_c,\cdot)$), the series counting arches and bridges in a T-strip satisfy*

$$\alpha A_T(x_c,y) + \beta(y) B_T(x_c,y) = 1. \tag{16}$$

Proof. Let us first prove that

$$\lim_L E_{T,L}(x_c,y) = 0 \quad \text{for } 0 \le y < y_T.$$

Indeed, $E_{T,L}(x_c,y)$ counts some self-avoiding walks of length at least L, starting from a, and confined to a T-strip. But the generating function of walks in the T-strip converges at (x_c,y) for $y < y_T$ (see Corollary 8), and thus its remainder of order L tends to 0 as L grows. This remainder is an upper bound on $E_{T,L}(x_c,y)$, which thus tends to 0 as well.

Taking the limit of (14) as $L \to \infty$ gives the proposition. ∎

4.4. Convergence of $B_T(x_c, 1)$ to 0

This is a key point in our argument, and also a result of independent interest.

Theorem 10. *The length generating function $B_T(x, 1)$ counting bridges in a strip of height T, taken at the critical value $x_c = 1/\sqrt{2 + \sqrt{2}}$, tends to 0 as T tends to infinity.*

The proof, of a probabilistic nature, is given in the appendix. Let us note that the fact that $B_T(x_c, 1)$ converges (and actually decreases) follows easily from the case $y = 1$ of (16). Indeed, $A_T(x_c, 1)$ increases with T, but remains bounded since $\beta(1)$ is positive. Thus $A_T(x_c, 1)$ has a finite limit when T increases, and this limit is the generating function $A(x_c)$ counting arches in a half-plane. It then follows from (16) that $B_T(x_c, 1)$ decreases as T grows, and

$$\lim_T B_T(x_c, 1) = 1 - \alpha A(x_c). \tag{17}$$

The appendix thus implies that $A(x_c) = 1/\alpha$.

Remarks
1. We can actually prove that $A_T(x_c, y) \to A(x_c)$ for $y < y^*$, but this will not be needed here. Returning to (16), this implies that $B_T(x_c, y) \to 0$ for $0 \le y < y^*$.

2. As discussed in [10, Remark 2], it follows from the SLE predictions of [17, Sec. 3.3.3 and 3.4.3] that $B_T(x_c, 1)$ is expected to decay as $T^{-1/4}$ as $T \to \infty$.

4.5. An upper bound on y_c

The series $A_{T+1}(x_c, y)$ counts arches of height at most $T+1$. This includes arches of height at most T, which have no contacts with the top boundary. Now consider an arch that has contacts with the boundary. By looking at its last contact, one can factor the arch into two bridges (see Fig. 11), and thus obtain

$$A_{T+1}(x_c, y) - A_T(x_c, 1) \le x_c B_T(x_c, 1) B_{T+1}(x_c, y).$$

This identity holds in the domain of convergence of the series it involves, that is, for $y < y_{T+1}$. Combine this with (16), first written for $T+1$ and $y < y_{T+1}$ and then for T and $y = 1$:

$$\alpha A_{T+1}(x_c, y) + \beta(y) B_{T+1}(x_c, y) = 1 = \alpha A_T(x_c, 1) + B_T(x_c, 1).$$

This gives, for $y < y_{T+1}$,

$$B_T(x_c, 1) - \beta(y) B_{T+1}(x_c, y) \le \alpha x_c B_T(x_c, 1) B_{T+1}(x_c, y),$$

or equivalently,

$$0 \le \frac{1}{B_{T+1}(x_c, y)} \le \alpha x_c + \frac{1}{B_T(x_c, 1)} \frac{y^* - y}{y(y^* - 1)}. \tag{18}$$

In particular, for $y < y_c = \lim_T y_T$,

$$0 \le \alpha x_c + \frac{1}{B_T(x_c, 1)} \frac{y^* - y}{y(y^* - 1)}.$$

Recall that $B_T(x_c, 1)$ tends to 0 (Theorem 10). This forces $y^* \ge y_c$, otherwise the right-hand side would become arbitrarily large in modulus and negative as $T \to \infty$ for $y^* < y < y_c$.

Together with (15), this establishes $y_c = y^* = 1 + \sqrt{2}$ and completes the proof of Theorem 2. ■

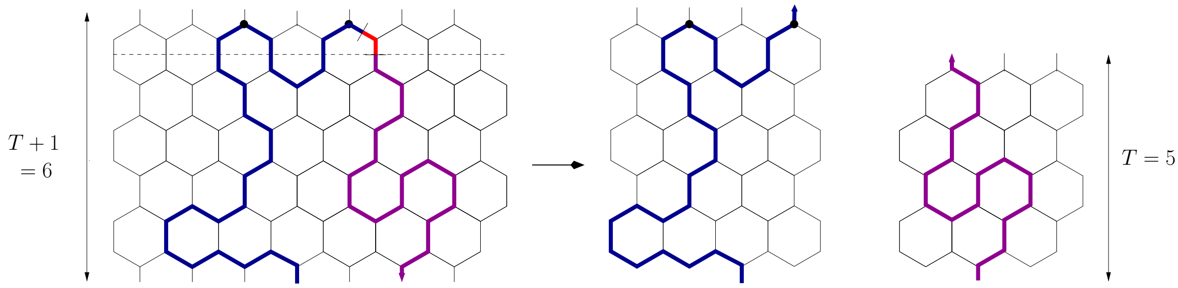

FIGURE 11. Factorisation of an arch of height $T+1$ into two bridges, of height $T+1$ and T respectively.

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Appendix. Proof of Theorem 10.

Before starting the proof, let us introduce some additional notation. The set of mid-edges of the honeycomb lattice is denoted by $\mathbb{H}$. The lattice has an origin $a \in \mathbb{H}$, at coordinates $(0,0)$. We denote by $(x(v), y(v))$ the coordinates of a point $v \in \mathbb{C}$ (that is, its real and imaginary parts). We consider self-avoiding walks that start and end at a mid-edge. A self-avoiding walk γ is denoted by the sequence $(\gamma_0, \ldots, \gamma_n)$ of its mid-edges. The length of γ, that is, the number of vertices of the lattice it visits, is denoted as before by $|\gamma| = n$. To lighten notation, we often omit floor symbols, especially in indices: for instance, γ_t should be understood as $\gamma_{\lfloor t \rfloor}$. The cardinality of a set A is denoted by $|A|$.

We have so far discussed bridges in a strip of height T (Fig. 8, right), which we call bridges *of height* T. In general, we call *bridge* any self-avoiding walk $\gamma = (\gamma_0, \ldots, \gamma_n)$ that is a bridge of height T for some T. Equivalently, $y(\gamma_0) < y(\gamma_i) < y(\gamma_n)$ for $0 < i < n$. The set of bridges of length n is denoted by SAB_n.

The set R_γ of *renewal points* of $\gamma \in \mathrm{SAB}_n$ is the set of points of the form γ_i with $0 \le i \le n$, for which $\gamma_{[0,i]} := (\gamma_0, \ldots, \gamma_i)$ and $\gamma_{[i,n]} := (\gamma_i, \ldots, \gamma_n)$ are bridges. We denote by $\mathbf{r}_0(\gamma), \mathbf{r}_1(\gamma), \ldots$ the indices of the renewal points. That is, $\mathbf{r}_0(\gamma) = 0$ and $\mathbf{r}_{k+1}(\gamma) = \inf\{j > \mathbf{r}_k(\gamma) : \gamma_j \in \mathsf{R}_\gamma\}$ for each k. When no confusion is possible, we often denote $\mathbf{r}_k(\gamma)$ by $\mathbf{r}_k$.

A bridge $\gamma \in \mathrm{SAB}_n$ is *irreducible* if its only renewal points are γ_0 and γ_n. Let iSAB be the set of irreducible bridges of arbitrary length starting from a. Every bridge γ is the concatenation of a finite number of irreducible bridges, the decomposition is unique and the set R_γ is the union of the initial and terminal points of the bridges that comprise this decomposition.

Kesten's relation for irreducible bridges (see [18, Section 4.2] or [15]) on the hypercubic lattice $\mathbb{Z}^d$ can be easily adapted to the honeycomb lattice. It gives

$$\sum_{\gamma \in \mathrm{iSAB}} x_c^{|\gamma|} = 1.$$

This enables us to define a probability measure $\mathbb{P}_{\mathrm{iSAB}}$ on iSAB by setting $\mathbb{P}_{\mathrm{iSAB}}(\gamma) = x_c^{|\gamma|}$. Let $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$ denote the law on semi-infinite walks $\gamma : \mathbb{N} \to \mathbb{H}$ formed by the concatenation of infinitely many independent samples $\gamma^{[1]}, \gamma^{[2]}, \ldots$ of $\mathbb{P}_{\mathrm{iSAB}}$. We refer to [18, Section 8.3] for details of related measures in the case of $\mathbb{Z}^d$. The definition of R_γ and the indexation of renewal points extend to this context (we obtain an infinite sequence $(\mathbf{r}_k)_{k \in \mathbb{N}}$).

Observe that a bridge γ of length n has height $\mathsf{H}(\gamma) = \frac{2}{3} y(\gamma_n)$ (since edges have unit length; in particular a bridge of length 2 has height 1). We define its *width* by

$$\mathsf{W}(\gamma) = \frac{1}{\sqrt{3}} \max\{x(\gamma_k) - x(\gamma'_k), 0 \le k, k' \le n\},$$

so that a bridge of length 2 has width 1/2.

We have proved in Section 4.4 that $B_T(x_c, 1)$ converges as $T \to \infty$. We provide here an alternative proof, and relate the limiting value to the average height of irreducible bridges.

Lemma 11. *As* $T \to \infty$,

$$B_T(x_c, 1) \to \frac{1}{\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma))}.$$

Proof. The result follows from standard renewal theory. We can for instance apply [18, Theorem 4.2.2(b)] to the sequence

$$f_T := \sum_{\gamma \in \mathrm{iSAB}:\ \mathsf{H}(\gamma) = T} x_c^{|\gamma|}.$$

Indeed, with the notation of this theorem, $v_T = B_T(x_c, 1)$ and $\sum_k k f_k = \mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma))$. ■

Thus Theorem 10 is equivalent to

$$\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma)) = \infty.$$

We will prove this by contradiction. Assuming $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma))$ is finite, we first show that $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma))$ is also finite. Then, we show that under these two conditions, an infinite bridge is very narrow. The last step consists in proving that this cannot be the case. The argument uses a *stickbreak* operation which perturbs a bridge by selecting a subpath and rotating it clockwise by $\frac{\pi}{3}$. The new path is a self-avoiding bridge for an adequately chosen subpath. But its width is relatively large, contradicting the fact that bridges are narrow. The strategy of proof is greatly inspired by a recent paper of Duminil-Copin and Hammond, where self-avoiding walks are proved to be sub-ballistic [9]. The additional difficulty here comes from the fact that Section 4 of [9] (which corresponds to the proof presented here) relies on the assumption $\mathbb{E}_{\text{iSAB}}(|\gamma|) < \infty$, which is stronger than the assumption $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < \infty$ that we have here. In particular, we need the following result.

Proposition 12. *If* $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < \infty$*, then* $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma)) < \infty$.

Proof. Consider the rectangular domain $R_{T,L}$ of $\mathbb{H}$ depicted in Fig. 12, with its boundary partitioned into four subsets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}^-$ and $\mathcal{E}^+$ (the mid-edges of $\mathcal{E}^+$ point up, those of $\mathcal{E}^-$ point down). We do not consider any kind of interactions here. As in Section 2, we define four generating functions counting self-avoiding walks in the rectangle, going from a to a mid-edge of the boundary. First, we set

$$\tilde{A}_{T,L}(x) := \sum_{\gamma: a \rightsquigarrow \mathcal{A}\setminus\{a\}} x^{|\gamma|},$$

and then the generating functions $\tilde{B}_{T,L}(x)$, $\tilde{E}^-_{T,L}(x)$ and $\tilde{E}^+_{T,L}(x)$ are defined similarly. We now use the local identity of Lemma 3 with $n = 0$, $y = 1$, $\theta = \pi/2$, $\sigma = 5\pi/24$, and $x_c^{-1} = 2\cos(\pi/8)$ to prove the following global identity, analogous to (14):

$$1 = \alpha\tilde{A}_{T,L}(x_c) + \tilde{B}_{T,L}(x_c) + \varepsilon^+\,\tilde{E}^+_{T,L}(x_c) + \varepsilon^-\,\tilde{E}^-_{T,L}(x_c), \tag{19}$$

where, as before, $\alpha = \cos(\frac{3\pi}{8})$, and now $\varepsilon^- = \cos(\frac{\pi}{4})$ and $\varepsilon^+ = \cos(\frac{\pi}{8})$.

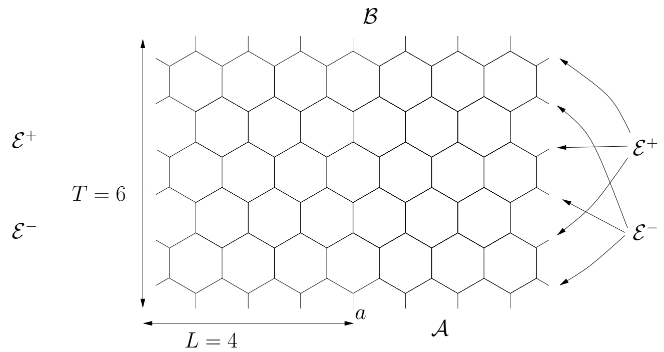

FIGURE 12. The rectangular domain $R_{T,L}$ with $T = 6$ and $L = 4$.

Convention. Since we always evaluate our generating functions at $x = x_c$, we will almost systematically omit the variable x_c, so that $\tilde{A}_{T,L}$ now means $\tilde{A}_{T,L}(x_c)$, and so on.

As in Section 4.3, we would like $\tilde{E}^\pm_{T,L}$ to tend to 0 as the size of the rectangle increases. This holds for fixed T as L increases, using the same argument as before, but now we want both T and L to tend to infinity, so the matter is a bit more delicate. Recall that an *arch* is a self-avoiding walk starting from a, confined to the upper half-plane, and ending on the line $y = 0$. For $L \in \mathbb{N}$, let $\mathfrak{a}_L(x)$ be the generating function of arches ending L cells to the right of a. We will bound $\tilde{E}^\pm_{T,L}$ in terms of $\mathfrak{a}_{2L}$.

For $m \in \mathbb{N}$, let $\mathfrak{e}_m^+(x)$ be the generating function of walks in $R_{T,L}$ ending on the right side of the rectangle, on the m^{th} row of $\mathcal{E}^+$, so that, by symmetry, $\tilde{E}_{T,L}^+ = 2\sum_{m\le\lfloor\frac{T}{2}\rfloor} \mathfrak{e}_m^+$. Using a reflection argument and the Cauchy-Schwarz inequality, we find

$$\left(\tilde{E}_{T,L}^+\right)^2 \le 4\lfloor\tfrac{T}{2}\rfloor \sum_{m\le\lfloor\frac{T}{2}\rfloor} (\mathfrak{e}_m^+)^2 \le 4\lfloor\tfrac{T}{2}\rfloor\; x_c^{-1}\mathfrak{a}_{2L}. \tag{20}$$

The second inequality comes from the fact that one can concatenate two walks contributing to $\mathfrak{e}_m^+$ (after reflecting the second one) by adding a step between them in order to create an arch contributing to $\mathfrak{a}_{2L}$. We obtain a similar upper bound for $\tilde{E}_{T,L}^-$ with $\lfloor\frac{T}{2}\rfloor$ replaced by $\lceil\frac{T}{2}\rceil$.

Assume that we couple $T \equiv T_k$ and $L \equiv L_k$ so that both tend to infinity as k grows, and $T\mathfrak{a}_{2L} \longrightarrow 0$. Then $\tilde{E}_{T,L}^+$ and $\tilde{E}_{T,L}^-$ tend to 0. Moreover, $\tilde{A}_{T,L}$ increases with L and T, and converges to $A \equiv A(x_c)$, where $A(x)$ is the generating function of arches (recall that $A(x_c)$ is finite, as argued in Section 4.4). Returning to (19) shows that $\tilde{B}_{T,L}$ must also converge, and gives

$$\begin{aligned}\lim_k \tilde{B}_{T_k,L_k} &= 1-\alpha A(x_c)\\ &= \lim_T B_T(x_c,1) \qquad \text{by (17)}\\ &> 0 \quad \text{by assumption.}\end{aligned} \tag{21}$$

Let us now return to random infinite bridges and use them to give an upper bound on $\tilde{B}_{T,L}$. Let $0<\delta<1/\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma))$. We have

$$\begin{aligned}\tilde{B}_{T,L} = &\sum_{\gamma:a\rightsquigarrow\mathcal{B}} x_c^{|\gamma|}\\ &\le \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\exists n\in\mathbb{N} : \mathsf{H}(\gamma_{[0,\mathbf{r}_n]}) = T \text{ and } \mathsf{W}(\gamma_{[0,\mathbf{r}_n]}) \le 2L\big)\\ &\le \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big) + \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\exists n \ge \delta T : \mathsf{H}(\gamma_{[0,\mathbf{r}_n]}) = T \text{ and } \mathsf{W}(\gamma_{[0,\mathbf{r}_n]}) \le 2L\big).\end{aligned}$$

Let $\gamma^{[i]}$ be the i^{th} irreducible bridge of γ. Since the $\gamma^{[i]}$s are independent, we obtain

$$\begin{aligned}\tilde{B}_{T,L} &\le \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big) + \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\forall i \le \delta T, \mathsf{W}(\gamma^{[i]}) \le 2L\big)\\ &= \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big) + \mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) \le 2L)^{\delta T}\\ &\le \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big) + \exp\left(-\delta T\,\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L)\right).\end{aligned}$$

Note that

$$\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) = \sum_{i=1}^{\delta T} \mathsf{H}(\gamma^{[i]}).$$

Hence the law of large numbers, together with the fact that $\delta\cdot\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < 1$, implies that $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big)$ tends to 0 as $T\to\infty$. Hence, if we can couple $T\equiv T_k$ and $L\equiv L_k$ in such a way that $T\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L)$ tends to infinity, then $\tilde{B}_{T,L}$ tends to zero.

We now argue *ad absurdum.* Assume that $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma)) = \infty$. Then

$$\limsup_{L\to\infty} \frac{\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L)}{\mathfrak{a}_{2L}} = \infty,$$

since $\mathfrak{a}_L$ is the term of a converging series (namely, the generating function $A(x_c)$ of arches) and $\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > L)$ is non-increasing in L and is the term of a diverging series (indeed, it sums to $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma)) = \infty$). Let $(L_k)_k$ be a sequence such that

$$\lim_{k\to\infty} \frac{\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L_k)}{\mathfrak{a}_{2L_k}} = \infty,$$

and take

$$T_k = \left\lfloor \frac{1}{\sqrt{\mathfrak{a}_{2L_k}\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L_k)}} \right\rfloor.$$

Then

$$T_k\,\mathbb{P}_{\mathrm{iSAB}}(\mathsf{W}(\gamma) > 2L_k) \to \infty \qquad \text{and} \qquad T_k\,\mathfrak{a}_{2L_k} \to 0.$$

According to our two estimates of $\tilde{B}_{T,L}$, this means that $\lim_k \tilde{B}_{T_k,L_k}$ is both zero and a positive number, an absurdity. Therefore, $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{W}(\gamma)) < \infty$. ■

Let Ω be the set of bi-infinite walks $\gamma : \mathbb{Z} \to \mathbb{H}$ such that $\gamma_0 = a$. Let $(\gamma^{[i]}, i \in \mathbb{Z})$ be a bi-infinite sequence of irreducible bridges sampled independently according to $\mathbb{P}_{\mathrm{iSAB}}$. Let $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}$ denote the law on Ω formed by concatenating the bridges $\gamma^{[i]}, i \in \mathbb{Z}$ in such a way that $\gamma^{[1]}$ starts at a. Let $\mathcal{F}$ be the σ-algebra generated by events depending on a finite number of vertices of the walk.

We extend the indexation of renewal points to these bi-infinite bridges (we obtain a bi-infinite sequence $(\mathbf{r}_n(\gamma))_{n\in\mathbb{Z}}$ such that $r_0(\gamma) = 0$). Let $\tau : \Omega \to \Omega$ be the *shift* defined by $\tau(\gamma)_i = \gamma_{i+\mathbf{r}_1(\gamma)} - \gamma_{\mathbf{r}_1(\gamma)}$ for every $i \in \mathbb{Z}$. (This is only defined if $\mathbf{r}_1$ exists, but this is the case with probability 1 under $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}$.) The shift translates the walk so that $\mathbf{r}_1(\gamma)$ is now at the origin a of the lattice. Note that $\mathbf{r}_i(\tau(\gamma)) = \mathbf{r}_{i+1}(\gamma) - \mathbf{r}_1(\gamma)$. Let σ denote the reflection in the real axis.

Proposition 13. *The measure $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}$ satisfies the following properties.*

- (P_1) *It is invariant under the shift τ.*
- (P_2) *The shift τ is ergodic for $(\Omega, \mathcal{F}, \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}})$.*
- (P_3) *Under $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}$, the random variables $(\sigma\gamma_n)_{n\le 0}$ and $(\gamma_n)_{n\le 0}$ are independent and identically distributed.*

Proof. Property (P_1) is fairly straightforward. Indeed, for every $n > 0$, the law of $\gamma_{[\mathbf{r}_{-n}(\gamma),\mathbf{r}_n(\gamma)]}$ determines, in the high-n limit, the law of γ (since we work with the σ-algebra $\mathcal{F}$). Now, the laws of $\tau(\gamma_{[\mathbf{r}_{-n+1}(\gamma),\mathbf{r}_{n+1}(\gamma)]})$ and $\gamma_{[\mathbf{r}_{-n}(\gamma),\mathbf{r}_n(\gamma)]}$ are the same by construction (both are the law of $2n$ concatenated independent irreducible bridges). Thus (P_1) follows by letting $n \to \infty$.

Let us turn to (P_2). Consider a shift-invariant event A. We want to show that $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A) \in \{0,1\}$. Let $\varepsilon > 0$. There exists $n > 0$ and an event A_n depending only on the vertices $\gamma_{-n}, \dots, \gamma_n$ such that $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A_n \,\Delta\, A) \le \varepsilon$, where Δ denotes the symmetric difference. In particular, $|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A) - \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A_n)| \le \varepsilon$. By extension, A_n depends only on vertices in $\gamma_{\mathbf{r}_{-n}}, \dots, \gamma_{\mathbf{r}_n}$. Invariance of A under τ implies that $A = \tau^{-2n}(A)$, so that

$$\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A) = \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\left(A \cap \tau^{-2n}(A)\right) . \tag{22}$$

Moreover,

$$\begin{aligned}&\left|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\big(A \cap \tau^{-2n}(A)\big) - \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\big(A_n \cap \tau^{-2n}(A_n)\big)\right| \\ &\qquad\qquad \le \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\big(A \,\Delta\, A_n\big) + \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\big(\tau^{-2n}(A) \,\Delta\, \tau^{-2n}(A_n)\big) \le 2\varepsilon .\end{aligned}$$

Using (22) and the independence between the walk before and after $\mathbf{r}_n$, this reads

$$|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A) - \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A_n)^2| \le 2\varepsilon,$$

which, combined with $|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A) - \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A_n)| \le \varepsilon$, implies

$$|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A) - \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A)^2| \le 4\varepsilon .$$

By letting ε tend to 0, we obtain that $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A) = \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A)^2$ and therefore $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(A) \in \{0,1\}$. Hence ($\mathrm{P}_2$) is proved.

Since the law of irreducible bridges is invariant (up to a translation) under reflection with respect to a horizontal line, (P_3) is straightforward. ■

Renewal points separate a walk into two parts, located below and above the point. We now introduce a more restrictive notion, illustrated in Fig. 13 (left). A mid-edge γ_k of a walk γ is said to be a *diamond point* if

- it lies on a vertical edge of the lattice,
- the walk is contained in the cone

$$\left((\gamma_k - \tfrac{i}{2}) + \mathbb{R}_+ e^{i\pi/3} + \mathbb{R}_+ e^{2i\pi/3}\right) \cup \left((\gamma_k + \tfrac{i}{2}) - \mathbb{R}_+ e^{i\pi/3} - \mathbb{R}_+ e^{2i\pi/3}\right)$$

(recall that edges have length 1). The set of diamond points of γ is denoted by $\mathbf{D}_\gamma$. Of course, it is a subset of $\mathbf{R}_\gamma$. The following proposition tells us that, under our assumption, a positive fraction of renewal points are diamond points.

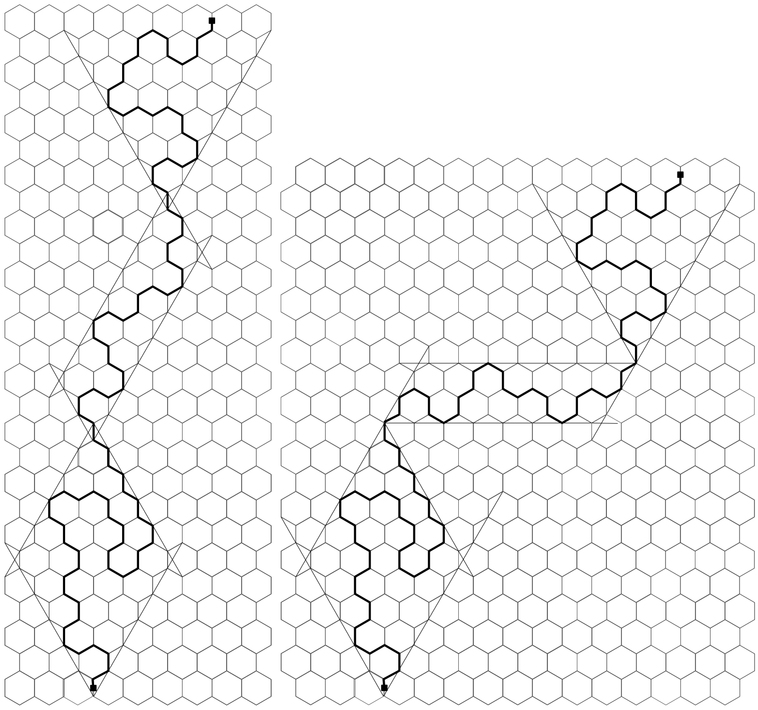

FIGURE 13. *Left*: A bridge having 3 diamond points. *Right*: A stickbreak operation applied to this bridge.

Proposition 14. *If* $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < \infty$*, then there exists* $\delta > 0$ *such that*

$$\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\left(\liminf_{n\to\infty} \frac{|\mathbf{D}_\gamma \cap \{0,\dots,\mathbf{r}_n\}|}{n} \geq \delta\right) = 1.$$

Let us first provide a heuristic argument. Since $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma))$ is finite, so is $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma))$ (Proposition 12). Then $\mathbb{E}_{\text{iSAB}}(x(\gamma_{|\gamma|})) = 0$, and the law of large numbers implies that the prefixes of an infinite bridge are tall and skinny – that is, height grows linearly, width grows sub-linearly. So the probability of a bridge staying within a cone as thin as one likes is positive, and a similar result holds going backwards. Thus, diamond points occur with positive density among renewal points.

Proof. Let us first prove that $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(\gamma_0 \in \mathbf{D}_\gamma) > 0$. Proposition 12 shows that $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma)) < \infty$. Hence $\mathbb{E}_{\text{iSAB}}(x(\gamma_{|\gamma|}))$ is well-defined, and is 0 since the law of an irreducible bridge is invariant under reflection with respect to the imaginary axis. The law of large numbers thus implies that, $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}$-almost surely, $x(\gamma_{\mathbf{r}_n})/n \longrightarrow 0$. Since the expected width of irreducible bridges is finite, a classical use of the Borel-Cantelli Lemma shows that $\mathsf{W}(\gamma_{[\mathbf{r}_n,\mathbf{r}_{n+1}]})/n \longrightarrow 0$ almost surely. Thus

$$\frac{1}{n}\left(|x(\gamma_{\mathbf{r}_n})| + \mathsf{W}(\gamma_{[\mathbf{r}_n,\mathbf{r}_{n+1}]})\right) \longrightarrow 0 \qquad \text{a.s.}$$

Since

$$\mathsf{W}(\gamma_{[0,\mathbf{r}_n]}) \le 2\max\{|x(\gamma_{\mathbf{r}_k})| + \mathsf{W}(\gamma_{[\mathbf{r}_k,\mathbf{r}_{k+1}]}),\ 0 \le k \le n-1\},$$

we find that, $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}$-almost surely, $\mathsf{W}(\gamma_{[0,\mathbf{r}_n]})/n \longrightarrow 0$.

Let us now apply the law of large numbers to $y(\gamma_{\mathbf{r}_n})$. We obtain that, $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}$-almost surely, $y(\gamma_{\mathbf{r}_n})/n \longrightarrow \frac{3}{2}\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) > 0$.

We deduce that

$$I(\gamma) := \inf_{k\ge 0}\Big(y(\gamma_k) - \sqrt{3}|x(\gamma_k)| + 1/2\Big)$$

is finite $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}$-almost surely. Note that for an infinite bridge $\gamma = (\gamma_0, \gamma_1, \ldots)$, the origin γ_0 is a diamond point if and only if $I(\gamma) \ge 0$. Let $K \in \mathbb{N}$ be such that $\rho_K := \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}(I(\gamma) \ge -K) > 0$. We are going to show that

$$\rho_0 \ge (2x_{\text{c}}^4)^K \rho_K > 0. \tag{23}$$

To prove (23), consider an experiment under which the law $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}$ is constructed by first concatenating K independent samples of $\mathbb{P}_{\text{iSAB}}$ (starting from a) and then an independent sample γ' of $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}$. If each of the K samples happens to be a walk of length 4 going from a to $a+3i$ and $I(\gamma') \ge -K$, then the complete walk γ satisfies $I(\gamma) \ge 0$. The probability that the i^{th} sample of $\mathbb{P}_{\text{iSAB}}$ is a walk of length four going from a to $a+3i$ is $2x_{\text{c}}^4$. Thus, the experiment behaves as described with probability $(2x_{\text{c}}^4)^K\rho_K$, and we obtain (23), that is, $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}(\gamma_0 \in \mathbf{D}_\gamma) > 0$.

Using Property (P_3) of Proposition 13, we deduce that

$$\delta := \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(\gamma_0 \in \mathbf{D}_\gamma) = \big(\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}(\gamma_0 \in \mathbf{D}_\gamma)\big)^2 > 0.$$

The shift τ being ergodic (cf. Property (P_2) of Proposition 13), the ergodic theorem, applied to $\mathbb{1}_{\gamma_0\in\mathbf{D}_\gamma}$, gives

$$\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}\left(\lim_{n\to\infty}\frac{|\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma)\}|}{n} = \delta\right) = 1\,.$$

Let γ be a bi-infinite bridge, and denote $\gamma^+ = \gamma_{[0,\infty)}$. Then for $n \ge 0$, $\mathbf{r}_n(\gamma) = \mathbf{r}_n(\gamma^+)$, and

$$\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma)\} = \mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma^+)\} \subset \mathbf{D}_{\gamma^+} \cap \{0,\ldots,\mathbf{r}_n(\gamma^+)\}$$

since all diamond points of γ are diamond points of γ^+. This implies that

$$\begin{aligned}\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}&\left(\liminf_{n\to\infty}\frac{|\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma)\}|}{n} \ge \delta\right)\\ &= \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}\left(\liminf_{n\to\infty}\frac{|\mathbf{D}_{\gamma^+} \cap \{0,\ldots,\mathbf{r}_n(\gamma)\}|}{n} \ge \delta\right)\\ &\ge \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}\left(\liminf_{n\to\infty}\frac{|\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma)\}|}{n} \ge \delta\right) = 1.\end{aligned}$$

This concludes the proof of the proposition. ∎

We are finally ready for the

Proof of Theorem 10. By Lemma 11, we want to prove that $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) = \infty$. We argue *ad absurdum.* Assume $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < \infty$ and let $\nu > \mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma))$. Also, let $0 < \varepsilon < \delta/20$, where δ satisfies Proposition 14.

Let Ω^+ denote the set of semi-infinite walks in the upper half-plane. That is, $\phi = (\phi_0, \phi_1, \ldots) \in \Omega^+$ if and only if $y(\phi_i) > 0$ for $i > 0$. For $\phi \in \Omega^+$ and γ a finite bridge, we denote $\gamma \lhd \phi$ if $\phi_{[0,|\gamma|]} = \gamma$ and $\phi_{|\gamma|}$ is a renewal point of ϕ. Note that

$$x_{\text{c}}^{|\gamma|} = \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}(\phi \in \Omega^+ : \gamma \lhd \phi)\,. \tag{24}$$

Let $\overline{\text{SAB}}_n$ denote the set of finite bridges γ with exactly $n+1$ renewal points (meaning that $\mathbf{r}_n(\gamma) = |\gamma|$) such that

(C_1) $\mathsf{H}(\gamma) \le \nu n$, (C$_2$) $|\mathbf{D}_\gamma| \geq \delta n/2$.

Let us define $\overline{\mathrm{SAB}}_n^+ = \{\phi \in \Omega^+ : \exists \gamma \in \overline{\mathrm{SAB}}_n \text{ such that } \gamma \lhd \phi\}$. That is, the prefix of ϕ consisting of its n first irreducible bridges satisfies (C$_1$) and (C$_2$). It follows from (24) that

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\overline{\mathrm{SAB}}_n^+\big) = \sum_{\gamma \in \overline{\mathrm{SAB}}_n} x_{\mathrm{c}}^{|\gamma|}\,. \tag{25}$$

We now prove that

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\overline{\mathrm{SAB}}_n^+\big) \longrightarrow 1 \text{ as } n \to \infty\,. \tag{26}$$

We consider Conditions (C$_1$) and (C$_2$) separately. Condition (C$_1$) for $\gamma \in \overline{\mathrm{SAB}}_n$ translates for $\phi \in \overline{\mathrm{SAB}}_n^+$ into $\mathsf{H}(\phi_{[0,\mathbf{r}_n(\phi)]}) \leq \nu n$. Since $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma)) < \nu$, the law of large numbers gives

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\Big(\phi \in \Omega^+ : \mathsf{H}(\phi_{[0,\mathbf{r}_n(\phi)]}) \leq \nu n\Big) \longrightarrow 1.$$

Let us now consider Condition (C$_2$), which translates into $|\mathbf{D}_{\phi_{[0,\mathbf{r}_n(\phi)]}}| \geq \delta n/2$. But

$$\mathbf{D}_{\phi_{[0,\mathbf{r}_n(\phi)]}} \supset \mathbf{D}_\phi \cap \{0, \dots, , \mathbf{r}_n(\phi)\},$$

since the truncation operation $\phi \to \phi_{[0,\mathbf{r}_n(\phi)]}$ can only create (and not annihilate) diamond points. Thus Proposition 14 yields

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(|\mathbf{D}_{\phi_{[0,\mathbf{r}_n(\phi)]}}| \geq \tfrac{\delta}{2} n\big) \longrightarrow 1,$$

and we have proved (26).

We are now going to prove that

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\Big(\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}(\phi)]}) > \varepsilon n\Big) \geq \left(\frac{\delta n x_{\mathrm{c}}}{10(\nu n + 2)}\right)^2 \mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\overline{\mathrm{SAB}}_n^+\big)\,. \tag{27}$$

Since $\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}(\phi)]})/n$ tends to zero $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$-almost surely, as follows from the beginning of the proof of Proposition 14, this contradicts (26) and proves that our assumption $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma)) < \infty$ cannot hold.

Consider $\gamma \in \overline{\mathrm{SAB}}_n$. Let $\mathbf{d}_i$ be the index of the ith diamond point of γ. For integers $i \in \big[\frac{\delta}{10} n, \frac{2\delta}{10} n\big]$ and $j \in \big[\frac{3\delta}{10} n, \frac{4\delta}{10} n\big]$, let $\mathsf{StickBreak}_{i,j}(\gamma)$ be the following walk (see Fig. 13, right):

$$\mathsf{StickBreak}_{i,j}(\gamma) = \gamma_{[0,\mathbf{d}_i]} \circ s \circ \rho\big(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}\big) \circ \tilde{s} \circ \gamma_{[\mathbf{d}_j,\mathbf{r}_n]}\,, \tag{28}$$

where $\circ$ stands for the concatenation of walks, ρ is the clockwise rotation of angle $\pi/3$, s is a single right turn, and $\tilde{s}$ is a single left turn. The definition of diamond points implies that $\mathsf{StickBreak}_{i,j}(\gamma)$ is not only self-avoiding, but also a bridge. Also, note that we used (C$_2$) in order to define $\mathsf{StickBreak}(\gamma)$ for all these values of i and j.

Let

$$\Phi \;=\; \big[\tfrac{\delta}{10} n, \tfrac{2\delta}{10} n\big] \;\times\; \big[\tfrac{3\delta}{10} n, \tfrac{4\delta}{10} n\big] \;\times\; \overline{\mathrm{SAB}}_n\,,$$

and denote

$$S := \sum_{(i,j,\gamma) \in \Phi} x_{\mathrm{c}}^{|\mathsf{StickBreak}_{i,j}(\gamma)|}\,.$$

One can express S in terms of $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\overline{\mathrm{SAB}}_n^+\big)$. Indeed, $|\mathsf{StickBreak}_{i,j}(\gamma)| = |\gamma| + 2$, and therefore

$$S = \sum_{(i,j,\gamma) \in \Phi} x_{\mathrm{c}}^{|\gamma|+2} = \left(\frac{\delta x_{\mathrm{c}} n}{10}\right)^2 \sum_{\gamma \in \overline{\mathrm{SAB}}_n} x_{\mathrm{c}}^{|\gamma|} = \left(\frac{\delta x_{\mathrm{c}} n}{10}\right)^2 \mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\overline{\mathrm{SAB}}_n^+\big)\,. \tag{29}$$

We used (25) for the last equality. We are now going to give an upper bound on S, which will imply (27).

Note that the walk $\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}$ contains at least $\delta n/10$ diamond points, and thus has height $h := \mathsf{H}(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}) \geq \delta n/10$. Rotating this walk by $\pi/3$ results in a walk of height at most h and width at least $h/2$. Hence $\mathsf{StickBreak}_{i,j}(\gamma)$ has width at least $\delta n/20 > \varepsilon n$. By ($C_1$), we also have $\mathsf{H}(\mathsf{StickBreak}_{i,j}(\gamma)) \leq \nu n + 1$ and therefore $\mathsf{StickBreak}_{i,j}(\gamma)$ has at most $\nu n + 2$ renewal points. Hence, for any $\phi \in \Omega^+$ such that $\mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi$, we have $\mathbf{r}_{\nu n+1}(\phi) \geq |\mathsf{StickBreak}_{i,j}(\gamma)|$ and therefore $\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}(\phi)]}) > \varepsilon n$. Thus, for any $(i,j,\gamma) \in \Phi$,

$$\begin{aligned} x_c^{|\mathsf{StickBreak}_{i,j}(\gamma)|} &= \mathbb{P}^{\otimes\mathbb{N}}_{\mathrm{iSAB}}\big(\phi \in \Omega^+ : \mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi\big) \\ &= \mathbb{P}^{\otimes\mathbb{N}}_{\mathrm{iSAB}}\big(\phi \in \Omega^+ : \mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi \text{ and } \mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}(\phi)]}) > \varepsilon n\big). \end{aligned}$$

Therefore,

$$\begin{aligned} S &= \sum_{(i,j,\gamma)\in\Phi} \mathbb{P}^{\otimes\mathbb{N}}_{\mathrm{iSAB}}\big(\phi \in \Omega^+ : \mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi \text{ and } \mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}(\phi)]}) > \varepsilon n\big) \\ &= \mathbb{E}^{\otimes\mathbb{N}}_{\mathrm{iSAB}}\Big(\big|\big\{(i,j,\gamma) \in \Phi : \mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi\big\}\big| \cdot \mathbb{1}_{\{\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}(\phi)]}) > \varepsilon n\}}\Big) \\ &\leq (\nu n+2)^2\, \mathbb{P}^{\otimes\mathbb{N}}_{\mathrm{iSAB}}\big(\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}(\phi)]}) > \varepsilon n\big). \end{aligned} \tag{30}$$

The last inequality follows from the fact that, for any given $\phi \in \Omega^+$, the number of elements (i,j,γ) of Φ such that $\mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi$ is at most $(\nu n+2)^2$. Indeed, the triple (i,j,γ) is completely determined if we specify in ϕ the renewal point that precedes the step denoted s in (28) and the one that follows the step $\tilde{s}$. As both points occur before $\mathbf{r}_{\nu n+1}$, as explained above, the bound (30) follows.

By combining (29) and (30) we obtain (27), which concludes the proof. ∎

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