Crossing probabilities in topological rectangles for the critical planar FK-Ising model

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Abstract

We consider the FK-Ising model in two dimension at criticality. We obtain RSW-type crossing probabilities bounds in arbitrary topological rectangles, uniform with respect to the boundary conditions, generalizing results of [DCHN10] and [CS09]. Our result relies on new discrete complex analysis techniques, introduced in [Che11].

We detail some applications, in particular the computation of so-called universal exponents, proof of quasi-multiplicativity properties of arm probabilities, and crossing bounds for the classical Ising model.

1 Introduction

The Ising model is one of the simplest and most fundamental models in equilibrium statistical mechanics. It was proposed by Lenz in 1920 [Len20], and then studied by Ising [Isi25], as a model for ferromagnetism, in an attempt to provide a microscopic explanation for the thermodynamical behavior of magnets. In 1936, Peierls [Pei36] showed that the model exhibits a phase transition at positive temperature in dimensions two and higher. After the celebrated exact derivation of the free energy of the two-dimensional model by Onsager in 1944 [Ons44], the Ising model became one of the most investigated models in the study of phase transitions and in statistical mechanics. See [Nis05, Nis09] for a historical review of the classical theory, for instance.

Recently, spectacular progress was made towards the rigorous description of the continuous scaling limit of 2D lattice models at critical temperature, in particular the Ising model [Smi10, CS09], notably thanks to the introduction of Schramm's SLE curves (see [Smi06] for a review of recent progress in this direction). In this paper, we develop tools that improve the connection between the discrete Ising model and the continuous objects describing its scaling limit.

Recall that the Ising model is a random assignment of ± 1 spins to the vertices of a graph G, where the probability of a spin configuration $(\sigma_x)_{x\in G}$ is proportional to $\exp\left(-\beta H\left(\sigma\right)\right)$; $\beta > 0$ is the inverse temperature and the H is the energy, defined as $-\sum_{i\sim j}\sigma_i\sigma_j$ (the sum is over all pairs of adjacent vertices). The Ising model favors local alignment of spins, and the strength of this effect is controlled by β. In the case we are interested in, namely the square grid $\mathbb{Z}^2$, an order/disorded phase transition occurs at the critical parameter value $\beta_c := \frac{1}{2} \ln \left(\sqrt{2}+1\right)$. In order to avoid confusion with the FK-Ising model defined below, we will call the Ising model the *spin-Ising model.*

In 1969, Fortuin and Kasteleyn [FK72] introduced a dependent bond percolation model, called *FK percolation* or *random-cluster* model, that gives a powerful geometric representation of a variety of models, among which the Ising model. The FK model depends on two positive parameters, usually denoted p and q. Given $p \in [0,1]$ and $q > 0$, the FK(p,q) model on a graph G is a model on random subgraphs of G containing all its vertices: the probability of a configuration $\omega \subset G$ is proportional to

$$\left(\frac{p}{1-p}\right)^{o(\omega)} q^{k(\omega)},$$

where $o\left(\omega\right)$ is the number of edges of ω and $k\left(\omega\right)$ the number of *clusters* of ω (connected components of vertices).

We call the FK model with $q=2$ the *FK-Ising model:* in this case, the model provides a graphical representation of the spin-Ising model, as is best seen through the so-called Edwards-Sokal coupling: if one samples an FK-Ising configuration on G, assigns a ± 1 spin to each cluster by an independent fair coin toss, and gives to each vertex of G the spin of its cluster, the configuration thus obtained is a sample of the spin-Ising model on G at inverse temperature $\beta = \frac{1}{2}\log(1-p)$. The value of the FK-Ising parameter p corresponding to the critical value of the spin-Ising parameter $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ hence equals $\frac{\sqrt{2}}{\sqrt{2}+1}$.

Via the Edwards-Sokal coupling, the FK-Ising model describes how the influence between the spins propagates across the graph: conditionally on the FK-Ising configuration, two spins of the Ising model are aligned if they belong in the same cluster and independent otherwise. For this reason, the spin-Ising and FK-Ising models are intimately related (including in their scaling limit).

In this paper, we will work with the *critical FK-Ising model*, hence the FK model with parameter values $p = \frac{\sqrt{2}}{\sqrt{2}+1}$ and $q = 2$.

1.1 Main statement

In this paper, we obtain uniform RSW-type crossing probabilities [Rus81, SW78] for the critical FK-Ising model on general topological rectangles.

Given a topological rectangle (Ω, a, b, c, d) (i.e. a bounded simply connected subdomain of $\mathbb{Z}^2$ with four marked boundary points) and boundary conditions ξ (see Section 2.2), denote by ϕ_{Ω}^{ξ} the critical FK-Ising probability measure on Ω with boundary conditions ξ and by $(ab) \leftrightarrow (cd)$ the event that there is a *crossing* between the arcs (ab) and (cd), i.e. that (ab) and (cd) are connected in the FK configuration.

Let us denote by $\ell_{\Omega}\left[(ab), (cd)\right]$ the discrete extremal length between (ab) and (cd) in Ω with unit conductances (see Section 3 for a precise definition). Informally speaking, $\ell_{\Omega}\left[(ab), (cd)\right]$ measures the distance between (ab) and (cd) from a random walk or electrical resistance point of view.

Our main result is a bound for FK-Ising crossing probabilities in terms of discrete extremal length only:

Theorem 1.1. *Let $M > 0$. There exists $\delta > 0$ such that for any topological rectangle (Ω, a, b, c, d) with $\ell_\Omega [(ab), (cd)] \leq M$, and for any boundary conditions ξ, we have*

$$\phi_\Omega^\xi [(ab) \leftrightarrow (cd)] \geq \delta.$$

Similarly, for any $m > 0$, there exists $\eta < 1$ such that for any topological rectangle (Ω, a, b, c, d) with $\ell_\Omega [(ab), (cd)] \geq m$, and for any boundary conditions ξ, we have

$$\phi_\Omega^\xi [(ab) \leftrightarrow (cd)] \leq \eta.$$

Such crossing probabilities bounds, uniform with respect to the boundary conditions, have been obtained in a (straight) rectangle in [DCHN10, Theorem 1]; asymptotic exact computations of crossing probability in arbitrary domains with specific boundary conditions have been derived in [CS09, Theorem 6.1]. In this paper, the crossing bounds hold in *general topological rectangles with general boundary conditions*, and are independent of the local geometry of the boundary. Roughly speaking, our result is a generalization of [DCHN10] to possibly "rough" discrete domains; this is for instance needed in order to deal with domains generated by random interfaces (which usually have fractal scaling limits).

As in [DCHN10], our result relies on discrete complex analysis: to connect the FK-Ising model with discrete complex analysis objects, we use the discrete analytic observable for the FK-Ising model introduced by Smirnov [Smi10] and crossing probability representation (in terms of harmonic measure) introduced by Chelkak and Smirnov [CS09]. To obtain the desired estimate, we adapt these results and use new harmonic measure techniques developed by Chelkak in [Che11].

1.2 Applications

Crossing probabilities estimates play a very important role in rigorous statistical mechanics, in particular for percolation models. They constitute the key argument enabling the use of the following techniques:

- Spatial decorrelation: probabilities of certain events in disjoint 'well separated' sets can be factorized at the expense of uniformly controlled constants. The main ingredients to do so are the spatial Markov property of the model (see Section 2.2) and the crossing probabilities bounds.

- Regularity estimates and precompactness: the crossing probabilities are instrumental to pass to the scaling limit. They imply a priori regularity estimates on the discrete random curves arising in the model.

- Discretization of continuous results: thanks to uniform estimates, one can connect the discrete models (at finite scales) to their continuous limits, and transfer results from the latter to the former.

While the RSW bounds of [DCHN10] already allow for a number of interesting applications (see for instance [CN09, LS10, CGN10, GP]), the stronger version of the RSW-type estimates provided by Theorem 1.1 increases the scope of applications. In particular, we get the following new consequences, that we describe in more details now.

In the rest of this paper, for two real-valued quantities A, B depending on a certain number of parameters, we will write $A \lesssim B$ if there exists a constant $c > 0$ such that $A \leq cB$ and $A \asymp B$ if there exists two constants $c_1, c_2 > 0$ such that $c_1 A \leq B \leq c_2 A$. While $A \lesssim B$ is in fact equivalent to $A = O(B)$, we prefer the first notation (we will be using sequences of inequalities, for which the first notation is more suitable).

Arm exponents. Thanks to crossing probabilities, the (microscopic) arm exponents for the FK-Ising model can be related to the (macroscopic) SLE arms exponents, which in turn can be computed using stochastic calculus techniques. The microscopic arm exponents are crucial to understand the fine structure of the phase transition of percolation [Kes87, Nol08], as well as as for interface regularity [AB99] and noise sensitivity [GP] questions.

Define $\Lambda_n := [-n, n]^2$ and $S_{n,N} = \Lambda_N \setminus \Lambda_n$. Dual edges are edges of $(\mathbb{Z}^2)^*$. Fix a finite sequence σ of o's (open) and c's (closed). We say that a path of (primal) edges is o-connected if its edges are open (in the FK-Ising model). We say that a path of dual edges is c-connected if it consists of dual-open edges: edges of $\mathbb{Z}^2$ in correspondence with dual edges of the path are closed (in the FK-Ising model). Fix $\sigma = \sigma_1 \dots \sigma_j$. For $n < N$, define $A_\sigma(n, N)$ to be the event that there are j *disjoint* paths from $\partial\Lambda_n$ to $\partial\Lambda_N$ with are σ_i-connected, for $i \leq j$ where the paths are indexed in counter-clockwise order. For instance, $A_o(n, N)$ is the one-arm event corresponding to the existence of a crossing from the inner to the outer boundary of $\Lambda_N \setminus \Lambda_n$.

The following theorem is crucial in the understanding of arm-exponents. The proof follows ideas going back to Kesten [Kes87]. Importantly, it relies heavily on Theorem 1.1 and previous results on crossing probabilities would not be sufficient to derive the theorem. Let $\phi_{\mathbb{Z}^2}$ denotes the unique infinite-volume measure at $q = 2$, $p = p_c(2)$.

Theorem 1.2 (Quasi-multiplicativity)**.** *Fix a sequence σ. For every $n_1 < n_2 < n_3$,*

$$\phi_{\mathbb{Z}^2}\big[A_\sigma(n_1, n_3)\big] \asymp \phi_{\mathbb{Z}^2}\big[A_\sigma(n_1, n_2)\big]\, \phi_{\mathbb{Z}^2}\big[A_\sigma(n_2, n_3)\big],$$

where the constants in $\asymp$ depend on σ only.

This important theorem has several consequences. We mention three of them.

let $I = (I_k)_{k \leq j}$ be disjoint intervals of size δ on the boundary of the square $Q = [-1, 1]^2$, found in counter-clockwise order following ∂Q. For a sequence σ of length j, let $A_\sigma^I(n, N)$ be the event that $A_\sigma(n, N)$ occurs and paths $\gamma_1, \dots, \gamma_j$ can be chosen in such a way that γ_k ends on I_k for every $k \leq j$.

Corollary 1.1. *Fix $j > 0$. For any choice of I, σ and $n < N$,*

$$\phi_{\mathbb{Z}^2}\big[A_\sigma^I(n, N)\big] \;\asymp\; \phi_{\mathbb{Z}^2}\big[A_\sigma(n, N)\big],$$

where the constants in $\asymp$ depend on σ and I only.

This leads to the computation of universal exponents.

Corollary 1.2 (Universal exponents)**.** *For every* $0 < k < n$*,*

$$\phi_{\mathbb{Z}^2}\left[A_{ocooc}(k,n)\right] \asymp (k/n)^2, \qquad \phi_{\mathbb{Z}^2}\left[A^{HP}_{oc}(k,n)\right] \asymp k/n, \qquad \phi_{\mathbb{Z}^2}\left[A^{HP}_{oco}(k,n)\right] \asymp (k/n)^2.$$

where $A^{HP}_\sigma(n,N)$ *is the existence of* j *paths in* $[-N,N]\times[0,N]\setminus[-n,n]\times[0,n]$ *from* $[-n,n]\times[0,n]$ *to* $([-N,N]\times[0,N])^c$*. Above, the constants in* $\asymp$ *are universal.*

Another implication is the following one, which can be used to prove convergence of critical FK-Ising interfaces to SLE(16/3).

Corollary 1.3. *There exists* $\alpha > 0$ *such that for every* $0 < k < n \le L_p$*,*

$$\phi_{\mathbb{Z}^2}\left[A_{ocococ}(k,n)\right] \lesssim (k/n)^{2+\alpha}$$
$$\phi_{\mathbb{Z}^2}\left[A_{ococ}(k,n)\right] \gtrsim (k/n)^{2-\alpha},$$

where the constants in $\lesssim$ *and* $\gtrsim$ *are universal.*

Crossing probabilities for the spin Ising model. Their conformal invariance was investigated numerically in [LPSA94]. Thanks to the Edwards-Sokal coupling, the FK-Ising and the spin-Ising models can be coupled. This enables us to derive crossing probabilities bounds for the spin Ising model from Theorem 1.1.

While it would be impossible to obtain crossing probabilities for the critical spin-Ising that are uniform with respect to the boundary conditions (the probability of crossing of + spins with − boundary conditions everywhere tends to 0 in the scaling limit, as can be seen using SLE techniques), it is possible to get nontrivial bounds that allow to deal with spin-Ising interfaces, notably in presence of free boundary conditions (which is the setup considered in [LPSA94, HK11]).

Corollary 1.4. *Let* $M > 1$*. Then there exists* $\delta \in \left(0, \frac{1}{2}\right)$ *such that the following holds:*

Let (Ω, a, b, c, d) *be a topological rectangle with* $\frac{1}{M} \le \ell_\Omega\left[(ab), (cd)\right] \le M$*. Consider the critical Ising model on* (Ω, a, b, c, d) *with free boundary conditions on* $(ab)\cup(cd)$ *and* + *boundary conditions on* $(bc)\cup(da)$*. Then we have*

$$\delta \le \mathbb{P}\left[\textit{There is a crossing of } - \textit{ spins } (ab) \leftrightarrow (cd)\right] \le 1-\delta.$$

By monotonicity of the spin-Ising model with respect to the boundary conditions (this is an easy consequence of the FKG inequality), this result implies that the probabilities of − crossings in topological rectangles with free boundary conditions (the setup considered in [LPSA94]) are also bounded away from below. By self-duality (for topological reason there cannot be both a − crossing between (ab) and (cd) and a + crossing between (bc) and (da)) and symmetry between − and + spins, such crossing probabilities are also bounded from above.

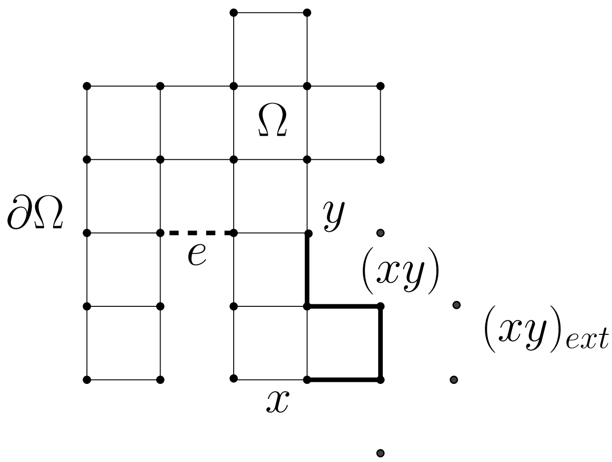

Figure 1: A domain Ω with two points x and y on its boundary. The set (xy) and $(xy)_{\text{ext}}$ are depicted. The edge e appears twice in $\partial\mathcal{E}_{\text{ext}}$.

Coupling of discrete and continuous interfaces. It is useful to couple the critical FK-Ising interfaces and their scaling limit SLE(16/3), in such a way that they are close to each other and that whenever the SLE(16/3) interface hits the boundary of the domain, so does the discrete interface with high probability. Such couplings are in particular useful in order to obtain the full scaling limit of discrete interfaces [CN06, KS12].

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2 Graph, FK-Ising model and notations

2.1 Graph

For a planar graph G, we denote by $\mathcal{E}(G)$ the set of its edges. Most of the time G will be identified with the set of its vertices, which we will also call *sites*. For any two vertices $x, y \in G$, we write $x \sim y$ if they are adjacent and we denote by $xy \in \mathcal{E}(G)$ the edge between them.

In this paper, we will consider finite connected and simply connected (meaning with connected complement) graphs that are made of the union of faces of the square grid $\mathbb{Z}^2$ (vertices are points of $\mathbb{Z}^2$ and vertices at distance 1 are linked by an edge). We will call these *discrete domains.*

For a discrete domain Ω, we denote by $\partial\Omega \subset \Omega$ its boundary (when we view Ω a domain consisting of the union of its faces); most of the time, we will identify $\partial\Omega$ with the set of its vertices, called *the boundary vertices.* We denote by $\mathrm{Int}(\Omega)$ the interior of the graph, defined as $\Omega \setminus \partial\Omega$. We denote by $\partial_{\text{ext}}\mathcal{E}(\Omega)$ the set of *external edge*s of Ω, defined as the set of edges of $\mathcal{E}(\mathbb{Z}^2) \setminus \mathcal{E}(\Omega)$ incident to a vertex of Ω, *counted with multiplicity*: if an edge of $\mathcal{E}(\mathbb{Z}^2) \setminus \mathcal{E}(\Omega)$ is incident to two vertices of Ω, it appears as two distinct elements of $\partial_{\text{ext}}\mathcal{E}(\Omega)$. We identify the edges of $\partial_{\text{ext}}\mathcal{E}(\Omega)$ with the set $\partial_{\text{ext}}\Omega$ of *external boundary vertices*, they are the formal endpoints in $\mathbb{Z}^2 \setminus \Omega$ of the edges of $\partial\mathcal{E}(\Omega)$.

For two points $x, y \in \partial\Omega$, we denote by $(xy) \subset \partial\Omega$ the counterclockwise arc of $\partial\Omega$ from x to y (including x and y); as usual we identify (xy) with the set of the vertices located on it; we will frequently identify $x \in \partial\Omega$ with the arc (xx); we denote by $(xy)_{\text{ext}}$ the set of vertices of $\partial_{\text{ext}}\Omega$ adjacent to (xy). We call a discrete domain Ω with four marked vertices $a, b, c, d \in \partial\Omega$ in counterclockwise order a *topological rectangle.*

We denote by $(\mathbb{Z}^2)^*$ the dual of $\mathbb{Z}^2$: the vertices of $(\mathbb{Z}^2)^*$ are the (centers) of the faces of $\mathbb{Z}^2$ and nearest neighbors are linked by an edge. Given a discrete domain Ω, the *dual domain* Ω^* is the induced subgraph of $(\mathbb{Z}^2)^*$ whose vertices correspond to faces of Ω. We denote by $\partial\Omega^*$ the set of vertices of Ω^* corresponding to faces of Ω sharing an edge with $\partial\Omega$. We denote by $\partial_{\text{ext}}\Omega^*$ the set of *external dual vertices*, corresponding to the faces of $\mathbb{Z}^2 \setminus \Omega$ adjacent to $\partial\Omega$, with multiplicity: it is in bijection with the edges of $\partial\Omega$.

2.2 FK-Ising model

In order to remain as self-contained as possible, some basic features of the random-cluster models are presented now. The reader can consult the reference book [Gri06] for additional details.

2.2.1 Definition of FK-measures

We define the *FK percolation* measure on arbitrary finite graphs. Let G be a finite graph, with a specified boundary ∂G. A *configuration* ω is a random subgraph of G given by the vertices of G, together with a subset of $\mathcal{E}(G)$. An edge of G is called *open* if it belongs to ω, and *closed* otherwise. Two sites $x, y \in G$ are said to be *connected* if there is an *open path* (a path composed of open edges only) connecting them. Similarly, two sets of vertices X and Y are said to be connected if there exist two sites $x \in X$ and $y \in Y$ that are connected; we use the notation $X \leftrightarrow Y$ for this event. We also write $x \leftrightarrow Y$ for $\{x\} \leftrightarrow Y$. Maximal connected components of the configuration are called *clusters.*

A set of *boundary conditions* $\xi = E_1, E_2, \ldots$ is a partition of ∂G into disjoint subsets $E_1, E_2, \ldots \subset \partial G$. For conciseness, singletons subsets are omitted from the notations. We say that two boundary vertices $x, y \in \partial G$ are *wired* if they belong to the same element of ξ; we call boundary vertices that are not wired to other vertices *free.* Informally speaking, the role of the boundary conditions conditions is to encode how sites are connected outside G (see Section 2.2.2).

We denote by $\omega \cup \xi$ the graph obtained from the configuration ω by artificially linking together any two pair of vertices $x, y \in \partial\Omega$ that are wired by ξ. Let $o(\omega)$ (resp. $c(\omega)$) denote the number of open (resp. closed) edges of ω and $k(\omega, \xi)$ the number of connected components of $\omega \cup \xi$. The probability measure $\phi^{\xi}_{p,q,\Omega}$ of the random-cluster model on G

with parameters p and q and boundary condition ξ is defined by

$$\phi^{\xi}_{p,q,G}(\{\omega\}) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}}{Z^{\xi}_{p,q,G}} \tag{2.1}$$

for every configuration ω on G, where $Z^{\xi}_{p,q,G}$ is the partition function (normalizing constant).

2.2.2 Domain Markov property

The state of the edges in the FK model being highly dependent, what happens in a given subgraph depends on the configuration outside the subgraph. The FK model possesses a 'screening effect' known as *domain* (or *spatial*) *Markov property*, which usually makes it possible to encode this dependence through boundary conditions. This property is used a number of times in this paper.

Take a graph G. For a subset of edges $F \subset \mathcal{E}(G)$, consider the graph G' having F as a set of edges, and the endpoints of F as a set of vertices. Then, for any set of boundary conditions ψ, $\phi^{\psi}_{p,q,\Omega}$ conditioned to match a configuration ω on $E \setminus F$ is equal to $\phi^{\xi}_{p,q,G'}$, where ξ is the set of connections inherited from ω (one wires in ξ the boundary vertices that are connected in $G \setminus G'$). In other words, the influence of the configuration outside G' is completely contained in the boundary conditions ξ.

2.2.3 FKG and monotonicity

The random-cluster model with parameters $p \in [0,1]$ and $q \geq 1$ on a finite graph G has the *strong positive association property.* More precisely, it satisfies the so-called Holley criterion [Gri06], a fact which has two important consequences. A first consequence is the well-known *FKG inequality*

$$\phi^{\xi}_{p,q,\Omega}(A \cap B) \geq \phi^{\xi}_{p,q,\Omega}(A)\, \phi^{\xi}_{p,q,\Omega}(B)$$

for any pair of *increasing* events A, B (increasing events are defined in the usual way [Gri06]) and any boundary conditions ξ. This correlation inequality is fundamental to the study FK percolation, for instance to combine several increasing events such as the existence of crossings in various rectangles.

A second property implied by the strong positive association is the following monotonicity with respect to boundary conditions, which is particularly useful when combined with the Domain Markov property. For any two sets of boundary conditions $\psi \leq \xi$ (any two vertices wired in ψ are wired in ξ), we have

$$\phi^{\psi}_{p,q,G}(A) \leq \phi^{\xi}_{p,q,G}(A) \tag{2.2}$$

for any increasing event A that depends only on Ω. We say that $\phi^{\psi}_{p,q,G}$ is stochastically dominated by $\phi^{\xi}_{p,q,G}$, and we write $\phi^{\psi}_{p,q,G} \leq \phi^{\xi}_{p,q,G}$.

Among all the possible boundary conditions, four play a specific role in our paper:

- The *free* boundary condition corresponds to the case when there are no extra edges connecting boundary vertices; we denote by $\phi^{0}_{p,q,\Omega}$ the corresponding measure.

- The *wired* boundary condition corresponds to the case when all the boundary vertices are pair-wise connected, and the corresponding measure is denoted by $\phi^1_{p,q,\Omega}$.
- For a discrete domain Ω with two boundary points $a, b \in \partial\Omega$, we call *Dobrushin* the boundary conditions where the vertices of (ab) are wired together, and the other vertices are free. The measure is denote by $\phi^{(ab)}_{\Omega}$.
- For a topological rectangle (Ω, a, b, c, d), we denote by $\phi^{(ab),(cd)}_{\Omega}$ the random-cluster measure with the vertices of (ab) wired together, the vertices of (cd) wired together and the rest of the vertices free. These boundary conditions are called alternating *free/wired/free/wired* boundary conditions.

Remark 2.1. *The free and wired boundary conditions are extremal for stochastic domination: for any boundary condition ξ and any increasing event A,*

$$\phi^0_{p,q,\Omega}(A) \;\leq\; \phi^\xi_{p,q,\Omega}(A) \;\leq\; \phi^1_{p,q,\Omega}(A). \tag{2.3}$$

Hence to get a lower bound (respectively an upper bound) on crossing probabilities that is uniform with respect to the boundary conditions, it is enough to get such a bound for free (respectively wired) boundary conditions.

2.2.4 Planar self-duality

Like the other critical FK models, the two-dimensional critical FK-Ising model is self-dual: given a discrete domain Ω, we can couple the critical FK-Ising models on Ω (called the primal model) and on Ω^* (called the dual model) in such a way that whenever an edge $e \in \mathcal{E}(\Omega)$ is open, the dual edge $e^* \in \mathcal{E}(\Omega^*)$ is closed, and vice versa. The boundary conditions of both models are dual to each other: if we consider wired boundary conditions on Ω, the boundary conditions of the dual model on Ω^* are free.

It can be shown that all critical FK models are self-dual [BDC10]. For more detail on planar duality, see [Gri06], for instance.

3 Discrete complex analysis

In the section, we introduce the discrete harmonic measures and random walk partition functions that will be used in this article. A number of their properties are provided, including factorization properties and uniform comparability results obtained in [Che11]. Finally, we relate certain elementary critical FK-Ising model probabilities to discrete harmonic measure, notably using the observables introduced in [Smi10, CS09]. These results will be brought together in the next section to prove Theorem 1.1.

3.1 Laplacians, harmonic measures and random walks

Let Ω be a discrete domain, with boundary vertices $\partial\Omega$ and external boundary vertices $\partial_{\text{ext}}\Omega$. Consider a collection of nonnegative conductances $\mathbf{C} = (\mathbf{c}_e)_e$ defined on the set of the edges $\mathcal{E}$ and of external boundary edges $\partial_{\text{ext}}\mathcal{E}(\Omega)$; we call the conductances on $\mathcal{E}(\Omega)$ the *bulk conductances* and the conductances on $\partial_{\text{ext}}\mathcal{E}(\Omega)$ the *boundary conductances*. **In this paper, the bulk conductances are always assumed to be** 1 **and the boundary conductances are in** $[\frac{1}{\mu}, \mu]$ **for some** $\mu > 1$.

With this set of conductances is associated a Laplacian $\Delta_{\mathbf{C}}$ defined (for a function $f : \Omega \cup \partial_{\text{ext}}\Omega \to \mathbb{R}$) by:

$$\begin{aligned} \Delta_{\mathbf{C}} f(x) &:= \frac{1}{\lambda_x} \sum_{y \sim x} \mathbf{c}_{xy} \left(f(y) - f(x) \right) \quad \forall x \in \Omega \\ \lambda_x &:= \sum_{y \sim x} \mathbf{c}_{xy} \end{aligned}$$

For $x, y \in \Omega$, we denote by $Z_{\Omega,\mathbf{C}}[x, y]$ the partition function of the random walks (RW) ω in Ω with conductances $\mathbf{C}$ from x to y, absorbed by $\partial_{\text{ext}}\Omega$. More formally, the possible realizations are the sequences $\omega_1, \ldots, \omega_n$ of vertices such that ω is adjacent to ω for each i, $\omega_1 = x$, $\omega_{2,\ldots,n-1} \in \Omega \setminus \{y\}$ and $\omega_n = y$. The partition function is defined by

$$\begin{aligned} Z_{\Omega,\mathbf{C}}[x, y] &:= \sum_{\omega : x \to y} \prod_{k=1}^{\text{length}(\omega)-1} \frac{\mathbf{c}_{\omega_k \omega_{k+1}}}{\lambda_{\omega_k}} \\ &= \mathbb{P}\left[\text{RW with generator } \Delta_{\mathbf{C}} \text{ starting from } x \text{ hits } y \text{ before } \partial_{\text{ext}}\Omega\right] \end{aligned}$$

When the context is clear, we will omit the set of conductances $\mathbf{C}$ in the subscripts.

Let $(cd) \subset \partial\Omega$ be a boundary arc. We define for $x \in \Omega$

$$\begin{aligned} Z_{\Omega,\mathbf{C}}[x, (cd)] &:= \sum_{y \in (cd)} Z_{\Omega,\mathbf{C}}[x, y] \\ &= \mathbb{P}\left[\text{RW with generator } \Delta_{\mathbf{C}} \text{ starting from } x \text{ hits } (cd) \text{ before } \partial_{\text{ext}}\Omega\right]. \end{aligned}$$

It is easy to check that $x \mapsto Z_{\Omega,\mathbf{C}}[x, (cd)]$ is a $\Delta_{\mathbf{C}}$-harmonic function on $\Omega \setminus ((cd) \cup \partial_{\text{ext}}\Omega)$ which has boundary conditions 1 on (cd) and boundary conditions 0 on $\partial_{\text{ext}}\Omega$.

If $(ab), (cd) \subset \partial\Omega$ are boundary arcs, we define

$$Z_{\Omega,\mathbf{C}}[(ab), (cd)] := \sum_{x \in (ab)} Z_{\Omega,\mathbf{C}}[x, (cd)].$$

Given a discrete domain Ω, we define in the same manner partition functions of random walks on Ω^*, taking $\partial\Omega^*$ and $\partial_{\text{ext}}\Omega^*$ instead of $\partial\Omega$ and $\partial_{\text{ext}}\Omega$.

3.2 Discrete extremal length

A very useful tool when dealing with discrete harmonic measures in topological rectangle is a discrete version of the extremal length. It measures the distance, from the discrete harmonic measures point of view, between two arcs of a domain, in a particularly robust manner. In this paper, we will mostly use it to compare partition functions of random walks on Ω and on the dual graph Ω^*.

Consider a topological rectangle (Ω, a, b, c, d) and a collection of conductances $\mathbf{C}$ (recall that bulk conductances are always 1). Denote by $\mathbf{C}^{\mathrm{DN}}_{(\Omega,a,b,c,d)}$ the set of conductances $\mathbf{C}$, except that the conductances to the edges incident to a vertex of $(bc) \cup (da)$ are set to 0: in other words, the Laplacian $\Delta_{\mathbf{C}^{\mathrm{DN}}_{(\Omega,a,b,c,d)}}$ is the generator of the random walk generated by $\Delta_{\mathbf{C}}$ reflected by the arcs (bc) and (da) (more precisely: reflected by the edges of $\partial\mathcal{E}(\Omega)$ incident to $(bc) \cup (da)$).

Following [Che11], we define the extremal length $\ell_{\Omega,\mathbf{C}}[(ab),(cd)]$ by

$$\ell_{\Omega,\mathbf{C}}[(ab),(cd)] := \left(\mathcal{Z}_{\Omega,\mathbf{C}^{\mathrm{DN}}_{(\Omega,a,b,c,d)}}[(ab),(cd)]\right)^{-1}.$$

When no set of conductances is specified, like in Theorem 1.1, all conductances are set to 1.

The discrete extremal length is particularly powerful because of its robustness: the discrete extremal lengths on a discrete domain with different boundary conductances are uniformly comparable. Also, the discrete extremal length on a rectangle and its dual are comparable (note that such a general result would not be true for partition functions of random walks with purely Dirichlet boundary conditions):

Theorem 3.1 ([Che11])**.** *Let $\mu > 1$. Let (Ω, a, b, c, d) be a topological rectangle and consider a set of conductances $\mathbf{C}$ on Ω with boundary conductances in $\left[\frac{1}{\mu}, \mu\right]$. Let Ω^* be the dual to Ω^* and let $\mathbf{C}^*$ be a set of conductances on Ω^* with boundary conductances in $\left[\frac{1}{\mu}, \mu\right]$. Then we have*

$$\ell_{\Omega,\mathbf{C}}[(ab),(cd)] \asymp \ell_{\Omega^*;\mathbf{C}^*}[(ab)^*,(bc)^*],$$

where the constants in $\asymp$ depend on μ only.

The next theorem asserts that if the extremal length is of order 1 (like in the statement of Theorem 1.1), then so are the partition functions of random walks with Dirichlet boundary conditions:

Theorem 3.2 ([Che11])**.** *Let $M > 1$ and $\mu > 1$. For any topological rectangle (Ω, a, b, c, d) and any set of conductances $\mathbf{C}$ with boundary conductances in $\left[\frac{1}{\mu}, \mu\right]$, if*

$$\frac{1}{M} \leq \ell_{\Omega,\mathbf{C}}[(ab),(cd)] \leq M$$

then

$$\ell_{\Omega,\mathbf{C}}[(bc),(da)] \asymp 1, \qquad Z_{\Omega,\mathbf{C}}[(ab),(cd)] \asymp 1, \qquad Z_{\Omega,\mathbf{C}}[(bc),(da)] \asymp 1,$$

where the constants in $\asymp$ depend on M and μ only.

3.3 Factorization results

In this section, we review the main results of [Che11] concerning factorization properties of discrete harmonic measure. While in the continuum such results are rather easy to derive (for instance using explicit expressions and conformal invariance), it requires a much more delicate analysis to obtain them (up to uniform constants) on the discrete level.

Theorem 3.3 ([Che11])**.** *Let $\mu > 1$. Let (Ω, a, b, c) be a topological triangle and consider a set of conductances $\mathbf{C}$ on Ω with boundary conductances in $\left[\frac{1}{\mu}, \mu\right]$. We have*

$$Z_{\Omega,\mathbf{C}}\left[a,(bc)\right] \asymp \sqrt{\frac{Z_{\Omega,\mathbf{C}}\left[a,b\right] Z_{\Omega,\mathbf{C}}\left[a,c\right]}{Z_{\Omega,\mathbf{C}}\left[b,c\right]}},$$

where the constants in $\asymp$ depend on μ only.

The following estimate will also be needed. It involves a discrete version of the cross-ratio (the left-hand side of 3.1):

Theorem 3.4 ([Che11])**.** *Let $M, \mu > 1$. Let (Ω, a, b, c, d) be a topological rectangle and consider a set of conductances $\mathbf{C}$ on Ω with boundary conductances in $\left[\frac{1}{\mu}, \mu\right]$. If $\ell_{\Omega,\mathbf{C}}\left[(ab),(cd)\right] \leq M$, then*

$$\sqrt{\frac{Z_{\Omega,\mathbf{C}}\left[a,d\right] Z_{\Omega,\mathbf{C}}\left[b,c\right]}{Z_{\Omega,\mathbf{C}}\left[a,b\right] Z_{\Omega,\mathbf{C}}\left[c,d\right]}} \asymp Z_{\Omega,\mathbf{C}}\left[(ab),(cd)\right], \tag{3.1}$$

where the constants in $\asymp$ depend on M and μ only.

3.4 Separators

A crucial concept in the following study is the notion of separators. These separators will allow us to perform efficient surgery of the discrete domains.

Informally speaking, separators of a domain are discrete curves that separate a domain in two pieces, in a "good" manner from harmonic measure point of view: the product of partition functions of random walks in the two pieces is of the same order as the partition function of random walks in the original domain.

If (Ω, a, b, c, d) is a topological rectangle, a *separating curve* between (ab) and (cd) is a simple discrete curve Γ in Ω from (bc) to (da) (it separates (ab) from (cd) in Ω). The connected components of $\Omega \setminus \Gamma$ containing (ab) and (cd) respectively are denoted by $\Omega_{\Gamma,(ab)}$ and $\Omega_{\Gamma,(cd)}$ respectively.

Theorem 3.5 ([Che11])**.** *Let $M, \mu > 1$. Let (Ω, a, b, c, d) be a topological rectangle and consider a set of conductances $\mathbf{C}$ on Ω with boundary conductances in $\left[\frac{1}{\mu}, \mu\right]$. Assume that $Z := Z_{\Omega,\mathbf{C}}\left[(ab),(cd)\right] \leq M$. Then for any $k \in \left[\frac{Z}{M}, \frac{M}{Z}\right]$, there exists a separating curve $\Gamma \subset \Omega$ between (ab) and (cd) such that we have*

$$\begin{aligned} Z_{\Omega_{\Gamma,(ab)},\mathbf{C}}\left[(ab),\Gamma\right] \cdot Z_{\Omega_{\Gamma,(cd)},\mathbf{C}}\left[\Gamma,(cd)\right] &\asymp Z_{\Omega,\mathbf{C}}\left[(ab),(cd)\right], \\ Z_{\Omega_{\Gamma,(cd)},\mathbf{C}}\left[\Gamma,(cd)\right] &\asymp k \cdot Z_{\Omega_{\Gamma,(ab)},\mathbf{C}}\left[(ab),\Gamma\right], \end{aligned} \tag{3.2}$$

where the constants in $\asymp$ depend on M and μ only.

We will call *separator* a separating curve satisfying (3.2). Let us give a corollary that will be particularly useful for us:

Corollary 3.1. *Let $M, \mu > 1$. Let (Ω, a, b, c, d) be a topological rectangle and consider a set of conductances $\mathbf{C}$ on Ω with boundary conductances in $\left[\frac{1}{\mu}, \mu\right]$. Then there exists $\varepsilon \in (0,1)$ (depending on M and μ only) such that for any topological rectangle (Ω, a, b, c, d) with $Z := Z_\Omega\left[(ab), (cd)\right] \leq M$ and any $\kappa \in \left[\frac{Z}{\varepsilon}, \varepsilon\right]$ there exists a separating curve $\Gamma \subset \Omega$ between (ab) and (cd) with*

$$\begin{aligned} Z_{\Omega_{\Gamma,(ab)},\mathbf{C}}\left[(ab), \Gamma\right] \cdot Z_{\Omega_{\Gamma,(cd)},\mathbf{C}}\left[\Gamma, (cd)\right] &\asymp Z_{\Omega,\mathbf{C}}\left[(ab), (cd)\right], \\ Z_{\Omega,\mathbf{C}}\left[\Gamma, (cd)\right] &\in [\varepsilon\kappa, \kappa], \end{aligned}$$

where the constant in $\asymp$ depends on M and μ only.

Proof By Theorem 3.5, there exists $C_1, C_2, C_3, C_4 > 0$ such that for any $k \in \left[\frac{Z}{M}, \frac{M}{Z}\right]$ we have

$$C_1 Z \leq Z_\Omega\left[(ab), \Gamma\right] \cdot Z_\Omega\left[\Gamma, (cd)\right] \leq C_2 Z \quad \text{and} \quad C_3 k \leq \frac{Z_\Omega\left[\Gamma, (cd)\right]}{Z_\Omega\left[(ab), \Gamma\right]} \leq C_4 k.$$

Hence, we obtain

$$\sqrt{C_1 C_3 k Z} \leq Z_\Omega\left[\Gamma, (cd)\right] \leq \sqrt{C_2 C_4 k Z}.$$

Take $\varepsilon := \min\{\sqrt{C_1 C_3/(C_2 C_4)}, \sqrt{C_2 C_4}/M\}$. If $\kappa \in \left[\frac{Z}{\varepsilon}, \varepsilon\right]$, we can choose $k := \frac{\kappa^2}{C_2 C_4 Z} \in \left[\frac{Z}{M}, \frac{M}{Z}\right]$ in Theorem 3.5 to get the result. ☐

We will also need the following corollary, which says that we can split a topological rectangle in "fair" shares:

Corollary 3.2. *Let $M, \mu > 1$. For any topological rectangle (Ω, a, b, c, d) and any set of conductances $\mathbf{C}$ on Ω with boundary conductances in $\left[\frac{1}{\mu}, \mu\right]$ such that*

$$M^{-1} \leq \ell_{\Omega,\mathbf{C}}\left[(ab), (cd)\right] \leq M,$$

there exists separating curve $\Gamma \subset \Omega$ between (ab) and (cd) such that

$$\ell_{\Omega_{(ab)},\mathbf{C}}\left[(ab), \Gamma\right] \asymp \ell_{\Omega_{(cd)},\mathbf{C}}\left[(cd), \Gamma\right] \asymp \ell_{\Omega,\mathbf{C}}\left[(ab), (cd)\right],$$

where the constants in $\asymp$ depend on M only.

Proof By Theorem 3.2, we have that $Z_{\Omega,\mathbf{C}}\left[(ab), (cd)\right] \asymp 1$ (where the constant depends on M only). Applying Theorem 3.5 with $k = 1$, we obtain a simple curve Γ separating (ab) from (cd) with

$$Z_{\Omega_{(ab)},\mathbf{C}}\left[(ab), (xy)\right] \asymp Z_{\Omega_{(cd)},\mathbf{C}}\left[(xy), (cd)\right] \asymp Z_{\Omega,\mathbf{C}}\left[(ab), (cd)\right],$$

where the constants in $\asymp$ depend on M only. Applying once more Theorem 3.2, we get the result. ☐

3.5 From FK-Ising model to discrete harmonic measure

In this section, we relate critical FK-Ising crossing probabilities with free/wired/free/wired boundary conditions to discrete harmonic measures. The main tool consists of the observables introduced in [Smi10, CS09], where the scaling limit of FK-Ising crossing probabilities (with free/wired/free/wired boundary conditions) is computed.

The probability that two wired arcs are connected (with free boundary conditions elsewhere) can be bounded from above in terms of discrete harmonic measure.

Let $\mathbf{C}_\circ^*$ denote the set of unit conductances on the edges of Ω_δ^* and let $Z_\circ$ be the corresponding random walk partition function. Let $\mathbf{C}_\bullet$ be the set of conductances on Ω_δ, where each bulk edge has conductance 1, the boundary edges incident to $(bc) \cup (da)$ have conductance 1 and the boundary edges incident to $(ab) \cup (cd)$ have conductance $\frac{2}{\sqrt{2}+1}$.

Proposition 3.3. *For any $M > 0$, for any (Ω, a, b, c, d) topological rectangle with $Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)] \leq M$, we have*

$$\phi_\Omega^{(ab),(cd)}[(ab) \leftrightarrow (cd)] \lesssim \sqrt{Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]},$$

where the constant in $\lesssim$ depends on M only.

The proof is given below. It follows the ideas of the proof of [CS09, Theorem 6.1], where the above crossing probability is computed in the scaling limit.

When we degenerate the arc (ab) to a singleton, the partition function $Z_{\Omega,\mathbf{C}_\bullet}$ becomes less than one and we obtain the following upper bound (see also [DCHN10, Proposition 6], where a double-sided estimate is derived):

Corollary 3.4. *With the notation of Proposition 3.3, we have*

$$\phi_\Omega^{(cd)}[a \leftrightarrow (cd)] \lesssim \sqrt{Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)]},$$

where the constant in $\lesssim$ is universal.

If we also degenerate the arc (cd) to a singleton, we have a double-sided harmonic measure estimate for the probability that two boundary vertices are connected with free boundary conditions.

Proposition 3.5. *Let Ω be a discrete domain. For any two sites $a, b \in \partial\Omega$, we have*

$$\sqrt{Z_{\Omega^*;\mathbf{C}_\circ^*}[a^*, b^*]} \lesssim \phi_\Omega^0(a \leftrightarrow b) \lesssim \sqrt{Z_{\Omega,\mathbf{C}_\bullet}[a,b]},$$

for any $a^* \in \partial\Omega^*$ at distance $\frac{\sqrt{2}}{2}$ from a and $b^* \in \partial\Omega^*$ at distance $\frac{\sqrt{2}}{2}$ from b. The constants in $\lesssim$ are universal.

This proposition is directly obtained from [DCHN10, Proposition 6], by looking at the case where the wired arc is reduced to a single vertex.

Proof of Proposition 3.3 Fix a domain (Ω, a, b, c, d) and consider the critical FK-Ising model with boundary conditions wired on (ab) and on (cd), and free elsewhere. In [CS09, Proof of Theorem 6.1], two discrete holomorphic observables F_1 and F_2 for this setup are introduced, and it is shown that there exists a unique linear combination F of F_1, F_2 and a unique $\kappa \in \mathbb{R}$ such that a discrete version H, defined on $\Omega \cup (ab)_{\text{ext}} \cup (bc)_{\text{ext}}$ of $\Im\mathfrak{m}\left(\int F^2\right)$ satisfies the following boundary conditions:

$$H = 0 \text{ on } (da)\,, H = 1 \text{ on } (cd) \text{ and } H = \kappa \text{ on } (ab)_{\text{ext}} \cup (bc)_{\text{ext}}\,.$$

This discrete function H is $\Delta_{\mathbf{C}_\bullet}$-subharmonic on $\Omega \setminus ((ab) \cup (cd))$. The constant $\kappa \in [0, 1]$ is shown to be in one-to-one continuous correspondence with $\phi_\Omega^{(ab),(cd)}\left[(ab) \leftrightarrow (cd)\right]$; from [CS09, Formula 6.6], we get in particular that

$$\sqrt{\kappa} \asymp \phi_\Omega^{(ab),(cd)}\left[(ab) \leftrightarrow (cd)\right], \tag{3.3}$$

where the constants are universal.

Let a_{in} be a vertex of $\text{Int}\,(\Omega)$ adjacent to a. By construction of H (see [CS09, Proof of Theorem 6.1]), we have that $H\left(a_{\text{in}}\right) \geq H\left(a_{\text{ext}}\right) = \kappa$, where a_{ext} is the vertex of $(ab)_{\text{ext}}$ closest to a. If we now consider the function $H - \kappa$, we obtain the following estimate in terms of discrete harmonic measure

$$\begin{aligned} 0 &= H(a_{\text{ext}}) - \kappa \leq H\left(a_{\text{in}}\right) - \kappa \\ &\leq Z_{\Omega,\mathbf{C}_\bullet}[a_{\text{in}}, (cd)] + \kappa Z_{\Omega,\mathbf{C}_\bullet}[a_{\text{in}}, (ab)_{\text{ext}}] + \kappa Z_{\Omega,\mathbf{C}_\bullet}[a_{\text{in}}, (bc)_{\text{ext}}] - \kappa \\ &= (1-\kappa)\, Z_{\Omega,\mathbf{C}_\bullet}\left[a_{\text{in}}, (cd)\right] - \kappa Z_{\Omega,\mathbf{C}_\bullet}\left[a_{\text{in}}, (bc)_{\text{ext}}\right], \end{aligned}$$

In the first equality, we used the boundary condition for H, in the second the fact that $H - \kappa$ is $\Delta_{\mathbf{C}_\bullet}$-subharmonic on Ω and the boundary conditions of $H - \kappa$. This leads to

$$\kappa \leq \frac{Z_{\Omega,\mathbf{C}_\bullet}\left[a_{\text{in}}, (cd)\right]}{Z_{\Omega,\mathbf{C}_\bullet}\left[a_{\text{in}}, (bc)_{\text{ext}}\right]} \lesssim \frac{Z_{\Omega,\mathbf{C}_\bullet}\left[a, (cd)\right]}{Z_{\Omega,\mathbf{C}_\bullet}\left[a, (bc)\right]}.$$

We used the fact that harmonic measures are comparable for neighboring vertices and neighboring arcs. Using the factorization for the harmonic measure given by Proposition 3.3, we get

$$\kappa \lesssim \frac{Z_{\Omega,\mathbf{C}_\bullet}\left[a, (cd)\right]}{Z_{\Omega,\mathbf{C}_\bullet}\left[a, (bc)\right]} \asymp \sqrt{\frac{Z_{\Omega,\mathbf{C}_\bullet}\left[a,c\right] Z_{\Omega,\mathbf{C}_\bullet}\left[a,d\right] Z_{\Omega,\mathbf{C}_\bullet}\left[b,c\right]}{Z_{\Omega,\mathbf{C}_\bullet}\left[a,b\right] Z_{\Omega,\mathbf{C}_\bullet}\left[a,c\right] Z_{\Omega,\mathbf{C}_\bullet}\left[c,d\right]}} = \sqrt{\frac{Z_{\Omega,\mathbf{C}_\bullet}\left[a,d\right] Z_{\Omega,\mathbf{C}_\bullet}\left[b,c\right]}{Z_{\Omega,\mathbf{C}_\bullet}\left[a,b\right] Z_{\Omega,\mathbf{C}_\bullet}\left[c,d\right]}}.$$

Using the assumption $Z_{\Omega,\mathbf{C}_\bullet}\left[(ab)\,,(cd)\right] \leq M$, we get by Theorem 3.4 that

$$\kappa \lesssim \sqrt{\frac{Z_{\Omega,\mathbf{C}_\bullet}\left[a,d\right] Z_{\Omega,\mathbf{C}_\bullet}\left[b,c\right]}{Z_{\Omega,\mathbf{C}_\bullet}\left[a,b\right] Z_{\Omega,\mathbf{C}_\bullet}\left[c,d\right]}} \asymp Z_{\Omega,\mathbf{C}_\bullet}\left[(ab)\,,(cd)\right].$$

Hence, (3.3) implies

$$\phi_\Omega^{(ab),(cd)}\left[(ab) \leftrightarrow (cd)\right] \asymp \sqrt{\kappa} \lesssim \sqrt{Z_{\Omega,\mathbf{C}_\bullet}\left[(ab)\,,(cd)\right]}.$$

$\square$

4 Proof of Theorem 1.1

In this section, we will be considering partition functions of random walks on a topological rectangles, and will omit the dependence on the domain in the notation when the context is clear. Recall that $Z_{\Omega,\mathbf{C}_\bullet}[\Gamma_1,\Gamma_2]$ is the partition function function of random walks on Ω as previously defined, with unit conductances everywhere, except on the external edges incident to $\Gamma_1 \cup \Gamma_2$, where the conductances are set to $\frac{2}{1+\sqrt{2}}$.

Lemma 4.1. *Let $M > 1$. For any (Ω, a, b, c, d) with $Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)] \le M$, we have*

$$\phi^{(cd)}\big(a \leftrightarrow (cd), b \leftrightarrow (cd)\big) \ \lesssim \ \sqrt{\frac{Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)]Z_{\Omega,\mathbf{C}_\bullet}[b,(cd)]}{Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]}},$$

where the constant in $\lesssim$ depends only on M.

Proof In this proof, constants in $\asymp$ and $\lesssim$ are depending only on M. Note that $Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)] \le Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)] \le M$. Fix $\varepsilon = \varepsilon(M) \in (0,1)$ as given by Corollary 3.1. Then we have two cases:

Case 1: $Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)] > \frac{\varepsilon}{3} Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]$ or $Z_{\Omega,\mathbf{C}_\bullet}[b,(cd)] > \frac{\varepsilon}{3} Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]$.

Suppose we are in the first case (the other case is symmetric). Then Corollary 3.4 implies

$$\begin{aligned}\phi_\Omega^{(cd)}\big(a, b \leftrightarrow (cd)\big) \ &\le \ \phi_\Omega^{(cd)}\big(b \leftrightarrow (cd)\big) \ \le \ \sqrt{Z_{\Omega,\mathbf{C}_\bullet}[b,(cd)]} \\ &\le \ \sqrt{\frac{3Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)]Z_{\Omega,\mathbf{C}_\bullet}[b,(cd)]}{\varepsilon Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]}}.\end{aligned}$$

Case 2: $Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)] \le \frac{\varepsilon}{3} Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]$ and $Z_{\Omega,\mathbf{C}_\bullet}[b,(cd)] \le \frac{\varepsilon}{3} Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]$.

By Corollary 3.1 (setting $\kappa := \frac{1}{3} Z_{\Omega,\mathbf{C}_\bullet}\,[(ab)\,,(cd)]$), there exists a separator Γ_a between a and (cd) such that

$$\frac{\varepsilon}{3} Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)] \ \le \ Z_{\Omega,\mathbf{C}_\bullet}[\Gamma_a,(cd)] \ \le \ \frac{1}{3} Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]. \tag{4.1}$$

Denote by Ω_a the connected component of $\Omega \setminus \Gamma_a$ containing a.

Similarly, there exists a separator Γ_b of b and (cd) such that

$$\frac{\varepsilon}{3} Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)] \ \le \ Z_{\Omega,\mathbf{C}_\bullet}[\Gamma_b,(cd)] \ \le \ \frac{1}{3} Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]. \tag{4.2}$$

Denote by Ω_b the connected component of $\Omega \setminus \Gamma_b$ containing b.

Note that the two separators do not intersect: $\Omega_a \cap \Omega_b = \emptyset$. Otherwise, their union would separate the whole arc (ab) from (cd), which is impossible, since

$$Z_{\Omega,\mathbf{C}_\bullet}[\Gamma_a \cup \Gamma_b,(cd)] \ \le \ Z_{\Omega,\mathbf{C}_\bullet}[\Gamma_a,(cd)] + Z_{\Omega,\mathbf{C}_\bullet}[\Gamma_b,(cd)] \ \le \ 2/3 \cdot Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)].$$

We are thus facing the following topological picture: the two arcs Γ_a and Γ_b are not intersecting and are separating a, b and (cd). Wiring the arc Γ_a and Γ_b, we find:

$$\phi_\Omega^{(cd)}[a, b \leftrightarrow (cd)] \;\le\; \phi_{\Omega_a}^{\Gamma_a}[a \leftrightarrow \Gamma_a]\phi_{\Omega_b}^{\Gamma_b}[b \leftrightarrow \Gamma_b]\phi_{\Omega\setminus(\Omega_a\cup\Omega_b)}^{(cd),\Gamma_a\cup\Gamma_b}[\Gamma_a \cup \Gamma_b \leftrightarrow (cd)].$$

Let us deal with the first term on the right-side. Using Corollary 3.4 and the fact that Γ_a is a separator between a and (cd), we obtain

$$\phi_{\Omega_a}^{\Gamma_a}[a \leftrightarrow \Gamma_a] \;\le\; \sqrt{Z_{\Omega,\mathbf{C}_\bullet}[a,\Gamma_a]} \;\asymp\; \sqrt{\frac{Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)]}{Z_{\Omega,\mathbf{C}_\bullet}[\Gamma_a,(cd)]}} \;\le\; \sqrt{\frac{Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)]}{Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]}},$$

where in the last inequality we used (4.1). Similarly:

$$\phi_{\Omega_b}^{\Gamma_b}[b \leftrightarrow \Gamma_b] \;\lesssim\; \sqrt{\frac{Z_{\Omega,\mathbf{C}_\bullet}[b,(cd)]}{Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]}}$$

For the last term, we get

$$\begin{aligned}\phi_{\Omega\setminus(\Omega_a\cup\Omega_b)}^{(cd),\Gamma_a\cup\Gamma_b}[\Gamma_a \cup \Gamma_b \leftrightarrow (cd)] \;&\le\; \phi_{\Omega\setminus(\Omega_a\cup\Omega_b)}^{(cd),\Gamma_a\cup\Gamma_b\cup(ab)}[\Gamma_a \cup \Gamma_b \cup (ab) \leftrightarrow (cd)]\\ &\le\; \sqrt{Z_{\Omega,\mathbf{C}_\bullet}[\Gamma_a \cup \Gamma_b \cup (ab),(cd)]}\\ &\le\; \sqrt{\tfrac{5}{3}Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]}\end{aligned}$$

where in the second inequality we used Proposition 3.3 and in the third one, (4.1) and (4.2). Putting everything together we find

$$\phi_\Omega^{(cd)}[a, b \leftrightarrow (cd)] \;\lesssim\; \sqrt{\frac{Z_{\Omega,\mathbf{C}_\bullet}[a,(cd)]Z_{\Omega,\mathbf{C}_\bullet}[b,(cd)]}{Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]}}.$$

□

Let us now reduce Theorem 1.1 to a lower bound for crossing probabilities, with free boundary conditions, assuming double-sided estimates for the discrete extremal length. In the following, let $\ell_\Omega[(ab),(cd)]$ be the discrete extremal length with conductances all equal to 1.

Lemma 4.2. *To prove Theorem 1.1, we only need to prove the following: for any $M > 1$, there exists $\delta > 0$ such that for any topological rectangle (Ω, a, b, c, d) with $M^{-1} \le \ell_\Omega[(ab),(cd)] \le M$, we have*

$$\phi_\Omega^0[(ab) \leftrightarrow (cd)] \ge \delta.$$

Proof Let us first notice that to prove both estimates of Theorem 1.1, we can assume that we have a double-sided control on the discrete extremal length of (Ω, a, b, c, d) (instead of the one-sided bounds in the assumptions of the theorem). Indeed, if we want a lower bound on the probability of a crossing $(ab) \leftrightarrow (cd)$ in Ω when $\ell_\Omega[(ab),(cd)] \le M$ (we can always take $M > 1$), we can also assume that $\ell_\Omega[(ab),(cd)] \ge M^{-1}$ by bringing a and b close enough to each other, if necessary (clearly, this will increase $\ell_\Omega[(ab),(cd)] \le M$, while decreasing the probability of a crossing $(ab) \leftrightarrow (cd)$). Similarly, if we want an upper bound on the probability of a crossing $(ab) \leftrightarrow (cd)$, under the assumption $\ell_\Omega[(ab),(cd)] \ge m$ (we can always take $m = M^{-1} < 1$), we can also assume $\ell_\Omega[(ab),(cd)] \le m^{-1}$, by bringing b and c close enough to each other if necessary.

Using the monotonicity with respect to the boundary conditions (Remark 2.1), in order to get a lower bound for the crossing probabilities that is uniform with respect to the boundary conditions, it is enough to get such a bound for free boundary conditions. Similarly, it is sufficient to get a uniform upper bound in the case of fully wired boundary conditions.

Using the self-duality of the model (see Section 2.2.4), we see that obtaining an upper bound for the probability of a crossing $(ab) \leftrightarrow (cd)$ on Ω (with wired boundary conditions) is equivalent to obtaining a lower bound for the probability of a crossing $(bc)^* \leftrightarrow (da)^*$ for the critical FK-Ising model on Ω^* (with free boundary condition). It is hence enough to bound from below the probability $\phi^0_{\Omega^*}[(bc)^* \leftrightarrow (da)^*]$ of a dual crossing from $(bc)^*$ to $(da)^*$ (by a constant depending on M only). The extremal length $\ell_\Omega[(ab)^*,(cd)^*]$ is of the same order as $\ell_\Omega[(ab),(cd)]$ by Theorems 3.1 and 3.2, so it is enough to prove the lower bound of Theorem 1.1, and this proves the lemma. $\square$

Proof of Theorem 1.1 Let $M > 1$. Once again, constants in $\asymp$, $\lesssim$ and $\gtrsim$ depend only on M. Fix a domain (Ω, a, b, c, d) with $\ell_\Omega[(ab),(cd)] \in [M^{-1}, M]$.

By Lemma 4.2, it is enough to prove that there exists $\delta > 0$ such that the probability of a crossing $(ab) \leftrightarrow (cd)$ with free boundary conditions is at least δ.

The proof relies on a second-moment estimate for the random variable

$$N \ := \sum_{u\in(ab),\ v\in(cd)} \phi^0_\Omega[u \leftrightarrow v]\ \mathbb{I}_{u\leftrightarrow v}. \tag{4.3}$$

Step 1: First moment of N.

Let us start by estimating the first moment:

$$\begin{aligned}\phi^0_\Omega[N] \ &= \sum_{u\in(ab),\ v\in(cd)} \phi^0_\Omega[u \leftrightarrow v]^2 \ \gtrsim \sum_{w\in(ab)^*, t\in(cd)^*} Z_{\Omega^*,\mathbf{C}^*_\circ}(w \leftrightarrow t)\\ &= \ Z_{\Omega^*,\mathbf{C}^*_\circ}[(ab)^*,(cd)^*] \ \asymp \ 1,\end{aligned}$$

Note that in order to obtain the first inequality, we used Proposition 3.5. For the last one, we used Theorems 3.3 and 3.4 to show that

$$1 \asymp \ell_\Omega[(ab),(cd)] \asymp \ell_{\Omega^*,\mathbf{C}^*_\circ}[(ab),(cd)] \asymp Z_{\Omega^*,\mathbf{C}^*_\circ}[(ab)^*,(cd)^*].$$

Step 2: Second moment of N.

Corollary 3.2 applied in (Ω, a, b, c, d) gives a separator $\Gamma \subset \Omega$ between (ab) and (cd) splitting Ω in two parts of comparable sizes (in terms of harmonic measure):

$$Z_{\Omega,\mathbf{C}_\bullet}[(ab),\Gamma] \asymp Z_{\Omega,\mathbf{C}_\bullet}[\Gamma,(cd)] \asymp Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)] \asymp 1. \tag{4.4}$$

We find:

$$\begin{aligned}\phi^0_\Omega[N^2] &= \sum_{u,v\in(ab),\ u',v'\in(cd)} \phi^0_\Omega[u\leftrightarrow v]\phi^0_\Omega[u'\leftrightarrow v']\phi^0_\Omega[u\leftrightarrow v, u'\leftrightarrow v'] \\ &\le \sum_{u,v\in(ab),\ u',v'\in(cd)} \phi^0_\Omega[u\leftrightarrow \Gamma]\phi^0_\Omega[u'\leftrightarrow \Gamma]\phi^0_\Omega[v\leftrightarrow \Gamma]\phi^0_\Omega[v'\leftrightarrow \Gamma]\phi^0_\Omega[u,u'\leftrightarrow\Gamma, v,v'\leftrightarrow\Gamma].\end{aligned}$$

Let Ω_1 and Ω_2 be the connected components of $\Omega\setminus\Gamma$ containing (ab), and (cd) respectively. Wiring the arc Γ, the right-hand side factorizes into the product of two terms

$$\begin{aligned}S_{\Omega_1} &= \sum_{u,v\in(ab)} \phi^\Gamma_{\Omega_1}[u\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[v\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[u,v\leftrightarrow\Gamma], \\ S_{\Omega_2} &= \sum_{u',v'\in(cd)} \phi^\Gamma_{\Omega_2}[u'\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_2}[v'\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[u',v'\leftrightarrow\Gamma].\end{aligned}$$

Assume for a moment that we possess the bounds

$$S_{\Omega_1} \lesssim Z_{\Omega,\mathbf{C}_\bullet}[(ab),\Gamma]^{3/2} \qquad \text{and} \qquad S_{\Omega_2} \lesssim Z_{\Omega,\mathbf{C}_\bullet}[\Gamma,(cd)]^{3/2}. \tag{4.5}$$

They imply, thanks to the definition of separators,

$$\phi^0_\Omega[N^2] \le \big(Z_{\Omega,\mathbf{C}_\bullet}[(ab),\Gamma]\cdot Z_{\Omega,\mathbf{C}_\bullet}[\Gamma,(cd)]\big)^{3/2} \lesssim Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)]^{3/2}.$$

Now, by Theorems 3.3 and 3.4, we get that

$$1 \asymp \ell_\Omega[(ab),(cd)] \asymp \ell_{\Omega^*,\mathbf{C}^*_\circ}[(ab),(cd)] \asymp \ell_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)] \asymp Z_{\Omega,\mathbf{C}_\bullet}[(ab),(cd)].$$

We conclude that $\phi^0_\Omega[N^2] \lesssim 1$.

Step 3: Proof of the two estimates in (4.5).

We only show the first one, since the second one is the same. Using Lemma 4.1 and Corollary 3.4, we find

$$\begin{aligned}S_{\Omega_1} &= \sum_{u,v\in(ab)} \phi^\Gamma_{\Omega_1}[u\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[v\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[u,v\leftrightarrow\Gamma] \\ &\lesssim \sum_{u,v\in(ab)} \frac{Z_{\Omega,\mathbf{C}_\bullet}(u,\Gamma)Z_{\Omega,\mathbf{C}_\bullet}(v,\Gamma)}{\sqrt{Z_{\Omega,\mathbf{C}_\bullet}[(uv),\Gamma]}}\end{aligned}$$

Note that for any sequence of positive real numbers $(u_n)_{n\geq 0}$, and $\alpha > 0$, a comparison between series and integral implies

$$\sum_{k=1}^{n} u_k \left(\sum_{j=1}^{k} u_j\right)^{\alpha-1} \leq \frac{1}{\alpha}\left(\sum_{k=1}^{n} u_k\right)^{\alpha}. \tag{4.6}$$

Say that $u \prec v$ if u and v are found in this order when going along the arc (ab) in the counterclockwise order. In our case, (4.6) implies that,

$$\begin{aligned}
\sum_{u,v\in(ab)} \frac{Z_{\Omega,\mathbf{C}_\bullet}[u,\Gamma]Z_{\Omega,\mathbf{C}_\bullet}[v,\Gamma]}{\sqrt{Z_{\Omega,\mathbf{C}_\bullet}[(uv),\Gamma]}} &\leq 2\sum_{u\prec v\in(ab)} \frac{Z_{\Omega,\mathbf{C}_\bullet}[u,\Gamma]Z_{\Omega,\mathbf{C}_\bullet}[v,\Gamma]}{\sqrt{Z_{\Omega,\mathbf{C}_\bullet}[(uv),\Gamma]}} \\
&= 2\sum_{v\in(ab)} Z_{\Omega,\mathbf{C}_\bullet}[v,\Gamma] \sum_{u\in(av)} \frac{Z_{\Omega,\mathbf{C}_\bullet}[u,\Gamma]}{\sqrt{Z_{\Omega,\mathbf{C}_\bullet}[(uv),\Gamma]}} \\
&\leq \sum_{v\in(ab)} Z_{\Omega,\mathbf{C}_\bullet}[v,\Gamma]\sqrt{Z_{\Omega,\mathbf{C}_\bullet}[(av),\Gamma]} \\
&\leq \sum_{v\in(ab)} Z_{\Omega,\mathbf{C}_\bullet}[u,\Gamma]\sqrt{Z_{\Omega,\mathbf{C}_\bullet}[(ab),\Gamma]} \\
&\leq Z_{\Omega,\mathbf{C}_\bullet}[(ab),\Gamma]^{\frac{3}{2}},
\end{aligned}$$

thus giving (4.5).

Step 4: Lower bound for crossing probabilities.

By the Cauchy-Schwarz inequality,

$$\phi_\Omega^0\big((ab)\leftrightarrow(cd)\big) = \phi_\Omega^0(N>0) = \phi_\Omega^0[(\mathbb{I}_{N>0})^2] \geq \frac{\phi_\Omega^0[N]^2}{\phi_\Omega^0[N^2]} \gtrsim 1,$$

where we used the two first steps. Our bound depends on M only. □

5 Arm exponents

5.1 Quasi-multiplicativity

Define $\Lambda_n(x) := x + [-n,n]^2$ and $\Lambda_n = \Lambda_n(0)$. Also set $S_{n,N}(x) = \Lambda_N(x) \setminus \Lambda_n(x)$ and $S_{n,N} = S_{n,N}(0)$.

A classical use of Theorem 1.1 implies the following lemma

Lemma 5.1. *For any σ, there exist $\beta_\sigma > 0$ and $\beta'_\sigma > 0$ such that*

$$(n/N)^{\beta_\sigma} \leq \phi[A_\sigma(n,N)] \leq (n/N)^{\beta'_\sigma}.$$

Another easy consequence of Theorem 1.1 (in fact of a weaker result, see Proposition 5.11 of [DCHN10]) is the following

Proposition 5.2 ([DCHN10]). *There exist $c, \alpha > 0$ such that for any $k \le n$,*

$$\big|\phi(A \cap B) - \phi(A)\phi(B)\big| \le c\left(\frac{k}{n}\right)^{\alpha} \phi(A)\phi(B)$$

for any event A (resp. B) depending only on the edges in the box Λ_k (resp. outside Λ_n).

An important consequence of this property is the following: up to uniform constants, the probability of existence of j arms does not depend on the boundary conditions. In particular,

$$\phi\big(A_\sigma(n,N) \;\big|\; \mathcal{F}_{\mathbb{Z}^2\setminus\Lambda_{2N}}\big) \asymp \phi(A_\sigma(n,N)) \quad \text{a.s.} \tag{5.1}$$

uniformly in n, N, where $\mathcal{F}_\Omega$ is the σ-algebra generated by (the state of) the edges in Ω.

Let us define the notion of well-separated arms. In words, well-separated arms extend slightly outside the boxes and their ends are at macroscopic distance of each others, see Fig. 2. More precisely, for small, fixed $\delta > 0$, j paths $\gamma_1, \ldots, \gamma_j$ with end-points $x_k = \gamma_k \cap \partial\Lambda_n$, $y_k = \gamma_k \cap \partial\Lambda_N$ are said to be *well-separated* if

- the points y_k are at distance larger than $2\delta N$ from each other.
- the points x_k are at distance larger than $2\delta n$ from each other.
- For every k, y_k is σ_k-connected up to distance δN of $S_{n,N}$ in $\Lambda_{\delta N}(y_k)$,
- For every k, x_k is σ_k-connected up to distance δn of $S_{n,N}$ in $\Lambda_{\delta n}(x_k)$.

Let $A_\sigma^{sep;\delta}(n,N) = A_\sigma^{\mathrm{sep}}(n,N)$ be the event that $A_\sigma(n,N)$ holds true and there exist arms realizing $A_\sigma(n,N)$ which are δ well-separated. The previous definition has several convenient properties.

Lemma 5.3. *Fix $\delta < 1$ small enough. For every $n_1 \le n_2$,*

$$\phi[A_\sigma^{\mathrm{sep}}(n_1,n_2)] \gtrsim \phi[A_\sigma^{\mathrm{sep}}(2n_1,n_2)].$$

Proof Condition on $A_\sigma^{\mathrm{sep}}(2n_1,n_2)$ and construct j disjoint tubes of width $\varepsilon = \varepsilon(\delta)$ connecting $(x_k + \Lambda_{2\delta n_1}) \setminus \Lambda_{2n_1}$ to disjoint boxed $\tilde{x}_k + \partial\Lambda_{\delta n_1}$ for every $k \le j$, where $\tilde{x}_k \in \partial\Lambda_{n_1}$. It easily follows from topological considerations that this is possible when δ is small enough. Via Theorem 1.1, the σ_k-paths connecting x_k to $\partial\Lambda_{2\delta n_1}(x_k) \cap \Lambda_{2n_1}$ to $\partial\Lambda_{n_2}$ can be extended to connect to $\partial\Lambda_{n_1}$ while staying in tubes with positive probability $c = c(\delta, p_0)$. □

Proposition 5.4. *Fix $\delta < 1$ small enough. For every $n_1 \le n_2 \le \frac{n_3}{2}$,*

$$\phi[A_\sigma^{\mathrm{sep}}(n_1,n_3)] \gtrsim \phi[A_\sigma^{\mathrm{sep}}(n_1,n_2)] \cdot \phi[A_\sigma^{\mathrm{sep}}(n_2,n_3)].$$

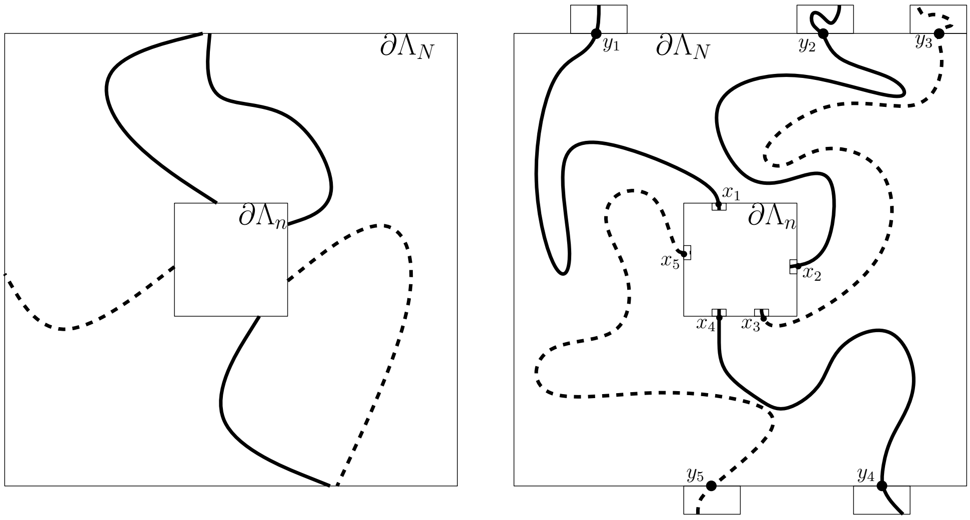

Figure 2: On the left, the five-arm event $A_{ocooc}(n,N)$. On the right, the event $A^{\rm sep}_{ocooc}(n,N)$ with well-separated arms. Note that these arms are not at macroscopic distance of each others inside the domain, but only at their end-points.

Proof of Proposition 5.4 We have

$$\begin{aligned}\phi\big[A^{\rm sep}_\sigma(n_1,n_2)\cap A^{\rm sep}_\sigma(2n_2,n_3)\big] &= \phi\big[A^{\rm sep}_\sigma(n_1,n_2)|A^{\rm sep}_\sigma(2n_2,n_3)\big]\cdot\phi\big[A^{\rm sep}_\sigma(2n_2,n_3)\big]\\ &\gtrsim \phi\big[A^{\rm sep}_\sigma(n_1,n_2)\big]\cdot\phi\big[A^{\rm sep}_\sigma(2n_2,n_3)\big]\\ &\gtrsim \phi\big[A^{\rm sep}_\sigma(n_1,n_2)\big]\cdot\phi\big[A^{\rm sep}_\sigma(n_2,n_3)\big]\end{aligned}$$

thanks to (5.1) and Lemma 5.3 and it suffices to prove that $\phi\big[A^{\rm sep}_\sigma(n_1,n_2)\cap A^{\rm sep}_\sigma(2n_2,n_3)\big]$ and $\phi\big[A^{\rm sep}_\sigma(n_1,n_3)\big]$ are comparable. To do so, condition on $A^{\rm sep}_\sigma(n_1,n_2)\cap A^{\rm sep}_\sigma(2n_2,n_3)$ and construct j disjoint tubes of width $\varepsilon=\varepsilon(\delta)$ connecting $(y_k+\Lambda_{\delta n_2})\setminus\Lambda_{n_2}$ to $(x_k+\Lambda_{2\delta n_2})\cap\Lambda_{2n_2}$ for every $k\le j$. It easily follows from topological considerations that this is possible when δ is small enough. Via Theorem 1.1, the σ_k-paths connecting x_k to $\partial\Lambda_{2\delta n_2}(x_k)\cap\Lambda_{n_2}$, and y_k to $\partial\Lambda_{\delta n_2}(y_k)\setminus\Lambda_{n_2}$ can be connected by a σ_k-path staying in a tube with positive probability $c=c(\delta,p_0)$. Therefore,

$$\phi(A^{\rm sep}_\sigma(n_1,n_3))\ge c\phi\big[A^{\rm sep}_\sigma(n_1,n_2)\cap A^{\rm sep}_\sigma(2n_2,n_3)\big],$$

which concludes the proof. □

This proposition, together with Lemma 5.1, has the following consequence. Fix $p\in(0,1)$ and $\delta<1$ small enough. There exists $\alpha=\alpha(\delta)>0$ such that for every $n_1\le n_2\le n_3$,

$$\phi\big[A^{\rm sep}_\sigma(n_1,n_2)\big]\lesssim\left(\frac{n_3}{n_2}\right)^\alpha\cdot\phi\big[A^{\rm sep}_\sigma(n_1,n_3)\big] \tag{5.2}$$

$$\phi\big[A^{\rm sep}_\sigma(n_2,n_3)\big]\lesssim\left(\frac{n_2}{n_1}\right)^\alpha\cdot\phi\big[A^{\rm sep}_\sigma(n_1,n_3)\big]. \tag{5.3}$$

Our main objective is now to prove the following:

Proposition 5.5. *Fix σ. For every $n < N$, we have*

$$\phi\big[A_\sigma^{\text{sep}}(n,N)\big] \asymp \phi\big[A_\sigma(n,N)\big].$$

Indeed, if $A_\sigma^{\text{sep}}(n,N)$ and $A_\sigma(n,N)$ have uniformly comparable probabilities, Theorem 1.2 follows from the previous statement:

Proof of Theorem 1.2 We have for $n_1 \le n_2 \le n_3$:

$$\begin{aligned}
\phi\big[A_\sigma(n_1,n_3)\big] &\le \phi\big[A_\sigma(n_1,n_2)|A_\sigma(2n_2,n_3)\big]\cdot\phi\big[A_\sigma(2n_2,n_3)\big]\\
&\asymp \phi\big[A_\sigma(n_1,n_2)\big]\cdot\phi\big[A_\sigma(2n_2,n_3)\big]\\
&\asymp \phi\big[A_\sigma^{\text{sep}}(n_1,n_2)\big]\cdot\phi\big[A_\sigma^{\text{sep}}(2n_2,n_3)\big]\\
&\lesssim \phi\big[A_\sigma^{\text{sep}}(n_1,n_2)\big]\cdot\phi\big[A_\sigma^{\text{sep}}(n_2,n_3)\big]\\
&\le \phi\big[A_\sigma(n_1,n_2)\big]\cdot\phi\big[A_\sigma(n_2,n_3)\big],
\end{aligned}$$

where in the second line we used (5.1), in the third, Proposition 5.5, and in the fourth, (5.3). Now,

$$\begin{aligned}
\phi\big[A_\sigma(n_1,n_3)\big] &\ge \phi\big[A_\sigma^{\text{sep}}(n_1,n_3)\big]\\
&\gtrsim \phi\big[A_\sigma^{\text{sep}}(n_1,n_2)\big]\cdot\phi\big[A_\sigma^{\text{sep}}(n_2,n_3)\big]\\
&\asymp \phi\big[A_\sigma(n_1,n_2)\big]\cdot\phi\big[A_\sigma(n_2,n_3)\big]
\end{aligned}$$

where in the first and third lines, we used Proposition 5.5, in the second Proposition 5.4. □

Therefore, we only need to prove Proposition 5.5. Let us start with the following two lemmas:

Lemma 5.6. *For any $\varepsilon > 0$, there exists $T > 0$ such that for every $n > 0$*

$$\phi^\xi_{S_{n,2n}}(\exists\ T\ \textit{disjoint crossings of}\ S_{n,2n})\ \le\ \varepsilon$$

uniformly in boundary conditions ξ.

Proof It is sufficient to show that for $\varepsilon > 0$, there exists $T > 0$ such that the probability of T disjoint vertical crossings of $[0,4n]\times[0,n]$ is bounded by ε uniformly in n and the boundary conditions. In fact, we only need to prove that conditionally on the existence of k crossings, the probability of existence of an additional crossing is bounded from above by some constant $c < 1$.

In order to prove this statement, condition on the k-th left-most crossing γ_k. Assume without loss of generality that γ_k is a dual crossing. Construct a subdomain of $[0,4n]\times[0,n]$ by considering the connected component of $[0,4n]\times[0,n]\setminus\gamma_k$ containing $\{4n\}\times[0,n]$. The configuration in Ω is a random-cluster configuration with boundary conditions ξ on the outside and free elsewhere (*i.e.* on the arc bordering the dual arc γ_k). Now, Theorem 1.1

implies that Ω is crossed from left to right by a primal *and* a dual crossing with probability bounded from below by a universal constant. Indeed, cut the domain Ω into two domains $\Omega_1 = \Omega \cap [0, 4n] \times [0, n/2]$ and $\Omega_2 = \Omega \cap [0, 4n] \times [n/2, n]$ and assume Ω_1 is horizontally crossed and Ω_2 is horizontally dual crossed). This prevents the existence of an additional vertical crossing or dual crossing, therefore implying the claim. $\square$

The previous proof harnesses Theorem 1.1 in a crucial way, the left boundary of Ω being possibly very rough, previous results on crossing estimates would not have been strong enough.

Lemma 5.7. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $2n \leq N$,*

$$\phi^{\xi}_{S_{n/2,2N}}(\text{ any set of crossings of } S_{n,N} \text{ can be made well-separated }) \;\geq\; 1 - \varepsilon$$

uniformly with respect to the boundary conditions ξ.

Proof Fix n and the boundary conditions ξ.

Consider T large enough so that the probability that there exist more than T disjoint crossings of $S_{n,2n}$ is less than ε.

Fix $\delta > 0$ such that in *any subdomain* of the annulus $S_{\delta r,r}$, $\partial\Lambda_{\delta r}$ is not connected or dual connected to $\partial\Lambda_r$ with probability $1-\varepsilon/T$, uniformly in the domain and the boundary conditions on $S_{\delta r,r}$. The existence of δ can be proved using Theorem 1.1.

With probability $1 - 4\varepsilon$, no crossing ends at distance less than δN of a corner of $S_{n,N}$. It is thus sufficient to work with vertical crossings in the rectangle $[-N, N] \times [n, N]$.

Now, condition on the left-crossing γ_1 of $[-N, N] \times [n, N]$ and set y to be the ending point of γ_1 on the top. As before, construct the domain Ω to be the connected component of $\{N\} \times [n, N]$ in $[-N, N] \times [n, N] \setminus \gamma_1$. With probability $1-\varepsilon/T$, no vertical crossing will land at distance δN of y by ensuring that $\Omega \cap S_{\delta^2 N,\delta N}$ contains open and dual-open circuits. Moreover, Theorem 1.1 allows to construct a path P in $\Lambda_{\delta^2 N}(y) \setminus ([-N, N] \times [n, N] \setminus \Omega)$ connecting γ_1 to the top of $\Lambda_{\delta^2 N}(y)$ with probability $c > 0$. The construction costs at most a factor of $c\varepsilon/T$ (in terms of probabilities) and γ_1 is guaranteed to be isolated from other crossings. Iterating the construction T times, we find the result.

The same reasoning applies to the interior side and we obtain the result. $\square$

Proof of Proposition 5.5 The lower bound $\phi[A^{\text{sep}}_{\sigma}(n, N)] \leq \phi[A_{\sigma}(n, N)]$ is straightforward. Let us prove the upper bound for $S_{2^n,2^N}$, first with only the separation on the exterior (the definition is an obvious extension of the definition of well-separated paths). Define $A^{\text{sep/ext}}_{\sigma}(2^n, 2^k)$ to be the event $A_{\sigma}(2^n, 2^k)$ with separation on the exterior only. Let B_k be the event that crossings in $S_{2^{k-1},2^k}$ can be made well separated. Lemma 5.7 ensures that $\phi(B^c_k) \leq \varepsilon$. Note that $A_{\sigma}(2^n, 2^k) \cap B_k \subset A^{\text{sep/ext}}_{\sigma}(2^n, 2^k)$. We thus have

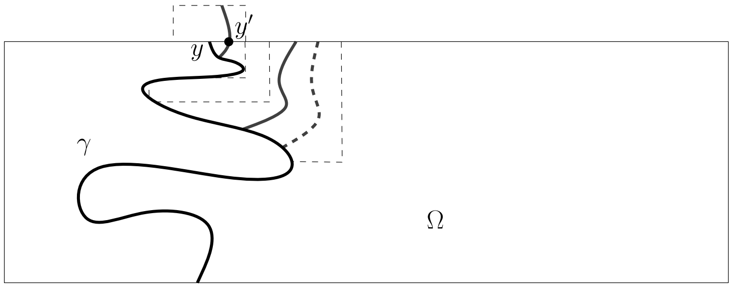

Figure 3: The construction of open and closed paths extending the crossing and preventing other crossings of finishing close to the path.

$$\begin{aligned}\phi\big[A_\sigma(2^n,2^N)\big] &\le \sum_{k=n}^{N-1}\phi\big[A_\sigma(2^n,2^k),B_k,B^c_{k+1},\dots,B^c_{N-1}\big]\\ &\le \sum_{k=n}^{N-1}\phi\big[A_\sigma(2^n,2^k),B_k,B^c_{k+2},B^c_{k+4},\dots\big]\end{aligned}$$

Since annuli are separated by macroscopic areas, we can use (5.1) repeatedly to find

$$\begin{aligned}\phi(A_\sigma(2^n,2^N)) &\le \sum_{k=n}^{N-1}\phi\big[A_\sigma(2^n,2^k),B_k\big]C\phi(B_{k+2})C\phi(B_{k+4})\dots\\ &\le \sum_{k=n}^{N-1}\phi\big[A_\sigma^{\text{sep/ext}}(2^n,2^k)\big]\ (C\varepsilon)^{(N-n)/2}\\ &\lesssim \left(\sum_{k=n}^{N-1}(2^{N-n})^\alpha(C\varepsilon)^{(N-n)/2}\right)\phi\big[A_\sigma^{\text{sep/ext}}(2^n,2^N)\big]\end{aligned}$$

where we used (5.2) in the third line. Choosing ε small enough, we obtain δ such that

$$\phi\big[A_\sigma(2^n,2^N)\big] \ \lesssim\ \phi\big[A_\sigma^{\text{sep/ext}}(2^n,2^N)\big]$$

One can then obtain the separation on the interior in the same way. Now, fix $n<N$ arbitrary. take $s,r\in\mathbb{N}$ such that $2^{s-1}<n\le 2^s$ and $2^r\le N<2^{r+1}$. We have

$$\phi\big[A_\sigma(n,N)\big]\ \le\ \phi\big[A_\sigma(2^s,2^r)\big]\ \asymp\ \phi\big[A_\sigma^{\text{sep}}(2^s,2^r)\big]\ \asymp\ \phi\big[A_\sigma^{\text{sep}}(n,N)\big]$$

using (5.2) and (5.3) a last time. □

5.2 Corollaries of Theorem 1.2

Proof of Corollary 1.1 The proof is classical and uses Proposition 5.5. □

Let $A^I_\sigma(n,N)$ be the event that there exist arms from the interior to the exterior of $S_{n,N}$, and such that γ_k ends on NI_k.

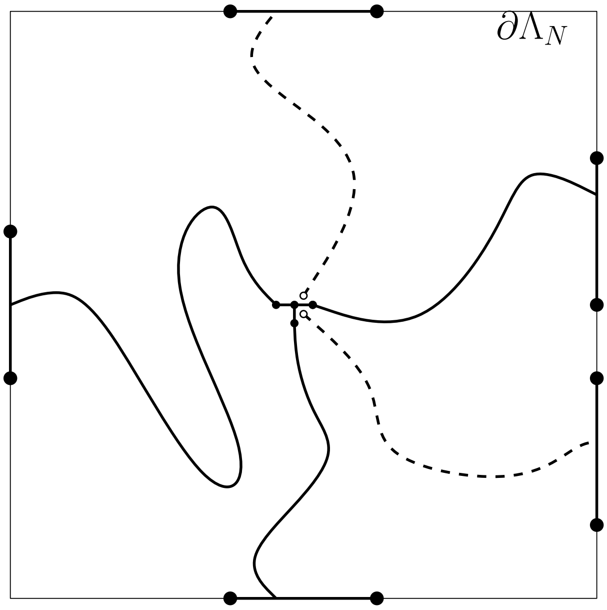

Figure 4: Only one site per rectangle can satisfy the following topological picture.

Proof of Corollary 1.2 We treat the first case only, as the other cases are similar and easier. By quasi-multiplicativity, we only need to look at the case $k = 1$.

Let us first prove the lower bound. Fix $n > 0$. Consider the following construction: assume there exist a horizontal crossing of $[-n, n] \times [-n/4, 0]$ and a dual horizontal crossing of $[-n, n] \times [0, n/4]$. This happens with probability bounded from below by $c > 0$ not depending on n. By conditioning on the lowest interface Γ between an open and a closed crossing of $[-n, n] \times [-n/4, n/4]$, the configuration above it is a random-cluster configuration with free boundary conditions. Let Ω be the connected component of $\Lambda_n \setminus \Gamma$ containing $[-n, n] \times \{n\}$. Assume that $[-n/4, 0] \times [-n, n] \cap \Omega$ is crossed horizontally by a closed path, and that $[0, n/4] \times [-n, n] \cap \Omega$ is crossed horizontally. The probability of this event is once again bounded from below uniformly in n, thanks to Theorem 1.1. Note that we need a strong form of crossing probabilities in order to guarantee the existence of the last crossing since the boundary of Ω can be very rough.

Summarizing, all these events occur with probability larger than $c' > 0$. Moreover, the existence of all these crossings implies the existence of a site in $\Lambda_{n/4}$ with five arms emanating from it. The union bound implies

$$(n/4)^2 \phi[A_{ocooc}(n/4)] \geq c'.$$

In order to prove an upper bound for $\phi[A_{ocooc}(n)]$, recall that it suffices to show it for well-separated arms for which we choose landing sequences. Consider the event described in Fig. 4. Topologically, no two sites in Λ_n can satisfy this event simultaneously, which implies the claim. □

Proof of Corollary 1.3 Fix $n < N$, we have

$$\phi(A_{ocococ}(n,N)) \asymp \phi(A_{ocococ}(n,N), \text{no arm finishing at the bottom}).$$

Conditioning on five arms (starting the exploration from the bottom for instance), it can be shown that

$$\phi(A_{ocococ}(n,N), \text{no arm finishing at the bottom}) \quad \leq \; \phi^0(A_c(n,N))\phi(A_{ococc}(n,N)).$$

The result follows from Corollary 1.2 and the fact that Theorem 1.1 implies

$$\phi^0(A_c(n,N)) \;\leq\; (n/N)^\alpha$$

for some $\alpha > 0$. The same proof works with *ococc* replacing *ocococ*. □

5.3 Spin-Ising crossing probabilities

The FK-Ising model ($q = 2$) and the spin-Ising model are coupled, through (a special case of) what is usually referred to as the *Edwards-Sokal coupling* [ES88].

Let us first recall that the Ising model (with free boundary conditions) on a discrete domain Ω (or more generally a graph) is a random assignment of ± 1 spins $(\sigma_x)_{x\in\Omega}$ to the vertices of Ω, where the probability of a spin configuration is proportional to

$$\exp\left(-\beta H\left(\sigma\right)\right),$$

where

$$H\left(\sigma\right) = -\sum_{i\sim j} \sigma_i \sigma_j,$$

and where $\beta > 0$ is the inverse temperature; the sum is over all pairs of adjacent vertices. We can specify boundary conditions by imposing (i.e. conditioning) that spins σ_x at vertices $x \in \partial\Omega$ take a specified ± 1 value (we speak of $\pm$ boundary conditions) or let them free (free boundary conditions).

Theorem 5.1. *Let Ω be a discrete domain, let $\xi = E$ be the set of boundary conditions, where the vertices in $E \subset \partial\Omega$ are wired together and the other vertices are free. Consider a realization ω of the FK-Ising model on Ω with boundary conditions ξ and parameter $p \in [0,1]$. Let $\sigma \in \{\pm 1\}^\Omega$ be the spin configuration obtained in the following manner:*

- *Set the spins of all the vertices belonging to the cluster containing E to $+1$.*
- *For each cluster K that is not the cluster containing E, sample an independent fair ± 1 coin toss, and give that value to the spins of all the vertices of K.*

Then σ has the law of an spin-Ising configuration, at inverse temperature $\beta = 1 - e^{-p}$, with $+$ boundary conditions on the wired vertices of E and free boundary conditions elsewhere.

As explained in the introduction, through this coupling, crossing probabilities are related to spin correlations (see [DCHN10], for example of applications).

We now make use of this coupling to prove Corollary 1.4.

Proof of Corollary 1.4 Let us show the lower bound only (the upper bound can be obtained by self-duality arguments).

By Theorem 5.1, we can couple this Ising model with an FK-Ising model with boundary conditions $(bc) \cup (da)$ (the sites on $(bc) \cup (da)$ are wired, and the sites on $(ab) \cup (cd)$ are free). Use Corollary 3.2 to split Ω into three "fair shares" $(\Omega_1, a, x_a, x_b, b)$, $(\Omega_2, x_b, x_a, x_c, x_d)$ and $(\Omega_3, c, d, x_d, x_c)$, with

$$\ell_{\Omega_1}\left[(ax_a), (x_b b)\right] \asymp \ell_{\Omega_2}\left[(x_b x_a), (x_c x_d)\right] \asymp \ell_{\Omega_3}\left[(cd), (x_d x_c)\right] \asymp 1$$

(the constants depend on M only). By Theorem 1.1 there exists $\alpha > 0$ such that with probability at least α, there *is no* FK crossing $(ax_a) \leftrightarrow (x_b b)$ in Ω_1, with probability at least α there *is no* FK crossing $(cd) \leftrightarrow (x_d x_c)$, with probability at least α there *is* an FK-Ising crossing $(x_b x_a) \leftrightarrow (x_d x_c)$. So, with probability at least α^3, we can ensure that there is an FK-Ising crossing $(ab) \leftrightarrow (cd)$ in Ω, that does not touch $(bc) \cup (da)$. Sampling a spin-Ising configuration from the FK-Ising model, we get that with probability at least $\frac{1}{2}\alpha^3$, there is an FK-Ising crossing with spin $-$. Note that we use the fact that this crossing is not connected to $(bc) \cup (da)$. □

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