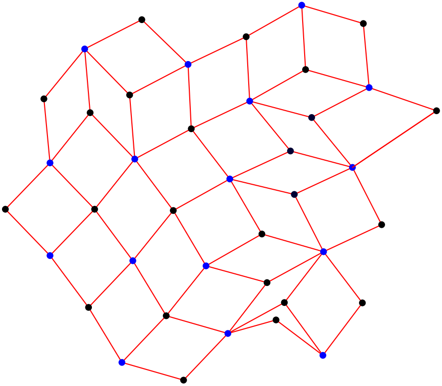
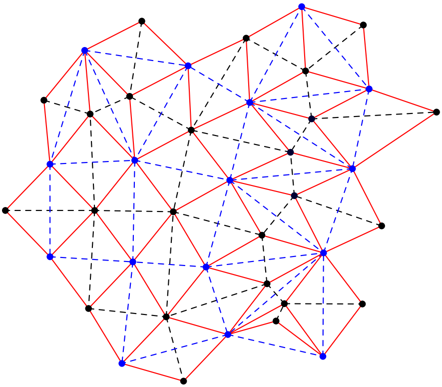

Felix Günther
September 5th
Discrete Riemann Surfaces
and Integrable Systems
TU berlin

Bipartite quad-graphs



Λ bipartite quad-graph
(strongly regular, locally finite)

BIPARTITE QUAD-GRAPHS



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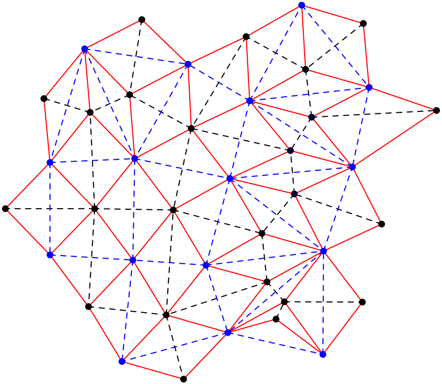
Γ graph of black diagonals

Γ^* graph of blue diagonals

Bipartite quad-graphs



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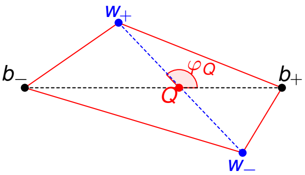
Dual graph $\diamond := \Lambda^*$

DISCRETE HOLOMORPHICITY



$f : V(\Lambda) \to \mathbb{C}$ *discrete holomorphic* at Q iff

$$\text{(dCR)} \quad \frac{f(b_+) - f(b_-)}{b_+ - b_-} = \frac{f(w_+) - f(w_-)}{w_+ - w_-}$$

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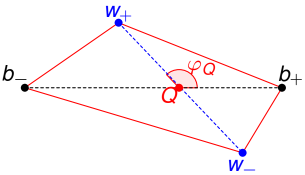

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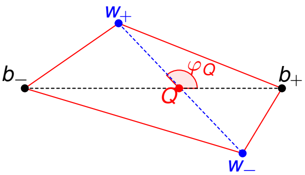

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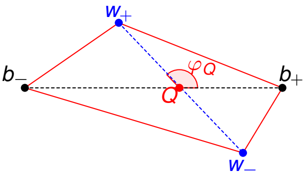
Properties:
f is discrete holomorphic iff $\bar{\partial}_\Lambda f = 0$.

Discrete holomorphicity



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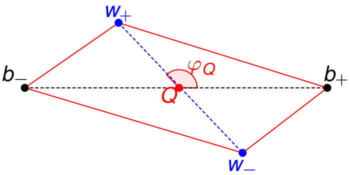
Properties:

If $f(v) = v$, then $\bar{\partial}_\Lambda f(Q) = 0$ and $\partial_\Lambda f(Q) = 1$.

Discrete holomorphicity



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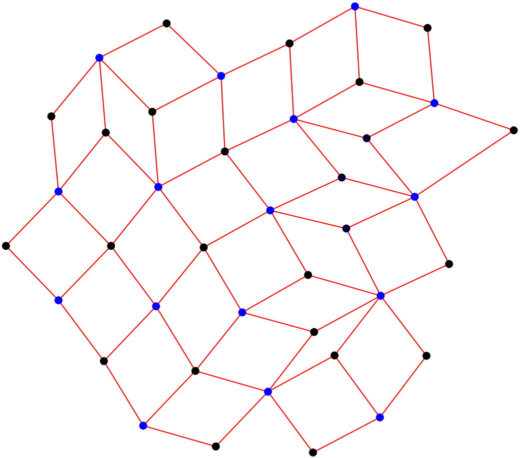
Properties:
If Q is a parallelogram and $f(v) = v^2$, then $\bar\partial_\Lambda f(Q) = 0$, $\partial_\Lambda f(Q) = 2Q$.

Medial graph



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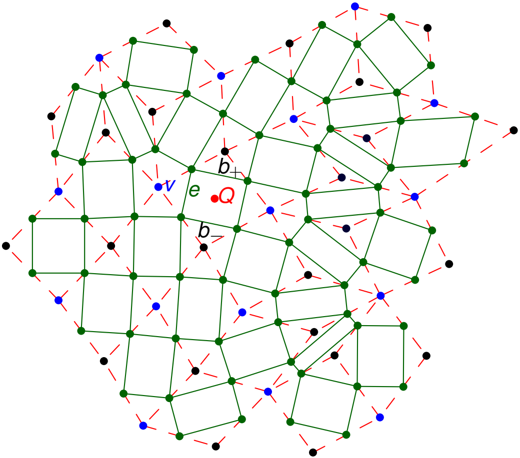

Λ bipartite quad-graph

X medial graph of Λ

edge e of X corresponds to pair $[Q, v] \in V(\diamond) \times V(\Lambda)$

$F(X) \cong V(\diamond) \dot{\cup} V(\Lambda)$

$$e = \pm \frac{b_+ - b_-}{2}$$

Discrete differential forms

- functions $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamondsuit) \to \mathbb{C}$ can be extended to functions $g : F(X) \to \mathbb{C}$ by 0 on yet undefined faces

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- products $g\omega$ and $g\Omega$ for $e = [Q, v]$ and face F of X:

$$\int_e g\omega = (g(Q) + g(v)) \int_e \omega$$

$$\iint_F g\Omega = g(F) \iint_F \Omega$$

DISCRETE DIFFERENTIAL FORMS

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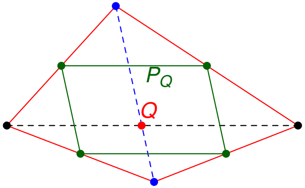
- ω is of *type* $\diamond$, if $\omega = pdz + qd\bar{z}$ for $p, q : V(\diamond) \to \mathbb{C}$

Discrete derivatives



$$\partial_\Lambda f(Q) = \frac{-1}{4i\,\mathrm{area}(F_Q)} \oint_{P_Q} f d\bar{z}$$

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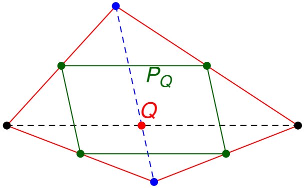

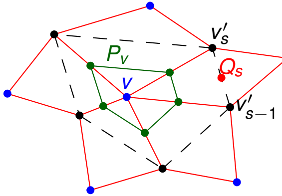

$$\partial_\Lambda f(Q) = \frac{-1}{4i\text{area}(F_Q)} \oint_{P_Q} f d\bar{z} \qquad \partial_\diamond h(v) := \frac{-1}{4i\text{area}(F_v)} \oint_{P_v} h d\bar{z}$$

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Discrete exterior derivative

Let $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$. Define df and dh by:

$$df := \partial_\Lambda f dz + \bar{\partial}_\Lambda f d\bar{z} \quad \text{and} \quad dh := \partial_\diamond h dz + \bar{\partial}_\diamond h d\bar{z}$$

DISCRETE EXTERIOR DERIVATIVE

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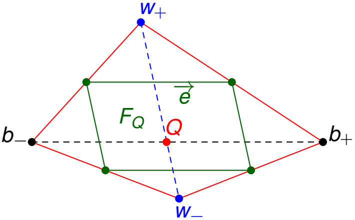

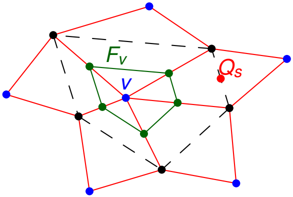

ω discrete one-form. Write $\omega = pdz + qd\bar{z}$ locally with functions p, q on vertices $b_\pm, w_\pm \sim Q$ or faces $Q_s \sim v$. Define $d\omega$ by:

$$d\omega|_{F_Q} := \left(\partial_\Lambda q - \bar{\partial}_\Lambda p\right) dz \wedge d\bar{z} \text{ and } d\omega|_{F_v} := \left(\partial_\diamond q - \bar{\partial}_\diamond p\right) dz \wedge d\bar{z}$$

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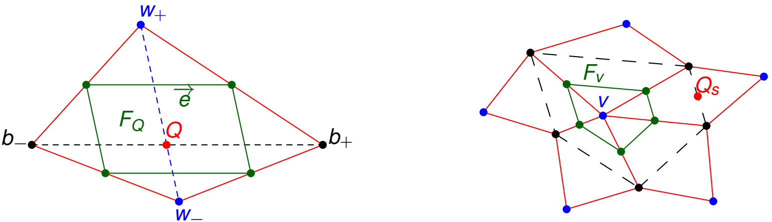

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Stokes' theorem: $\int_e df = \frac{f(w_+)+f(b_+)}{2} - \frac{f(w_+)+f(b_-)}{2}$ and $\iint_F d\omega = \oint_{\partial F} \omega$

Discrete holomorphic product

Let $f, g : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$. Then,

$$0 = ddf = \left(\partial_\diamond \bar{\partial}_\Lambda f - \bar{\partial}_\diamond \partial_\Lambda f\right) dz \wedge d\bar{z}$$

Thus, f discrete holomorphic iff $df = pdz$ for some $p : V(\diamond) \to \mathbb{C}$.

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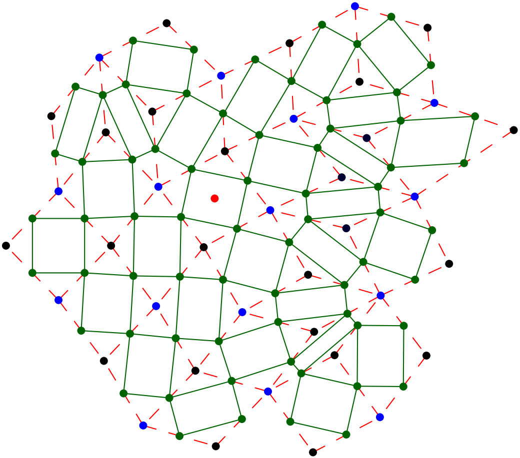

$fdg + gdf$ is closed

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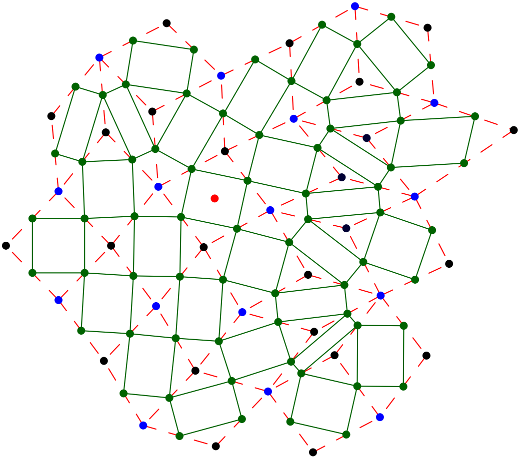

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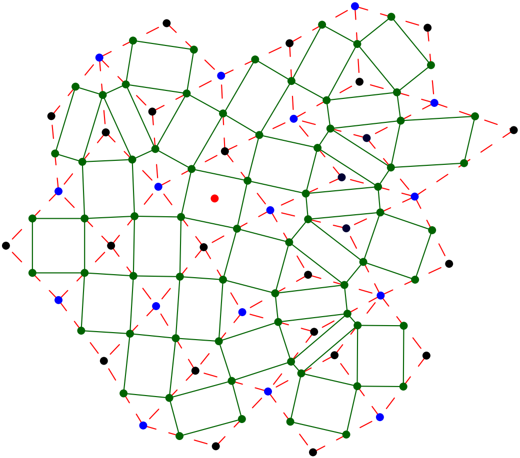

$fdg + gdf = (f\partial_\Lambda g + g\partial_\Lambda f)dz$ is closed

$fhdz$ is closed

Discrete wedge product

Let $\omega = p dz + q d\bar{z}$ and $\omega' = p' dz + q' d\bar{z}$ discrete one-forms of type $\diamond$, $p, p', q, q' : V(\diamond) \to \mathbb{C}$. *Discrete wedge product* $\omega \wedge \omega'$ defined by

$$
\begin{aligned}
(\omega \wedge \omega')|_{F_Q} &= \big(p(Q)q'(Q) - q(Q)p'(Q)\big)\, dz \wedge d\bar{z} && \text{for } Q \in V(\diamond) \\
(\omega \wedge \omega')|_{F_v} &= 0 && \text{for } v \in V(\Lambda)
\end{aligned}
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\end{aligned}$$

Theorem (Exterior derivative is derivation for $\wedge$)

Let $f : V(\Lambda) \to \mathbb{C}$ and ω discrete one-form of type $\diamond$. Then,

$$d(f\omega) = df \wedge \omega + f d\omega$$

Discrete Green's identities

Let $\diamondsuit_0 \subset \diamondsuit$ finite, and let $f, g : V(\Lambda_0) \to \mathbb{C}$.

1. $\langle f, \triangle g \rangle_{\diamondsuit_0} + \langle df, dg \rangle_{\diamondsuit_0} = \int_{\partial X_0} f \overline{\star dg}$

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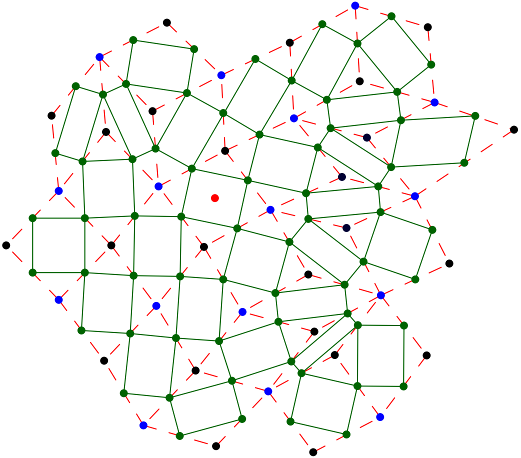
PF. $f \overline{\star(\star d \star dg)} + df \wedge \overline{\star dg} = d\left(f \overline{\star dg}\right)$



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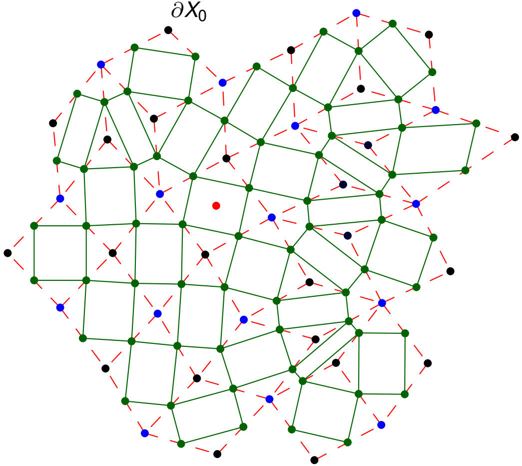
2. $\langle \triangle f, g\rangle_{\diamondsuit_0} - \langle f, \triangle g\rangle_{\diamondsuit_0} = \int_{\partial X_0} (f \star d\bar{g} - \bar{g} \star df)$



Discrete Cauchy's integral formulae

$Q_0 \in V(\diamond)$, $v_0 \in V(\Lambda)$. $K_{Q_0} : V(\Lambda) \to \mathbb{C}$ and $K_{v_0} : V(\diamond) \to \mathbb{C}$ are called *discrete Cauchy's kernels* iff for all $Q \in V(\diamond)$, $v \in V(\Lambda)$:

$$\bar{\partial}_\Lambda K_{Q_0}(Q) = \delta_{QQ_0} \frac{\pi}{2\text{area}(F_Q)} \quad \text{and} \quad \bar{\partial}_\diamond K_{v_0}(v) = \delta_{vv_0} \frac{\pi}{2\text{area}(F_v)}$$

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Let $f : V(\Lambda) \to \mathbb{C}$ and $h : V(\diamond) \to \mathbb{C}$ be discrete holomorphic. Then:

$$f(v_0) = \frac{1}{2\pi i} \oint_{C_{v_0}} f K_{v_0} dz$$

$$h(Q_0) = \frac{1}{2\pi i} \oint_{C_{Q_0}} h K_{Q_0} dz$$

DISCRETE CAUCHY'S INTEGRAL FORMULAE

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$$\partial_\Lambda f(Q_0) = -\frac{1}{2\pi i} \oint_{C_{Q_0}} f \partial_\Lambda K_{Q_0} dz$$

Asymptotics for parallelogram-graphs Λ

Theorem

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$$(\partial_\Lambda K(\cdot; Q_0))(Q) = -\frac{1}{(Q - Q_0)^2} - \frac{\tau(Q, Q_0)}{J(Q, Q_0)^2} + O\left(|Q - Q_0|^{-3}\right)$$

Discrete Riemann surfaces

Discrete Riemann surfaces

Theorem

Let Σ, Σ' compact discrete Riemann surfaces of genera g, g' and $f : V(\Lambda) \to V(\Lambda')$ N-sheeted discrete semi-holomorphic covering.

$$g = N(g' - 1) + 1 + b/2,$$

where $b = \sum_{v \in V(\Lambda)} b_f(v) + \sum_{Q \in V(\diamond)} b_f(Q)$ total branching number.

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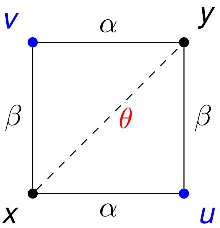
Theorem

Σ compact discrete Riemann surface of genus g, D admissible divisor. Then, $l(-D) = \deg D - 2g + 2 + i(D)$.

Discrete Laplace-type equations



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$$Q(x, u, y, v; \alpha, \beta) = 0$$

on faces (Q in ABS-list).

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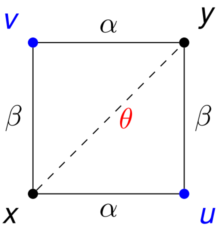

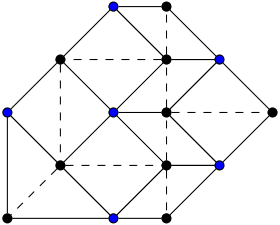

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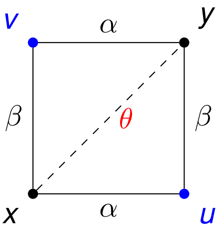

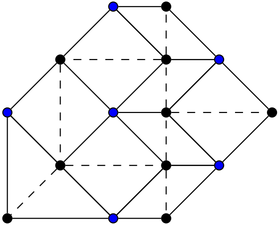

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Laplace equations for generalized Q-equations

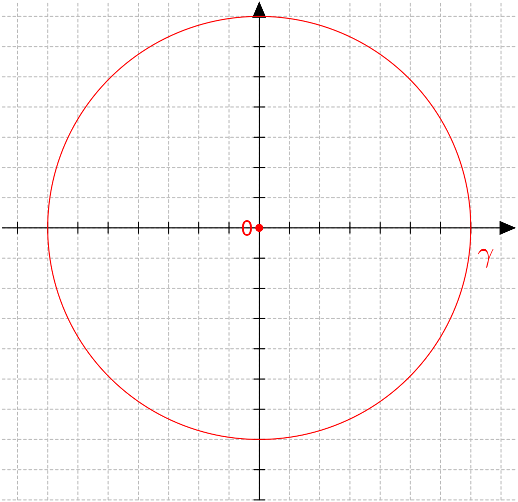
$$\sum_{e=(x,y_k)\in E(\Gamma)} \varphi(x, y_k; \theta_e) + \Theta_x = 0, \quad i\varphi = \log \Phi$$

Reality and convexity/concavity conditions

quad-equation	functional	space	concavity condition
$(Q1)_{\delta=0}$	(5.10)	U	$\theta_e \in \mathbb{R}^+$
$(Q1)_{\delta=1}$	(5.11)	U	$\theta_e \in \mathbb{R}^+$
(Q2)	(5.13)	$\mathbb{R}^{\lvert V(\Gamma)\rvert}$	$\theta_e \in \mathbb{R}^+$
$(Q3)_{\delta=0}$	(5.16)	U	$\exp(i\theta_e) \in S^1_+$
$(Q3)_{\delta=0}$	(5.20)	Ui	$\theta_e \in \mathbb{R}^+ i$
$(Q3)_{\delta=1}$	(5.23)	$\mathbb{R}^{\lvert V(\Gamma)\rvert}$	$\exp(i\theta_e) \in S^1_+$
$(Q3)_{\delta=1}$	(5.26)	$\mathbb{R}^{\lvert V(\Gamma)\rvert} i$	$\theta_e \in \mathbb{R}^+ i$
(Q4)	(5.29)	$\mathbb{R}^{\lvert V(\Gamma)\rvert}$	$\exp(i\theta_e) \in S^1_+$
(Q4)	(5.33)	$\mathbb{R}^{\lvert V(\Gamma)\rvert} i$	$\theta_e \in (0, r(t))i$

$$U = \left\{ \{X\}_{x \in V(\Gamma)} \subset \mathbb{R}^{\lvert V(\Gamma)\rvert} \Big| \sum_{x \in V(\Gamma)} X = 0 \right\}$$

Different view on Cauchy's integral formula



$$f(0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z} dz$$

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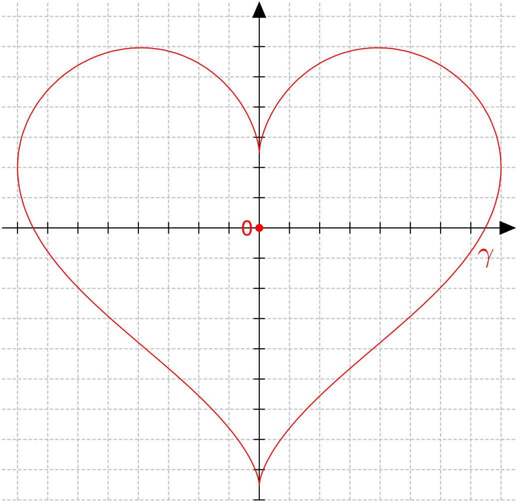

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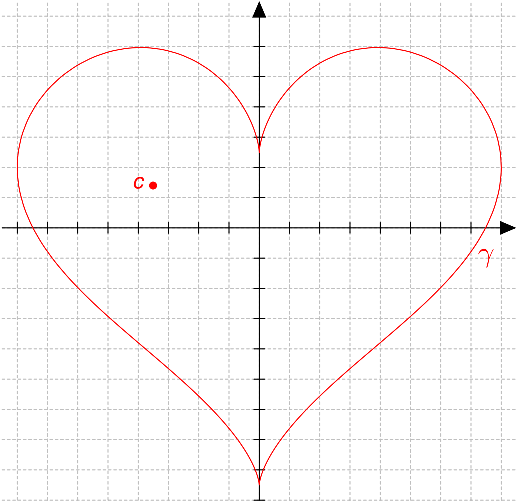

$$f(c) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - c} dz$$

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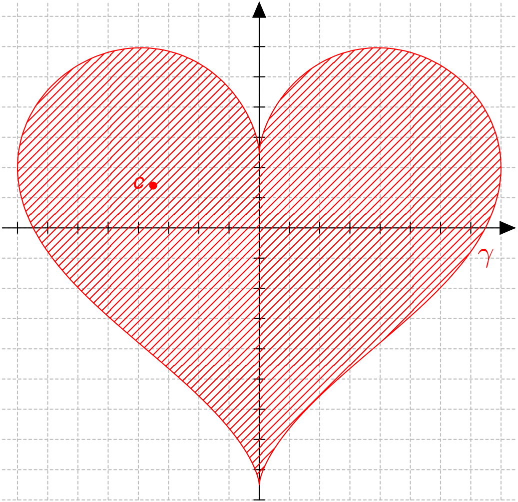

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SUPPORTING ORGANIZATIONS

Deutsche Telekom Foundation

German National Academic Foundation

CRC/Transregio 109

Berlin Mathematical School

Collection of main results

1. Discrete complex analysis on planar quad-graphs
   - $d(f\omega) = df \wedge \omega + fd\omega$
   - discrete Green's identities and Cauchy's integral formulae
   - existence of discrete Green's functions and Cauchy's kernels
2. Discrete Riemann surfaces
   - Riemann-Hurwitz formula
   - $l(-D) = \deg D - 2g + 2 + i(D)$
   - discrete Abel-Jacobi maps
3. Discrete complex analysis on planar parallelogram-graphs
   - asymptotics of discrete Green's function and Cauchy's kernels
4. Variational principles of real Laplace-type integrable equations
   - reality and convexity conditions for discrete Laplace-type equations
   - investigation of integrability

Asymptotics of discrete Green's function

Assume $\alpha \geq \alpha_0 > 0$ and $e/e' \geq q_0 > 0$ for all angles α and two side lengths e, e' of a parallelogram. Let $v_0 \in V(\Lambda)$ fixed.

Then, there exists discrete Green's function $G(\cdot; v_0)$ such that

$$G(v; v_0) = \frac{1}{4\pi} \log \left(|v - v_0| / |J(v, v_0)| \right) + O\left(|v - v_0|^{-2} \right),$$

$$G(v; v_0) = \frac{\gamma_{\text{Euler}} + \log(2)}{2\pi} + \frac{1}{4\pi} \log |(v - v_0) J(v, v_0)| + O(|v - v_0|^{-2})$$