CELL DIVISION AND HYPERBOLIC GEOMETRY

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by

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The idea of cell division comes, as everybody knows, from biology. Everything alive, from bacteria on, is built of cells and as the cells divide the living creatures grow. One might think that an individual cell divides according to a relatively simple program and this program is the same for all cells of a given type. Of course the execution of the program may depend on what happens around a given cell. In particular, it may depend on the neighbour cells located in the immediate vicinity of our cell.

So far nobody was able to find simple principles governing cell division which could explain the result of the division that is the variety of shape of the living matter we see around us.

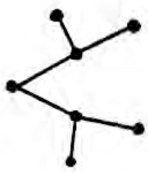
Now, I want to show how a pure mathematician would treat this problem. The essence of the mathematical approach is to extract what one could call *the idea* of the cell division. This idea must be at the same time formal, simple and viable. That is, the idea must have a potential to grow and develop into an aesthetically satisfying mathematical theory. (The true mathematics always brings unexpected turns at every step as one follows a development of an idea. One cannot predict in advance which direction brings you to a viable theory and which terminates with a dead end).

Ideas do not come cheap in mathematics. The price one should be ready to pay is the biological significance of the issuing theory. It may of course happen, by accident, that the mathematical development will be related to the real live but this is not a concern for a pure mathematician. A good way to start the search for an idea is to look at the simplest possible case. This is, of course, where our living thing consists of a single cell which divides into two cells identical to itself and the two baby cells have no link between them. It takes some time for the babies to grow to the adult size and then each of them divides again. The mathematics here is reduced to a dull sequence of numbers $1 \to 2 \to 4 \to 8 \to 16 \to$ ... In fact, the picture becomes not so dull for longer stretches of time. For example, let the division occurs every hour. Then in two hours we have 2 cells, in three hours there are 4 and so on. Does it take long to have more cells than there are elementary particles in the known universe? (Say in the ball of the radius twenty billions light years). Is it a day, a month, a century, or a billion of years? Of course, the answer may impress only a mathematically naive mind (month is the best estimate), but the very idea of unrestricted growth of time has proved very fruitful in many fields of pure (and not so pure) mathematics.

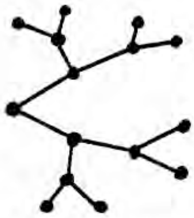
Now let us imagine that every daughter cell remembers its mother. It is amazing how much this "memory" enriches the mathematical structure. To visualize what happens we draw line segments between each child and its parent and then we see the following ancestral trees.

<u>Moment</u> 0 : there is a single cell, indicated by a point • ;

<u>Moment</u> 1 : The original cell divides into two, which are joined to the mother by two edges,  . Then, every next moment there come new branches



moment 2



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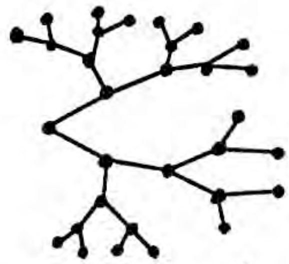

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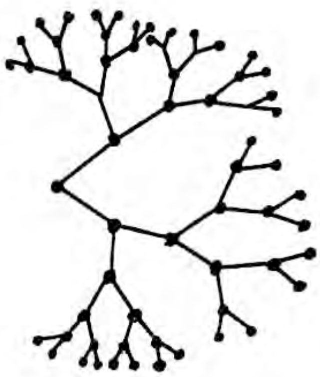

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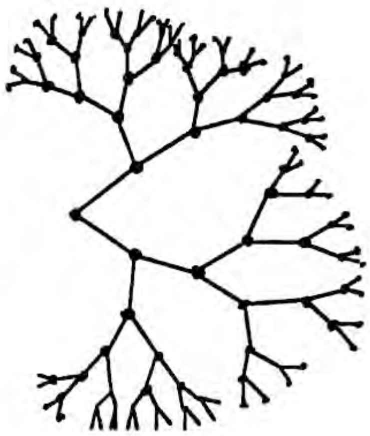
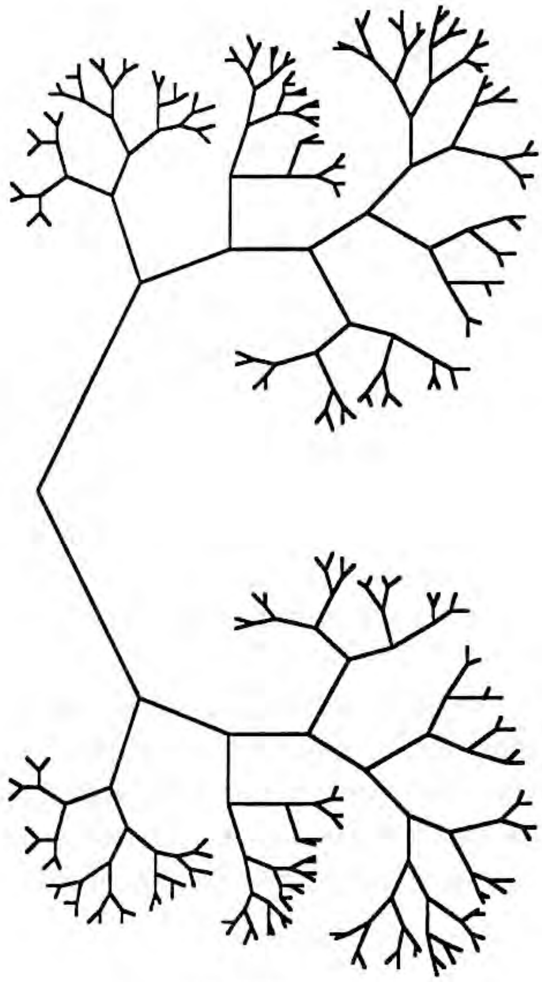


moment 6

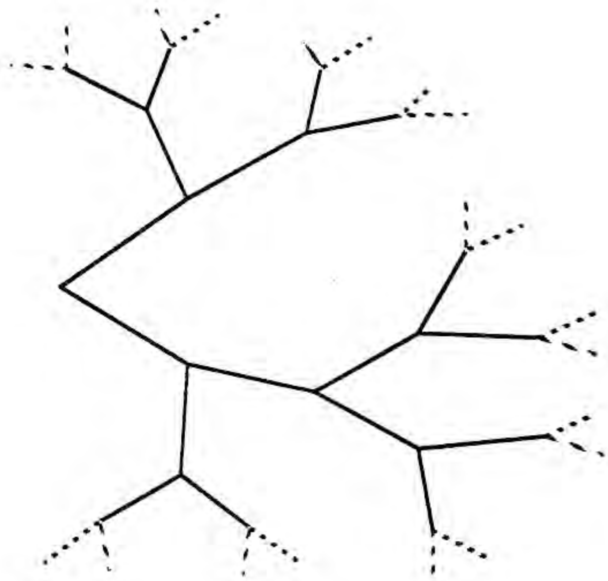
Here there are at least two different ways of seeing what happens as the time eventually goes to infinity. In one setting we imagine new branches getting smaller and smaller so that the limit tree, with infinitely many branches, is still contained in a bounded region of space. It looks something like this :



moment ∞

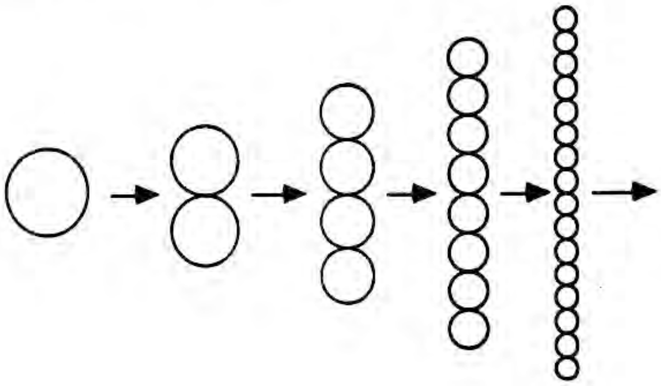
Every infinite branch here has an end and all these form what is called by mathematicians a *Cantor set* .

Another possibility for growth is where all edges remain of the same length. The resulting tree is infinite in size and its branches go to infinity as shown in the following picture.

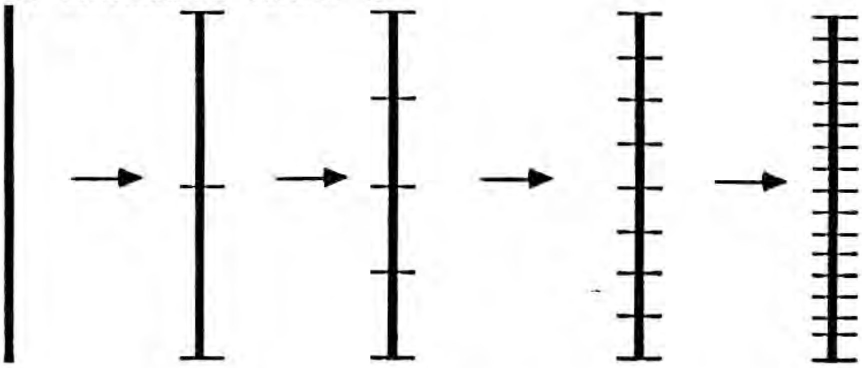


Second scenario for *time* $\rightarrow \infty$.

Our next objective is to understand what happens if the sister cells do not drift apart but remain close and thus after a stretch of time the descendants of a single cell form a true organism rather than a collection of disjoint cells. The simplest case of this is where the sister cells always have one point in common. Then the division process looks schematically as follows

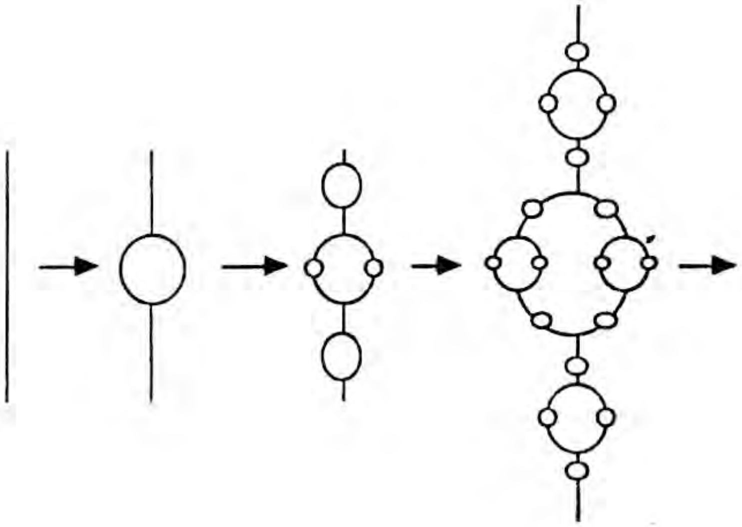


which can be schematized further to



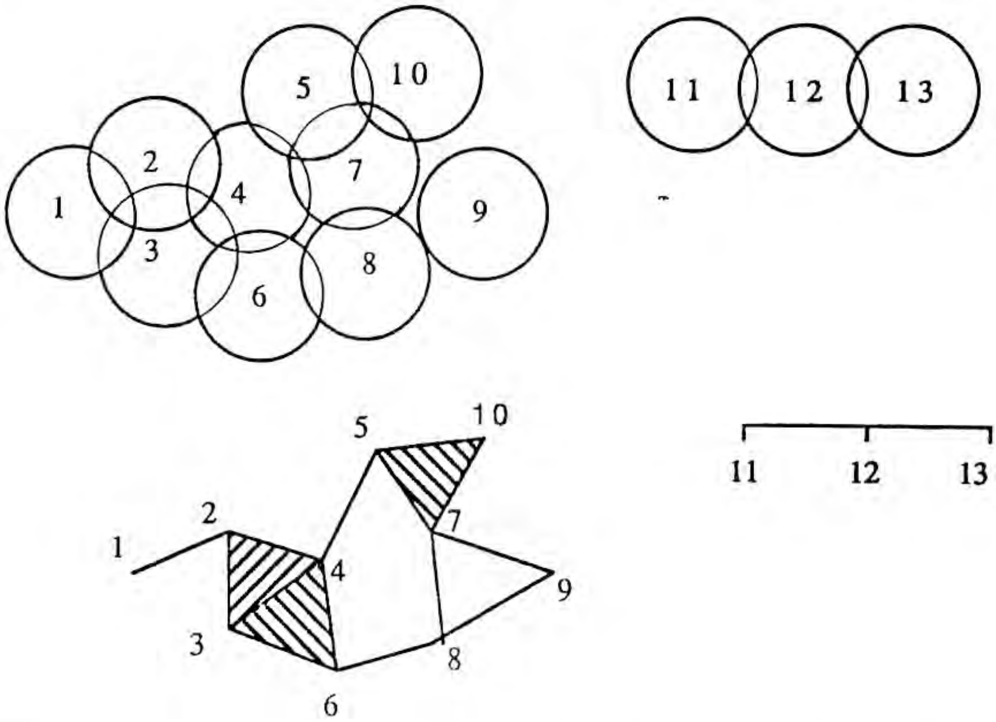
The limit object here (corresponding to the Cantor set in the free division picture) is an ordinary segment of real numbers, say [0,1] , where the division process is equivalent to the binary representation of the numbers.

Next, we look at the picture where the mother is represented by the unit interval and the babies are also unit intervals which overlap over two subintervals, say $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$ in [0,1] . The schematic division picture is as follows



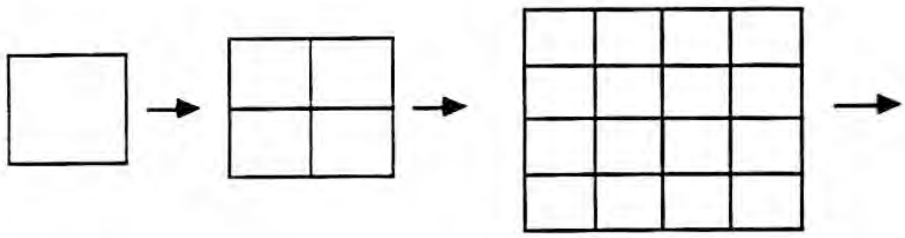
The limit object here is a rather complicated one-dimensional space, a kind of an infinite necklace.

Let us summarize the essential features of our examples. First of all our cells have no internal structure and so symbolically they are represented by points. Two or more cells in an organism may be neighbours and the *neighbourhood* is the only relation we introduce between the cells. Graphically, one joins every two neighbours by an edge, the triple neighbourhood is designated by a triangle etc. Here is an example :

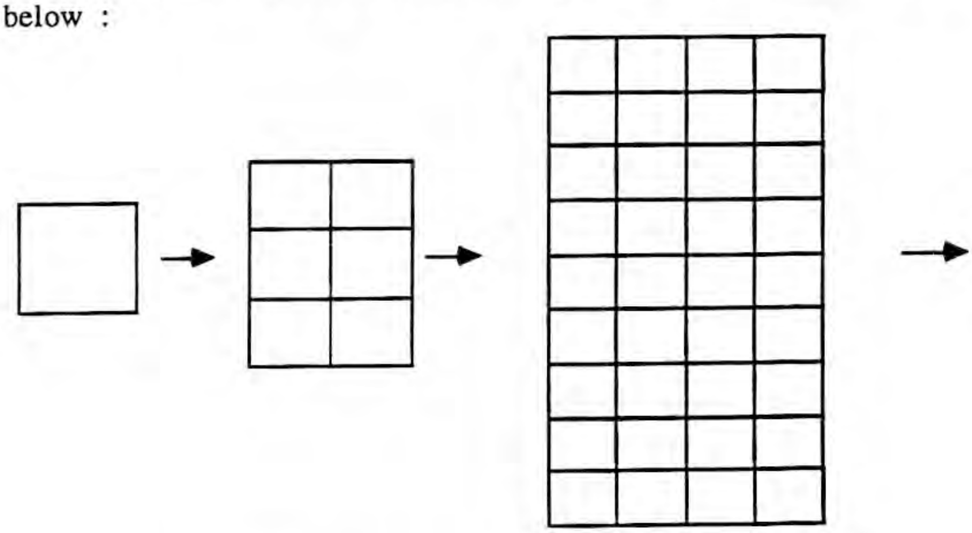


Next we want to introduce the notion of a *cell division law* , such that all cells divide according to this law. At this point it is convenient to attach an extra structure to our cells in order to allow several possibilities for division. For example, we may have two kinds of cells, say black and white, such that every black cell divides into two white ones while every white cell divides into one black and one white cell. Thus, mathematically speaking our cell organisms are represented by *colored simplicial complexes* ("simplex" is the generic name for segment, triangle, tetrahedron, etc). Then the division law assigns to each color a colored simplicial complex and the resulting division process of a given organism (colored complex) consists in replacing each vertex (vertices of a complex are just the points representing the cells) by the complex corresponding to the color of the vertex. Notice that here a cell may divide into more than two baby-cells. Also notice, that if we want the result of a division to be again a simplicial complex we must specify the neighbourhood relation between the new (daughter) cells. This requires a slightly more precise and elaborated definition of a division law which we do not explain here.

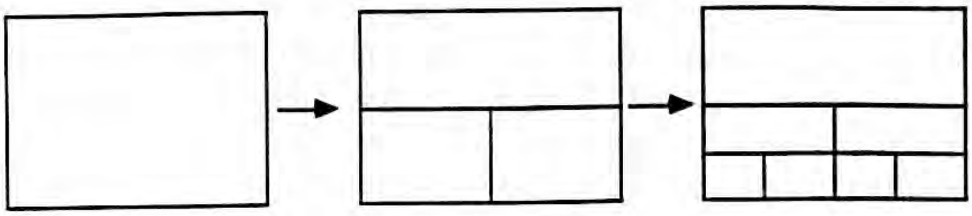
Now we think of a division law as a transformation acting on the set of colored complexes (or on some subset of these) and our problem is to understand what happens under infinite iteration of this transformation. In particular, one wants to classify the limit spaces resulting from infinitely many divisions of an original complex (organism). This general problem looks hopelessly complicated for a meaningful mathematical study. Yet there is an interesting class of division laws where one expects eventual answers to the basic questions. This class is constituted by what we call *hyperbolic* (or expanding) laws, where we, roughly speaking, require, that the division makes every organism grow "in all directions". All our earlier examples were hyperbolic. Here are some more :



The divided square above uniformly grows in both directions but the hyperbolicity allows some directions grow faster than others, as is seen below :



On the other hand the following division is non-hyperbolic



whenever we have a hyperbolic law we can define the limit space X similar to the Cantor set in our first example, but now it can be of an arbitrary dimension. This dimension is the basic invariant of the underlying division law and there are many more topological and geometric characteristics of the limit space X reflecting the (asymptotic) properties of a particular division law. In fact, there is an adequate description of the cell division creating X in terms of the space X itself. Namely, the division law gives rise to a sequence of finite partitions of X into smaller subspaces, where the elements of the first partition correspond to the colors. Then the division process is "generated" by finitely many (partially defined) homeomorphisms of X which have a certain expanding property. The most interesting examples here arise from *hyperbolic dynamical systems* and their *Markov partitions* and from *hyperbolic groups* acting on their *ideal boundaries.* Unfortunately, any meaningful discussion becomes technical at this point and we refer the reader to the papers [Bo], $[\mathrm{Gr}]_1$ and $[\mathrm{Gr}]_2$.

We conclude by indicating other models of cell division which also lead to interesting mathematics. One thing one can do is to allow the individual division to be a random rather than a deterministic event. One may think of this randomness as a kind of a secret color attached to a cell (now the color may take infinitely many values corresponding to the points of a probability space) and the division depends on this color.

Another possibility is to consider a continuous model, say a Riemannian manifold (e.g. the space we live in) and replace the division law by a differential evolution equation which would force an uniform expansion of the metric (which seems to happen to our universe). It is unclear however if a model of such kind can lead to a persistent creation of local topology as it happens in the discrete case (e.g. in the necklace example).

References

[Bo] R. Bowen, Markov partitions for axiom A diffeomorphisms, Am. J. Math. 92 (1970) 725-747.

$[Gr]_1$ M. Gromov, Hyperbolic manifolds groups and actions. Ann. Math. Studies 97 (1981), p. 183-215, Princeton.

$[Gr]_2$ M. Gromov, Hyperbolic groups, MSRI - publication. 8 (1987) p. 25-265. Springer Verlag.