6. E. Hecke, *Lectures on the theory of algebraic numbers*, Graduate Texts in Math., vol. 77, Springer-Verlag, Berlin and New York, 1981.

7. M. Lejeune and B. Teissier, *Clôture integrale des ideaux et équisingularité*, Sém. École Polytechnique, (1973–1974).

8. J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies, No. 61, Princeton, 1968.

9. H. C. Pinkham, *Deformations of algebraic varieties with G_m-action*, Astérisque **20** (1974).

10. J. G. Semple and G. T. Kneebone, *Algebraic curves*, Oxford, 1959.

11. S. Washburn, *Analytically trivial families of local rings*, J. Algebra **35** (1975), 528–533.

12. O. Zariski, *On the topology of algebroid singularities*, Amer. J. Math. **54** (1932), 453–465.

13. ——, *Studies in equisingularity*. I. *Equivalent singularities of plane algebroid curves*, Amer. J. Math. **87** (1965), 507–536.

14. ——, *Studies in equisingularity*. II. *Equisingularity in codimension* 1 (*and characteristic zero*), Amer. J. Math. **87** (1965), 972–1006.

15. ——, *Studies in equisingularity*. III. *Saturation of local rings and equisingularity*, Amer. J. Math. **90** (1968), 961–1023.

16. ——, *Characterization of plane algebroid curves whose module of differentials has maximum torsion*, Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 781–786.

SHERWOOD WASHBURN

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988

0273-0979/88 $1.00 + $.25 per page

Partial differential relations, by Mikhael Gromov. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 9, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris and Tokyo, 1986, ix + 363 pp., $60.00. ISBN 0-387-12177-3

In this important book, Gromov studies very general classes of partial differential equations and inequalities, many of which arise from problems in differential geometry. Using a variety of surprising and intricate techniques, he shows that in many cases these partial differential relations satisfy the "h-principle", i.e., they admit rich families of solutions whenever the appropriate topological obstructions vanish.

Most of the ideas presented here have their origins in a series of papers which Gromov wrote in Russian in the later 60s and early 70s, some alone and some in collaboration with Eliashberg and Rochlin. Thanks to the excellent lecture notes of Haefliger [H] and Poenaru [P], the earliest part of this work is reasonably well-known. However, this is just the tip of the iceberg: the later papers contain many more, totally original ideas. Unfortunately, these papers were sketchily written, and contained various references to other papers which never appeared. Gromov has devoted a great deal of effort over the past few years to working out these ideas. The end result is this magnificent book.

The core of the book is a series of abstract and powerful theorems. These include a sharp version of the Nash-Moser implicit function theorem which is specific to partial differential operators, as well as much more geometric results such as the main flexibility theorem and the theorems about convex integration. These serve as the basis for very interesting and far-reaching discussions of a wealth of applications and open questions. The book is also full of exercises for the reader, some with hints and/or references, which Gromov uses to suggest connections, new approaches and generalizations of his results. Thus this book not only contains a full and detailed description of a whole new theory, but also gives the reader a great deal of insight into the many new questions which it opens up.

To begin, I shall try to give you some idea of the scope of the applications. The foundational example is the celebrated immersion theorem of Smale-Hirsch (1959), which goes as follows.

THEOREM 1. *Let V and W be manifolds, and suppose that either* $\dim V < \dim W$, *or V is open. Then, a smooth map $f : V \to W$ can be homotoped into a smooth immersion if and only if it can be covered by a continuous bundle map $\varphi : TV \to TW$, which is injective on each fiber of TV.*

Here, as always, we assume that the manifolds are connected and without boundary, so that "open" is equivalent to "noncompact". Note that the force of this theorem lies in the "if" statement: the "only if" statement follows easily from the fact that the space of fiberwise injective bundle maps $TV \to TW$ [illegible] over the [illegible] of maps $V \to W$. As an example, let V be

an open, parallelizable n-dimensional manifold, and consider the trivial map $f : V \to \mathrm{pt} \in \mathbf{R}^n$. The parallelism defines a map $TV \to \mathbf{R}^n$ which is a linear isomorphism on each fiber. It therefore follows that *every* such V immerses into $\mathbf{R}^n$.

This book studies a vast range of similar questions. For example:

When can a smooth map $f : V \to W$ be homotoped into a submersion, or to a map which preserves some Riemannian metric, or which is transverse to a foliation?

When can a nondegenerate 2-form β on V be homotoped through a family of nondegenerate forms into a closed nondegenerate form ω? (Recall that a 2-form β on a $2n$-dimensional manifold V is said to be nondegenerate if the n-fold wedge product $\beta \wedge \cdots \wedge \beta$ never vanishes, and that a closed nondegenerate form is also called a symplectic form.)

The emphasis here is on general rather than particular cases, and the methods presented work in situations where solutions are abundant—in fact, as in Theorems 2 and 4 below, they may be dense in the relevant function space.

Here are some sample results. The first is an improvement of Nash's isometric immersion theorem, which clearly was another important influence on Gromov's work.

THEOREM 2. *Let* [illegible]

THEOREM 4. *Let (V, ω) and (W, Ω) be symplectic manifolds with* $\dim V \leq \dim W + 4$, *and suppose that $f_0: V \to W$ is a C^∞-embedding such that $f_0^*(\Omega)$ is cohomologous to ω. Then, there is a C^0-small isotopy of f_0 to an embedding which pulls Ω back to ω, if and only if f may be covered by a family of fiberwise injective bundle maps $\varphi_t : TV \to TW$ such that $\varphi_0 = df_0$ and $\varphi_1^*(\Omega) = \omega$.*

The book contains a host of similar theorems about Lagrangian and contact embeddings, maps with prescribed singularities (e.g., foldings), holomorphic functions and immersions, isometric C^1-embeddings of Riemannian manifolds (which behave quite differently from C^∞-embeddings), and so on, as well as theorems about certain general classes of differential equations and inequalities. They are all instances of the "h-principle" which I will describe later. For now, notice these characteristic points. In Theorem 4, as in Theorem 1, the condition on the pair (f, df) is purely topological: it just concerns the homotopy class of df considered as an element of the space of continuous fiberwise injective bundle maps over f. The other conditions which appear are dimensional: one always needs a little extra room to play with in order to make the constructions work. (In the present context an open n-dimensional manifold usually can be reckoned to have an effective dimension of $n-1$, since it deformation retracts onto a neighborhood of its $(n-1)$-skeleton. Of course, the solutions one obtains by exploiting this observation are usually very wild near ∞; in particular, they are never in any sense of the word proper.)

In many cases, the conditions on dimension that Gromov gives are the best possible, and there are obstructions to a solution in the critical dimension when the given conditions just fail. These obstructions are often rather poorly understood. On the other hand, the solutions which do exist in the critical [illegible]

ple, the situation with symplectic embeddings (cf. Theorem 4 above). Since symplectic manifolds always have even dimension, the critical codimension is 2. So far, there are no known methods for constructing compact symplec- [illegible]

EXAMPLE. In the case of immersions $V \to W$, let $p: X = V \times W \to V$ be the obvious projection. A section of p is then equivalent to a map $f: V \to W$. The 1-jet of f at the point $x \in V$ is the pair $(f(x), df_x)$, where $df_x: T_xV \to T_{f(x)}W$. Thus $X^{(1)}$ can be identified with the space of triples (x, y, α) where $(x, y) \in V \times W$ and $\alpha \in \mathrm{Hom}(T_xV, T_yW)$. A section of the bundle $X^{(1)} \to V$ consists of a map $g : V \to W$ together with a lifting φ of g to a bundle map $TV \to TW$. It is holonomic if and only if $\varphi = dg$. We define the immersion relation $\mathcal{IM}$ to be the subset consisting of all (x, y, α) such that α is injective. It is then clear that the solutions of $\mathcal{IM}$ are just the immersions of V in W.

As Gromov points out, one can divide the problem of solving $\mathcal{R}$ into two parts. One can first try to construct a continuous section of $\mathcal{R} \to V$, and then one can try to pass from this arbitrary section to a holonomic one. The first problem is purely topological, and is in theory well-understood. Therefore, this book concerns itself with the second, which is much more problematic. The most optimistic expectation is expressed in the following

HOMOTOPY PRINCIPLE. We say that $\mathcal{R}$ *satisfies the h-principle* if every continuous section of $\mathcal{R} \to V$ is homotopic through a family of continuous sections of $\mathcal{R} \to V$ to a holonomic section of $\mathcal{R} \to V$. (There are other versions of the h-principle—relative, or with parameters—but here I'll concentrate on its simplest form.)

EXAMPLE. Clearly, the immersion theorem (Theorem 1 above) may be expressed in this language by saying that the immersion relation $\mathcal{IM}$ satisfies the h-principle provided that $\dim V < \dim W$ or V is open. The other theorems may be also be expressed in a similar way.

One might imagine that very few relations satisfy the h-principle. One of the interesting things about this book is that it shows that a surprising number of geometrically significant relations do satisfy the h-principle, and that those which don't usually don't for some rather good reason.

In order to establish the h-principle for $\mathcal{R}$ it is often useful to consider the "continuous" sheaf Φ of solutions of $\mathcal{R}$. This is the sheaf over V whose space, $\Phi(U)$, of sections over an open $U \subset V$ is the space of all holonomic sections $f: U \to \mathcal{R}$, with the C^k-topology for some appropriate k. If C is compact, one defines $\Phi(C)$ to be the direct limit of the $\Phi(U)$ over all open neighborhoods U of C. This does not have a suitable topology, but it may be given a natural "quasitopology." In particular, a map φ from a polyhedron P to $\Phi(C)$ is called "continuous" if and only if there is a neighborhood U of C such that φ is the restriction of some map $P \to \Phi(U)$, which is continuous in the ordinary the lifting φ of ψ can be extended over $P \times [0,1]$ in such a way that the diagram still commutes. (Note that all maps are assumed to be "continuous" in the sense described above.) It is not too hard to show that, in many interesting cases (e.g., if the subset $\mathcal{R}$ is open), the relation $\mathcal{R}$ satisfies the h-principle whenever the corresponding sheaf Φ is flexible. But how can one show that a sheaf is flexible? Gromov divides this problem into two parts by introducing the notions of microflexibility and microfibration. A *microfibration* is defined in the same way as a fibration, except that one demands only that the lift φ can be extended over $P \times [0,\varepsilon]$, for some $\varepsilon > 0$ which may depend on P, φ and ψ as well as the pair (C, C'). One then calls a sheaf Φ *microflexible* if all its restriction maps $\Phi(C) \to \Phi(C')$ are microfibrations. For example, one can easily see that the sheaf corresponding to any open relation $\mathcal{R}$ is microflexible.

The main theorem which deduces flexibility from microflexibility is the following

FLEXIBILITY THEOREM. *Let Φ be a microflexible sheaf over V and let V_0 be a submanifold of V which is sharply movable by diffeomorphisms of V which lift to Φ. Then the sheaf $\Phi_0 = \Phi|V_0$ is flexible.*

The definition of "sharply movable" is too technical to give here. However, any submanifold of positive codimension is sharply movable by the group $\mathrm{Diff}(V)$ of all diffeomorphisms of V, and the same is true in the symplectic or contact categories. Thus, if $\mathcal{R}$ is any open relation over V which is invariant under the natural action of $\mathrm{Diff}(V)$, and if the submanifold V_0 has positive codimension in V, then the restriction to V_0 of the sheaf corresponding to $\mathcal{R}$ is flexible. Applying this to the immersion relation, one proves the immersion theorem for any manifold V_0 whose dimension is less than that of W.

This proof also works for symplectic and contact immersions, except that one needs an extra argument to show that the relevant sheaves are microflexible (for the relations are no longer open). In these two cases, one can establish microflexibility by using an appropriate version of Moser's stability theorem. (This says that a small perturbation of a contact structure, or of a symplectic form within its cohomology class, is diffeomorphic to the original one.) However, for more general sheaves, for example the sheaf of Riemannian isometric immersions, such a simple approach no longer suffices.

Gromov's deepest result in this connection concerns sheaves which may be defined as the solution sheaf $\Phi_{\mathcal{D}}$ of some partial differential operator $\mathcal{D}$. (All the sheaves mentioned in the previous paragraph have this form.) Here $\mathcal{D}$ is an operator which goes from the space $\mathcal{X}$ of C^r-sections of some fibration $X \to V$ to the space $\mathcal{G}$ of C^s-sections of some vector bundle $G \to V$, and the sheaf $\Phi_{\mathcal{D}}$ is defined by setting $\Phi_{\mathcal{D}}(U)$ equal to the set of solutions of $\mathcal{D}f = 0$ over $U \subset V$. Gromov establishes a version of the Nash-Moser implicit function [illegible]

theorem which holds for those $\mathcal{D}$ which are "infinitesimally invertible". It follows readily that $\Phi_{\mathcal{D}}$ is microflexible for these $\mathcal{D}$.

[illegible]

I have omitted various details which specify degrees of differentiability.) The condition is natural in that it does not involve auxiliary norms, and it is of- [illegible]

[illegible]

nonlinear partial differential operator is infinitesimally invertible. The main application of this implicit function theorem is to the isometric immersion problem (cf. Theorem 2 above). But, as usual, Gromov gives many other [illegible]

[illegible]

for arbitrary r. It is worth remarking here that Gromov's original paper on this subject [**G3**] only gave full details of the proof in the 1-dimensional case, leaving a large part of the proof for arbitrary n up to reader's imagination. This generalization looked plausible, but it wasn't at all clear how one might avoid a horribly messy argument. Gromov has now expressed his ideas much more fully, and has completed the proof in a very elegant, if abstract, way.

To end with, here are some comments on the book as a whole. It has been carefully written. The main theorems are clearly stated and their proofs, insofar as I have studied them, are accurate and quite detailed. The book is also essentially self-contained, so that it should be accessible to anyone who has a knowledge of the basics of differential topology and geometry. But one also needs a good deal of persistence, since it is easy to be overwhelmed by the wealth of new ideas and the many, very varied examples which accompany each theorem. And so one must make a considerable initial effort to understand the basic ideas and language and to learn one's way around. However, it's well worth it. The book is a wonderful treasure house of ideas.

References

[**G1**] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Invent Math. **82** (1985), 307–347.

[**G2**] ——, *Soft and hard symplectic geometry*, Talk at the Internat. Congr. Math., Berkeley, 1986.

[**G3**] ——, *Convex integration of differential relations*, Izv. Akad. Nauk SSSR **35** (1973), 329–343.

[**H**] A. Haefliger, *Lectures on the Theorem of Gromov*, Proc. Liverpool Sing. Symp. II (Wall, ed.), Lecture Notes in Math., vol. 209, Springer-Verlag, Berlin and New York, 1970, pp. 128–141.

[**P**] V. Poenaru, *Homotopy theory and differential singularities*, Manifolds—Amsterdam

Lecture Notes in Math., vol. 197, Springer-Verlag, Berlin and New York, 1970, pp. 106–133.

DUSA MCDUFF

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY