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HOMOTOPICAL EFFECTS OF DILATATION

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1. Statement of results

1.1. Geometrical and topological complexity. Let V and W be Riemannian manifolds, and X a space of mappings $V \to W$. For instance, X may consist of all smooth maps, or may be the space of imbeddings or immersions. We ask how to estimate a measure of the "topological complexity" of an $x \in X$ by geometry of x. We measure geometrical complexity of x by a positive functional $F: X \to \boldsymbol{R}_+$, say, by the dilatation of x or by an integral characteristic like the Dirichlet functional. The topological complexity of x may be measured by its degree (when the degree makes sense) or another numerical invariant.

The Morse theory suggests a different point of view. We take the levels $X_\lambda \subset X$, $X_\lambda = F^{-1}([0, \lambda])$, $\lambda \in \boldsymbol{R}_+$ and compare the numerical invariants of X_λ (say the number of components or the sum of all Betti numbers) with λ.

When $\lambda \to \infty$, the first asymptotic term of the topological complexity of X_λ is often independent of the particular choice of metrics in V and W (but depends, of course, on the particular type of F), and we come to a pure topological problem: how to express this asymptotic topology of X_λ in terms of usual invariants? When we study the asymptotic distribution of the critical values of F, what we need first is the asymptotic behavior of the Betti numbers $b_i(X_\lambda)$, $i, \lambda \to \infty$.

When we seek finer geometro-topological relations in X_λ depending on individual features of V and W, we enter a completely different field resembling geometry of numbers (such as minima of quadratic forms, packing $\boldsymbol{R}^n$ by balls, etc.).

This paper has a definite topological bias.

1.2. The number N of the homotopy classes and the homological dimension dm. We denote by $N(\lambda)$ the number of connected components of X intersecting X_λ, where $X_\lambda = F^{-1}([0, \lambda]) \subset X$.

We denote by dm (λ) the maximal integer d such that every map of an arbitrary d-dimensional polyhedron into X is homotopic to a map into X_λ.

1.3. Spectrum of the Laplacian. Consider, for example, the case when W is the real line and X is the projective space associated to the linear space of

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the smooth maps $V \to \boldsymbol{R}$. The ratio $\int_V |\text{grad } f|^2 \Big/ \int_V f^2$: $V \to \boldsymbol{R}$ defines a functional on X, and when V is closed dm (λ) is equal to the number of the eigenvalues of the Laplacian on V which are not greater than λ.

From now on all our manifolds are compact and connected.

1.4. Loop spaces. Let X be the space of all smooth loops in W based at $w_0 \in W$, and let $F(x) = \text{length}\,(x)$, $x \in X$.

Theorem. *If W is a closed manifold with finite fundamental group, then* dm(λ) $\approx \lambda$, $\lambda \to \infty$, *i.e.*, $C_1\lambda \geq \text{dm}\,(\lambda) \geq C_2(\lambda - 1)$ *where C_1 and C_2 are positive constants depending on W.*

Of course, the first inequality $C_1\lambda \geq \text{dm}\,(\lambda)$ is obvious and well known. The second inequality $\text{dm}\,(\lambda) \gtrsim \lambda$ implies that the Betti numbers $b_i(X_\lambda)$, $i < C_2(\lambda - 1)$, are not less than $b_i(X)$, and we come to the following improvement of the classical theorem of Morse:

If points $p, q \in W$ are not conjugate (for any geodesic passing through them), then the number of geodesic segments joining p and q and having length $< \lambda$ is not less than $B(C_2(\lambda - 1)) = \sum_{i=1}^{C_2(\lambda-1)} b_i(X)$.

Observe that in most cases $B(\lambda) \approx e^\lambda$, and we have exponentially many geodesics.

When $\pi_1(W)$ is infinite, the inequality dm $(\lambda) \gtrsim \lambda$ does not generally hold even if we replace X by one of its components, and the behaviour of $b_i(X_\lambda)$, $\lambda \to \infty$, becomes more attractive (and mysterious).

Observe also that the inequality dm $(\lambda) \gtrsim \lambda$ shows finitness of $b_i(X)$ and our proof from § 4.1 uses only one simple combinatorial trick, closely related to semisimplical ideas of Kan [5] (the author wishes to thank D. Sullivan for this observation), but no algebra (spectral sequences). Iterating this trick leads to a very short and elementary proof of the Serre-Kan theorem:

If W is simply connected, then all homotopy groups $\pi_i(W)$ are finitely generated and can be effectively computed. (The last statement supposes that we are given a triangulation with a reduction of the standard presentation of $\pi_1(W)$ to the trivial presentation.)

1.5. Closed geodesics. Take now for X the space of all smooth maps $S^1 \to W$. When $\pi_1(W)$ is finite we again have dm $(\lambda) \gtrsim \lambda$, and for the number of prime closed geodesics of length $< \lambda$ we get the lower estimate by $(\text{const.}/\lambda)B(\lambda) = (\text{const.}/\lambda) \sum_{i=1}^{\lambda} b_i(X)$ provided that the Riemannian metric in W is generic (bumpy). This is an improvement of the (easy generic case) Gromoll-Meyer theorem [3], [6]. (The author does not know how to eliminate the "bumpy" condition from our estimate.)

We except again that in most cases $b_i(X)$ grow esponentially, but there are only isolated (and unpublished) examples due to P. Trauber supporting this conjecture.

Some information about nonsimply connected manifolds is contained in [3].

1.6. Dilatation. Let X be the space of smooth maps $V \to W$. Denote by

$\operatorname{dil}(x)$, $x \in X$, the maximal value of the ratio $\operatorname{dist}(x(v_1), x(v_2))/\operatorname{dist}(v_1, v_2)$, $v_1, v_2 \in V$. Let $F(x) = \operatorname{dil}(x)$, $x \in X$.

Theorem. *If the fundamental group of W is finite, then $N(\lambda) \leq 1 + C\lambda^k$, where C is a positive constant depending on V and W, and k is a natural number depending only on the homotopy types of V and W.*

Proof is given in § 3.2.

This theorem shows that the number of homotopically distinct maps $V \to W$ grows at most polynomially as dilatation grows. Consider now an example where the behavior of $N(\lambda)$ can be described more precisely.

Let W be the standard n-dimensional ($n > 1$) sphere (sphere with metric of constant curvature), and let V be a closed orientable n-dimensional manifold. Then there exists the limit $L = \lim_{\lambda \to \infty} N(\lambda)/\lambda^n$ and $L = \tilde{C}_n \operatorname{Vol} V/\operatorname{Vol} W$, where $\tilde{C}_n \geq E_n > 0$, $\tilde{C}_n \leq D_n < 2$, and Vol denotes the volume of a manifold.

Proof immediately follows from statement A in § 2.3.

2. Dilatation and degree

2.1. A norm in the homotopy groups. Fix a point $w_0 \in W$, and denote by $A(\alpha)$, $\alpha \in \pi_n(W, w_0)$, the volume of the minimal possible (metrical) ball $B \subset \boldsymbol{R}^n$ for which there exists a map $x: (B, \partial B) \to (W, w_0)$ representing α and having $\operatorname{dil}(x) \leq 1$. One can easily prove that there exists the limit $\|\alpha\|_\infty = \lim_{p \to \infty} A(p\alpha)/|p|$ having the following properties:

$$\|\alpha\|_\infty \geq 0\,, \quad \|\alpha + \beta\|_\infty \leq \|\alpha\|_\infty + \|\beta\|_\infty\,, \quad \|K\alpha\|_\infty = |K|\,\|\alpha\|_\infty\,.$$

2.2. Let V be an n-dimensional closed oriented manifold, and let W be $(n-1)$-connected. The set of the homotopy classes of maps $V \to W$ can be identified with $\pi_n(W, w_0)$. Denote by $\operatorname{dil}[x]$, $x: V \to W$, the minimal possible dilatation of a map homotopic to x.

Theorem. $(\operatorname{dil}[x])^n = \|[x]\|_\infty/\operatorname{Vol} V + C([x])$, *where* $C([x])/\|[x]\|_\infty \to 0$ *as* $\|[x]\|_\infty \to \infty$, *and* $[x]$ *denotes both the homotopy class of x and the corresponding element from $\pi_n(W, w_0)$.*

Proof. To show that $\limsup (\operatorname{dil}[x])^n/\|[x]\|_\infty \leq (\operatorname{Vol}(V))^{-1}$, $\|[x]\|_\infty \to \infty$, we cover V by small round balls and constract sufficiently "short" map $V \to S^n$ by representing the generator from $\pi_n(S^n)$ by maps supported on these balls. The opposite inequality $\liminf (\operatorname{dil}[x])^n/\|[x]\|_\infty \geq (\operatorname{Vol}(V))^{-1}$ is equivalent to the following.

Lemma. *For a map x of any triangulated manifold into W with* $\operatorname{dil}(x) = d$, *there exists a homotopic map $\tilde{x}$ mapping the $(n-1)$-skeleton to $w_0 \in W$ and satisfying the condition* $\operatorname{dil}(\tilde{x}) = d + C(d)$, *where $C(d)/d \to 0$ as $d \to \infty$.*

Proof. Because W is $(n-1)$-connected, the first condition on $\tilde{x}$ can be replaced by the following: $\tilde{x}$ maps K^{n-1} to the $(n-1)$-skeleton of a given triangulation of W. To construct such map (keeping the dilatation almost undisturbed), we subdivide K^{n-1} properly, replace $x|_{K^{n-1}}$ by its simplical approximation, and extend the approximating map to the whole manifold.

2.3. Maps into spheres. For closed oriented manifolds V and W of the same dimension, we denote by dil $\{d\}$ the minimal possible dilatation of a map $V \to W$ of degree d.

Statements. *Let W be the standard sphere S^n.*

(A) *If V is n-dimensional and oriented, then* $\mathrm{dil}\{d\} \sim C_n |d|^{1/n} (\mathrm{Vol}\, W/\mathrm{Vol}\, V)^{1/n}$, *where the constant C_n depends only on n, and $C_n > 1$.*

(B) *If diam $W = 1$, where diam (.) denotes the diameter of a Riemannian manifold, and V is a flat torus, then* $\mathrm{dil}^{-1}\{1\}$ *is equal to the injectivity radius of V.*

(C) *If V is also the standard sphere of the same size as W, then dil $\{d\} \geq 2$ for $|d| \geq 2$.*

Proof. Statement (A), with the exception of the inequality $C_n > 1$, follows from § 2.2. The inequality $C_n > 1$ follows from the next theorem (see § 2.4.). Statement (B) is obvious. Statement (C), when d is even, was proven by R. Oliver (see [8], and [7], [9] for further information). We shall prove the following generalization of (C).

Lemma. *Let V and W be Riemannian manifolds with the following properties: for every point $w \in W$ there exists an "opposite" point $w' \in W$ with* dist $(w', w) > 1$; *the complement of any unit ball in V is convex, i.e., every two points of the complement can be joined by the unique shortest geodesic lying in the complement. Then for any map x of V onto W with* dil $(x) \leq 1$ *there exists a map $y: W \to V$ such that the composition $y \circ x: V \circlearrowright$ is homotopic to the identity.*

Proof. The inverse image $x^{-1}(A)$, where $A \subset W$ is sufficiently small, belongs to a convex set, and so any map y defined originally only on the 0-skeleton of an appropriate triangulation of W can be extended to W with the required properties.

Remark. Obviously, there exist maps $S^m \to S^m$ with dilatation equal to 2 and with degree $1, 2, \cdots, 2^h, h = [\frac{1}{2}(m + 1)]$.

2.4. For oriented manifolds V and W of the same dimension n, the geometric degree of a map $x: V \to W$ is defined as the integral $\int_V x^*(\omega)$, where ω is the oriented volume form. This definition does not suppose the manifolds to be closed. It is obvious that geom. $\deg (x) \leq (\mathrm{dil}\,(x))^n$ Vol V, and equality holds only for locally isometrical mappings. Let us prove the asymptotic version of this remark.

Lemma. *Let $x_i: V \to W$ be a sequence of mappings uniformly converging to a map $x: V \to W$. If* dil $(x_i) \leq 1$, *and geom. deg* $x_i \xrightarrow[i \to \infty]{}$ Vol (V), *then x is a locally isometrical map.*

Proof. The obvious localization argument reduces the problem to the special case where V and W are flat balls. In this case the lemma follows from the isoperimetric inequality for balls.

Theorem. *Let V and W be closed oriented manifolds of dimension n with*

Vol V = Vol W. *If* $\lim_{d\to\infty}\inf[(\text{dil}\{d\})^n/|d|] = 1$, *then* V *and* W *are flat Riemannian manifolds.*

Proof. The localization argument reduce the situation to the case where V is a flat ball, and then flatness of W follows from the lemma. Applying the lemma again, we conclude that V is also flat.

Remarks. (A) If V and W are flat tori of unit volume, then $\lim_{d\to\infty}(\text{dil}\{d\})^n/|d| = 1$.

(B) If W is a flat torus, rank $H_1(V) = n$, and there exists a map $V \to W$ of degree one, then there exists $\lim_{d\to\infty}(\text{dil}\{d\})^n/|d|$. This limit certainly depends on V. Cf. Statement (A) in § 2.3.)

Proof. The first remark is obvious, and the second follows from the first.

3. The Hopf invariant

3.1. Let W be a sphere of even dimension n, and V a sphere of dimension $2n - 1$. Denote by dil $\{h\}$ the minimal possible dilatation of a map $V \to W$ with the Hopf invariant equal to h.

Theorem. $C_1|h| \leq (\text{dil}\{h\})^{2n} \leq C_2|h|$, *where* C_1 *and* C_2 *are positive constants depending on* V *and* W.

Proof. The second inequality $(\text{dil}\{h\})^{2n} \leq C_2 h$ follows from the existence of maps $W \circlearrowleft$ with degree proportional to $(\text{dil})^n$

To prove the first inequality we fix an n-form ω on W with $\int_W \omega = 1$. The Hopf invariant $h(x)$ of a map $x: V \to W$ is equal to the integral $\int_V x^*(\omega) \wedge \eta$, where η is any $(n - 1)$-form satisfying the equation $d\eta = x^*(\omega)$. Now the theorem follows from the following obvious fact.

Lemma. *Fix a norm* $\|\ \|$ *in the space of all continuous forms on* V. *There exists such constant* C *that for any exact form* ω *on* V *one can choose the form* η *with* $d\eta = \omega$ *satisfying the inequality* $\|\eta\| \leq C\|\omega\|$.

3.2. Let W be a simply connected manifold. According to D. Sullivan (see [10]), any functional $\theta: \pi_k(W) \to \boldsymbol{R}$ can be obtained by generalization of previous construction for the Hopf invariant. This generalization involves forms ω_i on W, forms $x^*(\omega_i)$, where $x: S^k \to W$ is the map representing given element of $\pi_k(W)$, integrals of forms $x^*(\omega)$, products of resulting forms, etc. The value $\theta[x]$ is equal to the integral over S^k of the k-form obtained by such a procedure. Combining this fact with the previous lemma and using the notation in § 1.6, we reach

Theorem. *If* W *is simply connected and* V *is a homotopy* k-*sphere, then* $N(\lambda) \leq 1 + C\lambda^{rl}$, *where* C *is a constant depending on* V *and* W, r *is the rank of the group* $\pi_k(W)$, *and* l *is an integral number depending only on* k. (*One can take* $l = 2(k - 1)$.)

Proof of the theorem in § 1.6. Induction by the skeletons of a triangulation of V reduces the simply connected version of the theorem in § 1.6 to the above theorem. The general case follows from the simply connected one.

4. The functionals of length and volume

4.1. *Proof of the theorem in* § 1.4. Choose a triangulation of W, and replace X by the space $\tilde{X} \subset X$ of piecewise linear loops. $\tilde{X}$ possesses the natural cell decomposition: a cell is the product of simplexes of the triangulation which form a sequence where every two consecutive terms are the faces of one simplex.

Suppose that W is simply connected, and consider a smooth map $\alpha\colon W \to W$ homotopical to the identity and contracting the 1-skeleton of the triangulation to a point. The associated map $\tilde{\alpha}\colon \tilde{X} \to X$ sends each i-skeleton of the cell decomposition into the set $F^{-1}[0, Ci] \subset X$, where C is a constant depending on W and α. This finishes the proof for the simply-connected case, and the general case follows immediately from the simply-connected one.

4.2. Consider the space X of maps $V \to W$. Let $\dim X = k$, and let $F(x)$ be the k-volume of the map x, i.e., the volume of V with the metrics induced by x. The above argument shows

Theorem. *If W admits a cell decomposition without k-cells, then* $\mathrm{dm}\,(\lambda) \geq C(\lambda - 1)$, *where C is a positive constant depending only on W.*

5. Additional remarks

5.1. Immersions. Denote by $\mathrm{dil}_I\,[x]$ the infimum of dilatations of smooth immersions $V \to W$ homotopic to x.

Theorem. *If V and W are parallelizable, and* $\dim W > \dim V$, *then* $\mathrm{dil}_I\,[x] = \mathrm{dil}\,[x]$ (*see notation in* § 2.2.).

Proof can be easily obtained by using convex integration (see [2]).

5.2. Imbeddings. For an imbedding $x\colon V \to W$ denote by distor (x) the maximal value of the sum

$$\frac{\operatorname{dist}(v_1, v_2)}{\operatorname{dist}(x(v_1), x(v_2))} + \frac{\operatorname{dist}(x(v_1), x(v_2))}{\operatorname{dist}(v_1, v_2)}\,, \qquad v_1, v_2 \in V\,.$$

Theorem. *If W is simply connected and* $\dim W > \frac{3}{2}\dim V + 2$, *then the number of distinct imdebdings $V \to W$ (up to an isotopy) grows at most polynomially as distortion grows.*

Proof. The theorem follows from the theorem in § 1.6. and the Haefliger imbedding theorem (see [4]).

Remark. When the group of knots $S^n \to S^q$ is infinite, then there exist infinitely many knots with uniformly bounded distortion.

5.3. The Dirichlet functionals. A linear map $\mathscr{D}\colon \boldsymbol{R}^n \to \boldsymbol{R}^q$ is uniquely characterized (up to orthogonal transformations of $\boldsymbol{R}^n$ and $\boldsymbol{R}^q$) by numbers $\lambda_1(\mathscr{D}) \geq \lambda_2(\mathscr{D}) \geq \cdots \geq \lambda_n(\mathscr{D}) \geq 0$. (These numbers are the diagonal elements of the diagonal matrix corresponding to $\mathscr{D}$ under the proper choice of orthonormal basises in $\boldsymbol{R}^n$ and $\boldsymbol{R}^q$.) For a map $x: V \to W$ we denote by $\lambda_i(x): V \to \boldsymbol{R}_+$ the function $\lambda_i(x)(v) = \lambda_i(\mathscr{D}_v(x))$, where $\mathscr{D}_v$ denotes the differential of x at $v \in V$. Let us note that $\operatorname{dil}(x) = \max_{v \in V} \lambda_1(x)(v)$.

Denote by $\sigma_j(x): V \to \boldsymbol{R}_+$, $j = 1, 2, \cdots$, the j-th symmetric function of $\lambda_i(x)$, and by $D_j^r(x)$ the integral $\int_V (\sigma_j(x))^r$. For some of the functionals $D_j^r(x)$ the previous argument can be applied to establish polinomial estimates for the growth of $\operatorname{dm}(\lambda)$ and $N(\lambda)$.

Theorem. *Let X be the space of maps $V \to W$, and let $F(x) = D_j^r(x)$.*

(A) *If W is k-connected, $j \leq k$, and* $\dim V \leq rj$, *then $N(\lambda)$ grows at most polynomially* (*cf.* § 1.6).

(B) *If W admits a cell decomposition without cells of dimensions $k, k+1, \cdots,$* $\dim V$, *$j \geq k$, and* $\dim V \geq rj$, *then* $\operatorname{dm}(\lambda)$ *grows at least as $C\lambda$, $C > 0$* (*cf.* § 4.2).

5.4. Density. Consider a map $x: V \to W$ and the smallest number $\varepsilon > 0$ such that the ε-neighborhood of the image of x is dense in W. Let us denote $\operatorname{dens}(x) = 1/\varepsilon$.

Theorem. *Let X be the space of maps $V \to W$ and let $F(x) =$* $\operatorname{dens}(x)$. *Let i be the inclusion of the space of all maps $V \to W \setminus w_0$, $w_0 \in W$, into X. If there exists a cohomology class $\alpha \in H^r(X, A)$, $r > 0$, with $\alpha^j \neq 0$, $j = 1, 2, \cdots$, and with $i^*(\alpha) = 0$, where A is any ring, then* $\operatorname{dm}(\lambda)$ *grows at most as $C\lambda^n$, $n = \dim W$.*

Proof. Consider points $w_1, w_2 \cdots w_d \in W$ and a map $y: K \to X$ with $y^*(\alpha) \neq 0$. It is clear that there exists such $k \in K$ that all points $w_1, \cdots, w_d$ belong to the image of the map $x = y(k): V \to W$. To finish the proof, it is enough now to choose sets $\{w_1, \cdots, w_d\}$ forming the ε-nets with $\varepsilon \sim d^{-1/n}$.

Remarks. (A) The theorem can be applied, for example, to the loop space of a sphere.

(B) The argument in § 4 shows that for $F(x) = \operatorname{dens}(x)$ the invariant $\operatorname{dm}(\lambda)$ always grows at least as $C\lambda^l$. $l = \dim W - \dim V$, $C > 0$.

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