L'Enseignement Mathématique, t. 46 (2000), p. 391–402

THREE REMARKS ON GEODESIC DYNAMICS AND FUNDAMENTAL GROUP

by Mikhaïl GROMOV

§ 1. HOMOTOPIC STABILITY

For a Riemannian manifold V we denote by $S = S(V)$ the space of all unit tangent vectors. We denote by $G(V)$ the geodesic foliation on S: leaves are orbits of the geodesic flow (i.e. liftings to S of geodesics from V).

GEODESIC RIGIDITY. If V, W are closed manifolds of negative curvature with isomorphic fundamental groups, then the spaces $S(V)$ and $S(W)$ are homeomorphic. Moreover the geodesic foliations $G(V)$ and $G(W)$ are homeomorphic (i.e. there is a homeomorphism $S(V) \to S(W)$ sending leaves from $G(V)$ into leaves from $G(W)$).

It is unknown whether V and W are homeomorphic. The last question was discussed several times by Cheeger, Gromoll, Meyer and myself. In the end Cheeger constructed (see below) a homeomorphism between the Stiefel fiberings over V and W. Later Veech suggested to me that geodesic rigidity would be a better geometric substitute for the Mostow rigidity theorem than existence of a homeomorphism $V \to W$. Unfortunately the geodesic rigidity theorem is too simple and superficial and it does not lead to deep corollaries of the Mostow theorem. For example, finiteness of the group $\mathrm{Aut}(\pi_1(V))/\mathrm{Conj}(\pi_1(V))$ can not be derived (at least directly) from geodesic rigidity. ("Aut" means the group of automorphisms of the fundamental group, "Conj" means the group of inner automorphisms. The case $\dim V = 2$ is excluded.)

THE CHEEGER HOMEOMORPHISM. Denote by $\mathrm{St}_2(V)$ the space of all tangential orthonormal 2-frames in V. If V and W are as before, then the spaces $\mathrm{St}_2(V)$ and $\mathrm{St}_2(W)$ are homeomorphic.

The Cheeger construction is more canonical than our geodesic homeomorphism. In particular, $\mathrm{St}_2(V)$ can be viewed as a functorial object over $\pi_1(V)$.

STABLE HOMEOMORPHISM. As we have already mentioned, existence of a homeomorphism $V \to W$ is still a problem, but existence of a stable homeomorphism (homeomorphism $V \times \mathbf{R}^N \to W \times \mathbf{R}^N$ with large N) follows immediately from the topological equivalence between unit tangent bundles of V and W (see below).

CONSTRUCTIONS AND PROOFS

Ideas and notions involved in the constructions below are well known and due to M. Morse (see Appendix 2 for details).

For a complete simply connected manifold X of negative curvature we denote by $\mathrm{Cl}(X)$ its compactification (closure) and by $\partial(X)$ the complement $\mathrm{Cl}(X) \setminus X$. The space ∂X is homeomorphic to the $(n-1)$-sphere, $n = \dim X$, and it can be viewed as the set of all asymptotic classes of geodesic rays in X.

Consider a group Γ of isometries acting on X. Such an action can be continuously extended to $\partial(X)$. When Γ is discrete and the factor X/Γ is compact, the space $\partial(X)$ and the action of Γ in $\partial(X)$ depend (functorially) only on Γ. When $\Gamma = \pi_1(V)$ and X is the universal covering of V, then the unit tangent bundle of V is topologically equivalent to the bundle associated to the covering $X \to V$ with fiber $\partial(X)$. This immediately yields the topological equivalence of tangent bundles and so the stable homeomorphism theorem.

For a geodesic ray $r \subset X$ we denote by $\partial(r) \in \partial(X)$ the asymptotic class it represents. For an oriented geodesic g we denote by $\partial^+(g) \in \partial(X)$ and $\partial^-(g) \in \partial(X)$ the asymptotic classes of its positive and negative directions. When X has strictly negative curvature (i.e. the upper limit of sectional curvature is negative), the map $g \mapsto \big(\partial^+(g), \partial^-(g)\big) \in \partial(X) \times \partial(X)$ establishes a homeomorphism between the set of all oriented geodesics in X and the complement $\partial^2(X) = \big(\partial(X) \times \partial(X)\big) \setminus \Delta$, where Δ is the diagonal.

THE CHEEGER HOMEOMORPHISM

Denote by $\partial^3(X)$ *the set of triples* (x_1, x_2, x_3), $x_1, x_2, x_3 \in \partial(X)$ *with* $x_i \neq x_j$ *for* $i \neq j$. *If* X *has strictly negative curvature, then* $\mathrm{St}_2(X)$ *is canonically homeomorphic to* $\partial^3(X)$.

Proof. Realize an $s \in \mathrm{St}_2(X)$ by a pair (g, r) where $g \subset X$ is an oriented geodesic and $r \subset X$ is a geodesic ray starting from a point $x \in g$ and normal to g. Set $\mathrm{Chee}(s) = \mathrm{Chee}(g, r) = \big(\partial^+(g), \partial^-(g), \partial(r)\big)$. This is a homeomorphism because the normal projection $P = P_g \colon X \to g$ can be continuously extended to $\mathrm{Cl}(X) \setminus \{\partial^+(g), \partial^-(g)\}$.

REMARK. The original construction of Cheeger is more symmetric: he realizes an $s \in \mathrm{St}_2(X)$ by a triple of rays (r_1, r_2, r_3) all starting from the same point $x \in X$ with angles $120°$ between every two of them.

Applying the above construction to the universal covering X of a compact manifold V we get a homeomorphism between $\mathrm{St}_2(V)$ and the factor of $\partial^3(X)$ by the diagonal action of $\Gamma = \pi_1(V)$. This proves the Cheeger homeomorphism theorem.

GEODESIC RIGIDITY

Realize points from $S(X)$ by pairs (g, x), where g is an oriented geodesic and $x \in g$. When X and Y are the universal coverings of V and W, an isomorphism $I \colon \pi_1(V) \to \pi_1(W)$ induces a homeomorphism $D \colon \partial^2(X) \to \partial^2(Y)$. View D as a homeomorphism between the sets of oriented geodesics in X and Y. Take a smooth equivariant map $f_0 \colon X \to Y$ (i.e. the lifting of a smooth homotopy equivalence $V \to W$ corresponding to I) and define a map $F_0 \colon S(X) \to S(Y)$ as follows: $F_0(g, x) = (h, y)$, $h = D(g)$, $y = y(x) = P_h \circ f_0(x)$. (We use in Y the same representation of points from $S(Y)$ as in X and P_h means the normal projection $Y \to h$.) The map F_0 preserves the foliations $\mathrm{G}(X)$ and $\mathrm{G}(Y)$ but it is not necessarily a homeomorphism: it can identify points lying on the same geodesic. Choose natural parameters (length) in all geodesics and average F_0 along geodesics by the formula:

$$F_c(g, x) = \Big(h\,, \ \frac{1}{c}\int_x^{x+c} y(t)\, dt\Big),$$

where $h = D(g)$, $t, x \in g$, $y(t) \in h$, $y(t) = P_h \circ f_0(t)$. When c is large enough, the map F_c is a homeomorphism and it is obviously equivariant. Returning to V and W we get the geodesic homeomorphism $S(V) \to S(W)$.

GENERALIZATIONS

The hyperbolic ideas of Morse were successfully applied to discrete type systems by Shub (*expanding endomorphisms*, see [Sh]) and Franks (π_1*-diffeomorphisms*, see [Fr]). Their results are discussed (and slightly generalized) in Appendix 3.

From a global geometric point of view generalizations of totally hyperbolic systems must include manifolds of nonpositive curvature and correspondingly semihyperbolic systems. (See Appendix 4.)

In differential dynamics most attention has always been paid to "local" versions of hyperbolicity (stability, Anosov's systems, Axiom A diffeomorphisms of Smale). We do not touch here upon that more analytical line of development of Morse's ideas.

§2. ENTROPY

Take a closed Riemannian manifold V, consider its universal covering X and denote by $\mathrm{Vol}_x(R)$, $x \in X$, the volume of the ball of radius R centered at x. Set $H(V) = \lim_{R\to\infty} \log \mathrm{Vol}_x(R)$. The limit obviously exists and does not depend on x. Denote by $h(V)$ the topological entropy of the geodesic flow in $S(V)$.

ENTROPY ESTIMATE. *We have* $h(V) \geq H(V)$.

COROLLARY. *If the fundamental group* $\pi_1(V)$ *can be presented by* k *generators and one relation and* $\mathrm{Diam}(V) \leq 1$ (Diam means the *diameter* of V), *then* $h(V) \geq \log(k-1)$.

The entropy estimate immediately follows from the Covering lemma.

COVERING LEMMA. *Take a compact manifold S and consider a regular (normal) covering $T \to S$ with the action of $\Gamma = \pi_1(S)/\pi_1(T)$. Fix a fundamental domain $D \subset T$ and denote by $N(U)$, $U \subset X$, the number of motions $\gamma \in \Gamma$ such that the intersection $\gamma(U) \cap D$ is not empty. Consider an action of the group $\mathbf{R}$ of reals in S and its lifting to T.*

The entropy h of the action of $\mathbf{R}$ in S satisfies

$$h \geq \liminf_{r \to \infty} \frac{1}{|r|} \log N(r(D)),$$

where $r(D)$ denotes the image of D under the lifted action of $r \in \mathbf{R}$ in T.

Proof. Use the definition of entropy involving coverings.

This lemma (and the proof) holds for discrete time systems and immediately implies Manning's estimate of the topological entropy of an $f: S \to S$ in terms of the spectral radius of $f_*: H_1(S; \mathbf{R}) \to H_1(S; \mathbf{R})$. See [Ma], [Pu]. In Appendix 5 we show how to make use of the whole group $\pi_1(S)$.

§3. PERIODIC ORBITS

For maps $f: S \to S$ there are several ways to estimate from below the number $\mathrm{card}(\mathrm{Fix}(f^m))$ of all points of period m. Denote by $L(f)$ the Lefschetz number $\sum_{i=0}^{i=n}(-1)^i \,\mathrm{Trace}(f_{*i})$, where $i = \dim S$ and $f_{*i}: H_i(S; \mathbf{R}) \to H_i(S; \mathbf{R})$.

(***L***) *If all periodic points are nondegenerate (say, f is smooth and generic), then* $\mathrm{card}(\mathrm{Fix}(f^m)) \geq |L(f)|$ (Lefschetz).

(***Sh-S***) *If f is smooth and $\lim_{m\to\infty} |L(f^m)| = \infty$, then*

$$\lim_{m \to \infty} \mathrm{card}(\mathrm{Fix}(f^m)) = \infty$$

(Shub and Sullivan, see [Sh-S]).

(***Nie***) *Generally there is no way to extend the* (***L***)*-estimate to all maps, but in the presence of the fundamental group one can apply the Nielsen theory of fixed-point classes* (see [Nie] and Appendix 6). *This theory yields in many cases the estimate*

$$\mathrm{card}(\mathrm{Fix}(F)) \geq \mathit{const}\, |L(f)|,$$

and sometimes even $\mathrm{card}(\mathrm{Fix}(f^m)) \geq |L(F^m)|$, *where f is an arbitrary continuous map.*

EXAMPLE. Let S be a compact cell complex with homotopy type of a torus and let $f: S \to S$ be a continuous map such that $f_{*1}: H_1(S; \mathbf{R}) \to H_1(S; \mathbf{R})$ is hyperbolic (no eigenvalues with norm 1). Then

$$\operatorname{card}(\operatorname{Fix}(f^m)) \geq |L(f^m)| \geq C^m - 1$$

for some $C > 1$, and the closure of the union $\bigcup_{m=1}^{\infty} \operatorname{Fix}(f^m)$ contains a Cantor set.

REMARK. This example allows one to detect periodic points in Smale's horseshoe by homological means. A horseshoe is a space X with three subspaces A, B, Z and a map $f: X \to X$ with the properties:

(a) f sends $A \cup B$ into A and Z into B;

(b) Z separates A from B, i.e. there exists a function $a: X \to \mathbf{R}$ which is positive on A, negative on B and with $a^{-1}(0) \subset Z$.

(*Sm*) *If X, A and B are closed balls, then* $\operatorname{card}(\operatorname{Fix}(f^m)) < \frac{2^m - 1}{2}$ (Smale).

Proof. Take another copy X' of X and identify each point $x \in A \cup B$ with the corresponding point $x' \in A' \cup B' \subset X'$. Denote by Y the factor of $X \cup X'$ with that identification and construct a map $g: Y \to Y$ as follows:

- if $y \in X \subset Y$ and $a(y) \geq 0$, then $g(y) = f(y)$;
- if $y \in X$ and $a(y) < 0$, then $g(y) = (f(y))'$, where $()'$ means the involution permuting X and X' in Y;
- if $y \in X'$, then $g(y) = (g(y'))'$.

Since Y has the homotopy type of the circle, $|L(g^m)| = 2^m - 1$, and thus $\operatorname{card}(\operatorname{Fix}(g^m)) \geq 2^m - 1$. Projecting $Y \to X$ represents f as a factor of g that gives Smale's estimate.

This representation of f explains also (via Manning's estimate) why horseshoes have positive entropy.

CLOSED GEODESICS

Dealing with closed geodesics in a closed Riemannian manifold V we replace the Lefschetz numbers by the Betti numbers b_i of the space of maps from the circle S^1 to V. We set $M_m = M_m(V) = \frac{1}{m} \sum_{i=0}^{m} b_i$. The Morse theory provides the following estimate for the number $N_m = N_m(V)$ of simple closed geodesics of length $\leq m$:

(***Mo***) *If V is simply connected and all closed geodesics are nondegenerate (generic case), then $N_m \geq C M_m - 1$ for some $C > 0$* (see [Gr]).

(Probably, for most manifolds, M_m grows exponentially.)

In the degenerate case the situation is much more difficult, but still:

(***G-M***) $\limsup_{i\to\infty} b_i = \infty$ *implies* $\lim_{m\to\infty} N_m = \infty$ (Gromoll and Meyer, see [G-M]).

(About recent progress, see Klingenberg's lectures [Kl].)

The Nielsen theory collapses to a triviality in the geodesic case:

In each class of free homotopy of maps $S^1 \to V$ there is a closed geodesic; if it represents an indivisible element in $\pi_1(V)$, then every closed geodesic from that class is simple.

The estimate for N_m contained in this statement is exact for manifolds of negative curvature. For such manifolds $N_m \geq C^m - 2$ for some $C > 1$ (Sinai, see Appendix 7). But even for manifolds homeomorphic to the 2-torus it is unknown whether the estimate $N_m \geq C m^2 - 1$ is the best possible.

We give now three examples having no (?) discrete time analogs and demonstrating further connections between fundamental groups and closed geodesics. Proofs are more or less obvious and so omitted.

1. Suppose that the group $\pi_1(V)$ contains a (noncommutative) nilpotent subgroup Γ without torsion. Take a $\gamma \in [\Gamma, \Gamma]$, where $[\Gamma, \Gamma]$ denotes the commutator subgroup of Γ and γ is not the identity element, and denote by Z the (free cyclic) group generated by γ. Denote by N_m^Z the number of closed geodesics of length $\leq m$ representing elements from Z. Then $N_m^Z \geq C m - 1$ and there are infinitely many divisible elements in Z represented by simple closed geodesics; these geodesics can be chosen shortest, each in its homotopy class.

2. There is a non-empty class B of finitely presented groups such that if $\pi_1(V) \in B$, then there exists an infinite sequence g_i of simple closed contractible geodesics in V such that each g_i provides local minimum to the length functional and length$(g_i) \to \infty$ as $i \to \infty$. For example, B contains all groups with unsolvable word problem. (See Appendix 8 for further information.)

3. In order to make use of π_1 in locating other (not locally minimal) closed geodesics without non-degeneracy condition, one has to extend [Gr] to the non simply connected situation. When V is homeomorphic to $V_0 \times S^1$ and V_0 is simply connected, we can apply [Gr] directly and get $N_m(V) \geq C \log(m)$ for some $C > 0$. (Probably this is true when $H_1(V)$ is infinite or at least when $\pi_1(V) = \mathbf{Z}$.) The last estimate can be sharpened and we show this here for the simplest example when V_0 is the sphere S^3 and the proof is obvious [Gr].

Let V be homeomorphic to $S^3 \times S^1$. Then there exist closed geodesics $g_j^i \subset V$ (not necessarily simple) such that

1. *Each g_j^i, $i,j = 1,2,\ldots$, represents $\gamma^i \in \pi_1(V)$ where γ is a generator in $\pi_1(V)$.*
2. *For each i the geodesic g_1^i is the shortest in its homotopy class.*

Denote by $|g_j^i|$ the length of g_j^i.

3. $|g_1^{i+k}| + C \geq |g_1^i| + |g_1^k| \geq |g_1^{i+k}|$, *where $C \geq 0$ and $i,k = 1,2,\ldots$*
4. $|g_j^i| + C \geq |g_{j+1}^i| \geq |g_j^i|$ *for some $C > 0$ and $i,j = 1,2,\ldots$*
5. $\left||g_j^i| - |g_j^k|\right| \leq C|i-k|$ *for some $C > 0$ and $i,j,k = 1,2,\ldots$*
6. $|g_j^i| \geq \frac{j}{C}$ *for some $C > 0$ and $i,j = 1,2,\ldots$*

COROLLARY. *If V is as above, then* $\limsup_{m\to\infty} \frac{N_m(V)}{m^2} \geq$ *const* > 0.

All our estimates give a rather poor approximation to the (unknown) reality. Probably, in most cases N_m grows exponentially. That is so, of course, for "$\mathcal{C}^0$-generic" manifolds ("$\mathcal{C}^0$-generic" is used for C^0-generic manifolds having uncountably many closed geodesics).

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WHY THE APPENDICES WERE NOT WRITTEN; AUTHOR'S APOLOGIES TO THE READERS

APPENDIX 2. The stable homeomorphism suggests a geometric link between the homotopy and topological invariance of Pontryagin classes, at least for manifolds with negative curvature but I did not manage to forge this to my satisfaction till 1996 (see [Gro_2]); also see [Fa-Jo] for a deeper analysis.

APPENDIX 3. One can define a notion of hyperbolicity for an automorphism α of an arbitrary finitely generated group Γ, such that (Γ, α) functorially defines a Bowen-Franks hyperbolic system (see [Gro_1]). Unfortunately, this class of (Γ, α) is rather limited, e.g. is not closed under free products and does not include hyperbolic automorphisms of surface groups. I still do not know what the right setting is.

APPENDIX 4. An obvious example of semi-hyperbolicity is provided by non-strictly expanding endomorphisms, where the geometric picture is rather clear. However, I still do not see a functorial description, in the spirit of the symbolic dynamics, of more general semi-hyperbolic systems, not even for the geodesic (or Weil chamber) flows on locally symmetric spaces (compare [B-G-S] and [Br-Ha]).

APPENDIX 5. The section on entropy was inspired by Manning's paper [Ma_1], but I was unaware of the prior paper by Dinaburg (see [Din]) that essentially contained the entropy estimate for geodesic flows (also discussed in [Ma_2]). On the other hand, estimating the entropy of an endomorphism (or an automorphism) f in terms of $f_*\colon \pi_1 \to \pi_1$ appears now much less clear than it seemed to me back in 1976. It is not hard to bound the entropy from below via the "asymptotic stretch" of $f_*: \pi_1 \to \pi_1$ with respect to the word metric in π_1 (see [Bow]). But this is not sharp even for linear automorphisms of tori T^n, where the entropy is expressed by the "k-dimensional stretch" on H_1 for some $k \leq n$ that equals the spectral radius of f_* on H_k. Such k-stretch can be defined, in general, in terms of $f_*: \pi_1(S) \to \pi_1(S)$ and the classifying map $S \to K(\pi_1, 1)$ (refining the spectral radius of f_* on H^k coming from $K(\pi_1, 1)$), but my obvious "proof" of the lower bound on the entropy by this k-stretch missed a hidden trap. This was also overlooked in [Ma$_3$] (for $f_*: H_1 \to H_1$, where a proper identification of the "k-stretch" with the spectral radius needs extra work), as was pointed out to me much later by David Fried. (The difficulty already appears for closed subsets S in the torus T^n invariant under linear automorphisms f of T^n, where one wishes to estimate the entropy of $f|S$ in terms of f_* acting on the *spectral* cohomology of S coming from T^n. On the other hand, the case of $T^n \to T^n$ is settled in [Mi-Pr].)

APPENDIX 6. Probably, the recent progress in Nielsen theory allows a description of the cases, where card(Fix$f$) is well controlled from below by some twisted Lefschetz number (see [Fel]).

APPENDIX 7. Nothing interesting to say.

APPENDIX 8. Minima of geometric functionals related to the logical complexity have been studied in depth by A. Nabutovski (see [Na] and references therein). Yet I do not feel ready yet to write this Appendix. For example, I do not see what is the actual influence of a suitable (?) complexity measure of $\pi_1(V)$ on the Plateau problem in V.

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Mikhaïl Gromov

Institut des Hautes Études Scientifiques
35, route de Chartres
F-91440 Bures-sur-Yvette
France