Geometric Reflections on the Novikov conjecture

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ABSTRACT. The simplest manifestation of rough geometry in the Novikov conjecture appears when one looks at a fiberwise rough (coarse) equivalence between two vector bundles over a given base, where the fibers are equipped with metrics of negative curvature. Such an equivalence induces a fiberwise homeomorphism of the associated sphere bundles and hence, by Novikov's theorem, an isomorphism of the rational Pontrjagin classes of the bundles.

What is relevant of the negative curvature, as we see it nowadays, is a certain largeness of such spaces. More generally, if X^n is a contractible manifold admitting a cocompact group of isometries (or more general uniformly contractible space) one expects it to be rather large in many cases, e.g. admitting a proper Lipschitz map into $\mathbb{R}^n$ of non-zero degree. An example of an X without such maps to $\mathbb{R}^n$ may eventually lead to a counterexample to the Novikov conjecture. On the other hand, there is a growing list of spaces where such a map is available.

A purely analytico-geometric counterpart of the Novikov conjecture for X is the claim that the non-reduced L^2-cohomology $L^2H^*(X)$ does not vanish. A similar conjecture can be stated for the Dirac operator (instead of the de Rham complex): the square of the Dirac operator on X contains zero in its spectrum. Both properties express the idea of the "spectral largeness" of X, and the latter is closely related to the non-existence of a metric with positive scalar curvature quasi-isometric to X. The non-existence of a positive-scalar-curvature metric on X is yet another version of the Novikov conjecture which is often somewhat easier than the original Novikov conjecture, as one can combine here operator-theoretic techniques with the minimal surface approach of Schoen and Yau.

Most of the work on the Novikov conjecture (see [FRR]) is an outgrowth of the original ideas of Lusztig [Lu] and Mishchenko [M] when they first started working on the problem. In its usual formulation, the Novikov conjecture is a problem about the topology of compact manifolds with large fundamental group, but one is inevitably led to the study of certain aspects of the geometry of the (non-compact) universal cover. There are many different code-words for these ideas: asymptotic geometry, coarse geometry, quasi-isometry, pseudo-isometry, etc. We begin by indicating (through a representative example, similar to that treated in [FH], and not as complicated as the most general ones for which one can verify the conjecture) how these ideas come into play. Suppose $M' \xrightarrow{h} M$ is a homotopy equivalence

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of compact (necessarily aspherical) hyperbolic manifolds (without boundary). Let Γ be the common fundamental group. Then associated to h is the commutative diagram

$$\begin{array}{ccc} \tilde{M}' \times_\Gamma \tilde{M}' & \xrightarrow{\tilde{h}} & \tilde{M} \times_\Gamma \tilde{M} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ M' & \xrightarrow{h} & M, \end{array}$$

which can be viewed as a map of bundles where the fibers are copies of hyperbolic space H^n. These bundles are equivalent to the (topological) tangent bundles of M' and of M, respectively. Thus a strong version of the Novikov conjecture is equivalent to the statement that p' and h^*p have the same rational Pontrjagin classes. One way of proving this is to first use the fact that the exponential map $\mathbb{R}^n \to H^n$ is a diffeomorphism, whose inverse $\log : H^n \to \mathbb{R}^n$ is a Lipschitz map to Euclidean space. One can use this to compactify the fibers of the bundles by adding spheres at infinity, and $\tilde{h}$ extends to a fiberwise homeomorphism of the sphere bundles. Now apply to the sphere bundles Novikov's theorem on the topological invariance of rational Pontrjagin classes. We conclude that the sphere bundles of p' and h^*p have the same rational Pontrjagin classes, and thus p' and h^*p have the same rational Pontrjagin classes. (For further discussion of the relationship between Novikov's theorem on the topological invariance of rational Pontrjagin classs and the Novikov conjecture, see [FW].) A similar idea can be used in a combinatorial or discrete setting: if X_1 and X_2 are hyperbolic spaces (in the sense of [GdlH, Ch. 1]) on which a group Γ acts freely and properly with compact quotients, and if $h : X_1/\Gamma \to X_2/\Gamma$ is a homotopy equivalence, then h lifts to a quasi-isometry $\tilde{h} : X_1 \to X_2$ of the universal covers, sending some ε_1-dense net in X_1 to an ε_2-dense net in X_2, and by extending to the boundary (constructed in [GdlH, Ch. 7]), one sees $\tilde{h}$ gives a homeomorphism "at infinity."

In discussions with Blaine Lawson a number of years ago (see [GrL]), we realized there is a close parallel between the Novikov conjecture and the positive scalar curvature problem. We pause now to discuss the latter and then we will return to the subject of the Novikov conjecture. A conjecture parallel to the Novikov conjecture (see [GrL]) is that a closed aspherical manifold $V = X/\Gamma$ should not admit a Riemannian metric of positive scalar curvature. In the Novikov conjecture we (ostensibly) study properties of the group Γ, but here it's really the coarse geometry of the simply connected open manifold X that is immediately relevant. If we can show that the Dirac operator on X is non-invertible, this will imply (for reasons to be explained in a moment) that X cannot have a Riemannian metric of uniformly positive scalar curvature, and thus that V cannot have a Riemannian metric of positive scalar curvature. A similar conjecture is that the non-reduced L^2 cohomology of X must be non-zero. (The *non-reduced L^2 cohomology* $L^2H^*(X)$ is the quotient of the L^2 closed forms by the L^2 exact forms which are exterior derivatives of L^2 forms. The *reduced L^2 cohomology* is defined similarly, but dividing out by the *closure* of the L^2 exact forms which are exterior derivatives of L^2 forms. These are in general different; for example, $X = \mathbb{R}$ has vanishing reduced L^2 cohomology but non-vanishing non-reduced L^2 cohomology in degree 1, since not every function in $L^2(\mathbb{R})$ has an antiderivative in $L^2(\mathbb{R})$.) As pointed out by Lott [Lo, §8 in the preprint version], this conjecture about non-reduced L^2 cohomology can be deduced from a suitable strengthening of the Novikov conjecture.[1] But these conjectures about spectral geometry of X (say for X an open, uniformly contractible Riemannian manifold) are still valid even when there is no group around, or if the group has no non-trivial homology (so that there are no higher signatures to discuss).

The connection between the Dirac operator and positive scalar curvature comes from the simple formula

$$\text{(Lichn)} \qquad \mathcal{D}^2 = \Delta + \frac{1}{4}S,$$

first discovered by Lichnerowicz [Li] (see also [LaM, Theorem 8.8]), valid on any Riemannian spin manifold. Here $\mathcal{D}$ is the Dirac operator acting on the spinor bundle, S is the scalar curvature, and Δ is a non-negative elliptic operator that in local coordinates looks like the Laplacian. Thus if S is

[1] Here is a quick version of the argument. Indeed, the non-reduced L^2 cohomology on any complete Riemannian manifold X is non-zero if and only if the signature operator $d + d^*$ on X has 0 in its spectrum (for its action on L^2 forms with respect to the Riemannian metric). (The "only if" direction is clear, and the "if" direction follows from the fact that if 0 is in the spectrum, then either 0 is an eigenvalue, in which case even the reduced L^2 cohomology is non-zero, or else $d+d^*$ does not have a gap in its spectrum near zero, and the non-reduced L^2 cohomology is non-Hausdorff and thus infinite dimensional.)

Now suppose that X is the universal cover of a compact aspherical manifold $V = X/\Gamma$ and that the "Strong Novikov Conjecture" (SNC) holds for Γ in the sense of [R]. For simplicity assume V is oriented. Arguing as in [R], but with the signature operator in place of the Dirac operator, we see that if 0 is not in the spectrum of the signature operator on X, then its generalized index in $K_*(C^*_r(\Gamma))$ must vanish. But then the SNC gives as in [R, §2]:

$$\langle \mathbf{L}(V) \cup id^*(x), [V] \rangle = 0$$

for any cohomology class x on $B\Gamma = V$. Taking $x \neq 0$ in the top-degree cohomology, we get a contradiction. The orientability condition can be removed as in [R, §3].

bounded below by a positive constant $s > 0$, $\mathcal{D}^2 \geq \frac{s}{4} > 0$, and so $\mathcal{D}$ has a bounded inverse. If S is strictly positive but not uniformly so, then at least we immediately see from (Lichn) that $\mathcal{D}$ cannot have a non-zero kernel in the L^2 spinors.

There are basically two known approaches to proving non-existence of positive scalar curvature metrics: the *Dirac operator method* and the *minimal hypersurface method*. (For a complete survey of these and of some of the results one can obtain from them, see [RS].) The minimal hypersurface method of [SY] is in some sense parallel to the codimension-one splitting methods for attacking the Novikov conjecture (see notably [C]), whereas the Dirac operator method parallels the work on the Novikov conjecture using higher index theory of the signature operator. The idea of the minimal hypersurface method is based on the fact that one can show, using the stability condition, that a stable minimal hypersurface in a manifold of positive scalar curvature can be given a new metric (conformally equivalent to the induced metric) in which it has positive scalar curvature. Thus sometimes a manifold with suitable codimension-one submanifolds which do not admit metrics of positive scalar curvature cannot admit such a metric, either. This method has the advantage that it applies even in the absence of a spin structure, but big technical problems arise from the fact that minimal currents in high-dimensional manifolds usually have singularities, so that when one tries to represent a homology class in codimension one by a stable minimal current, one often does not get a smooth submanifold. It is for this reason that many results obtained from the minimal hypersurface method are stated in the literature only for manifolds of dimension ≤ 7. While Schoen and Yau ([S], [Y]) have announced that they can get around the dimensional restriction in the method, no details have appeared yet.

Let us return again to the operator-theoretic approach. To show that a spin manifold cannot have a Riemannian metric of positive scalar curvature, we need to show that the Dirac operator is non-invertible, and to do this one usually needs to perturb the operator a bit by adding a suitable "vector potential." How to make this idea precise is suggested by a theorem of Vafa and Witten [VW]. We briefly review their result and then discuss how similar techniques can be applied to *non-compact* manifolds.

Theorem (Vafa and Witten [VW]**).** *Let V be a compact even-dimensional Riemannian spin manifold (without boundary), let S be the spinor bundle on V (with its usual connection coming from the Riemannian metric), and let $\mathcal{D}$ denote the Dirac operator of V (acting on sections of S). Then there is a constant $\varepsilon(V)$, depending only on V and its Riemannian metric, with the property that if E is an* **arbitary** *vector bundle on V with a connection and a metric, and if $\mathcal{D}_E$ denotes the twisted Dirac operator* *acting on sections of $S \otimes E$, then $\mathcal{D}_E$ has an eigenvalue with absolute value $\leq \varepsilon(V)$.*

Proof (Quick sketch). Recall that since we're assuming that V is even-dimensional, the spinor bundle has a canonical splitting $S = S^+ \oplus S^-$ into "half-spinor bundles," and that $\mathcal{D}$ sends sections of S^+ to sections of S^- and vice versa. Also, the Dirac operator is self-adjoint and elliptic, and so has a good spectral decomposition. (For details of all these facts, see [LaM, Chapter II].) The Atiyah-Singer Index Theorem gives a formula for the index of $\mathcal{D}_E$ (as an operator from sections of $S^+ \otimes E$ to sections of $S^- \otimes E$):

$$\text{(A-S)} \qquad \operatorname{ind} \mathcal{D}_E = \langle \hat{A}(V) \cup \operatorname{Ch}[E], [V] \rangle.$$

If this quantity is non-zero, then $\mathcal{D}_E$ must have zero as an eigenvalue and we're done. However, there will be some bundles E for which the formula (A-S) gives 0, in which case there is no reason why $\mathcal{D}_E$ should have a non-zero kernel. It's for these bundles that we need an estimate on the spectrum.

The idea for getting this estimate is that if a differential operator has spectrum far away from zero, then a small perturbation in the operator cannot suddenly create a non-zero kernel. Therefore, assuming that $\operatorname{ind} \mathcal{D}_E = 0$, we first find another bundle E' such that

$$\langle \hat{A}(V) \cup \operatorname{Ch}[E] \cup \operatorname{Ch}[E'], [V] \rangle \neq 0.$$

Because of (A-S), this means that $\operatorname{ind} \mathcal{D}_{E \otimes E'} \neq 0$, so that $\mathcal{D}_{E \otimes E'}$ has zero as an eigenvalue. Now if the bundle E' were topologically trivial, $E \otimes E'$ would just be a direct sum of $\dim E'$ copies of E, but with a different connection (since we're not assuming E' is flat). So $\mathcal{D}_{E \otimes E'}$ would be a perturbation of a direct sum of $\dim E'$ copies of $\mathcal{D}_E$ by lower-order terms. Thus, since $\mathcal{D}_{E \otimes E'}$ has zero as an eigenvalue, $\mathcal{D}_E$ cannot have too large a gap in its spectrum around zero.

Of course, we've oversimplified too much, since if $\operatorname{ind} \mathcal{D}_E = 0$ and ind $\mathcal{D}_{E \otimes E'} \neq 0$, then E' could not be topologically trivial. However, we can always find another bundle E'' such that $E \oplus E''$ is topologically trivial, and then

$$\mathcal{D}_{E \otimes (E' \oplus E'')} = \mathcal{D}_{E \otimes E'} \oplus \mathcal{D}_{E \otimes E''}.$$

So giving $E \oplus E''$ the direct sum of the connection on E' with a connection on E'', we see that $\mathcal{D}_{E \otimes (E' \oplus E'')}$ has zero as an eigenvalue, and we can argue as before. □

Now let's discuss how we could apply the same ideas to open manifolds. (For an example of an application of these ideas in the different context of Kähler geoemtry, see [Gr2].) The same philosophy ought to apply, but since an elliptic operator on a non-compact manifold need not have discrete spectrum, we need to be much more careful about what constitutes a "small" perturbation of an operator. Before we get to this, we'll pursue some more consequences of these ideas with respect to compact manifolds. Suppose that $V = X/\Gamma$ is compact and aspherical, with fundamental group Γ. Intuitively, if we had a bundle E on V which were "almost flat," and for which we could compute by (A-S) that $\operatorname{ind} \mathcal{D}_E \neq 0$, then $\mathcal{D}_E$ would have to have non-zero kernel. On the other hand, since E is "almost flat," the formula (Lichn) (with $\mathcal{D}$ replaced by $\mathcal{D}_E$) would not be off by very much, and also $\mathcal{D}_E$ would be only a small perturbation of $\dim E$ copies of $\mathcal{D}$. So if V had a Riemannian metric of positive scalar curvature, we could conclude that $\mathcal{D}$ would have to have an eigenvalue very close to 0, contradicting the estimate coming from (Lichn). So V could not have a positive scalar curvature metric after all.

This argument can indeed often be made to work, but the correct notion of "almost flat bundle" [CGM] is not really a single vector bundle but rather a sequence of $\mathbb{Z}/2$-graded vector bundles $E_i = E_i^0 \oplus E_i^1$, such that the formal differences $[E_i^0] - [E_i^1]$ all represent the same element of $K^0(V)$, and such that all the bundles are equipped with connections whose curvatures tend to 0 in the right sense as $i \to \infty$. One will also usually have $\dim E_i \to \infty$ if this is the case, since there is a limit on how flat we can make a representative of bounded dimension of a fixed K-theory class. (This limitation comes from Chern-Weil theory, which relates the curvature of the bundle to the Chern classes. Thus if any of the Chern classes is non-zero, the curvature must be non-zero.)

As an example of how to make this precise, suppose the fundamental group Γ of a compact non-positively curved manifold V^{2n} is residually finite (i.e., Γ has a faithful family of homomorphisms to finite groups). This condition is satisfied, for example, by arithmetic groups. Then we can use the finite quotients of Γ to construct a tower of finite coverings V_i of V which "converge" in some sense to the universal covering X. Pull the non-positively curved Riemannian metric on V back to each V_i, so that $\operatorname{diam} V_i \to \infty$. Choose smooth degree-one maps $V_i \to S^{2n}$ which have Lipschitz constants tending to 0 on bigger and bigger balls. (One can do this because of the non-positive curvature condition, since the inverse of the exponential map at a basepoint is well-defined and has Lipschitz constant 1 on bigger and bigger balls as $i \to \infty$. See for example [GL, §5].) Then we can pull back a fixed non-trivial bundle on S^{2n} by means of these maps, and we get a sequence of bundles (of course each defined on a different manifold V_i) whose curvatures tend to 0 in the operator norm (on L^2 sections). Pushing the bundles back down by means of the covering maps $V_i \to V$, we get an almost flat sequence of bundles (now with ranks tending to infinity) on V, which show that V cannot also admit a metric of positive scalar curvature. In fact, if one does this argument carefully, one can get an estimate on the Novikov-Shubin invariants of V, in other words, of the spectral density near 0 of the Dirac operator or Laplacian on the universal cover.[2]

Heuristically, we would expect a similar argument to prove the Novikov conjecture in similar cases, by using the analogue of (A-S) for the signature operator with coefficients in E:

$$\operatorname{ind} D_E^{\mathrm{sign}} = \langle \mathcal{L}(V) \cup \mathrm{Ch}[E], [V] \rangle$$

and controlling the error terms in the expansion of $(D_E^{\mathrm{sign}})^2$ coming from the curvature of E. Some arguments of this sort are carried out in [HiS]. However, there are complications in carrying out the details—one needs the connection, and not just the curvature, to be small, and one needs some "spectral purity" condition in order to define "almost Betti numbers."

In fact, if one generalizes the notion of an almost flat K-theory class,[3] one can show that the class of any complex line bundle over $B\Gamma$ is almost flat. This implies the Novikov conjecture for any higher signature coming from a 2-dimensional cohomology class. (Of course, Novikov himself [N] proved the homotopy invariance of any higher signature coming from a 1-dimensional cohomology class.) To prove this, consider a compact manifold V with a map to $B\Gamma$. The line bundle pulls back on the non-compact universal cover X of V to a topologically trivial line bundle L, and thus we can take arbitrary "roots" $L^{\frac{1}{n}}$ of L which provide the necessary approximating sequence $(L^{\frac{1}{n}})^n$. See [Gr].

In the final analysis, then, the Novikov conjecture seems to be a statement about the "largeness" of the universal covers X of compact aspherical manifolds V. It seems to be important to quantify this; in particular to determine in what sense X "dominates" Euclidean space. The best situation would be if we could always show that X is strongly hyper-Euclidean, i.e., that X admits a Lipschitz map to a Euclidean space of the same dimension.[4]

[2] The possible width of a spectral gap is estimated by the norm of an index-changing perturbation. Similar ideas appear in the study of the "non-commutative isoperimetric function" in [Hu], and in §3 of [Lü].

[3] See the comment in [CGM]: "il est d'ailleurs nécessaire dans les applications ... de reformuler cette notion [de fibré presque plat] en l'adaptant aux fibrés de dimension infinie munis d'une superconnexion en un sens convenable."

[4] Note added since the original lecture: Dranishnikov, Ferry, and Weinberger have now constructed a uniformly contractible manifold with no such map, but it is not clear if one can arrange for such a uniformly contractible manifold to be the universal cover of a compact aspherical manifold.

One very weak partial result in this direction is that if X is both complete and uniformly contractible in some Riemannian metric, then it has to have infinite volume. But it's hard to quantify its "growth" without additional assumptions. See [Gr1] for a discussion of different notions of "largeness."

What would be the way to construct a counterexample to the Novikov conjecture? Evidently we would need a rather pathological group. Most of the standard group-theoretical constructions rely on amalgamated free products and HNN extensions, which would lead to strange geometry in dimensions 1 and 2. But we know that the Novikov conjecture is by its nature a high-dimensional problem (the conjecture is true for cohomology classes on $B\Gamma$ in dimensions 1 and 2), so quite different techniques are needed.

We could try to measure the rate of contractibility of $E\Gamma$ in various dimensions, that is, to see how big a chain is needed to bound a given cycle. In dimension 1, this question is related to the solvability of the word problem in Γ; however, we're interested in this problem in higher dimensions. In most cases for which one can compute anything, the contractibility rate seems to grow exponentially. Perhaps to get a counterexample to the Novikov conjecture one should look for a case of very non-uniform growth in different directions.

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