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RANDOM WALK IN RANDOM GROUPS

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This paper compiles basic components of the construction of random groups and of the proof of their properties announced in [G10]. Justification of each step, as well as the interrelation between them, is straightforward by available techniques specific to each step. On the other hand, there are several ingredients that cannot be truly appreciated without extending the present framework. We shall indicate along the way possible developments, postponing full exposition to forthcoming articles expanding the following points touched upon in the present paper.

I. *Notions of randomness inside and outside infinite groups.*
II. *Small cancellation theories for rotation families of groups.*
III. *Diffusion, codiffusion, relaxation constants and Kazhdan's T.*
IV. *Entropies of random walks, Hausdorff–Gibbs limit of mm spaces, and mean hyperbolicity.*
V. *Non-geodesic metric spaces, Gibbs' hulls and fractal hyperbolicity.*
VI. *Entropies of displacements.*
VII. *Families of expanders.*

Acknowledgements and apologies. The line of thought presented in this paper was triggered by a conversation held with A. Dranishnikov, in the summer of 1999, who pointed out to me that the final remark (b) in § 7.E_2 of [G2], starting, "There is no known...", should have been replaced by, "There is a well-known example due to P. Enflo...", who constructed a sequence of finite graphs (of growing degrees) admitting no *uniform embedding* (see 1.2) into the Hilbert space $\mathbb{R}^\infty$ (see [En]). Dranishnikov was concerned with *infinitely* generated groups *uniformly non-embeddable* into $\mathbb{R}^\infty$ (see [DrGLY]). When one recognizes Enflo's graphs as expanders, and recalls the relation between the first eigenvalue λ_1 and the uniform Hilbertian embeddings, one realizes that *Pinski–Margulis–Selberg expanders* (see 3.12) can be incorporated into an (obviously) generalized hyperbolic small cancellation theory in conjunction with (a rough version of) the randomization from § 9 in [G2]. This delivers, with a few details to check out, *finitely generated* (and, eventually, *finitely presented*) groups Γ admitting

no uniform embedding into $\mathbb{R}^\infty$: the above I–VII transform into a few technical lemmas, with a straightforward half-page proof each, and an extra half-page to derive the above Γ from the lemmas. However, it took me two years to (partially) uncover the proper contexts for the lemmas rendering them "inevitable" and the proofs "tautological" rather than "technical". I try to explain below some of what I understood along the way with the unavoidable limitations imposed by the length of the paper. (A reader may find it amusing to play the game backward by reducing the present paper to seven pages of formal statements and proofs.) Finally, I want to thank Misha Kapovich and the anonymous referee for useful remarks, and L. Silberman for writing down an addendum providing a better perspective on some issues touched upon in the paper.

1 Random Groups Associated to Graphs

Consider a (finite or infinite) graph, i.e. a 1-dimensional cellular (e.g. simplicial) complex (V,E), where V stands for the set of vertices and E is the set of (non-oriented) edges. We denote by $\overleftrightarrow{E}$ the set of *oriented* edges of $V=(V,E)$. This $\overleftrightarrow{E}$ comes with the natural (forgetting orientation) map $\overleftrightarrow{E}\to E$ (that is a 2-to-1 map) and the map $\overleftrightarrow{E}\to V\times V$ assigning to each $\overrightarrow{e}\in\overleftrightarrow{E}$ its ends. If (V,E) is a *simplicial* complex, then this latter map is an *embedding* landing outside the diagonal in $V\times V$; thus the graph is defined by a subset $\overleftrightarrow{E}\subset(V\times V)\setminus\operatorname{diag}$ invariant under the involution $(v,v')\leftrightarrow(v',v)$ and $E=\overleftrightarrow{E}/\{+1,-1\}$ (where -1 stands for this involution).

Given a group Γ and a map $\alpha:\overleftrightarrow{E}\to\Gamma$, denote by $W_\alpha\subset\Gamma$ the subset consisting of the products $\alpha(\overrightarrow{e}_1)\alpha(\overrightarrow{e}_2)\dots\alpha(\overrightarrow{e}_1)$ for all *i-cycles* of directed edges and all $i=1,2\dots$, i.e. all closed paths in (V,E). For example, if the graph is connected with a distinguished marked point, the map α is *symmetric*, i.e. $\alpha(\overrightarrow{e})=\alpha^{-1}(\overleftarrow{e})$ for all $e\in E$, then α defines a homomorphism α_* from the (free!) fundamental group $\pi_1(V)$ to Γ and the *normal closure* of $\alpha_*(\pi_1(V))\subset\Gamma$ equals that of W_α,

$$\big[\alpha_*(\pi_1(V))\big]=[W_\alpha]\subset\Gamma\,.$$

Next, let μ be a probability measure on Γ and denote by μ^E the (probability) product measure on the space $\mathcal{A}$ of symmetric maps $\overleftrightarrow{E}\to(\Gamma,\mu)$. In what follows, μ is (usually assumed) symmetric under $\gamma\leftrightarrow\gamma^{-1}$ and the (power) notation μ^E for $E=\overleftrightarrow{E}/\{+1,-1\}$ is justified.

A *random group* Γ_α in this context refers to the factor group $\Gamma/[W_\alpha]$ for a "typical" $\alpha \in \mathcal{A}$, where "typical" events (properties of α in this context) are those which occur with *non-zero* probability p. In fact, this probability p will usually be of the form $p = p_n \geq 1 - (1-\varepsilon)^n$ for a fixed $\varepsilon \in (0,1)$ and $n \to \infty$; one says in this case that the corresponding event takes place with *overwhelming probability.*

1.1 Standard example. Let Γ be the free group on k generators, $\Gamma = F_k$, $k \geq 2$, and the *standard measure* assign equal weights $= 1/2k$ to the generators and their reciprocals, say $g_i^{\pm 1}$, $i = 1, \dots, k$. Then the group Γ_α is presented by g_i's subject to the relations corresponding to the cycles in the graph V. In particular, Γ_α is *finitely presented* if V is finite.

If (V,E) contains too many short simple cycles (e.g. $\geq (2k-1)^i$ of cycles of length $\leq i$ for large i), then the group Γ_α is trivial (i.e. $= \{\mathrm{id}\}$) with high (and convergent to 1 for $i \to \infty$) probability p_{triv}. Our major concern is to bound this probability, or even better to bound from below the probability of the group Γ_α being infinite,

$$p_{\mathrm{inf}} = \mathrm{Prob}\left\{|\Gamma_\alpha| = \infty\right\},$$

where, clearly, $p_{\mathrm{triv}} + p_{\mathrm{inf}} \leq 1$ (and in most cases $p_{\mathrm{triv}} + p_{\mathrm{inf}}$ is very close to 1). Furthermore, we wish the natural map from the graph (V,E) to the Cayley graph of Γ_α to be "essentially" one-to-one with high probability. This property will be established later on in § 4 (along with a lower bound on p_{inf}) under a suitable *thinness* assumption on (V,E) ensuring, in particular, the bound

$$1 + \mathrm{card}\{\text{simple } i\text{-cycles}\} \leq \exp(\tau k i)\,, \quad i = 1, 2, \dots \qquad (*)_\tau$$

for some (rather small) $\tau > 0$, where our major tool is *small cancellation theory* adapted to the present situation. We shall use in applications a sequence of finite graphs V_g, $g = 1, 2, \dots$, where the girth of V_g, i.e. the length of the shortest non-contractible cycle, equals g and where $(*)_\tau$ is satisfied for $k = 2$ and a fixed small τ, say $\tau = 10^{-3}$. We take these V_g from $V_{i,j}$ as in 3.13 by setting $V_g = V_{g,j_0}$ for a large j_0, say $j_0 = 1000$.

The small cancellation theory tells us (see §2) that *the group* $\Gamma_\alpha = \Gamma_\alpha(V_g)$ is *infinite with overwhelming probability for* $g \to \infty$. Moreover, Γ_α is *a non-elementary torsion free word hyperbolic group, and the presentation* $\{g_1, \dots, g_k \mid W_\alpha\}$ *is essentially aspherical*, i.e. there is a subset $W' \subset W_\alpha$ with $[W'] = [W_\alpha]$, where $\{g_1, \dots, g_k \mid W'\}$ is truly aspherical (and which corresponds to a generating subset of independent cycles in V).

1.2 Expanders, uniform embeddings and fixed point properties. Let $E_v \subset E$ denote the set of edges $[v, v']$ issuing from $v \in V$ and for a function $f : V \to \mathbb{R}$ set

$$\Delta f(v) = f(v) - (\operatorname{card} E_v)^{-1} \sum_{E_v} f(v') \,.$$

The first non-zero eigenvalue of the resulting Laplace operator Δ on (the L_2-space of) functions on V is denoted $\lambda_1(V)$ and the graph is called a λ-*expander* if $\lambda_1(V) \geq \lambda$.

The above graphs V_g, according to 3.12 are λ-expanders for some $\lambda > 0$, say $\lambda = 10^{-7}$. This implies (see §3) that the group $\Gamma_\alpha = \Gamma_\alpha(V_g)$ satisfies *with overwhelming probability*, the following

1.2.A Fixed point property. *Let Y be a regular* CAT(0) *space, i.e. a complete (finite or infinite) dimensional simply connected manifold with non-positive sectional curvature. Then every isometric action of* Γ_α *on Y has a fixed point.*

This, for $Y = \mathbb{R}^\infty$, amounts to *Kazhdan's T-property* for Γ_α.

Remark. If one allows torsion, one can easily arrange a similar Γ with the fixed point property that admits a discrete cocompact simplicial isometric action on a 2-dimensional *polyhedron* with negative curvature, but I have not worked out in detail potential torsion free examples.

1.2.B Definition. A *Lipschitz* map $f : X \to Y$ between metric spaces is called a *uniform embedding* if

$$\big|f(x_1) - f(x_2)\big|_Y \geq \varphi\big(|x_1 - x_2|_X\big)$$

for some function $\varphi = \varphi_f(d) \underset{d \to \infty}{\longrightarrow} \infty$, where $|x - y|$ stands for the distance between the points in spaces in question.

One knows (this was observed by Sela), that each of the above groups $\Gamma_\alpha(V_g)$ admits a uniform embedding into the Hilbert space $\mathbb{R}^\infty$, but the corresponding φ depends on g. In fact, if one takes an *infinite* sequence of the graphs, $V_{g_1}, V_{g_2}, \dots$, and make Γ_α with the disjoint union

$$V_G = V_{g_1} \sqcup V_{g_2} \sqcup \dots \,,$$

where G stands for $\{g_1, g_2, \dots\}$, one shows (see §3) that

1.2.C. *The group $\gamma_\alpha = \Gamma_\alpha(V_G)$ admits no uniform embedding into a regular* CAT(0)*-space* (e.g. into a *Hilbert space*)*, provided g_i grow sufficiently fast with $i = 1, 2, \dots$.* (Actually, $g_i \geq 10^{10} i$ is fast enough, but we are content with $g_i \geq 1000^i$ in this paper.)

Remarks. (a) It is well known (and almost obvious) that the graph V_G itself admits no uniform embedding into $\mathbb{R}^\infty$ (and into any regular CAT(0)-space for this matter). The main point in the proof of 1.2.C is showing that V_G *quasi-uniformly* (as is explained in §3) embeds into the Cayley graph of Γ_α with overwhelming probability (for $g_i \to \infty$).

(b) What makes $\Gamma(V_G) = \Gamma_\alpha(V_G)$ for a typical α non-embeddable into a *regular* CAT(0)-space is the presence of (arbitrarily) complicated singularities at the centers (apexes) of the *cones* over V_g for large g due to the expander property of V_g. In fact, one can show that $\Gamma(V_G)$ admits no uniform embedding into a CAT(0)-space Y with "bounded" singularities, e.g. if the *tangent cone* of Y at each point $y \in Y$ is isometric to a (finite or infinite) Cartesian (Pythagorian) product of conical spaces $Y_i = \operatorname{Cone} B_i$ for a *Hausdorff precompact* family of CAT(1)-spaces B_i. Moreover, the CAT(0)-property can be replaced by a much weaker one concerning (semi)-local quasi-isometric embeddings of Y to $\mathbb{R}^\infty$. Thus one can arrange, for example, two (or uncountably many, if one wishes) G's, say G and G', such that $\Gamma(G)$ admits no uniform embedding into the countable ℓ_2-Cartesian (i.e. Pythagorian) power $\Gamma(G')^{\{\mathbb{N}\}}$ and vice versa (where one can "add" finitely many arbitrary word hyperbolic groups Γ_i to $\Gamma(G')$, i.e. take $(\Gamma(G') \times \Gamma_1 \times \ldots \times \Gamma_k)^{\mathbb{N}}$ and still have no uniform embedding of $\Gamma(G)$ into it).

1.3 Finitely presented groups. The random groups $\Gamma_\alpha(V_G)$ are infinitely presented (for infinite $G = \{g_1, \dots\}$) but "random" can be replaced by "pseudorandom" (i.e. recursive) due to the (elementary) nature of the probabilistic ingradient in the proof (see §4). This delivers, via the Higman embedding theorem, *finitely presented* groups Γ uniformly non-embeddable into regular CAT(0)-spaces. Furthermore, since $\Gamma_\alpha(V_G)$ come with a (natural) *aspherical* presentation (see §2), some of these Γ also admit such presentations according to a recent result by Rips and Sapir. Then the standard reflection construction provides *closed aspherical 4-manifolds M, such that $\Gamma_1 = \pi_1(M) \supset \Gamma \supset \Gamma_\alpha(V_G)$; hence Γ_1 is not uniformly embeddable into a regular* CAT(0)*-space.*

Question. Can one make such a Γ_1 satisfy T as well?

It is not hard to embed Γ_1 into a finitely presented T-group Γ_2: by adding suitable relations to the free product $\Gamma_1 \times F_2$. This Γ_2 serves as the fundamental group of an aspherical 4-polyhedron P^4, however, this P^4 is far from being a manifold. In fact, it seems that all *known* T-groups Γ that serve as fundamental groups of *closed aspherical manifolds* are *arithmetic.* (Test cases are fundamental groups of *ramified covers* of arithmetic manifolds and of *quotients* of these by finite groups.)

1.4 Factorization of hyperbolic groups. One can replace in the previous discussion the free group by an arbitrary non-elementary word hyperbolic group $\Gamma = \Gamma^0$ with a fixed symmetric generating subset $\{g_1^{\pm 1}, \dots, g_k^{\pm 1}\} \in \Gamma^0$. In fact, even for $\Gamma^0 = F_k$, we approach the group Γ^i associated with $V_G = V_{g_1} \sqcup V_{g_2} \sqcup$ by adding relations associated to $\alpha : \overset{\leftrightarrow}{E}_{g_i} \to \Gamma^{i-1}$ to the (hyperbolic!) group Γ^{i-1} obtained on the previous step with $V_{g_1}, \dots, V_{g_i - 1}$. What is important here, is, roughly, to keep a *uniform* lower bound on the required τ in $(*)_\tau$, since this, a priori, depends on the group Γ^{i-1} as we add the relations W_α to Γ^{i-1} for $\alpha : \overset{\leftrightarrow}{E}_{g_i} \to \Gamma^{i-1}$ (see §4).

One can generalize further by starting with an infinite *non-word* hyperbolic group Γ^0 with a *faithful* action of Γ^0 on the ideal boundary $\partial_\infty \Gamma$ of Γ^0, but here it is harder to keep track of τ and I failed to work out a sufficiently general condition on Γ^0 encompassing the available examples (including free products and some amalgamated products, many reflection groups, non-cocompact lattices and others, where the small cancellation techniques perform, so far, only on the case-by-case basis).

1.5 Factorization with subgroups of dimension > 1. If the starting group Γ is realized as the fundamental group of some space $\underline{X}$ then the above "V-graphical" quotients of Γ appear as fundamental groups of spaces obtained by attaching mapping cones cone $\underline{\alpha}$ to $\underline{X}$ for (random) maps $\underline{\alpha} : V \to \underline{X}$. Now let X be a symmetric space with non-positive curvature and Γ an isometry group acting on X. The principal (but not the only!) examples are those where $\operatorname{rank}_{\mathbb{R}} X = 1$ (i.e. $K(X) < 0$) and Γ is a cocompact arithmetic group with no torsion. Here $\underline{X} = X/\Gamma$ is a compact (real, complex, quaternion or Cayley) hyperbolic manifold and the arithmeticity condition (see below) allows $\underline{X}$ to have many closed totally geodesic (immersed or embedded) submanifolds $\underline{X}' \subset \underline{X}$. One denotes by $\Gamma' \subset \Gamma$ the normal closure of the fundamental groups of the connected components of $\underline{X}'$ and observe that $\Gamma/\Gamma' = \pi_1(\underline{X} /\!\!/ \underline{X}')$ where $\underline{X} /\!\!/ \underline{X}'$ stands for the union of mapping cones of the connected components of $\underline{X}'$ immersed to $\underline{X}$. (If $\underline{X}'$ is a connected truly embedded submanifold, then $\underline{X} /\!\!/ \underline{X}'$ is homotopy equivalent to $\underline{X}/\underline{X}'$ obtained by shrinking $\underline{X}' \subset \underline{X}$ to a point.)

Remark. It is sometimes worthwhile to consider the image $\underline{X}'' = i(\underline{X}') \subset \underline{X}$ and honestly shrink it to a point. For example, if $\underline{X}$ and $\underline{X}'$ are *complex hyperbolic* and the self-intersection of $\underline{X}'$ in $\underline{X}$ is "mild" (e.g. empty) then such $\underline{X}/\underline{X}''$ is a (singular) complex space. Typically this space is non-projective, but if the volume of $\underline{X}'$ is small compared to that of $\underline{X}$, then $\underline{X}/\underline{X}''$ stands a good chance of admitting a projective embedding.

The small cancellation theory (see §2) provides a criterion for the space $\underline{X}/\!\!/\underline{X}'$ (and sometimes for $\underline{X}/\underline{X}''$) to be aspherical with the group $\pi_1(\underline{X}/\!\!/\underline{X}')$ being word hyperbolic and if $\dim \underline{X}'/\dim \underline{X}$ is small ($\ll 1/2$) then one can show that this criterion is satisfied for "generic" $\underline{X}'$.

Let us assign probability weights $p(\underline{X}')$ to all (relevant) $\underline{X}'$ of the form $p(\underline{X}') = \exp -\beta \operatorname{diam}(\underline{X}')$ with large β. (Here "diam" refers to the intrinsic diameter of $\underline{X}'$; if $\dim \underline{X}' \geq 2$, then the relevant arithmetically defined $\underline{X}'$ have $\mathrm{Vol}(\underline{X}') \approx \exp \alpha \operatorname{diam} Y'$ for some $\alpha = 0$ and the injectivity radii of Y' are $\approx \operatorname{const} \operatorname{diam} Y'$. The number of different Y' of a given volume V is $\approx V^p$ for some $p > 0$). If one shrinks each Y' to a point at random with probability given by the corresponding weight, then, with probability $p > 0$, the resulting space will be aspherical with (necessarily) infinite fundamental group. (Probably the relevant "smallness" of $\dim X'/\dim X$ should refer to the *Hausdorff* dimensions of the *ideal boundaries* of X and X' with natural metrics, where the conjectural sharp inequality reads: $\dim_H \partial X'/\dim_H \partial X < 1/2$ as motivated by random groups in [G2].)

The arithmetricity condition essentially says that Γ (or a subgroup of finite index in Γ) embeds into the group $SL_N(\mathbb{Z})$ of orientation preserving automorphisms of the Abelian group $\mathbb{Z}^N$. The group $SL_N(\mathbb{Z})$ isometrically acts on the symmetric space $X = SL_n(\mathbb{R})/SO_n$ (of $\mathbb{R}$-rank $= N-1$) and $SL_n(\mathbb{Z})$ itself has no infinite quotient groups for $N \geq 3$. But it contains many (arithmetic and non-arithmetic) non-free infinite subgroups Γ and $\Gamma' \subset \Gamma$, where $\Gamma/[\Gamma']$ is a non-elementary hyperbolic and sometimes a word hyperbolic group. Here are major examples of subgroups $\Gamma \subset SL_N\mathbb{Z}$.

(a) Take a totally geodesic submanifold $X \subset SL_N(\mathbb{R})/SO_N$. (Recall that every symmetric space of non-compact type comes this way.) If the stabilizer Γ of X in $SL_N(\mathbb{Z})$ is "moderately large" then X/Γ has finite volume and Γ is arithmetic. (Notice that such a Γ has no non-trivial infinite quotient group unless $\mathbb{R}$-rank $X = 1$ by Margulis' theorem.)

(b) There are many non-arithmetic reflection groups Γ in $SL_N\mathbb{Z}$ admitting lots of infinite quotients. For example such are reflection groups Γ acting on the hyperbolic space H^n, both of finite and infinite covolumes. Besides these, many reflection groups act on polyhedra of non-positive curvature. Most (all?) of them also have such quotients.

(c) $SL_N\mathbb{Z}$ comes along with distinguished subgroups of finite index, called *congruence subgroups*, such as the kernels of the canonical homomorphisms $SL_N(\mathbb{Z}) \to SL_N(\mathbb{F}_p)$, for all prime p.

(d) Given the above (classes of) subgroups Γ one can obtain more by the following operations.

(i) Intersections $\Gamma = \Gamma_1 \cap \Gamma_2$;

(ii) generation of Γ by finitely or infinitely many Γ_i in $SL_N\mathbb{Z}$;

(iii) conjugation over $\mathbb{Q}$ for $\Gamma_1 = (A\,\Gamma A^{-1}) \cap SL_N\mathbb{Z}$ with $A \subset SL_N(\mathbb{Q})$.

Thus one obtains a large pool of pairs $\Gamma' \subset \Gamma \subset SL_N\mathbb{Z}$ where the small cancellation theory often applies (to $\Gamma/[\Gamma']$ and that can be combined with a suitable randomization of subgroups $\Gamma' \subset \Gamma$.

We shall abandon the pursuit of higher dimensional random (sub)-groups at this point till another paper. Just notice that "randomness" here is tied up to arithmetics more than to combinatorics (governing random subgroups in free groups). This confirms the belief that (almost?) all "decent" sufficiently high dimensional groups are obtained by relatively simple combinatorial constructions out of basic "arithmetic molecules".

1.6 Resilient (properties of) groups. All of the above random quotient groups can be viewed as probability measures σ on the space 2^Γ of subsets W in a given group Γ. An actual quotient group is $\Gamma/[W]$ where $W \in 2^\Gamma$ is a "sample" subset ($[W]$ denotes the normal subgroup generated by W in Γ) and where "typical properties of $\Gamma/[W]$" are valid, by definition, with positive probability. This can be expressed with the truth value function on 2^Γ for a given property (predicate) P, denoted $\chi = \chi_p : 2^\Gamma \to \{0,1\}$, where "0" stands for "false" and "1" for "true".

There is a pairing on 2^Γ: taking unions of subsets in Γ, that is $(W_1, W_2) \mapsto W_1 \cup W_2$ or $U : 2^\Gamma \times 2^\Gamma \to 2^\Gamma$. Given two properties P_1 and P_2 of $G/[W]$ characterized by functions $\chi_1, \chi_2 : 2^\Gamma \to \{0,1\}$, their simultaneous truth is expressed by the function $\chi : 2^\Gamma \times 2^\Gamma \to \{0,1\}$ for $\chi(W_1, W_2) = \min(\chi(W_1), \chi(W_2))$ also denoted $\chi_1 \wedge \chi_2$.

The property we were mostly concerned with was the quotient group $\Gamma/[W]$ being *infinite* and we enlisted certain probability measures σ with respect to which this was typical. In all these cases small cancellation theory delivers more than just "typicality", that is the inequality,

$$\sigma\{\chi = 1\} > 0\,,$$

for the corresponding truth function $\chi^{in} : 2^\Gamma \to \{0,1\}$ corresponding to "being infinite" but also

$$(\sigma_1 \times \sigma_2)\{\chi^{in} \wedge \chi^{in}\} > 0\,,$$

where $\sigma_1 \times \sigma_2$ denotes the product measure on $2^\Gamma \times 2^\Gamma$ and where σ_1 and σ_2 are taken from our list. It follows that if we take a σ_1-typical group

$\Gamma_1 = \Gamma/[W_1]$ and then add a σ_2-typical set of relations, the resulting group $\Gamma_2 = \Gamma_1/[W_2] = \Gamma/W_1 \cup W_2]$ is $\sigma_1 \times \sigma_2$- typically infinite. This also applies to finite collections of our measures and shows, in particular, that each class of *infinite* groups $\Gamma/[W]$ is *resilient* in a sense that it is invariant, with positive probability, under adding extra "typical" relations to a given "typical" W. (It would be more proper to say that the predicate "being infinite" is resilient.)

All this is quite trivial, of course, modulo the randomized hyperbolic small cancellation theory; however, it is not so clear when an *individual* group $\Gamma_1 = \Gamma/[W_1]$ is resilient, since small cancellation theory does not apply to *non-hyperbolic* groups, such as Γ_1 for an *infinite* subset $W_1 \subset \Gamma$. We shall indicate in the next section some features of such Γ_1's relevant to the resiliency.

1.7 Lacunarity, mesoscopic curvature, and fractal hyperbolicity. Let us concentrate, for notational simplicity, on groups $\Gamma = F_2/[w_1, w_2, \dots, w_i, \dots]$, where w_i is a sequence of cyclically irreducible words of length $\ell_1 < \ell_2 < \dots < \ell_i < \dots$. *Typically*, such a Γ is infinite, by small cancellation theory and has a rather particular asymptotic geometry with respect to the word metric that is especially transparent for *lacunary sequences*, where $\ell_{i+1}/\ell_i \to \infty$ for $i \to \infty$. Namely, if we scale Γ, i.e. the word metric Γ, by numbers ε_i "deeply between" ℓ_i and ℓ_{i+1}, namely such that $\varepsilon_i \ell_i \to 0$ and $\varepsilon_i \ell_{i+1} \to \infty$ for $i \to \infty$, then the Hausdorff (ultra) limit of $\varepsilon_i \Gamma$, $i \to \infty$ is a tree, i.e. a path-connected 1-dimensional space with no simple cycles. But if $\varepsilon_i \ell_i \to c$ for some constant $0 < c < \infty$, then such a limit Γ_∞ has a (unique) up to isometry simple non-contractible cycle of length c. (Accidentally, this shows, that the asymptotic growth of ℓ_i is a quasi-isometry invariant. Also one can show that for two such typical sequences $\{w_i\}$ and $\{w'_i\}$ the corresponding groups are not quasi-isometric, even if $\mathrm{length}(w_i) = \mathrm{length}(w'_i)$ for all i compare [TV].) In fact, this Γ_∞ looks like a (renormalized) Cayley graph of a $1/n$-cancellation group for $n \to \infty$. In particular Γ_∞ is locally isometric to an $\mathbb{R}$-tree.

One gets a more informative picture by attaching disks to all simple cycles in the Cayley graph of Γ. Actually it is better to proceed slightly differently: start with the Cayley graph $X_0 \supset \Gamma$ of Γ and isometrically attach disks to all (shortest) cycles of length ℓ_1 in X, where each disk D is given the metric of constant curvature $= -\ell_1^{-2}$ and it has $\mathrm{length}(\partial D) = \ell_1$. Denote the resulting (geodesic metric) space by X_1, take the shortest closed geodesics c there (they have length somewhat smaller than ℓ_2), attach to them disks with boundaries of length $\ell_2' = \text{length}(c)$ and curvature $-(\ell_2^1)^{-2}$. Thus we arrive at $X_2 \supset X_1 \supset X_0$ and eventually at

$$X = \bigcup_i X_i \qquad \text{for} \qquad \ldots\ X_i \supset \ldots \supset X_2 \supset X_1 \supset X\,,$$

where Γ acts by isometrics. Now, for every sequence $\varepsilon_i \to 0$ the (ultra)limit spaces $X_\infty = \lim \varepsilon_i X$ are CAT(0). Each X_∞ contains a geodesically convex *core* $C_\infty \subset X_\infty$ where X_∞ is obtained by attaching half-planes to geodesics in C_∞ and where everything is equivariant under the limit isometry group Γ_∞ acting on X_∞. (This core (roughly) equals $\lim_{i\to\infty} \varepsilon_i X_i$ and it is CAT $(\kappa < 0)$ if $\varepsilon_i \ell_i \to c$.)

The geometry of the core can be seen in the scaling limit of the (non-word!) metric on Γ induced from X by the obvious embedding $\Gamma \subset X_0 \subset X$, where the negativity of the curvature of C_∞ emerges from that of the rescaled *mesoscopic* curvature of $(\Gamma, \text{dist}_X \Gamma)$: the *mesoscopic curvature* (in the sense of [G6]) of $\varepsilon B(id,\ \varepsilon^{-1})$ is $\leq \kappa < 0$ on the scale $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$ (where $B(id,\ \varepsilon^{-1})$ stands for the ε^{-1}-ball in Γ, and where one should adjust the definition from [G6] to non-geodesic metrics, and also take care of possible minor problems at ∂B).

Non-lacunary case. Suppose that $\delta = \ell_i/\ell_{i+1}$ is small but yet separated away from zero. Then each space $\ell_i^{-1} X_i$ has negative mesoscopic curvature $\leq \kappa < 0$ on the scale $\varepsilon = \varepsilon(\delta)$, where κ is independent of δ while $\varepsilon(\delta) \to 0$ for $\delta \to 0$. Furthermore, the "local" metric distortion of the embedding $X_i \subset X$ is bounded in the following way

$$|x - x'|_X \geq \big(1 - \varepsilon'(\delta)\big)|x - x'|_{X^i}\,,$$

provided $|x - x'|_{X^i} \leq C(\delta)\ell_i$ where $\varepsilon'(\delta) \to 0$ and $C(\delta) \to \infty$ for $\delta \to 0$.

These properties can be expressed in terms of the (non-geodesic) metric on Γ induced from X, where one has hyperbolicity (or mesoscopic negative) kind of properties with the following features:

(i) scale invariance (unlike the ordinary δ-hyperbolicity);

(ii) only approximate middle point or geodesic property.

Observe that the latter provides chains of points $x_0 = x,\ x_1, \ldots, x_n = x'$ for given x and x', such that $|x_i - x_{i+1}|$ are close to $|x - x'|/n$, while "hyperbolicity" says in effect, that every two such sequences $x_1, \ldots, x_{n-1}$ and $x_1', \ldots, x_{n-1}'$ must be mutually close, i.e. $|x_i - x_{i+1}'|$ must be small.

With this in mind, one makes the following

Tentative definition of fractal δ-hyperbolicity for a metric space X. (i) The restriction of the metric $|\quad|_X$ to every subset $Y \subset X$

of diameter $\mathcal{D}$ is $\delta\mathcal{D}$-hyperbolic.

(ii) For every two points x, $x' \in X$ there exists $x_1 \in X$, such that

$$\max\left(|x-x_1|, |x_1-x'|\right) \le \left(\tfrac{1}{2}+\delta\right)|x-x'| \ .$$

There are several ways of strengthening and weakening this definition:

(a) require the $\delta\mathcal{D}$-hyperbolicity only for particular k-tuples, e.g. 4-tuples of points $\{y_1, y_2, y_3, y_4\} \in Y$, say, where y_2 and y_3 are approximate middle points between y_1 and y_4.

(b) insisting on the existence of intermediate points $x_1, \dots, x_{n-1}$ with $|x_i - x_{i+1}| \le \left(\frac{1}{n}+\delta\right)|x-x'|$ for *all* $n \le n(\delta)$, where $n(\delta) \to \infty$ for $\delta \to 0$; or, on the contrary, allowing this kind of chain (possibly with spread ratios $|x_i - x_{i+1}|/|x_{i'} - x_{i'-1}|$) only for *certain* $n \ge 2$.

In order to have a satisfactory theory one needs to enlist *all* possible definitions in the context of the *first order metric language* (see [GLP]) and:

- study their mutual relationships;
- find most general conditions sufficient for proving "fractal counterparts" of the standard hyperbolic (and/or mesoscopic with $\kappa < 0$) properties, such as the Cartan–Hadamard theorem);
- find a most general and most invariant mechanism for setting such (invariant) metrics on finitely generated groups;
- evaluate critical δ where the fractal δ-hyperbolicity becomes productive (for large δ the fractal δ-hyperbolicity is a vacuous condition);
- identify this hyperbolicity for known classes of finitely generated groups.

All of this is beyond the scope of the present paper.

1.8 Remarks on the notion of "random group". (a) The definition of a random quotient of Γ could be made more invariant by introducing the (closed!) subset $\mathcal{N} \subset 2^\Gamma$ consisting of all normal subgroups $\Delta \subset \Gamma$, where "random Γ/Δ" is understood as a probability measure μ on $\mathcal{N}$, that is such a measure on 2^Γ with the support in $\mathcal{N}$. For example, if Γ happens to act on a probability space P, we can assign the fixed point set $\mathrm{Fix}_\Delta \subset P$ to each $\Delta \subset \Gamma$ (which makes sense for all subsets in Γ, not only for normal subgroups) and then set

$$\mathrm{Fix}(\mathcal{M}) = \bigcup_{\Delta\in\mathcal{M}} \mathrm{Fix}_\Delta \subset P$$

for all subsets $\mathcal{M} \subset \mathcal{N}$. This leads to a measure on $\mathcal{N}$ coming from that on P, namely

$$\mu(\mathcal{M}) \underset{def}{=} \mu_P \,\mathrm{Fix}(\mathcal{M})\,. \qquad (*)$$

Thus Γ-spaces P can be thought of as "decorated" random quotients of Γ, where the "decoration" comes from the dynamics of the action of Γ on P, while the aspect of the action needed for our (narrow minded?) concept of randomness is reflected in the *numbers* that are the measures of the unions of (particular) subsets $\mathcal{M} \subset 2^\Gamma$ defined by $(*)$.

(b) From an abstract measure theoretic point of view there is a close link between *all* measures on 2^Γ and those on $\mathcal{N} \subset 2^\Gamma$. In fact, there is a *Borel* map from 2^Γ to $\mathcal{N}$, where every $W \subset \Gamma$ goes to the normal subgroup $\Delta = [W]$ spanned by W and so every measure descends from 2^Γ to $\mathcal{N}$. On the other hand, in probability theory, one deals (almost) exclusively with the *product* measures, i.e. *independent* random variables, and their pushforwards, i.e. random structures *parametrized* by *independent* variables where the nature of the parametrization is essential. Measure theoretically, a random group appears as the group of measurable sections of the "fibration" over a measure space where the fibers are groups, e.g. quotients of a fixed free group.

(c) If the (finite or infinite) set G of generators of Γ carries a geometric structure, being a kind of a manifold (smooth, algebraic, linear, combinatorial, etc). Then the set of relations of length is required to be compatible with this structure, i.e. to be a submanifold $W \subset G^n$. Then randomness is expressed with a probability measure on the space $\mathcal{W}$ of such W's (e.g. of algebraic subvarieties of a given degree d where one sees clearly the role of the critical dimension $\dim W = \frac{1}{2}\dim G$). Besides the space G^n and $W \subset G^n$ may carry their own probability (or algebraic genericity) structure. All this is compatible with small cancellation theory, but the latter does not (?) acquire truly new significant features in the geometric environment.

1.9 On determination of (asymptotic) geometric invariants of random groups. An *invariant*, or a *property* of groups $\Gamma = \Gamma_\alpha$ can be viewed as a real (e.g. two valued true/false) random variable on the probability space $\mathcal{A} \ni \alpha$ that is entirely determined by the combinatorics of the underlying graph (V, E). The general problem is that of evaluating the expectation, dispersion, phase transitions, etc., for these variables. Here are some of them where a satisfactory answer seems possible.

(i) Topology/geometry of $\partial_\infty \Gamma$, e.g. the dimension of Γ at infinity (Pansu's quasiconformal for hyperbolic Γ);
(ii) L_p-cohomology etc.;
(iii) Simplicial, spherical and more general norms on cohomology;
(iv) Existence/nonexistence of (many) non-free subgroups in Γ.

(v) (Geometric and/or algebraic) embeddability of Γ_α to $\Gamma_{\alpha'}$;
(vi) Something C^*-algebraic about Γ.

2 Small Cancellation

The main problem of small cancellation theory can be stated as follows. Given a group Γ and subgroups $\Gamma_j \subset \Gamma$, $j \in J$, study the quotient group Γ/Γ_*, where $\Gamma_* \subset \Gamma$ denotes the normal closure of the group generated by all Γ_j. This Γ_* can be obtained in two steps: firstly take all conjugates $\gamma\Gamma_j\gamma^{-1}$, $\gamma \in \Gamma$, thus obtaining a larger family, say Γ_i, $i \in I$, now invariant under conjugation in Γ; secondly define Γ_* as the subgroup generated by Γ_i, $i \in I$.

Traditional small cancellation theory is concerned with free groups Γ and infinite cyclic subgroups Γ_j, each generated by a word $w_j \in \Gamma$. This theory provides algebraic information on Γ/Γ_* in terms of the geometry of the family $\{\Gamma_i\} = \Gamma\{\Gamma_j\}\Gamma^{-1}$. The freedom of Γ is not indispensable: the basic notions and proofs of small cancellations extend to an arbitrary word hyperbolic group. Also, the conception of small cancellation makes sense for families of groups Γ_i acting on a general hyperbolic space X, e.g. on a group $\Gamma \supset \Gamma_i$ (see below), where the proofs of basic properties of X/Γ_* follow by a straightforward translation of the classical argument into the present language (compare §2).

Remark. The small cancellation theory for non-cyclic subgroups Γ_i does not formally reduce to the cyclic case, not even for free Γ, albeit each Γ_i can be generated by some words $w_{ij} \in \Gamma_i$ and $\Gamma/\Gamma_* = \Gamma/[w_{ij}]$. This is because the small cancellation condition(s) for Γ_i's does *not* imply that for w_{ij}'s. On the other hand the essential *arguments* of small cancellation theory readily apply to non-cyclic subgroups Γ_i thus significantly enlarging the scope of applications of this theory.

2.1 Rotation families of groups. A *rotation schema of groups* is a set of groups $\{\Gamma_i\}_{i\in I}$ with an action of each Γ_i on I and isomorphisms $e_\gamma : \Gamma_j \to \Gamma_{\gamma j}$ for all i, $j \in I$ and $\gamma \in \Gamma_i$ *compatible* with the actions of Γ_i on I in the following sense. $e_{\gamma-1} = e_\gamma^{-1}$ and $e_{\gamma\gamma'} = e_\gamma e_{\gamma'}$ for all $i \in I$ and $\gamma, \gamma' \in \Gamma_i$, $\gamma i = i$ for all $\gamma \in \Gamma_i$ and all $i \in I$ and $e_\gamma\gamma' = \gamma\gamma'\gamma^{-1}$ for all $\gamma' \in \Gamma_i$. In other words, each Γ_i fixes i and acts on itself by conjugation.

We say that Γ_i make *a rotation family* of isometrics on a metric space X if each Γ_i isometrically and faithfully acts on X, such that the isomorphisms

e_γ become conjugations in the isometry group Iso X,

$$e_\gamma\gamma' = \gamma\gamma'\gamma^{-1} \quad \text{for all } \gamma \in \Gamma_i\,,\ \gamma' \in \Gamma_j \text{ and } i,j \in I\,.$$

Besides, we include in the definition of a rotation family, a distinguished collection of subsets in X indexed by $i \in I$, say $U_i \subset X$, such that $\gamma U_j = U_{\gamma j}$ for all $i, j \in I$ and $\gamma \in \Gamma_i$.

In our applications, we usually *start* with a collection $\mathcal{U}$ of subsets $U_i \subset X$ and subgroups $\Gamma_i \subset \operatorname{Iso} X$ indexed by these subsets, where, automatically, the indexing map $I \to 2^X$ is one-to-one. If this is the case, we speak of (rotation) $\mathcal{U}$*-families* of groups denoted $\{U_i, \Gamma_i\}$ and denote by $\Gamma_{\mathcal{U}} = \Gamma_I$ the subgroup generated by all Γ_i in Iso X.

EXAMPLES. (a) Let Γ be the free group F_n with the usual action on the $(2n-1)$-array tree and w_j, $j = 1,\dots k$, be cyclically irreducible words in F_n. Each w_j defines an action of $\mathbb{Z} = \mathbb{Z}_j \subset F_n$ on X and a (unique) line U_j invariant under this action. The family (U_i, Γ_i) consists of the lines $\gamma(U_j) \subset X$, $\gamma \in F_n$, $j = 1,\dots,k$, and cyclic subgroups $\gamma\mathbb{Z}_j\gamma^{-1}$.

(b) Let $\mathcal{U}$ be a collection of hyperplanes in the hyperbolic space H^n. If $\mathcal{U}$ is invariant under the reflections $\gamma_U : H^n \to H^n$, $U \in \mathcal{U}$, then $(\mathcal{U}, \{id, \gamma_U\})$ make a rotation (reflection in this case) family.

2.2 Polyhedra $P_\rho = P_\rho(\mathcal{U})$ and $1/k$-families. Given a $\mathcal{U}$-family $\{U_i, \Gamma_i\}$ we consider the *nerve* of the collection of the ρ*-neighbourhoods* $U_i + \rho \subset X$, denoted $P_\rho(\mathcal{U})$: with the natural actions of Γ_i on $P_\rho(\mathcal{U})$. Recall that $P_\rho(\mathcal{U})$ has I for the vertex set where $i_0,\dots,i_n$ span an n-simplex if the ρ-neighbourhoods of $\mathcal{U}_i$, $\ell = 0,\dots,n$, have a *non-empty* intersection. For example, i_0 and i_1 are joined by an edge (1-simplex) iff $\operatorname{dist}(U_0, U_1) \le 2\rho$, provided X is a *geodesic* space. (A metric space is called *geodesic* if every two points can be joined by a geodesic segment of the length equal the distance between the points.)

Observe that $(I, \{\Gamma_i\})$ make an I-family on $P = P_\rho(\mathcal{U})$, where Γ_i *fixes* the vertex $i \in P_\rho$ for all $i \in I$ (and where P_ρ given the standard path metric built of the unit Euclidean simplices).

Combinatorial $1/k$-condition. This is expressed in terms of Γ_i acting on the *links* $L_i \subset P$ (by definition, L_i consists of the simplices in P with the vertices at distance one from $i \in P$): for every $i \in I$, every vertex $j \in I$ in the link $L_i \subset P$, and every $\gamma \in \Gamma_i$ different from *id* (the identity element) the shortest path of edges between j and $\gamma(j)$ in L_i has length $\ge k$, i.e. the distance between j and $\gamma(j)$ in the 1-skeleton of L_i is $\ge k$. We say that the family $\{U_i, \gamma_i\}$ is *combinatorially* $\frac{1}{k+\rho}$ if $(I, \{\Gamma_i\})$ make an

$1/k$-family on $P_\rho(\mathcal{U})$. This agrees with the traditional $1/k$-condition where $\rho = 0$ (and where, sometimes, this is called $\frac{1}{k-1}$-condition).

The *geometric* $\frac{1}{k+\rho}$-condition relates the *maximal overlap* between the ρ-neighbourhoods of U_i,

$$\mathrm{Ov}_\rho(\mathcal{U}) \overset{def}{=} \sup_{i\neq j} \mathrm{diam}(U_i + \rho) \cap (U_j + \rho)\,,$$

and the minimal displacement $\underline{\mathrm{dis}}\{\Gamma_i \mid X\backslash U_i\}$ of Γ_i on the complements of U_i, that is the supremum of the numbers d, such that

$$\big|x - \gamma(x)\big|_X \geq d$$

for all non-identity $\gamma \in \Gamma_i$, $i \in I$ and $x \in X\backslash U_i$. This geometric condition reads

$$\mathrm{dis}\big\{\Gamma_i \mid X\backslash U_i\big\} \geq k\, \mathrm{Ov}_\rho\{U_i\}\,.$$

Clearly geometric $\frac{1}{k+\rho} \Rightarrow$ combinatorial $\frac{1}{k+\rho}$.

2.3 1/6-theorem. *Let $\{U_i, \Gamma_i\}_{i\in I}$ be a $\mathcal{U}$-family of groups acting on a geodesic δ-hyperbolic metric space X, where the subsets $U_i \subset X$, $i \in I$, are σ-convex. (A subset U in a geodesic metric space is called* σ-convex *if every minimizing geodesic segment with the end points in U is contained in the σ-neighbourhood $U + \sigma$.) If the combinatorial $\frac{1}{6+\rho}$-condition is satisfied with $\rho \geq \sigma + 10^3\delta$, then*

- *(i) The rotation schema $(I, \{\Gamma_i\})$ is free: there exists a subset $J \in I$, such that Γ_i are freely independent and the family $(I, \{\Gamma_i\})$ is isomorphic to the family of the conjugates of the Γ_j's.*
- *(ii) The family $\{U_i, \Gamma_i\}$ is injective: the obvious map $U_i/\Gamma_i \to X/\Gamma_{\mathcal{U}}$ is injective for all $i \in I$.*
- *(iii) Denote by $\overline{P}^1_\rho \subset P_\rho$ the polyhedron with the same vertices and edges (1-simplices) as P_ρ and where, by definition, a finite set of vertices spans a simplex iff every two vertices in this subset are joined by an edge in P_ρ. If the subsets U_i cover X, i.e. $\bigcup_{i\in I} U_i = X$, then the quotient polyhedron $\overline{P}^1_\rho/\Gamma_{\mathcal{U}}$ is contractible. (This, for all $\Gamma_i = \{id\}$, generalizes I. Rips' contractibility theorem.)*

About the proof. The classical 1/6-theorem reduces to the above for X being a tree with cyclic subgroups Γ_i acting on X, each admitting an invariant line $U_i \subset X$. In fact, the traditional argument, when rephrased in the present language, immediately extends to the general case via approximation of hyperbolic spaces by trees as in [De]. Since we do not use the full 1/6-theorem as stated above, we omit details and pass to the more relevant

1/7-theorem. *Let the family $\{U_i\Gamma_i\}$ be combinatorially $\frac{1}{7+\rho}$* (this is stronger than the above $\frac{1}{6+\rho}$). *If U_i cover X, then the $\Gamma_{\mathcal{U}}$-quotient of the 1-skeleton P^1_ρ, that is $P^1_\rho/\Gamma_{\mathcal{U}'}$ is δ_1-hyperbolic for $\delta_1 \le 10^4$.*

Actually, this is still too refined for the needs of the present paper. The following coarse corollary appealing to the *geometric* $1/k$-condition under extra assumptions suffices.

Denote by $\underline{\mathrm{dis}}\{\Gamma_i \mid X\}$ the infimum of the displacements by all *non-identity* $\gamma \in \Gamma_i,\ i \in I$, that is the triple infimum

$$\inf_{i\in I}\inf_{\gamma\in\Gamma_i}\inf_{x\in X}\left|x-\gamma(x)\right|,$$

where $\gamma \ne id$ in Γ_i. Clearly

$$\mathrm{dis}\{\Gamma_i \mid X\} \ge \mathrm{dis}\{\Gamma_i \mid X\backslash U_i\}\,.$$

$1/10^6$-COROLLARY. *Let $\underline{\mathrm{dis}}\{\Gamma_i \mid X\} \ge 10^6\delta$, the quotients $U_i+\rho/\Gamma_i$ are bounded with $\sup_{i\in I}\mathrm{Diam}(U_i/\Gamma_i) \le \mathcal{D}$ and $\mathrm{Ov}_\rho(\mathcal{U}) \le 10^{-6}\underline{\mathrm{dis}}\{\Gamma_i \mid X\}$ for $\rho \ge \sigma + 10^6\delta$. Then the group $\Gamma_{\mathcal{U}}$ has minimal displacement $\underline{\mathrm{dis}}(\Gamma \mid X) \ge 10^{-6}\underline{\mathrm{dis}}\{\Gamma_i \mid X\}$, the quotient space $X/\Gamma_{\mathcal{U}}$ is unbounded δ_1-hyperbolic for $\delta_1 \le 10^6\,\mathcal{D}$, while $\overline{P}^1_\rho/\Gamma_{\mathcal{U}}$ is contractible.*

Let us indicate the proof of this, independent of the 1/6 and 1/7-theorems (from which the above follows immediately).

If X is a tree, the above follows from the unfolded small cancellation theorem from [G9] with a few additions mimicking point by point the classical case. The general case follows from that by appealing to approximation of hyperbolic spaces by trees. This delivers all the needed (hyperbolic) inequalities for the quotient X/Γ (and the actions of Γ_i on X) involving arbitrarily large finite subsets in X, whose cardinalities are fixed in advance, and then the hyperbolic Cartan–Hadamard theorem delivers the global hyperbolicity.

An alternative approach consists in applying the Dehn diagram techniques to the skeleton $\overline{P}^2_\rho$ with growing ρ, where one fills-in circles in $\overline{P}^2_\rho$ by disks in $\overline{P}^2_{\rho'}$ for some ρ' (slightly) greater than ρ and where one returns from $\overline{P}^2_{\rho'}$ to $\overline{P}^2_\rho$ by subdividing and retracting. This works for disks with a given bound on the area and then the length/area criterion applies.

All this is standard (see [G5], [BrH]).

REMARKS. (a) Our main application is where $X=\Gamma$ is a word hyperbolic group and $\Gamma_i \subset \Gamma$ are quasiconvex subgroups. Here we (sometimes tacitly) assume that Γ is *faithfully* hyperbolic, i.e. the action of Γ on the ideal boundary $\partial_\infty\Gamma$ is faithful. In this case one sees easily, that $\Gamma/\Gamma_{\mathcal{U}}$ is also *faithfully* hyperbolic.

(b) Our assumptions in the above theorems imply that the intersections $(U_i+\rho)\cap(U_j+\rho)$ are *bounded* for all $\rho\geq 0$ and $i\neq j$. However, this is not the case, for hyperbolic reflection groups, for example, where the reflection hyperplanes in H^n typically have unbounded intersections for $n\geq 3$. The above small cancellation theorems can be generalized to allow unbounded intersections along the following lines. Define *rotation families of depth d* by induction as follows: for $d=1$ it is what we had before where, in essence, the overlaps are bounded. Then the depth $d+1$ signifies that for each i_0, the intersections, $(U_i+\rho)\cap(U_{i_0}+\rho)$, $i\neq i_0$, and the corresponding groups make a rotation family of depth d, where U_{i_0} takes the place of X (and where details are filled in by recording what one observes for hyperbolic reflection groups, for instance).

(c) The small cancellation techniques rely not so much on the hyperbolicity of X itself but rather on that of the polyhedron $\overline{P}^1_\rho$, and in some cases this is directly available for non-hyperbolic X, e.g. for some CAT(0)-spaces X, where curvature is *strictly* negative on large parts of X.

(d) Ultimately, *all* small cancellation conditions are expressed by inequalities imposed on distances certain points in X and their translates by some $\gamma\in\Gamma_i$. If one is consistent, one does not even need to assume X geodesic and U_i quasiconvex, though this can often be achieved by passing to *Gibbs' completion* of X (see below). However, such passage is undesirable if one cares (we do not at all in this paper) for the best constants in the theorems.

(e) The classical theory harbours more combinatorial geometry than just a metric, e.g. a careful look at the 1/6-groups, distinguishes *two* metric-like structures associated to shortest paths of edges and paths of 2-cells. A generalization of this leads to better constants in the process of adding relations, relevant, for example, for evaluating the minimal Burnside exponent. (I. Rips recently suggested a new approach, as was privately communicated to me by M. Sapir.)

(f) *Gibbs' max-completions.* Depart from the (*isometric!*) *Kuratowski embedding* of a metric space X to the space $\mathbb{R}^X$ of functions $d:X\to\mathbb{R}$ defined by $x\longmapsto d_x(y)\overset{def}{=}|x-y|$, where $\operatorname{dist}(d_1(y),d_2(y))\overset{def}{=}\sup_{y\in X}(d_1(y)-d_2(y))$. Furthermore, the corresponding map $X\to\mathbb{R}^X/\mathbb{R}=\{\text{const}\}$, where one factors away constant functions, remains isometric.

For every bounded function $c(x)$ on X, let

$$m\,c\,d(y) \overset{def}{=} \sup_{x\in X} \big(d_x(y) + c(x)\big) \in \mathbb{R}^X .$$

Take the union of these *mcd*'s for all bounded $c(x)$ and let $X^\vee \subset \mathbb{R}^X/\{\text{const}\}$ be the image of this union under the projection $\mathbb{R}^X \to \mathbb{R}^X/\{\text{const}\}$. This $X^\vee$ is a geodesic metric space for the sup-metric induced from $\mathbb{R}^X/\{\text{const}\}$ via $(\mathbb{R}^X, |\quad|_{\sup})$, where moreover, there is a *distinguished* geodesic between every two points: $d_1(x)$ and $d_2(x)$ are joined by the path

$$t \mapsto \max\nolimits_{1,2} \big(d_1(x)+t, d_2 + r - t\big)/\{\text{const}\}$$

for $r = |d_1(x) - d_2(x)|_{\sup}$ and $t \in [0,r]$.

This leads to a fully-fledged notion of convexity and convex hulls in $X^\vee$, where the isometry group $\operatorname{Iso} X$ naturally acts on $X^\vee$ preserving distinguished geodesics. This allows equivariant convex hulls in $X^\vee$ for subsets $Y \subset X \subset X^\vee$. Next one shows by an elementary argument:

If X is δ-hyperbolic then $X^\vee$ is 6δ-hyperbolic.

This improves upon [G5] (where one has 12 instead of 6) and is similar to [BS] (where there is no constant at all but no equivariance either).

Example. If X equals the set of leaves of a finite tree T, then, canonically, $X^\vee = T$, and the above convex hull of a subset in $X^\vee$ equals the minimal *subtree* containing this subset.

(g) **$\perp$-convexity**. The geodesic σ-convexity (as defined in [G5]) used in small cancellation theory is not quite satisfactory as the intersections of quasiconvex subsets are *not*, in general, quasiconvex. This can be remedied by passing to $X^\vee$ or, alternatively, as follows. Let $E_{x_1,x_2}(x)$ denote the demi-excess function in a metric space X,

$$E_{x_1,x_2}(x) = \tfrac{1}{2}\big(|x-x_1| + |x-x_2| - |x_1-x_2|\big)\,,$$

and define $\perp_\sigma^\rho\{x_1,x_2\} \subset X$ by the two conditions

$$E_{x_1,x_2}(x) \le \rho \qquad (\perp^\rho)$$

and

$$\max\big(E_{x,x_1}(x_2), E_{x,x_2}(x_1)\big) \ge \sigma \qquad (\perp_\sigma)\,.$$

A subset $Y \subset X$ is *$\perp_\sigma^\rho$-convex* if $\perp_\sigma^\rho\{y_1,y_2\} \subset Y$ for all $y_1,\ y_2 \in Y$.

This notion essentially agrees with the geodesic σ-convexity for σ-hyperbolic spaces, allows convex hulls, and does not need to make a ρ-neighbourhood to work with, provided $\rho \gg \delta$, where one uses the following simple inequality for the quintuple of points $\{x_1, x_1, x_2, y_1, y_2\}$ in

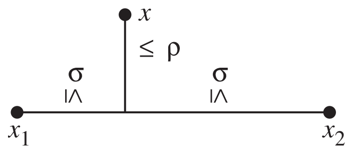

Figure 1

Say that $f(x)$ is $\backslash\!\_/_\sigma$-convex if $f(x) \le f^{\backslash\_/}_{x_1,x_2}(x) + \sigma$ for all $x_1, x_2 \in X$.

The demi-excess function $E_{y_1,y_2}(x)$ is $\backslash\!\_/_{3\delta}$-convex.

This is proven by a straightforward (and boring) computation.

Eventually, one needs this and/or similar notions to formulate constant-wise sharp(er) versions of the $\frac{1}{k+\varepsilon}$-theorems.

(h) **Axial hulls.** In some case there is a preferred choice of (quasiconvex) subsets U_i invariant under Γ_i. In general, let Γ isometrically act on a δ-hyperbolic space X, denote by $\Gamma_{ax} \subset \Gamma$ the subset of the *axial* (i.e. acting freely and properly on X with exactly two fixed points on $\partial_\infty X$) isometries and consider the union of their "axes"

$$\{\underline{di}_\gamma \le 10\delta\} \overset{def}{=} \big\{x \in X \;\big|\; |x - \gamma(x)| \le 10\delta\big\}$$

over all $\gamma \in \Gamma_{ax}$, denoted

$$Ax_\Gamma \overset{def}{=} \{\underline{di}_{\Gamma_{ax}} \le 10\delta\} = \bigcup_{\gamma \in \Gamma_{ax}} \{\underline{di}_\gamma \le 10\delta\} \subset X\,.$$

If X is geodesic δ-hyperbolic then $U = Ax_\Gamma$ is 30δ-convex for each Γ isometrically acting on Γ.

In fact, if γ_1 and γ_2 are axial, then so are also $\gamma_1^m\gamma_2^n$ for suitable m, n with large $|m|$ and $|n|$ and each segment between their axes L_{γ_1} and L_{γ_2} can be approximated by segments on L_γ for $\gamma = \gamma_1^{\pm m}\gamma_2^{\pm n}$ (see Fig. 2 and compare [G5]).

(h′) It may happen that L_{γ_1} and L_{γ_2} keep close as they go to infinity in some (or both) direction(s) and then $\gamma_1^m\gamma_2^n$ is not necessarily axial; but then $L_{\gamma_1} \cup L_{\gamma_2}$ is 30δ-convex by itself, see Fig. 3.

(Here γ is axial for $\gamma = \gamma_1^{-m}\gamma_2^n$ but not necessarily for $\gamma_1^m\gamma_2^n$.)

(h″) The above is most relevant for Γ_i, where all $\gamma \in \Gamma_i$ are axial (that is usually assumed in the traditional small cancellation theory) and one can use ordinary Dehn diagrams "terminating" on the axes of γ's.

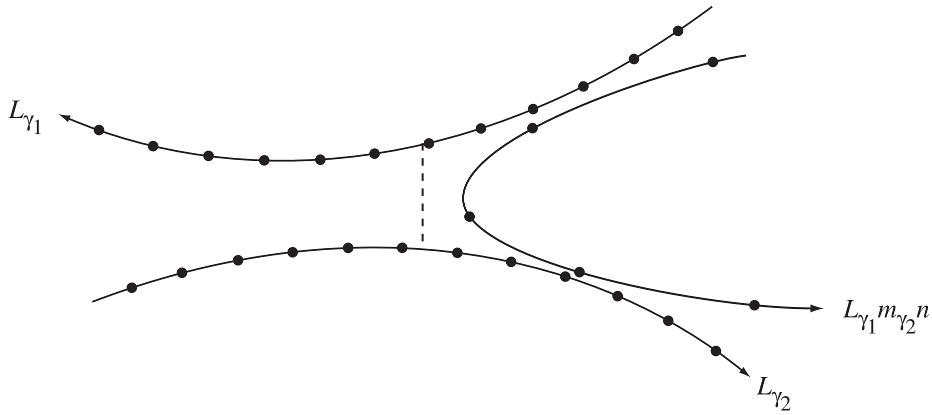

Figure 2

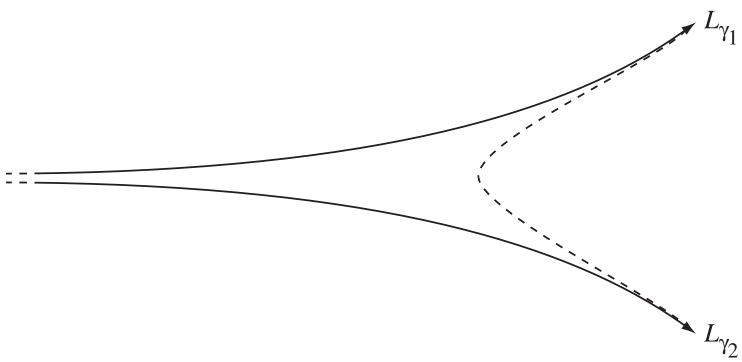

Figure 3

3 Diffusion and Contraction

A *random walk* on a countable space X is defined as a map from X to the space of probability measure on X, denoted $x \mapsto \mu(x \to)$, where $\mu(x \to y) \overset{def}{=} \mu(x \to)(y)$ represents the probability of a (single) step from x to y. Given two walks μ and μ', one defines their composition (convolution) $\mu = \mu * \mu'$ by

$$\mu(x \to z) = \sum_{y \in X} \mu(x \to y) \mu'(y \to z)$$

and abbreviates

$$\mu^n \overset{def}{=} \underbrace{\mu * \mu * \ldots * \mu}_{n} .$$

Thus $\mu^n(x \to y)$ stands for the probability of reaching y from x in n independent μ-steps.

REMARK. A random walk can be viewed as a morphism in a randomized category of (countable in the present context) sets. Recall, that a *random(ized) category* has each set $\mathrm{Hom}(X \to Y)$ equipped with a probability measure say $\tilde{\mu} = \tilde{\mu}(X,Y)$, where the composition map $\mathrm{Hom}(X \to Y) \times \mathrm{Hom}(Y \to Z) \to \mathrm{Hom}(X \to Z)$ pushes forward $\tilde{\mu}(X,Y) \times \tilde{\mu}(Y,Z)$ to $\tilde{\mu}(Y,Z)$. A random walk μ on X obviously defines a (rather special) random map $\tilde{\mu} : X \to X$ for $\tilde{\mu} = \times_{x \in X} \mu(x \to)$.

A random walk μ is called *$\underline{\nu}$-symmetric* for a measure $\underline{\nu}$ on X, or $\underline{\nu}$ is called a *stationary measure* for μ if

$$\underline{\nu}(x)\mu(x \to y) = \underline{\nu}(y)\mu(y \to x)$$

for all $x, y \in X$, where plain *"symmetric"* refers to the measure $\underline{\nu}$ assigning $\underline{\nu}(x) = 1$ for all $x \in X$.

3.1 Examples. (a) Every measure ν on $X \times X$ with $\nu(x \times X) < \infty$, $x \in X$, defines a random walk $\mu = \mu_\nu$ on X by

$$\mu(x \to y) = \big(\nu(x \times X)\big)^{-1} \nu(x,y)\,.$$

If ν is *balanced*, i.e. if

$$\nu(x \times X) = \nu(X \times x),\ \ x \in X\,, \tag{$*$}$$

then μ_ν is $\underline{\nu}_*$-symmetric for the *pushforward* $\underline{\nu}_*$ of ν under the projection $X \times X \to X$, where, by definition

$$\underline{\nu}_*(x) = \nu(x \times X) = \mu(X \times x)\,.$$

Clearly, every *symmetric* ν, where

$$\nu(x,y) = \nu(y,x), \quad x, y \in X\,,$$

is balanced.

(b) If (V,E) is a graph with at most finitely many edges with given ends v_1 and v_2 in V (e.g. $\overset{\leftrightarrow}{E} \subset V \times V$) then the *graph measure* $\nu = \nu_E$ on $V \times V$, assigns, by definition, this number $\mathrm{card}(\mathrm{edges}(v_1 \to v_2)) = \mathrm{card}(\mathrm{edges}(v_2 \to v_1))$ to each pair $(v_1, v_2) \in V \times V$ (if $\overset{\leftrightarrow}{E} \subset V \times V$ this is given by the characteristic function of the subset $\overset{\leftrightarrow}{E}$). The corresponding *standard* random walk μ on V is defined if the degrees of all (vertices) v in V are finite and this random walk is $\underline{\nu}_*$-symmetric for $\underline{\nu}_*(v) = \deg(v)$. In particular, if Γ is a group with a finite *symmetric* generating set $G \subset \Gamma$, then the *standard random walk* on Γ associated to the Cayley graph (Γ, E) of Γ is symmetric, since the corresponding measure $\underline{\nu}_*$ on Γ is Γ-invariant.

(c) **Measurable graphs.** If V is a space with a geometric structure where certain subspaces $E \subset V \times V$ come with natural measures, then every such E gives rise to a random walk on V. Basic examples of such V are *symplectic* (especially *algebraic Kähler*) manifolds and homogeneous spaces. The simplest instance of this is the disjoint union V of the projective plane P with its dual, where the edges correspond to incidences between points and lines in P.

Diffusion and convolution. If one assigns the δ-measures $\delta(x)$ to each $x \in X$, and interprets $\mu(x \to)$ as the map from δ-measures to probability measures on X for $\delta(x) \mapsto \mu(x \to)$, then one can linearly extend this to a map from measures to measures, denoted $\nu \mapsto \mu * \nu$, where the composition of two such maps, called *diffusions*, say μ and μ' is also denoted $\mu * \mu'$. If μ and μ' are Γ-invariant diffusions on a group Γ this corresponds to the convolution of measures, $\mu(id \to) * \mu'(id \to)$ on Γ. The convolution powers of a given μ, that are $\mu^n = \underbrace{\mu * \mu * \ldots * \mu}_{n}$, make a $\mathbb{Z}_+$-semigroup. One often encounters $\mathbb{R}_+$-semigroups μ^t, $t \in [0, \infty]$, where $\mu^{t_1+t_2} = \mu^{t_1} * \mu^{t_2}$, such as the *Gaussian diffusion* on (finite and infinite dimensional) linear spaces and the Riemannian diffusion on Riemannian manifolds.

There is a natural *Cartesian product* in the category of diffusion spaces allowing *infinite countable* products. This agrees with the Cartesian (Pythagorian) products of Riemannian manifolds. In the case of infinitely many factors, $X = X_1 \times X_2 \times \ldots \times X_i \times \ldots$ the product metric is infinite almost everywhere and X "foliates" into "leaves", where the metric is finite on each leaf. The product diffusion *does not* preserve leaves but, for the rest, behaves as the ordinary Riemannian diffusion. Similar product diffusion exists (and has been extensively studied) on *adelic* spaces.

The category of diffusion spaces is poor in morphism: one needs a surjective (Borel) map $f : X \to \underline{X}$ where the pushforward of the measure $\mu(x \to)$ is constant on the pull-backs $f^{-1}(\underline{x}) \ni x$, $\underline{x} \in X$. A basic example is a Γ-*invariant diffusion* for Γ acting on (X, μ) that descends to a diffusion on $\underline{X} = X/\Gamma$. More generally, if each fiber $f^{-1}(\underline{x})$ comes with a probability measure, then every $\mu(x \to)$ on X averages to $\underline{\mu}(\underline{x} \to)$ on $\underline{X}$, but this, in general, is *not* a semigroup homomorphism. An instance of this "averaged morphism" $\mu \to \underline{\mu}$ is associated with the canonical (partition) measures on $f^{-1}(\underline{x})$ induced by a μ-stationary measure ν on X.

3.2 Codiffusion. A *codiffusion* on a (complete metrizable) space Y is, by definition, a map c from the space Σ of probability measures on Y back to Y, such that each δ-measure $\delta(y)$ goes to y and the pull-back $c^{-1}(y) \in \Sigma$ is *convex* for every point $y \in Y$.

EXAMPLES. (a) Every affine (e.g. linear) space Y, such as $Y = \mathbb{R}$, comes along with the natural affine codiffusion, called the *center of mass*

$$c : \tilde{\sigma} \mapsto \int_Y y\tilde{\sigma}dy\,.$$

(b) Let Y be a metric space embedded into the space $\mathbb{R}^Y$ of functions $Y \to \mathbb{R}$ by

$$y' \mapsto |y - y'|^2\,.$$

The affine codiffusion on $\mathbb{R}^Y$ composed with this map sends each measure σ on Y to the function

$$|\sigma - y|^2 \overset{def}{=} \int_Y |y - y'|^2 \sigma dy'\,.$$

If this function has a *unique* minimum point $y_{\min} \in Y$, this is called the (Riemannian) *center of mass* of σ and denoted

$$c(\sigma) \overset{def}{=} \int_Y y\sigma dy \overset{def}{=} y_{\min}\,.$$

A sufficient condition for the uniqueness of $y_{\min}$ is the *strict convexity* of the functions $|y - y'|^2$ in the variable y on the geodesics segments in Y where Y should be a *geodesic* metric space in this case.

For example, if Y is a CAT(0)-space, then *the second derivative of* $|Y - Y'|^2$ *along each geodesic* is ≥ 2 by the definition of CAT(0). In other words the *difference* $|y - y_1|^2 - |y - y_2|^2$ *is convex on each segment in* Y *containing* y_2 (and concave on the segments containing y_1). Consequently, *the function* $|\sigma - y|^2$ *has the same 2-convexity property*: $\frac{d^2|\sigma - y|^2}{d^2 y} \geq 2$ *on all geodesics in* Y. This convexity on the segments $[c(\sigma), y]$ yields *the* (Wirtinger type) *inequality*

$$\big|\sigma - c(\sigma)\big|^2 \leq \tfrac{1}{2} \iint\limits_{Y \times Y} |y - y'|^2 \sigma \times \sigma\, dydy' \qquad (\times)$$

for all probability measures σ *on* CAT(0)*-spaces* Y*.*

REMARK. If Y is Hilbertian then $(\times)$ becomes an equality while for general metric spaces one has

$$\big|\sigma - c(\sigma)\big| \geq \tfrac{1}{4} \iint |y - y'|^2 \sigma \times \sigma dydy'$$

by the triangle inequality.

***L_2*-contraction by *c*.** Given two measures σ, $\sigma' \in \Sigma$, consider the measures $\tilde{\sigma}$ on $Y \times Y$, such that the projections to the first and the second factor send $\tilde{\sigma}$ to σ and σ' correspondingly and define the *L_2-metric* on Σ (*Monge–Kantorovich*) *as*

$$\|\sigma - \sigma'\|_{L_2} \overset{def}{=} \inf_{\tilde{\sigma}} \left(|y - y'|_Y^2 \tilde{\sigma} dy dy'\right)^{1/2}.$$

If Y is CAT(0), then, by the 2-convexity, the map $c : \Sigma \to Y$ is *distance decreasing*. Moreover, if $c(\sigma) = c(\sigma')$, then the measures σ and σ' are *parallel* in Y in the following sense: there exists a measure isomorphism $(Y, \sigma) \leftrightarrow (Y, \sigma')$, such that for almost all $y_1, y_2 \in Y$ the quadruple $(y_1 \leftrightarrow y_1', y_2 \leftrightarrow y_2')$ is isometric to a Euclidean parallelogram.

Parametrized contraction. Let P be an abstract probability space (isomorphic to $[0,1]$) and Φ denote the space of Borel maps $\varphi : P \to Y$ with the L_2-metric

$$\|\varphi_1 - \varphi_2\|_{L_2} \overset{def}{=} \left(\int_P \left|\varphi_1(p) - \varphi_2(p)\right|_Y^2 dp\right)^{1/2}.$$

If Y is CAT(κ) for a given $\kappa \in (-\infty, \infty)$ then Φ is also CAT(κ) and for $\kappa \le 0$ the codiffusion map $c_* : \Phi \to Y$, for $c_* = c\left(\varphi_*(dp)\right)$, is *distance decreasing*. Furthermore, the group of measurable automorphisms isometrically acts on Φ and $\Phi/\operatorname{Aut} = \Sigma$

REMARK. The space Σ is *not* CAT(κ). But, on the other hand, if Y is an *Alexandrov's* space with curvature $\ge k$, then Σ also has curvature $\ge k$.

WARNING. The Riemannian codiffusion (unlike the affine one) is non-commutative and does not satisfy the Fubini theorem, unless Y is flat. This means that the product measures $\sigma = \sigma_1 \times \sigma_2$ on Y (and maps of product measure space into Y) do not necessarily satisfy the relations

$$\textstyle\iint y\sigma = \int \sigma_1 \int y\sigma_2 \qquad \text{(Fubini)}$$
$$\textstyle\int \sigma_1 \int y\sigma_2 = \int \sigma_2 \int y\sigma_1 \qquad \text{(commutativity)}$$

(where the failure of commutativity can be evaluated by the curvature of Y).

Cartesian products. A Cartesian product of codiffusion spaces carries a natural product codiffusion structure. In particular, the space of maps $A \to Y$ comes with a natural codiffusion induced from Y. This agrees with the Pythagorian product of metric spaces for the center of mass codiffusion.

3.3 Heat operator(s), Laplacian and harmonicity. Given $X = (X, \mu)$ and $Y = (Y, c)$ define the *heat operators* $H^{\overset{\varepsilon}{\to}}$, $\varepsilon \in [0,1]$, acting on maps $f : X \to Y$ by

$$H^{\overset{\varepsilon}{\to}} f(x) = c\big(f_*(\varepsilon\mu(x \to) + (1-\varepsilon)\delta(x))\big)$$

and observe that $H^{\overset{0}{\to}} = id$ while $H \overset{def}{=} H^{\overset{1}{\to}} = cf_*\mu$, where this μ is understood as the map from X to measures on X for $x \mapsto \mu(x \to)$. (For an affine codiffusion, $H^{\overset{\varepsilon}{\to}} = \varepsilon H + (1-\varepsilon)id$). A map f is called *harmonic* if $Hf = f_i$ this is (obviously) equivalent to $H^{\overset{\varepsilon}{\to}}(f) = f$ for *all* $\varepsilon \in [0,1]$.

If Y is a smooth manifold, then the μ-Laplacian $\Delta f = \Delta_\mu f$ is the vector field,

$$\Delta f \overset{def}{=} \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon}\big(f - H^{\overset{\varepsilon}{\to}}(f)\big)\,,$$

(provided the limit exists and the difference between maps is taken in local coordinates). In general, $\Delta f(x)$ takes values in the (suitably defined) "tangent cone" $\mathrm{Con}_{f(x)} Y$. If Y comes with a metric we set

$$\big|\Delta f(x)\big| \overset{def}{=} \liminf_{\varepsilon \to 0} \varepsilon^{-1}\big|f(x) - H^{\overset{\varepsilon}{\to}} f(x)\big|_Y\,.$$

If Y is CAT(0), then the 2-convexity of the squared distance functions implies (for the center of mass codiffusion) that

$$\big|\Delta f(x)\big| \geq \varepsilon^{-1}\big|f(x) - H^{\overset{\varepsilon}{\to}} f(x)\big|_Y\,.$$

Thus, the above "lim inf″ can be replaced by plain "lim" and harmonicity is equivalent to $|\Delta f| = 0$.

REMARKS. (a) Besides the length, a "tangent vector" $\delta \in \mathrm{Con}_{y_0} Y$ can be assigned its "scalar products" with geodesic segments $[y_0, y_1] \subset Y$ issuing from y_0 defined via the derivatives (variations) of the squared distance functions $y \mapsto |y - y_1|^2$ at $y = y_0$ along δ; alternatively, one can use the variation of the length of δ for y_0 sliding along $[y_0, y_1]$. If Y smooth, or more generally has curvature bounded from above, e.g. Y is CAT(0), then the two scalar products coincide by a theorem of Alexandrov that easily follows from the 2-convexity of the distance in the CAT(0) case.

(b) In some cases, e.g. on manifolds with (somewhere) positive curvature, the codiffusion c is defined on *a subset* of measures on Y. This limits the definition of $H^{\overset{\varepsilon}{\to}}$ to the maps with $f_*(\varepsilon\mu(x \to) + (1+\varepsilon)\delta(x))$ in the domain of c. Yet this suffices for the definition of the heat flow on all smooth maps between Riemannian manifolds, for example, since the diffusion μ^t on X is employed in the limit $t \to 0$.

3.4 Growth inequalities. If Y is a Hilbert space, then for every probability measure σ on Y and each vector $y_1 \in Y$ one has

$$\int \|y\|^2 \sigma dy = \|y_1\|^2 + \int \|y - y_1\|^2 \sigma dy + 2\int \langle y_1, y - y_1\rangle \sigma dy\,.$$

In particular, if $y_1 = c(\nu) \overset{def}{=} \int y\sigma dy$, then,

$$\int \|y\|^2 = \|y_1\|^2 + \int \|y - y_1\|^2 \sigma dy\,.$$

These relations extend to *inequalities* for CAT(0)-spaces, where we adopt the following notation. Given points $x_0, y_1,\ y \in Y$, parametrize the geodesic segment $[y_0, y_1]$ isometrically by $t \in [0, |y_0 - y_1|]$ and define the "scalar product" $\langle y_1 - y_0, y - y_1\rangle$ as the minus one half of the t-derivative of the function $|y - y_1(t)|^2$ at $t = 1$,

$$\langle y_1 - y_0,\ y - y_1\rangle \overset{def}{=} -\tfrac{1}{2}\tfrac{d}{dt}\big|y - y_1(t)\big|^2_{t=1}$$

(that agrees with what happens in Hilbert spaces Y). Then, by the 2-convexity of $|y(t) - \sigma|^2$, one has

$$\int |y_0 - y|^2 \sigma dy \ge |y_0 - y_1|^2 + \int |y_1 - y|^2 \sigma dy + 2\int \langle y_1 - y_0,\ y - y_1\rangle \sigma dy\,. \quad (\star)$$

It follows that every map $f : X \to Y$ satisfies, for arbitrary diffusion μ and probability measure $\mu_\bullet$ on X and each $y_0 \in Y_1$,

$$\begin{aligned}\int_X \big|y_0 - f(x)\big|^2_Y \mu * \mu_\bullet dx \ge &\int_X \big|y_0 - f(x)\big|^2_Y \mu_\bullet dx \qquad (\star)_*\\ &+ \int_X \mu_\bullet dx_1 \int_X \big|f(x_1) - f(x)\big|^2_y \mu(x_1 \to x)dx\\ &\quad + 2\int_X \mu_\bullet dx_1 \int_X \big\langle f(x_1) - y_0, f(x) - f(x_1)\big\rangle \mu(x_1 \to x)dx\,,\end{aligned}$$

where $(\star)_*$ becomes an *equality* for Hilbert spaces Y. In the latter case

$$\int_X \big\langle f(x_1) - y_0, f(x) - f(x_1)\big\rangle \mu(x_1 \to x)dx = \big\langle f(x_1) - y_0, \Delta f(x_1)\big\rangle$$

for the Hilbertian scalar product and we use this identity as the *definition* of $\langle f(x_1) - y_0, \Delta_\varepsilon(f(x_1)\rangle$ for non-Hilbertian Y, where Δ_ε refers to $1 - H^{\overset{\varepsilon}{\to}}$. Namely, we set $\mu(x_1 \overset{\varepsilon}{\to} x) = (1-\varepsilon)\delta(x_1) + \varepsilon\mu(x_1 \to x)$ for the δ-measure $\delta(x_1)$ and let

$$\big\langle f(x_1) - y_0, \Delta_\varepsilon f(x_1)\big\rangle \overset{def}{=} \int_X \big\langle f(x_1) - y_0, f(x) - f(x_1)\big\rangle \mu(x_1 \overset{\varepsilon}{\to} x)dx\,.$$

For smooth (e.g. Hilbertian) spaces X, the equality $|\Delta_\varepsilon f(x_1)| \overset{def}{=} |f(x_1) - H^{\overset{\varepsilon}{\to}} f(x_1)| = 0$ implies that $\langle f(x_1) - y_0, \Delta_\varepsilon f(x_1)\rangle = 0$, while for (possibly singular) CAT(0)-spaces (as well as for all CAT($\kappa > +\infty$) for this matter) one (obviously) has the *inequality*

$$\big\langle f(x_1) - y_0, \Delta_\varepsilon f(x_1)\big\rangle \geq 0$$

for all μ-harmonic maps f and all $y_0 \in Y$. Therefore, μ-harmonic maps into CAT(0)-spaces satisfy the following

Harmonic growth inequality.

$$\int_X \big|y_0 - f(x)\big|_Y^2 \mu * \mu_\bullet dx \geq \int_X \big(|y_0 - f(x)|_Y^2 + |df|_\mu^2(x)\big)\mu_\bullet dx\,, \qquad (*)$$

where

$$|df|_\mu^2(x) \overset{def}{=} \int \big|f(x) - f(x')\big|^2 \mu(x \to x')dx'$$

and $\mu_\bullet$ is an arbitrary measure on X.

Corollaries.

(a) $\int_X |y_0 - f(x)|_Y^2 \mu * \mu_\bullet dx$
$\geq \int_X (|c(\mu_\bullet) - f(x)|_Y^2 + |df|_\mu^2(x))\mu_\bullet dx + |y_0 - \mu_\bullet|^2,$

(b) The measures $\mu_0^n = \underbrace{\mu * \mu * \ldots * \mu}_{n}(x_0 \to x)$ *satisfy*

$$\int_X \big|y_0 - f(x)\big|_Y^2 \mu_0^n dx \geq \big|y_0 - f(x_0)\big|^2 + \int_X (\mu_0^0 + \mu_0^1 + \ldots + \mu_0^{n-1})\|df\|_\mu^2(x)dx\,, \quad (+)_n$$

where μ_0^0 is the δ-measure $\delta(x_0)$ (and where $(+)_n$ becomes an equality for Hilbertian Y).

(c) $\int_X \big|c(\mu_0^n) - f(x)\big|^2 \mu_0^n dx \geq n \inf_{x\in X} |df|_\mu^2(x) + \sum_{i=2}^n \big|c(\mu_0^i) - c(\mu_0^{i-1})\big|^2.$

3.4.A Continuous remarks. (a) Let μ^t, $t \in (0,\infty)$ be a *diffusion semigroup* on X, i.e. $\mu^{t_1+t_2} = \mu^{t_1} * \mu^{t_2}$ with $\mu^0(x \to) = \delta(x)$, and set

$$|\Delta f| \overset{def}{=} \liminf_{t\to 0}\, t^{-1}|f - H^t f|\,.$$

Then $|\Delta f| = 0$ implies that

$$\frac{d}{dt}\int_X \big|y_0 - f(x)\big|_Y^2 \mu^t(x_0 \to x)dx \geq \int_X |df|^2(x)\mu^t(x_0 \to x)dx \qquad (+)$$

for all CAT(0) spaces Y, and for $y(t) = c_*(\mu^t(x_0{\to}x)) \overset{def}{=} c(f_*(\mu^t(x_0{\to}x)))$ one has

$$\tfrac{d}{dt}\int_X \big|y(t) - f(x)\big|_Y^2 \mu^t(x_0 \to x)dx \geq \big|\tfrac{d}{dt}y(t)\big|^2 + \int_X |df|^2(x)\mu^t(x_0 \to x)dx\,, \qquad (+)'$$

where

$$|df|^2(x) \overset{def}{=} \limsup_{t\to 0} t^{-1} \int_X \big|f(x)-f(x')\big|_Y^2 \mu^t(x\to x')dx'$$

and

$$\big|\tfrac{d}{dt}y(t)\big| \overset{def}{=} \limsup_{t\to 0} \varepsilon^{-1}\big|y(t+\varepsilon)-y(t)\big|_Y .$$

(b) The above is useful to confront with the *Poincaré inequalities* (compare 3.9 and [L]) for $(X,\, \mu_0^t=\mu^t(x_0\to))$. For example, *if* $X=\mathbb{R}^n$ *with the standard* (*Gaussian*) *diffusion, then*

$$\tfrac12 \iint\limits_{X\times X} \big|f(x)-f(x')\big|_Y^2 \mu_0^t\times\mu_0^t dxdx' \le t\int\limits_X |df|^2(x)\mu_0^t dt \qquad (P)$$

for maps $f:\mathbb{R}^n\to Y$ *into an arbitrary metric space* Y, *where the equality holds iff* f *is a composition of an affine map* $\mathbb{R}^n\to\mathbb{R}^n$ *followed by an isometric map* $\mathbb{R}^n\to Y$. (This reduces to the obvious case $X=\mathbb{R}^1$ by integrating along the lines in $\mathbb{R}^n$.) Therefore, *if* Y *is* CAT(0), *then*

$$\int_X \big|y(t)-f(x)\big|_Y^2\mu^t(x_0\to x)\le t\int_X |df|^2(x)\mu(x_0\to x)dx\,. \qquad (P_0)$$

It follows from $(+)'$ with $y_0=x_0$, that the function $\varphi(t)=t^{-1}\int_X|f(x)-f(x_0)|^2\mu_0 t\,dx$ is monotone increasing in t and hence $|df|^2(x_0)\overset{def}{=}\limsup_{t\to0}\varphi(t)\le\varphi(t)$ for all $t>0$. Thus we arrive at the *Lipschitz* property of harmonic maps of $\mathbb{R}^n$ into CAT(0)-spaces.

In fact, this is known (see [GS]) for arbitrary *smooth Riemannian* manifolds X, where one can employ a similar argument, since the Poincaré inequality (obviously) remains valid in the following form:

$$\tfrac12 \iint\limits_{X\times X} \big|f(x)-f(x')\big|_Y^2 \mu_0^t\times\mu_0^t dxdx' \le t\big(1+\varepsilon(t)\big)\int\limits_X |df|^2(x)\mu_0^t dt \qquad (P_\varepsilon)$$

for some $\varepsilon(t)=\varepsilon(t;X,x_0)\underset{t\to0}{\longrightarrow}0$, where, moreover, $\int_0^{t_0}\varepsilon(t)t^{-1}dt<\infty$ (and where the actual asymptotic behaviour of $\varepsilon(t)$ for $t\to0$ can be fully accounted for by the curvature tensor of X at x_0).

(c) By letting $t\to0$ in (P_0) one concludes to the subharmonicity of the function $|df|^2(x)$ on $X=\mathbb{R}^n$

$$|df|^2(x_0)\le\big(H^t(|df|^2)\big)(x_0)$$

for all $x_0\in X$ and $t\ge0$ and for the heat (flow) operators the H^t acting on functions $\mathbb{R}^n=X\to\mathbb{R}$.

(d) The definition of Δ for μ^t only needs a codiffusion on Y defined in the "infinitesimal neighbourhoods" of the points $y\in Y$, given, for example, by a (symmetric) *affine connection* on a *smooth* Y.

3.5 Smoothing by heat. Let $f_1 = H(f_0)$ for $f_0 : X \to Y$ and write down the (obvious) generalization of $(*)$,

$$\int_X \left|y_0 - f_0(x)\right|_Y^2 \mu * \mu_\bullet dx \ge \int_X \left(|y_0 - f_1(x)|_Y^2 + |d' f_0|_\mu^2(x)\right)\mu_\bullet dx\,, \qquad (*)_1$$

where

$$|d' f_0|_\mu^2(x) \overset{def}{=} \int \left|f_1(x) - f_0(x')\right|_Y^2 \mu(x \to x')dx'\,.$$

This holds true for all CAT(0)-spaces Y and it becomes an *equality* for Hilbertian spaces.

Turning to the continuous (diffusion) case, set $f_t = H^t(f_0),\ t \in [0,T]$, and observe with $(\star)_*$ the following

Parabolic growth inequality.

$$\tfrac{d}{dt}\int_X \left|y_0 - f_t(x)\right|_Y^2 \mu^{T-t}(x_0 \to x)dx \ge \int_X |df_t|^2(x)\mu^{T-t}(x_0 \to x)dx \quad (+)_T$$

for all $t \le T$, where, again, this becomes an *equality* for Hilbertian Y.

Similarly, for $y(t) = c((f_t)_*(\mu^{T-t}(x_0 \to x)))$, we have

$$\tfrac{d}{dt}\int_X \left|y(t) - f_t(x)\right|_Y^2 \mu^{T-t} \ge \left|\tfrac{d}{dt}y(t)\right|^2 + \int_X |df_t|^2(x)\mu^{T-t}dx\,, \qquad (+)'_T$$

for $\mu^{T-t} = \mu^{T-t}(x_0 \to x)$, where $y(t)$ is constant in t for Hilbertian Y and $(+)'_T$ becomes an *equality*. This, combined with (P_0), implies for $X = \mathbb{R}^n$ that the function $\varphi(t) = t^{-1}\int_X |f_t(x) - y(t)|^2 \mu^{T-t}dx$ is *monotone increasing* in t; in particular

$$|df_T|^2(x_0) \le \int \left|f_0(x) - f_0(x_0)\right|^2 \mu^T dx\,.$$

Furthermore, if X is an arbitrary Riemannian manifold, then the maps f_t are Lipschitz for all $T > 0$, where the implied Lipschitz constants are controlled by $|df_0|_{\mu^T}^2$ and Poincaré's $\varepsilon(T)$ from (P_ε).

REMARK. The smoothing effect of the *initial* heat flow (obviously) remains valid for (possibly singular) and/or infinite dimensional spaces with the curvature bounded from above, $K(Y) \le \kappa < \infty$ (e.g. for CAT($\kappa < \infty$)-spaces), where the Lipschitz bound at x_0 depends on the maximal δ, such that the ε-ball in$X$ around x_0 is sent by f to the ball of radius $r \le \frac{\pi}{2}\sqrt{\kappa} - \delta$ and Poincaré's $\varepsilon(t)$ for $\mu(x_0 \to x)$ in X.

3.5.A Smoothing kernel. Given two probability measures ν_1 and ν_2 on X we denote by $|\nu_1 - \nu_2|_{L_1}$ the infimum of the sums of the total masses of measures δ_+ and δ_-, such that $\nu_1 + \delta_- = \nu_2 + \delta_+$. Then, for every map $f : X \to Y$, the L_2-distance between the pushforwards of these ν_i in Y is obviously bounded by

$$\big\| f_*(\nu_1) - f_*(\nu_2) \big\|_{L_2} \leq \iint\limits_{X\times X} \big\| f(x) - f(x') \big\| \delta_+^2(x)\delta_-(x)dxdx' \,,$$

that is, in turn, bounded by

$$\tfrac{1}{2}|\nu_1 - \nu_2|_{L_1} \sup_{x,x'} \big\| f(x) - f(x') \big\|^2$$

for x, x' running over the supports of ν_1 and ν_2. If Y is CAT(0), then, one gets the same bound for the squared distance $|c(f_*(\nu_1)) - c(f_*(\nu_2))|_Y^2$ by the contraction property of c. Thus the map $H_\mu f$ is Lipschitz if the corresponding diffusion $x \mapsto \mu(x \to)$ is Lipschitz for the L_1-metric is the space $\Sigma(X)$ of measures on X defined above and the image of (the relevant part of) f is bounded in Y.

Remarks. (a) The smoothing effect of H is used in conjunction with bounds on $|f - Hf|_Y$ that are derived from the Poincaré inequalities for the spaces $(X,\ \mu(x \to))$, $x \in X$.

(b) The above bound on $\|f_*(\nu) - f_*(\nu_2)\|_{L_2}$ can be sharpened for "infinitesimally close" measures ν_1 and ν_2. In fact let ν_s be a 1-parametric family of probability measures that is differentiable in s for the L_1-metric in $\Sigma(X)$ and denote by $\nu'_s = \frac{d}{ds}\nu'_s \in \Sigma'(X)$ its derivative in the linear span $\Sigma'(X) \supset \Sigma(X)$. Assume for the moment that X is a compact Riemannian manifold and the measures ν_1 and ν' have continuous densities with respect to the Riemannian dx, written $\nu_s(x)dx$ and $\nu'_s(x)dx$, where $\int_X \nu'_s(x)dx = 0$. Let $h_s(x)$ be the solution of the Laplace equation $\Delta h = \nu'_s$ and observe that the gradient of h provides the L_2-optimal transport of ν_s to the infinitesimally close measure ν_{s+r}. Thus the derivative of ν_s in the L_2-space of measures $\Sigma(X)$ is bounded by

$$\|\nu'_s\|_{L_2}^2 \leq \int_X \|\operatorname{grad} h_s\|^2 dx = \int_X \nu'_s(x)h_s(x)dx \leq \lambda_1^{-1} \int \big|\nu'_s(x)\big|^2 dx$$

for the first eigenvalue λ_1 of the Laplacian on X. Next, for a map $f : X \to Y$, the L_2-derivative of $f_*(\nu'_s)$ in $\Sigma(Y)$ is bounded by

$$\int_X \big\| d_f(\operatorname{grad} h) \big\|_Y^2 dx \leq \lambda_1^{-1} \|d_f\|_{L_2} \|\nu'_s\|_{L_2} \,.$$

Therefore, given a smooth function $\nu(x, x')$ on X the L_2-norm of the differential of the map

$$f_*\nu : X \to \Sigma(Y) \text{ for } x \mapsto \nu(x, x')dx \overset{f_*}{\to} \sigma \in \Sigma(Y)$$

is bounded by

$$\|df_*\nu\|_{L_2}(x) \le \lambda_1^{-1}\|d_f\|_{L_2}\left(\int_X \left\|d_x\nu(x,x')\right\|^2 dx'\right)^{1/2} \qquad \binom{*}{*}$$

where, observe $d_x\nu(x,x') = d_{x}, \nu(x,x')$ for symmetric ν.

This $\binom{*}{*}$ makes sense for maps f of an arbitrary diffusion space (X,μ^t) to a metric space Y, where the notations should be understood as follows.

"dx" is a stationary measure for μ^t,

λ_1 refers to the Laplacian Δ related to the operators H^t on $L_2(X,dx)$ by $H^t = \exp -t\Delta$,

the L_2-norms of the differentials are taken with $\mu^t,\ t\to 0$, as earlier.

With these conventions, $(*)$ automatically extends to all diffusion spaces $(X,\mu t)$ mapped into metric spaces Y. Furthermore, if Y is CAT(0), then the same estimate applies to $H_\mu f$ since $H_\mu = c(f_x\mu)$ and

$$\|dH_\mu f\|^2 \le \|df_*\mu\|^2$$

by the contraction property of c.

(c) The inequality $\binom{*}{*}$ can be (sometimes) improved by replacing the measure "dx'" by "$\nu(x,x')dx'$" and modifying the definition of λ_1 and $\|df\|_{L_2}$ accordingly.

3.6 Harmonic maps and harmonic stability. Take a map $f_0 : X \to Y$ and let $f_i = H^{\overset{\varepsilon}{\to}}(f_{i-1})$ for $i = 1,2,\dots$ and some ε in the interval $0<\varepsilon<1$, e.g. $\varepsilon = 1/2$. If Y is CAT(0), then

$$\sup_{x\in X}\left|\Delta_\varepsilon f_i(x)\right| \le \sup_{x\in X}\left|\Delta_\varepsilon f_{i-1}(x)\right| \qquad \binom{**}{*}$$

according to 3.2, where the equality in $\binom{**}{*}$ is indicative of "approximate parallelism" of the f_{i-1} and f_i on the support of the measure $\mu(x\to)$ at some x. (This is also true and especially clear for *uniformly* convex Banach spaces Y with affine codiffusion.) If the operator $H^{\overset{\varepsilon}{\to}}$ *eventually contracts in Δ_ε-direction*, i.e. if

$$\sup_{x\in X}\left|\Delta_\varepsilon f_{i+j}(x)\right| \le \lambda_j \sup_{x\in X}\left|\Delta_\varepsilon f_i(x)\right|,$$

where $\lambda_j\to 0$ for $j\to\infty$, we regard f_0 as (harmonically) *stable.* If f_0 is stable and Y is complete (as a metric space) then, obviously, the maps f_i converge to a harmonic map $f : X \to Y$ lying within finite distance from f_0, i.e.

$$\sup_{x\in X}\left|f(x) - f_0(x)\right| < \infty\,.$$

If f_0 is *not* stable, it still may give rise to a *non-constant* harmonic map f, if not between spaces X and Y themselves but between some *limit spaces* X_∞ and Y_∞, where $f = f_\infty = \lim_{i\to\infty}(f_i : X_i \to Y_i)$ for $X_i = (X, x_i)$ and $Y_i = d_i^{-1}Y$ for suitable reference points $x_i \in X$ and scaling factors d_i^{-1}. To be specific, we limit ourselves to the case where X is a discrete metric space, where the supports of the measures $\mu(x \to)$ have bounded cardinalities and diameters uniformly in $x \in X$, and where Y is either a CAT(0)-space or a *uniformly* convex Banach space. We start with a map $f_0 : X \to Y$, where $\sup_{x\in X} |\Delta_\varepsilon f(x)| < \infty$ and then the failure of stability necessarily makes some maps f_i and f_{i+j} almost parallel on some balls $B(x_i, r_i) \subset X$ with $r_i \to \infty$ for $i, j \to \infty$. Rescale the spaces Y by d_i^{-1} for $d_i = |df(x_i)|_\mu$ (provided $d_i \neq 0$), and assume that the maps $f_i : (X, x_i) \to d_i^{-1}Y_i$ have $|df_i(x)|_\mu \leq \varphi(|x - x_i|_X)$ on the balls $B(x, r_i)$ for some function $\varphi(r)$ (independent of i and x_i). Then we can pass to the limit (or at least a sublimit) $f_\infty : X_\infty \to Y_\infty$ where Y_∞ remains CAT(0) being a Hausdorff (ultra) limit of CAT(0)-spaces. (If Y is a Banach space it does not change under scaling limits.) It is clear that $H^{\overset{\varepsilon}{\to}}$ is parallel to f_∞. If $H^{\overset{\varepsilon}{\to}} f_\infty = f_\infty$ this f_∞ is the promised harmonic map; otherwise we factor away Y (or rather the convex hull of $f_\infty(X_\infty) \subset Y$) by the direction $\Delta_\varepsilon f_\infty$ and end up with a harmonic map $\underline{f}_\infty$ of X_∞ into the resulting quotient of Y_∞. This $\underline{f}_\infty$ is non-constant unless f_∞ lies in a geodesic line in Y. (A map with parallel Δ_ε may land in a line, such as a horofunction mapping the hyperbolic space to $\mathbb{R}$, for example.)

Remark. In the continuous case of a semigroup μ^t one makes $f_i = H^{t_i} f_{i-1}$ for a sequence of t_i slowly converging to zero, e.g. for $t_i = i^{-1}$ and conclude to the existence of harmonic maps in a similar way.

3.7 L_2-estimates in Γ-spaces. Let X be a Γ-space, i.e. X is acted upon by a group Γ, and ν be a Γ-invariant measure on X for which the action is a.e. *free* and *proper*, thus descending to a (unique!) measure $\underline{\nu}$ on the quotient space $\underline{X} = X/\Gamma$. Then for every Γ-invariant real function φ on X one denotes by $\underline{\varphi} = \varphi : \Gamma$ the corresponding function on X/Γ and sets

$$\|\underline{\varphi}\| \overset{def}{=} \|\varphi : \Gamma\| \overset{def}{=} \|\varphi : \Gamma\|_\nu \overset{def}{=} \|\varphi : \Gamma\|_{L_2^\nu} \overset{def}{=} \left(\int_{\underline{X}} \underline{\varphi}^2(\underline{x})\underline{\nu} d\underline{x}\right)^{1/2}$$
$$= \left(\int_D \varphi^2(x)\nu dx\right)^{1/2}$$

where $D \in X$ is a fundamental domain for Γ. If Γ is a finite group freely acting on Γ, then, obviously, $\|\varphi : \Gamma\|^2 = \|\varphi\|^2/\operatorname{card}\Gamma$ that justifies the above notation for infinite Γ. This notation will also be used for functions on $X\times X$ for the *diagonal* action of Γ, where the relevant invariant measures on $X\times X$ are not necessarily of the form $\nu\times\nu$ for ν on X. In particular, if Y is a metric Γ-space one defines, with a given property Γ-invariant measure ν on $X \times X$, the "norm"

$$\|\underline{df}\| \overset{def}{=} \|\,|df|\,\| \overset{def}{=} \|df : \Gamma\|^{\nu}_{L_2^{\nu}}\,,$$

where

$$\big|df(x_1,x_2)\big| \overset{def}{=} \big|f(x_1)-f(x_2)\big|_Y\,,$$

and for two such maps $f_1,\ f_2 : X \to Y$ one has the L_2-*distance*

$$\|f_1 - f_2\|_{L_2} = \big\||f_1(x)-f_2(x)|_Y : \Gamma\big\|_{L_2^{\nu}}\,.$$

This distance may be infinite for some f_1 and f_2: the space of Γ-invariant maps $X \to Y$ is "foliated" into "leaves", where this distance is finite. If Y is CAT(0), then each such "leaf" is also CAT(0) and if Y is Hilbertian then so are the "leaves".

Remark on non-free actions. If an action of Γ on (X,ν) is proper but not a.e. free, one may pass to another Γ-space $(\tilde X,\tilde\nu)$, where the action is free as well as proper and such that there exists an equivariant measure preserving map $p : \tilde X \to X$ where the pull-back $p^{-1}(x) \in \tilde X$ is identified with the isotropy subgroup $\Gamma_x \subset \Gamma$ for $x \in X$. Then the definition of the above norm can be delegated to $\tilde X$.

3.7.A Laplacian as the gradient of the energy. Given a metric space Y with the center of mass codiffusion (possibly partially defined) and X with a random walk μ one defines the μ-*energy* $E(f)$ of an f as $\frac14\|\,|df|_\mu\|^2_{L_2^{\nu}}\nu$ for a μ-stationary measure ν on X. If X and Y are Γ-spaces and f is equivariant, then

$$E(f) = E_\mu(f) \overset{def}{=} \tfrac14\big\||df|_\mu : \Gamma\big\|^2_{L_2^{\nu}}\,.$$

If Y is a Hilbert space, then

$$|df(x)|^2_\mu = \big\langle df(x), df(x)\big\rangle_\mu\,,$$

where

$$\big\langle df(x), dg(x)\big\rangle_\mu \overset{def}{=} \int \big\langle f(x)-f(x'), g(x)-g(x')\big\rangle \mu(x\to x')dx'\,.$$

Furthermore, if $\Gamma = \{id\}$ (or, more generally, Γ is finite) then one has the usual identity

$$\tfrac12\big\|\langle df(x), dg(x)\rangle_\mu^{1/2}\big\|^2_{L_2^{\nu}} = \big\langle \Delta_1 f(x), g(x)\big\rangle_{L_2^{\nu}} \overset{def}{=} \int \big\langle \Delta_1 f(x), g(x)\big\rangle_Y \nu dx\,.$$

It follows, that $\varepsilon^{-1}\Delta_\varepsilon$ for all $\varepsilon > 0$ equals the gradient of the energy (function) with respect to the Hilbertian metric structure on the space of maps $f : X \to Y$ given by $\langle f, g\rangle = \int \langle f(x), g(x)\rangle_Y \nu dx$.

All of the above obviously generalize to arbitrary (infinite!) group Γ where $f : X \to Y$ is an equivariant map and g is a *difference* of two such maps. Thus,

$$\big\| \, |d(f+g)|_\mu : \Gamma \big\|^2_\nu = \big\| \, |df|_\mu : \Gamma \big\|^2_\nu + 4 \big\| \langle \Delta_1 f, \ g\rangle^{1/2}_Y : \Gamma \big\|^2_\nu + \big\| \, |dg|_\mu : \Gamma \big\|_\nu$$

and for $g = -\Delta_\varepsilon f$ we obtain

$$E(H^{\overset{\varepsilon}{\to}}(f)) = E(f) - \big\| |\langle \Delta_1 f, \Delta_\varepsilon f\rangle^{1/2}_Y : \Gamma \big\|_\nu + \tfrac{1}{2} \big\| \langle \Delta_\varepsilon f, \Delta^2_\varepsilon f\rangle_Y : \Gamma \big\|_\nu .$$

Since $\Delta_\varepsilon = \varepsilon \Delta_1$ for *Hilbertian* Y, we see that the energy on an orbit of the $-\Delta_1$-flow, say f_t, satisfies

$$\tfrac{d}{dt} E(f_t) = - \big\| \|\Delta_1 f_t\|_Y : \Gamma \big\|^2_\nu .$$

This remains (obviously) valid for arbitrary smooth Riemannian manifolds Y with $\Delta = \lim_{\varepsilon \to 0} \varepsilon^{-1} \Delta_\varepsilon$ substituted for Δ_1 while for CAT(0) spaces one easily sees with 2-convexity (or just using Wir_4 from [G3]) that

$$\tfrac{d}{dt} E(f_t) \le - \big\| |\Delta f_t| : \Gamma \big\|^2_\nu .$$

Thus harmonic maps are identified as minima of the energy in this case. (We shall encounter later on the non-isometric action of Γ on Y where this interpretation of harmonicity is not appropriate.)

3.7.A′. Let us make the above more transparent by looking at the space F of the Γ-invariant maps $f : X \to Y$ within bounded L_2-distance from some $f_0 : X \to Y$,

$$F = \big\{ f : X \to Y \mid \ \| |f(x) - f(y)_Y : \Gamma \|_{L_2} < \infty \big\} .$$

If Y is CAT(0) then F is also CAT(0) and the energy function E on F is convex. Denote by $F^\varepsilon \subset F$ the (convex!) subspace of maps f with $E(f) \le \varepsilon^2$ and then, for all $f \in F$, consider the squared distance (function) from f to F^ε denoted $|f - F^\varepsilon|^2$. Let $\varepsilon_{\mathrm{inf}}$ be the infimum of $\sqrt{E}$ on F and $|f - F^{\mathrm{inf}}|^2$ stand for $\sup |f - F^\varepsilon|^2$ over $\varepsilon > \varepsilon_{\mathrm{inf}}$.

If $F \underset{def}{\overset{inf}{=}} F^{\varepsilon_{inf}}$ is non-empty, i.e. if there exists a harmonic Γ-equivariant map $X \to Y$, then this $F^{\varepsilon_{inf}} \subset F$ is convex, provided Y is CAT(0), and therefore there exists a unique normal projection $P : F \to F^{\varepsilon_{inf}}$. If Y is Hilbertian, then $P(f) = \lim_{t\to\infty} H^t f$ but this is not so in general. (The flow H^t essentially corresponds to $-\operatorname{grad} |f - F^\varepsilon|^2$ for $\varepsilon = E(f) - \delta$ with an infinitesimal $\delta > 0$.) Furthermore the subset $F^{\mathrm{inf}} = F^{\mathrm{inf}}_\mu \subset F$ may change for non-Hilbertian Y if we replace μ by a convolution power μ^n, where, in particular, one is concerned with the (sub)limit of $F^{\mathrm{inf}}_{\mu^n}$ for $n \to \infty$.

On harmonic stability. Let us indicate a (well known and *not* most general) criterion for harmonic stability of a Γ-equivariant map $X \to Y$ in terms of the action of Γ on Y.

DEFINITIONS. Call (Y,Γ) (compactly) *stable* if for every function $\delta(\gamma) \geq 0$, $\gamma \in \Gamma$ the subset $Y_\delta \subset Y$ defined by the *displacement inequalities*

$$d_\gamma(y) \stackrel{def}{=} \big|y - \gamma(y)\big|_Y \leq \delta(\gamma)\,, \quad \gamma \in \Gamma \backslash \{id\}\,,$$

is *compact* mod Γ, i.e. its image in Y/Γ is compact.

Say that (Y,Γ) is *semistable* if there exists a closed Γ-invariant geodesically convex subset $Y_0 \subset Y$ that admits an isometric Γ-equivariant splitting $Y_0 = Y_1 \times Y_2 \times \ldots \times Y_{k'}$ such that the action of Γ on each Y_i, $i = 1, \ldots, h$, is stable (one may allow splittings into *infinitely* many Y_i and splittings into *semi*stable Y_i).

EXAMPLES. (a) If $\Gamma = \{id\}$ then (Y,Γ) is (tautologically) stable.

(b) If the action of Γ on Y is discrete c(Γ)-5.7(com)-6.4(5 54.3510.)-347.0 Tm5 54.3ein, tri,(

3.7.B Remarks. (a) **Parabolic and quasi-parabolic (non)-stability.** If f_0 is unstable, and the maps $f_t = H^t f$ do not converge for $t \to \infty$, one faces two possibilities.

(i) **Parabolic case.** Here $E(f_t) \to 0$ for $t \to \infty$ and by the Lipschitz property all displacements go to zero as well: $d_\gamma(f_t(x)) \to 0$ for each $x \in X$ and $t \to \infty$.

(ii) **Quasiparabolicity.** This signifies that $E(f_t) \underset{t\to\infty}{\longrightarrow} E_{\inf} \neq 0$.

In both cases the maps f_t are "asymptotically harmonic", i.e. the L_2-norm of the Laplacian $\to 0$. Moreover, since $E'(f_t) = -\Delta f_t$, the Laplacian decays faster than the energy,

$$\big\| \, |\Delta f_t| : \Gamma \big\|^2_{L_2} / E(f_t) \to 0\,, \qquad t \to \infty\,.$$

In the case (i) the Lipschitz constant of f_t goes to zero, i.e. f_t is "asymptotically constant", while in the case (ii) f_t converges to a *non-constant* harmonic map in the (some kind of) ideal boundary of Y, obtained by taking the Hausdorff (ultra) limit of the marked metric spaces $(Y, f_t(x_0))$, $t \to \infty$.

(b) The role of the smoothing is to erase irrelevant local analytical problems and bring into focus the essential features: the (local) geometry of singularities in Y and of the action of Γ on Y seen at infinity of Y/Γ. This analysis (apart from the quantitative characteristics of the smoothing) has little to do with the sign of the curvature of Y and trivially generalizes to maps between (almost) arbitrary metric spaces. To save the space, state it in the case of maps between compact Riemannian manifolds $V \to W$ (where V plays the role of X/Γ and W of Y/Γ).

Smoothing lemma (responding to a question by S. Kuksin). *There exists a function $C(E) = C(E,p,V,W) < \infty$ for $E \geq 0$, $p \geq 1$, such that every continuous map $f_0 : V \to W$ with finite L_p-energy $E_p(f_0) \overset{def}{=} \int_V \|df_0(v)\|^p dv$ admits a homotopy $\overline{f} : V \times [0,1] \to W$ of finite L_p-energy, where the map $f_1(v) = \overline{f}(v,1)$ is C^1-smooth and*

$$E_p(\overline{f}) + \sup_{v\in V} \big\| df_1(v) \big\| \leq C(E_p(f_0))\,.$$

Furthermore, given two homotopic maps f_0 and f_0', there exists a homotopy $\tilde f$ between them with $E_p(\tilde f) \leq C(\tilde E_p(f_0) + \tilde E_p(f_0'))$.

The (standard) proof follows by induction on skeletons of some triangulation of $\overline{V} = V \times [0,1]$. The important dimension is the maximal $k \leq \dim V$, where E_p controls the continuity modulus of functions on k-dimensional spaces, i.e. the maximal $k < p$ (except $p = 1$, where $k = 1$).

One uses a generic (depending on f) triangulation of $V_0 = V \times 0 = V$, where the L_p-energy is bounded on the k-skeleton,

$$E_p(f_0 \mid V^k) = \int_{V^k} df(v)dv \le c_{p,V} E_p(f_0)\,.$$

Since the continuity modulus of $f_0 \mid V_0^k$ is bounded by $\mathrm{const}_V\, E_p(f_0)$, the existence of the required $\overline{f}$ on $\overline{V}^{k+1}$ follows by Ascoli's compactness theorem (where the "compactness constant" depends on W as well as V), while the extension to the $(m+1)$-simplices for $m \ge k$ is obtained with the radial projections of their interiors to the boundaries: these have bounded L_p-energies. (Every contractible map φ of the unit m-sphere into W extends to a continuous map $\overline{\varphi}$ of the unit $(m+1)$-ball, such that $E_p(\overline{\varphi}) \le c_{p,m} E_p(\varphi)$, provided $p \le m+1 \ge 2$.)

Problems. Express $C(E, p\ V, W)$ in terms of explicit (geometric and topological) invariants of V and W (compare [G8]).

3.7.C Integrated growth inequalities. Since convolution of random walks (measures) $\mu * \mu'$ induces composition of the heat kernels, denoted $H_* \overset{def}{=} H_{\mu*\mu'} = HH'$, the corresponding Laplacians satisfy

$$\langle \Delta_* f, f\rangle = \big\langle (\Delta + \Delta')f, f\big\rangle - \langle \Delta f, \Delta' f\rangle$$

whenever the scalar products are defined (and where non-commutativity of composition of operators is compensated by their symmetry). Therefore, if μ and μ' are both Γ-invariant and ν serves as a stationary measure for μ and μ', then the $\mu * \mu'$-energy $E_*(f)$, for Hilbert space valued f, satisfies

$$E_*(f) = E(f) + E'(f) - \tfrac{1}{2}\big\| \langle \Delta f, \Delta' f\rangle_Y^{1/2} : \Gamma \big\|_\nu^2 .$$

If f is either μ or μ'-harmonic, then the energy is additive under convolution of measures. In particular,

$$E_{\mu^n}(f) = nE_\mu(f)\,, \quad n = 1, 2, \dots \qquad (*)_n$$

for harmonic maps f, and $(*)_n$ is *equivalent* to the harmonicity of f. This is most visible for $n = 2^m$, since

$$E_{\mu^n}(f) = nE_\mu(f) - \tfrac{1}{2} \sum_{i=0}^{m-1} \big\| \|\Delta_{(i)} f\|_Y : \Gamma \big\|_\nu^2\,,$$

where (i) stands for μ^{2^i}. This generalizes as earlier to CAT(0) spaces Y, where

$$E_{\mu^n}(f) \ge nE_\mu(f)$$

for harmonic Γ-equivariant maps $f : X \to Y$ and

$$E_{\mu^n}(f) \ge nE_\mu(f) - \tfrac{1}{2} \sum_{i=0}^{m-1} \big\| |\Delta_{(i)} f| : \Gamma \big\|_\nu^2$$

for non-harmonic f. Notice in passing that Alexandrov's spaces Y with $K \geq 0$ satisfy the opposite inequality, $E_{\mu^n}(f) \leq nE_\mu(f)$ for all Γ-equivariant $f : X \to Y$.

3.7.D Radial growth. Let us apply the above to the free group F_k with the standard random walk μ and where the role of μ' is played by the measure μ_n supported on the sphere of radius n around $id \in F_k$ and equally distributed on this sphere. Since

$$\mu * \mu_n = (2k)^{-1}\big(\mu_{n-1} + (2k-1)\mu_{n+1}\big)$$

for $n \geq 1$ and

$$\mu * \mu_0 = \mu = \mu_1$$

we conclude that the energy $E'_n = E_{\mu_n}$ satisfies

$$E'_{n+1}(f) \overset{def}{=} E_{n+1}(f) - E_n(f) = \frac{1}{2k-1}E'_n(f) + \frac{2k}{2k-1}E_1(f)$$

and thus

$$(E'_{n+1} - E_\bullet) = \frac{1}{2k-1}(E'_n - E_\bullet)$$

for $E_\bullet = \frac{2k}{2k-2}E_1$ and each $k \geq 2$. In particular, for $n \to \infty$, one has

$$E_n(f) = \frac{2nk}{2k-2}E_1(f) + \mathcal{O}(1)\,.$$

In other words, $E_n(f)$ is asymptotic to $E_{\mu^m}(f)$ for $n = \frac{2k-2}{2k}m$ that agrees with the *law of large numbers* saying that most of the measure $\mu_n(id \to)$ is contained near the sphere of radius $r = \frac{n(2k-2)}{2k}$, where "near" refers to the band of width $\mathcal{O}(\sqrt{r})$ (compare [Wo]).

3.7.D$'$ Smooth case. Let X be a Riemannian manifold, where the energy of a smooth map f is defined by integrating $\frac{1}{2}\|df\|^2 \overset{def}{=} \frac{1}{2}\mathrm{trace}(df)^* df$. Assume that the r-spheres around $x_0 \in X$ have constant mean curvature $m(r)$ and observe that the (positive) Laplacian of a radial function $\varphi(x) = \varphi(r(x))$ equals

$$\Delta\varphi = -\varphi''(r) - m(r)\varphi'(r)\,.$$

On the other hand, every smooth Hilbert space valued map $f : X \to \mathbb{R}^\infty$ satisfies,

$$\Delta\|f\|^2 = 2\langle\Delta f, f\rangle - 2\|df\|^2.$$

Therefore, if f is harmonic ($\Delta f = 0$) and $\|f(x)\|$ depends only on $r = r(x)$ then the average of $\|df\|^2$ over the r-sphere, that is $2E(r)$, satisfies,

$$4E(r) = \varphi''(r) + m(r)\varphi'(r)$$

for $\varphi(r) = \|f(r)\|^2$, hence, if $m(r) \underset{r\to\infty}{\longrightarrow} m > 0$, and $E(r) \to E > 0$, then this $\varphi(r)$ is asymptotic to $4m^{-1}Er$. For example if f is *Riemannian isometric* as well as harmonic, ten $\varphi(r)$ is asymptotic to $(2m^{-1}\dim X)r$, since $E(r) = \frac{1}{2}\dim X$. On the other hand, if X is Riemannian homogeneous and f is *equivariant isometric* (harmonic or non-harmonic) then $\varphi(r) = \|f\|^2$ grows *no faster* than $(2m^{-1}\dim X)r$ by the earlier discussion. ("Equivariant" is essential. For example, if $K(X) \leq 0$ and the inverse exponential map $f_0 : X \to T_{x_0}(X) = \mathbb{R}^d$, $d = \dim X$, is contracting, then f_0 can be approximated by an isometric C^1-embedding f in the ambient $\mathbb{R}^{d+1}$ by the Nash–Kuiper theorem. Here $\|f\|^2$ grows quadratically in r. If one wishes to smooth such an f, one needs $\mathbb{R}^N \supset \mathbb{R}^{d+1}$ with $N = \frac{d(d+1)}{2} + 2d + 3$, see [G7]).

EXAMPLES. (a) If X is a k-dimensional hyperbolic space (of constant curvature -1) then $m = k - 1$ and every *equivariant* (and thus isometric) map $f : X \to \mathbb{R}^\infty$ has

$$\|f\|^2(r) \lesssim \frac{2kr}{k-1}, \text{ for } r \to \infty$$

with the *asymptotic equality* for *harmonic* maps f.

(b) If X is complex hyperbolic (with $-1/4 \geq K(X) \geq -1$) of dimension $2k$, then $m = k$ and harmonic isometric maps f have

$$\|f\|^2(r) \sim 4r\,.$$

It follows that such an equivariant f is necessarily pluri-harmonic, i.e. harmonic on all complex lines in X (as is well known).

(c) If X is quaternion hyperbolic of dimension $4k$, then $m = 2k + 1$ and harmonic isometric f have $\|f\|^2(r) \sim (\frac{8k}{2k+1})r$. Since $\frac{8k}{2k+1} > \frac{8}{3}$ for $k > 1$, this growth is faster than that on quaternion lines in X and thus *no equivariant isometric* map $f : X \to \mathbb{R}^\infty$ is harmonic for $k \geq 2$. This (trivially) implies (see below) *Kazhdan's T-property* for the isometry group $Sp(k,1)$ of such an X.

One can give a *lower bound* on the *mean curvature* of a *local* (or infinitesimal) isometric (not necessarily equivariant) embedding f of an X to $\mathbb{R}^\infty$ by just writing down the *Gauss formula* expressing the Riemannian curvature of X in terms of the second fundamental form of f. Similarly, one obtains such a bound for all irreducible symmetric spaces of non-positive curvature with the usual exceptions of $H^k_{\mathbb{R}}$ and $H^{2k}_{\mathbb{C}}$, thus establishing Kazhdan's property for the corresponding Lie groups.

REMARKS. (i) Non-existence of equivariant harmonic (minimal) maps $f : X \to \mathbb{R}^\infty$ can be detected by looking at the integrals of $\|f\|^2$ over the spheres of a *given* radius r (or by integrating $\|f\|^2$ against measures $\mu^r(x_0 \to\ ))$. If $r \to 0$, one proceeds by using the Gauss formula (which amounts in this case to the Bochner formula on 1-forms) while for $r \to \infty$, one finds oneself in the asymptotic framework of Furstenberg–Mostow–Margulis (super)rigidity theory. These approaches lead to equivalent conclusions for (equivariant maps of) *symmetric* spaces X (to $\mathbb{R}^\infty$ and more general CAT(0)-spaces), but the corresponding inequalities for different r lead to non-equivalent generalization for *non-symmetric* spaces X.

(ii) Most symmetric spaces X admit no isometric harmonic maps to $\mathbb{R}^\infty$ but each such X admits an equivariant (and isometric) harmonic (and thus minimal) map f to the Hilbert sphere S^∞. One may obtain a harmonic map f by starting with a suitable isometric action of Iso X on S^∞ (i.e. unitary representation) and then apply the heat flow H^t in S^∞ to an orbit f_0 of such an action. For example, if one departs from the obvious action of Iso X on $L_2(X)$ and apply the heat flow to the orbit of a (spherical) function on X invariant under the isometries fixing a point $x_0 \in X$, one ends up, in the limit for $t \to \infty$, with a minimal orbit in S^∞ for another action, namely the one corresponding to the unitary representation in the space of 1/2-densities on the Furstenberg boundary of X.

Besides the sphere S^∞, one may use actions on other infinite dimensional (symmetric) spaces S and identify minimal orbits in S with an emphasis on $\mathbb{R}$-*mass minimizing* ones (in the space of equivariant current) in the spirit of the *simplicial volume* defined via $L_1(X)$ and the *spherical volume* associated to $L_2(X)$ (see [G1] and references therein). A specific example is that of S being the complex projective space associated to the Hilbert space of *holomorphic* L_2-forms of top degree on the universal covering X of an algebraic manifold X. If, for instance, X is contractible and the canonical bundle of X is generated by L_2-sections (that are holomorphic top degree L_2-forms on X), then X embeds into the projectivized space of the L_2-cohomology of the Galois group Γ acting on X, say $Y = PH_{L_2}(\Gamma) \supset S$, and X equivariantly and holomorphically goes to $S \subset Y$ so that $\underline{X}$ appears as a complex subvariety in S/Γ. The question is, how much of (the complex structure of) $\underline{X}$ is reconstructable out of the image of the fundamental class of $\underline{X}$ in $H_*(Y/\Gamma) \supset H_*(S/\Gamma)$ (where the quotient homology must be properly redefined if the action of Γ on Y is non free).

(iii) One may study (not necessarily equivariant) *harmonic isometric* (minimal) maps of Riemannian manifolds X into (infinite dimensional) spaces Y_κ of constant curvature κ, and more generally, isometric maps

$f : X \to Y_\kappa$ minimizing the L_2-norm of the *mean curvature* $f(X) \subset Y_\kappa$ (that equals $\|\Delta f\|^2_{L_2}$ for *isometric* maps and where a traditional problem is that of minimization of $\|\Delta f\|^2_{L_2}$ under the integral constraint $E(f) = E_0$). This seems to be related to the work by Calabi on *holomorphic isometric* maps, while the general $\|\Delta f\|^2_{L_2}$-minimization problem for isometric maps could be treated in the context of isometric $\mathcal{C}^2$-immersions with prescribed curvature (see [G7]).

(iv) It is tempting to replace the "harmonic approach" to the rigidity theory by the "minimal approach" where one tries to identify *stable minimal subvarieties* X in a (now finitely dimensional) locally symmetric space S and where one seeks conditions necessarily making such X totally geodesic or a union of totally geodesic subvarieties. (The Mostow rigidity theorem concerns subvarieties in $S = S_1 \times S_2$ with $\pi_1(S_1) = \pi_1(S_2)$ where the minimal $X \subset S$ represents the "homotopy diagonal" in S; in general, one should, probably concentrate on subvarieties above the middle dimension in S).

3.7.E Growth of harmonic maps in metric spaces. There is no well shaped codiffusion in general metric spaces Y but the energy $E_\mu(f)$ for Γ-invariant maps $f : X \to Y$ is defined just the same. Then one may speak of (minimal) *μ-harmonic maps* $f : X \to Y$, where, by definition this energy assumes its minimum.

Hyperbolic example. Let Y be a δ-hyperbolic geodesic metric Γ-space, and $\Gamma_\rho(y) \subset \Gamma$, for $\rho > 0$, denote the subset of those $\gamma \in \Gamma$ where $|y - \gamma(y)|_Y \le \rho$. Consider a random walk μ on Γ and suppose that $\mu^n(\Gamma_\rho(y)) \le \epsilon_n(\rho)$ and all $y \in Y$ for $\epsilon_n(\rho) \underset{n\to\infty}{\longrightarrow} 0$, for all $\rho < \infty$, where μ^n stands for the measure $\mu^n(\mathrm{id} \to\ )$. Then *the E_{μ^n} energy of the μ^m-harmonic orbit (maps) $f_m : \Gamma \to Y$ for $n = km$ satisfy*

$$E_{\mu^n}(f) \ge \mathcal{C}_m\, k E_{\mu^n}(f)\,, \quad k = 1, 2, \dots\,,$$

where $\mathcal{C}_m \to 1$ for $m \to \infty$.

In fact, the argument employed for CAT($\kappa \le 0$)-spaces extends to the hyperbolic case (with some care taken of δ-errors in convexity of squared distance functions).

Remark. A similar growth bound can be established for certain *semi-hyperbolic* spaces Y, i.e. for Cartesian product of hyperbolic spaces, but the full picture is yet to be clarified (e.g. for a (sub)group Γ acting on a larger *group*).

3.8 Around Kazhdan's *T* (compare [D], [Gu] and [HV]). Given diffusion and metric codiffusion Γ-spaces $X = (X, \mu)$ and $Y = (Y, c)$ one defines the *Kazhdan* (*relaxation*) *constant* κ as the rate of contraction of the energy of Γ-equivariant maps by the heat operator (flow) as follows,

$$\kappa^{\rightarrow}(f) = \kappa^{\rightarrow}_{\mu}(f) = 1 - \tfrac{1}{2}\epsilon^{-1}\big(1 - E(H^{\rightarrow}(f))/E(f)\big)$$

and

$$k^{\rightarrow}(X, Y) = \sup_f \kappa^{\rightarrow}(f)$$

over all Γ-equivariant maps $f : X \to Y$.

At some point one needs to choose a particular value of ϵ, e.g. $\epsilon = 1/2$ or let $\epsilon \to 0$. This makes little difference, especially for Hilbertian Y, where the picture is the clearest for a continuous diffusion μ^t on X, since the function $\ell(t) = \ell_f(t) = \log E(H^t(f))$ is *convex* by the Schwartz inequality. Here one sets

$$\kappa^{(0)}(f) = -\tfrac{d}{dt}\ell(t) = 0\,,$$
$$\kappa^{(\infty)}(f) = \limsup_{t\to\infty} \big(E(H^t(f))\big)^{1/t},$$

and then defines the corresponding $\kappa^{(0)}(X, Y)$ and $\kappa^{(\infty)}(X, Y)$ by taking suprema over all $f : X \to Y$. These numbers coincide for Hilbertian Y (due to the convexity of $\ell(t)$) and are denoted $\kappa(X, Y)$ in this case, while in general, one should distinguish between $\kappa^{(0)}$ defined as $\limsup_{\epsilon\to 0} \kappa^{\rightarrow}$ for the discrete diffusion and the asymptotic $\kappa = \kappa_\mu(X, Y)$ defined with

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \big((H_{\mu}^{\overset{k}{\rightarrow}})^{kn}(f)\big)^{1/n},$$

non-ambiguously for the discrete and the continuous cases alike, where clearly $\kappa \le \kappa^{(0)}$ in all cases. Eventually we are interested in the *asymptotic* behaviour (decay) of $H^n(f) = (H^{\overset{1}{\rightarrow}})^n(f)$, and also of $H_{\mu^n}^{\overset{1}{\rightarrow}}(f)$, (different from $H^n_\mu(f)$ for non-Hilbertian Y) that is expressed in terms of $\kappa^{(\infty)}$. In practice, it is easier to evaluate $\kappa^{(0)}$ as this is essentially a local invariant.

3.8.A Expanders. Let $X = (X, \mu, \Gamma)$ be a diffusion Γ-space and $\mathcal{Y}$ be a class of metric codiffusion Γ-spaces $Y = (Y, c, \Gamma)$. Define $\kappa = \kappa(X, \mathcal{Y}) = \sup_{Y\in\mathcal{Y}} \kappa(X, Y)$ and say that X is κ-*contracting* in $\mathcal{Y}$ for this κ. We use the word "*contracting*" as a reference to "κ-contracting" with some $\kappa < 1$ and if $\mathcal{Y}$ equals the class $\mathcal{H}$ of all Hilbert Γ-spaces we say that X is a κ-*contractor* or λ-*expander* for $\lambda = 1 - \kappa$, and write $\kappa(X)$ for $\kappa(X, \mathcal{H})$. A space X is called an *expander*, if it is a λ-expander for some $\lambda > 0$. Similarly, a sequence of spaces $\{X_i\}_{i\in I}$ is called an expander if $1-\kappa(X_i) \geq \lambda > 0$. It is easy to see that for finite graphs X with the standard diffusion this reduces to the traditional definition of an expander, while for infinite $X = \Gamma$ this is equivalent to *Kazhdan's T-property* as defined below.

3.8.B *T* and f.p. properties. We say that an action of a (semi-) group Γ on a metric space Y *almost fixes a point* if there exists a sequence $y_i \in Y$, such that $|\gamma(y_i) - y_i| \underset{i\to\infty}{\longrightarrow} 0$ for all $\gamma \in \Gamma$, where such $\{y_i\}$ is referred to as a *representative* of a.f.p. An action is called *Kazhdan* if each representative of a.f.p. contains a subsequence converging to an actual fixed point. (This definition is well tuned to finitely generated (semi)groups, and to compactly generated ones with a mutable assumption of continuity of actions, while for infinitely generated Γ one should replace the fixed point set of the action by a descending sequence of subsets $Y_j \subset Y$, $j = 1, 2 \dots$, such that every $\gamma \in \Gamma$ fixes Y_j for all sufficiently large j).

A group Γ is called *Kazhdan T* if every isometric action of Γ on the unit Hilbertian sphere $S^\infty \subset \mathbb{R}^\infty$ satisfies a.f.p. property. (Kazhdan established this property in 1966 for lattices in semisimple Lie groups of $\mathbb{R}$-rank ≥ 2 in the following terms: the trivial representation makes a *closed* point in the space $\mathcal{U}(\Gamma)$ of unitary representations of Γ with the week topology, where, observe, the space $\mathcal{U}(\Gamma)$ is often non-Hausdorff).

Given a sequence of marked metric spaces $(Y_i, y_i) =$ and a sequence of numbers $s_i > 0$, one may pass to a Hausdorff (ultra)limit $\lim_{i\to\infty}(s_iY_i, y_i)$, that is again a metric space (see [G2]). The class of all these limits for all Y_i isometric to a fixed Y is denoted $\mathcal{Y}^\updownarrow = \mathcal{Y}^\updownarrow_Y$. If we restrict to sequences $s_i \to \infty$, the resulting class is denoted $\mathcal{Y}^\uparrow$; for example, if we take $\lim_{i\to\infty}(s_iY_i, y)$ for *fixed* $y \in Y$ we obtain the *tangent cone* of Y at y.

(Similarly, one introduces $\mathcal{Y}^\downarrow \subset \mathcal{Y}$ corresponding to sequences $s_i \to 0$ that for a fixed y gives one the asymptotic cone of Y, compare [G2]).

If spaces Y_i are acted upon by (semi)groups Γ_i one can often pass to the Hausdorff (ultra)limit in the category of metric Γ-spaces. There are (at least) two kinds of limit group actions, *geometric* and *algebraic* ones (according to the Kleinian group terminology) where in the latter case one speaks of (semi)groups Γ_i with *marked* generators, say $\gamma_{ij} : Y_i \to Y_i$. We consider here only uniformly finitely generated groups, i.e. j runs over $1, \dots, k$ for a fixed $k < \infty$, where we assume the set of maps $\{\gamma_{i1}, \dots, \gamma_{ik}\}$ to be symmetric under taking inverse maps. In order go to the limit of the transformation groups Γ_i on Y_i generated by γ_{ij} (remaining in the category of *standard* metric spaces) one needs two *boundedness conditions*,

(i) the transformation γ_{ij} are α-Lipschitz for some $\alpha = \infty$ and all i, j.

(ii) there are marking $y_i \in Y_i$ such that
$$|y_i - \gamma_{ij}(y_i)| \leq \text{const} < \infty$$
for all i, j.

Now, suppose that these y_i are *almost fixed* under Γ_i, i.e. $|y_i - \gamma_{ij}(y_i)|_{Y_i} \leq \epsilon_i \underset{i\to\infty}{\longrightarrow} 0$ for all $j = 1, \dots, k$, and the actions of Γ_i have no fixed points in the vicinities of $y_i \in Y$. Namely, let $\epsilon(y_i) = \sup_j |y_i - \gamma_{ij}(y_i))$, take a sequence $a_i \to \infty$, where $\delta_i(y_i) = a_i\epsilon_i(y_i) \to 0$ for $i \to \infty$, and assume that no point in the ball $B(y_i, \delta(y_i)) \subset Y_i$ is jointly fixed by $\gamma_{i1}, \dots, \gamma_{ik}$. If the metric space Y is complete, then, for each i, where $a_i \geq 2$, there exists a point $y_i' \in B(y_i, \delta(y_i))$, such that $\epsilon(y) \geq \frac{1}{2}\epsilon(y_i')$ for all y in the ball $B(y_i')\frac{1}{2}\delta(y_i') \subset Y$: otherwise one would obtain, starting from $y_i^1 = y_i$, a (Cauchy) sequence $y_i^{k+1} \in B(y_i^k, \delta(y_i^k))$, such that $y^{k+1} \in B\big(y_i^k, \frac{1}{2}\delta(y_i^k)\big)$ and $\epsilon(y_i^{k+1}) \leq \frac{1}{2}\delta(y_i^k)$, necessarily converging to a fixed point. Now, we scale the spaces Y_i by $s_i = 2\epsilon_i(y_i')$ and pass to the limit $Y_\infty = \lim_{i\to\infty}(s_iY_i, y_i) \in \mathcal{Y}^\uparrow$. Clearly, *the action of the limit group Γ_∞, generated by the limit transformations $\gamma_{\infty,j}$, $j = 1, \dots, k$ has no fixed point; moreover,*
$$\sup_{j=1,\dots,k} \big|y - \gamma_{\infty,j}(y)\big|_{Y_\infty} \geq 1\,.$$
Thus, there is no almost fixed point either. In particular, if Γ is not Kazhdan T, then it admits an isometric action on the Hilbert space without almost fixed points, since the tangent cones of S^∞ are isometric to $\mathbb{R}^\infty$ (this result is due to A. Guichardet, while the applicability of the scaling limit argument in the context of T-groups was pointed out to me by Rick Schoen).

It is known (a theorem by Delorme) that the converse is also true:

If Γ is Kazhdan's T, then every isometric action of Γ on a Hilbert space has a fixed point.

We shall not prove and use this result in the present paper; instead we shall incorporate it in the following

DEFINITION (compare [W1,2]). Let $\mathcal{Y}$ be a class of complete metric spaces Y with some *admissible* actions of groups (or semigroup) of Lipschitz transformations $Y \to Y$. A (semi)group Γ is said to satisfy the f.p. *property* with respect to $\mathcal{Y}$ if every admissible action of Γ on each $Y \in \mathcal{Y}$ has a fixed point. Similarly "a.f.p." is defined as the existence of an almost fixed point for such an action. One has the following obvious implication(s):

If the class $\mathcal{Y}$ is stable under scaling Hausdorff limits (i.e. $\mathcal{Y}_Y^\updownarrow \subset \mathcal{Y}$ for all $Y \in \mathcal{Y}$) then

$$\text{f.p} \Longleftrightarrow \text{a.f.p.}$$

Furthermore, *if a finitely generated (semi)group* Γ *satisfies* f.p.*, then there exists a surjective homomorphism* $\Gamma_0 \to \Gamma$*, where* Γ_0 *is finitely presented and is also* f.p. (compare ([Z2], [S]).

Here are some relevant classes of spaces

(i) the class $\mathcal{H}$ of all Hilbert spaces,
(ii) the class $\mathcal{C}_{\mathrm{CAT}}$ of all CAT(0)-spaces.
(iii) the class $\overline{\mathcal{C}}_{\mathrm{reg}}$, that is the minimal class of spaces containing all *smooth* CAT(0)-spaces and that is closed undertaking (totally geodesic) *subspaces* under *Hausdorff limits.*

Observe that $\overline{\mathcal{C}}_{\mathrm{reg}}$ contains certain singular spaces, such as tree and some (not all) Bruhat–Tits buildings. Yet most CAT(0)-spaces are neither Hausdorff approximable by smooth ones nor are they isometrically embeddable into Hausdorff limits of smooth CAT(0)-spaces, where the primary examples are Euclidean cones over CAT(1)-expanders.

The list of *actions* starts with *isometric* ones. Then one may include (uniformly) Lipschitz affine transformations. Groups of such transformations on Hilbert (and Banach) spaces are, in general, far from isometry groups (see [P]). On the other hand, the existence of non-isometric affine (i.e. middle point preserving) transformations seems γ rare (how rare?) for non-flat geodesic spaces. (We are concerned at this point with infinite dimensional CAT(0)-spaces.)

3.8.C Remarks. (a) If Y is a *bounded* space with *uniformly convex* squared distance function, then the function

$$b(y) \stackrel{def}{=} \sup_{y' \in Y} |y - y'|_Y^2$$

is uniformly convex on Y and thus, for complete Y, has a unique minimum point $y_0 \in Y$, called the *Mazur center* of Y. Clearly, this y_0 is fixed under the full isometry group of Y. For example, if Γ is a group of uniformly Lipschitz affine transformations of a uniformly convex Banach space Y, where some (and, hence, every) orbit $\Gamma(y) \subset Y$ is bounded, then there exists a fixed point y_0 in the convex hull of Y, where the relevant metric is $\sup_{\gamma\in\Gamma} |\gamma(y) - \gamma(y)|_Y$). It follows, that f.p. *property for isometric* (and uniformly Lipschitz) *actions on uniformly convex Banach* (*and more general metric*) *spaces is equivalent to boundedness of the orbits.*

(b) If a locally compact group G acts on a *codiffusion* space Y, such that some lattice $\Gamma \subset G$ has a fixed point $y \in Y$, then the center of the pushforward measure from G/Γ to Y for $g \mapsto g(y)$ is, clearly, fixed under G. Thus the f.p. *property passes from* Γ *to* G.

(c) If G and Γ are as above, where Γ isometrically acts on Y, then G acts on the space $Y^{G/\Gamma}$ of Γ-equivariant maps $G \to Y$. If the latter action has a fixed point then so does the former. In particular, *the* f.p. *property for* Γ *implies that for* Γ if the class $\mathcal{Y}$ in question is closed under infinite Cartesian L_p-powers for some $p \geq 1$. (Notice that infinite Cartesian powers like $Y^{G/\Gamma}$ are Hausdorff limits of finite powers.) This applies, with $p = 2$, to the classes $\mathcal{H}, \mathcal{C}_{\mathrm{CAT}}, \mathcal{C}_{\mathrm{reg}}$ and $\overline{\mathcal{C}}_{\mathrm{reg}}$. (The $Y \mapsto Y^{G/\Gamma}$ construction, called "induced representation" in the unitary Hilbertian category, was used by Kazhdan for the reduction of T from G to Γ.)

3.8.D Fixed points in codiffusion spaces (compare [W2]). Let (Y, c) be a codiffusion metric Γ-space as earlier and μ be a symmetric invariant diffusion on Γ where the support of $\mu(\mathrm{id} \to\ )$ generates Γ and thus the Haar measure ν is stationary for μ. Consider a Γ-equivariant map, i.e. an orbit $f : \Gamma \to Y$ and observe that *the inequality* $\kappa^{(\infty)}(f) < 1$ *implies that* Γ has an almost fixed point in Y. In fact, if $\mu_1, \mu_2, \dots,$ is a sequence of diffusions on Γ such that the operators $H_1 = H_{\mu_1}, \dots, H_i = H_{\mu_i} \circ H_{i-1}$ contract the μ-energy to zero, $E_\mu(H_i f) \to 0$ for $i \to \infty$, then, obviously, the points $(H_i f)(\mathrm{id}) \in Y$ represent an *almost fixed point* in Y. Furthermore, if Y is complete and $\|\Delta_\mu f\|_{L_2} \leq \mathrm{const}\, E_\mu(f)$ for all orbits f (as is trivially true for discrete groups Γ), then *the heat flow on orbits* (obviously) *converges to a fixed point.*

Corollary. *The inequality* $\kappa((\Gamma, \mu), Y) < 1$ *implies f.p. for every class of codiffusion complete metric space* Y (*with an obvious continuity requirement on codiffusion satisfied for all our examples*).

The converse is also true for classes $\mathcal{Y}$ stable under scaling Hausdorff limits. In fact, if $\kappa^{(0)}(\Gamma, \mathcal{Y}) = 1$, then by the scale-limit argument, there exists a *non-constant harmonic* orbit (map) $f_\infty : \Gamma \to Y$ for some $Y \subset \mathcal{Y}$, that is necessarily energy minimizing if Y is CAT(0). Thus Γ has no fixed point on Y.

Remarks. (a) If $\kappa^{(0)}(\Gamma, Y) = \kappa < 1$, then the scaling limit argument delivers a Γ-space $Y' \subset \mathcal{Y}_Y^{\uparrow}$ and an orbit $f' : \Gamma \to Y'$, where $\kappa^{(0)}(f') = \kappa = \kappa^{(0)}(\Gamma, Y')$, called an *extremal* κ-orbit (map). For example if Y is a Hilbert space, then Y' is also Hilbert and $\Delta f' = (1 - \kappa) f'$; if Y is a complete CAT(0), then Y' equals the tangent cone of Y at a fixed point $y_0 \in Y$ and again $\Delta f' = (1 - \kappa) f'$, meaning that the heat flow *radially* contracts f'

with the rate κ.

(b) The scaling argument (trivially) extends to an arbitrary complete Γ-space Y (*no* codiffusion on Y) with a Γ-invariant function (energy) on (Γ-orbits in) Y, denoted $E : Y \to \mathbb{R}_+$, such that

$$E(y) = 0 \Longleftrightarrow y \text{ is a fixed point.}$$

One distinguishes *the contracting case*, where there exist numbers $r < \infty$ and $\kappa < 1$, such that for each $y \in Y$

$$\inf_{y' \in B(y,r)} E(y') \leq \kappa E(y) . \qquad (*)$$

In this case, there obviously exists a point $y_0 \in Y$, where $E(y_0) = 0$ that is necessarily fixed by Γ.

If $(*)$ fails to be true for all $r < \infty$ and $\kappa < 1$ then there are two possibilities

(i) $\inf_{y \in Y} E(y) > 0$. Here the action has no a.f. point and there exists a space $(Y', y') = \mathrm{Hauslim}_{i \to \infty}(Y, y_i)$, such that E assumes its minimum on Y' at some $y' \in Y$ with $E(y') > 0$.

(ii) $\inf_{y \in Y} E(y) = 0$. Then there exists $(Y', y') \in \mathcal{Y}_Y^{\uparrow}$, where $E(y') = 1$ and $E(y'') \geq 1$ for all $y'' \in Y'$, and no almost fixed point either.

(c) Instead of orbits, one may work with Γ-equivariant maps $f : X \to Y$ for a (rather) general diffusion Γ-space $X = (X, \mu)$, e.g. where X/Γ has finite total mass for the (Γ-invariant) stationary measure ν on X. One can bound the average displacement $\int_{X/\Gamma} |f(x) - \gamma f(x)|^2 \nu dx$ by $E(f)$ for all $\gamma \in \Gamma$, and if, for example, the heat flow eventually brings the energy to zero, one gets an almost fixed point (under mild assumptions on (X, μ, ν, Γ). Alternatively, one can reduce to the case $X = \Gamma$ by taking the space $Y^{X/\Gamma}$ in place of Y.

3.9 Poincaré spaces and constants. Consider a Γ-space X with two measures μ and ν on $X \times X$ invariant under the diagonal action, called a *Poincaré* Γ*-space* and let Y be a metric space with an isometric action of Γ. We have two Γ-energies on the space F of Γ-equivariant maps $f : X \to Y$, that are $E_\mu(f) = \frac{1}{4} \| |f(x) - f(x')|_Y : \Gamma \|_{L_2^\mu}$ and similar $E_\nu(f)$; the issue is a bound on $E_\nu(f)$ in terms of $E_\mu(f)$ for all $f \in F$. (A more general *non-linear spectral problem* is concerned with the subsets $F_{\alpha,\beta} \subset F$ defined by the inequalities $E_\mu(f) \leq \alpha$, $E_\nu(f) \geq \beta$. Here one is keen on the *spectral lines* in the (α, β)-plane $\mathbb{R}_+^2$, such that the topology and/or global geometry of $F_{\alpha,\beta}$ undergoes drastic transitions as (α, β) crosses these lines; compare [G4] and [G8]).

Basic example. Let μ be the "infinitesimal measure" supported on tangent vectors of a smooth manifold X, e.g. the normalized Riemann–Liouville measure on a riemannian X, where $E_\mu(f) = \frac{1}{2}\int_X \|df(x)\|^2 d\mu(x,x)$ and ν is of the form $\underline{\nu} \times \underline{\nu}$ for a probability measure $\underline{\nu}$ on X. If $Y = \mathbb{R}$, or an arbitrary Hilbert space for this matter, the Poincaré inequality takes the familiar form

$$\iint \left|f(x) - f(x')\right|_Y^2 d\underline{\nu}(x) d\underline{\nu}(x') \leq 2\lambda_1^{-1} \int_X \left\|df(x)\right\|^2 d\mu(x,x)\,.$$

Since the definition of the energies involves Y only via the induced metric $|f(x) - f(x')|_Y$, one can reformulate the problem by distinguishing a class $\mathcal{G}$ of Γ-equivariant (possibly degenerate) metrics on X and seek for (Poincaré) inequalities of the form $E_\nu(g) \leq \pi(E_\mu(g))$, where the energies here refer to the square of the (Γ-reduced) L_2-norm of the function $g : X \times X \to \mathbb{R}_+$ with respect to μ and ν. (Accordingly, one modifies the non-linear spectral problem by replacing F by $\mathcal{G}$.) The relevant classes of metric are

(i) the space $\mathcal{H}$ of Hilbertian metric, i.e. those induced by maps $X \to \mathbb{R}^\infty$.
(ii) the metrics induced from CAT(0)-spaces, denoted $\mathcal{C}_0 \supset \mathcal{H}$.

Notice that unlike (i), the relevant CAT(0)-space Y may not have an isometric Γ-action. If this is enforced, we have a smaller class of metrics, denoted $\mathcal{C}_{0,\Gamma} \subset \mathcal{C}_0$.

Observe, that all three classes $\mathcal{G}$ are stable under scaling and Cartesian (Pythagorian) products. In particular, if we have a family of such metrics g_p on X parametrized by a probability space $P \ni p$ one can *square average* over P, by taking $g = (\int_P g_p^2 dp)^{1/2}$.

EXAMPLE. Let (X, μ, ν) be acted upon by a locally compact group $G \supset \Gamma$, where $\mathrm{Vol}(G/\Gamma) \leq \infty$. Then one can square average $g \in \mathcal{G}$ over G/Γ without changing the energies and remaining in the (Pythagorian!) class $\mathcal{G}$. Thus each Poincaré's inequality for Γ-*invariant* metrics $g \in \mathcal{G}$ is equivalent to that for G-*invariant* metrics. In particular for $\Gamma = \{\mathrm{id}\}$ one can average over every compact group acting on (X, μ, ν).

Let us add two more Pythagorian classes $\mathcal{G}$ to the list: the space of *all* metrics, denoted $\mathcal{A}$, and of all riemannian metrics, called $\mathcal{R}$.

3.9.A Critical spaces. Suppose, that besides μ and ν, X carries an equivariant metric g_0 from some class $\mathcal{G}$. Say that $X = (X, g_0) = (X, g_0, \mu, \nu, \Gamma)$ is $\mathcal{G}$-*critical* if the ratio

$$E_\nu / E_\mu : \mathcal{G} \to \mathbb{R}_+ \cup \infty$$

assumes its maximum on the ray of the metrics $\{\lambda g_0\}$, $\lambda > 0$.

EXAMPLES. (a) Let $X = (X, g_0)$ be a Riemannian homogeneous space where the isotropy subgroup $I_x \subset G = \operatorname{Iso} X$, $x \in X$, acts *irreducibly* on the tangent space $T_x(X)$, e.g. X is a (compact or non-compact) irreducible symmetric space. If Γ is a lattice in G, then, by averaging, we see that X is $\mathcal{R}$*-critical* for each pair of G-invariant μ and ν. In fact, the two energies are functionally related for these X:

$$E_\nu(g) = F_{\mu,\nu}(E_\mu(g)) \text{ for all } g \in \mathcal{R} \,.$$

(b) Let μ be the Riemann–Liouville (infinitesimal) measure and ν is as above. Then X is $\mathcal{A}$*-critical*, provided it is *two-point homogeneous*, i.e. symmetric with $\mathbb{R}$-rank $= 1$. (This fails to be true for most symmetric spaces of $\mathbb{R}$-rank ≥ 2 as is seen in flat tori, for instance, where the Poincaré extremal metrics tend to be Finsler.)

(c) Let X be a compact irreducible symmetric space, μ, ν as above, and g_0 the metric induced by the *Veronese* embedding $X \to \mathbb{R}^N$, e.g. $S^n \subset \mathbb{R}^{n+1}$ or $P^n \subset \mathbb{R}^N$ for $N = \frac{(n+1)(n+2)}{2} - 1$. Then (X, g_0) is $\mathcal{H}$-critical by the Wirtinger inequality (which is true almost by definition since the Veronese embedding is given by the first eigenfunctions of the Laplacian on X). Moreover, (X, g_0) is $\mathcal{C}_0$-critical. This follows from Reshetnyak's theorem as shown in [G3] for $\mathbb{R}$-rank$X = 1$ by integrating over closed geodesics in X. If $\mathbb{R} - rank = k \geq 2$, then the problem similarly reduces to that for the maximal tori $T^k \subset X \subset \mathbb{R}^N$, and Reshetnyak's theorem applies to the closed *supersingular* geodesics mapped into a CAT(0)-space. (*Split* tori are covered by [G3] and the general case can be handled in a similar way.)

REMARKS AND SPECULATIONS. (a) In typical examples, the measure μ is (rather) concentrated "near" the diagonal while ν is spread (rather) uniformly on $X \times X$. This allows one to use the triangle inequality and bound the L_2-spread of g over all of X in terms of what happens to nearby (pairs of) points. Such a bound can be sharpened if the metrics g

of metrics closed under Cartesian L_p-products. Also one may look for *Poincaré–Sobolev* inequalities relating L_p and L_q-energies, but this is, usually, technically harder. Also one may generalize by considering more than two measures on $X \times X$, where, sometimes it is useful to work with infinite families of measures.

(c) The eigenvalue estimates for (Laplace) operators on vector bundles fit into the (X, μ, ν)-framework with suitable choices of μ, ν and $\mathcal{G}$, where the metrics $g \in \mathcal{G}$ should be assigned to the total spaces of bundles in question. However, geometrically significant operators (Hodge–deRham, Dirac, $\overline{\partial}$) suggest a more general setting, where the possible approaches are as follows:

(1) Pass to higher Cartesian powers $\underbrace{X \times X \times \ldots \times X}_{k}$, with functions $g(x_1, \ldots, x_k)$ satisfying some kind of "triangle inequalities", e.g. the $(k-1)$-volume of the convex hull (simplex) of k-points in $\mathbb{R}^N$ (where the "triangle inequality" refers to subdivision of simplices and where the "volume" may be either the Euclidean volume or the absolute value of the integral of a $(k-1)$-form).

(2) Replace pairs of points $(x, x') \in X \times X$ by $(k-2)$-cycles $V \subset X$ with a suitable "filling volume" instead of g, where one may start with (measures on) the spaces X^{S^1} of maps of the circle to X.

Here is a specific

QUESTION. Let X be a graph (i.e. 1-dimensional polyhedron) embedded into the Hilbert space $\mathbb{R}^\infty$. What are unavoidable relations between the filling areas of the cycles $\mathcal{C} \subset X$ in $\mathbb{R}^\infty$ (besides the "triangle inequalities": $\operatorname{FillArea} \mathcal{C} \le \sum_i \operatorname{FillArea} \mathcal{C}_i$ for $\sum_i \mathcal{C}_i = \mathcal{C}$)?

(d) All of the above can be tried in the presence of an extra geometric structure on X (symplectic, conformal or complex, for instance) that can be used to limit a class of metric (or generalized metrics) on X, such as Kähler metrics on complex manifolds or those induced by Donaldson's embeddings of symplectic manifolds to $\mathbb{C}P^N$. Also, such a structure may distinguish particular "cycles" in X, e.g. circles in complex manifolds bounding holomorphic disks.

3.9.B Bounds on Poincaré constants by integral geometry. Let (X'_s, μ'_s, ν'_s) be a family of Poincaré spaces parametrized by a measure space $\mathcal{S}$, where $\mathcal{S}$ and the union $\widetilde{X} \overset{def}{=} \bigcup_{s \in \mathcal{S}} X'_s$ are acted upon by a group $\widetilde{\Gamma}$ commuting with the projection $\widetilde{X} \to \mathcal{S}$ and preserving all measures in question. Denote by $\widetilde{\mu}$ and $\widetilde{\nu}$ the integrated measures $\int_{\mathcal{S}} \mu'_s ds$ and $\int_{\mathcal{S}} \nu'_s ds$ on $\widetilde{X} \times \widetilde{X}$ (that are supported on the fiber product $\widetilde{X} \underset{\mathcal{S}}{\times} \widetilde{X} \subset \widetilde{X} \underset{\mathcal{S}}{\times} \widetilde{X}$) and let $\Gamma'_s \subset \widetilde{\Gamma}$ stand for the isotropy subgroup of Γ'_s-spaces (X'_s, μ'_s, ν'_s). The $\widetilde{\Gamma}$-energies of each metric $\widetilde{g} : \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ with respect to $\widetilde{\mu}$ and $\widetilde{\nu}$ clearly satisfy

$$E_{\widetilde{\mu}}(\widetilde{g}) = \int_{\mathcal{S}} E_{\mu'}(g'_s) ds \text{ and } E_{\widetilde{\nu}}(\widetilde{g}) = \int_{\mathcal{S}} E_{\nu'}(g'_s) ds$$

where $g'_s = \widetilde{g} \mid X'_s$ and the energies on the fibers X'_s are taken in the Γ'_s-sense. Thus Poincaré's inequalities in the fibers, say $E'_{\nu'}(g'_s) \leq \pi(E'_{\mu'}(g'_s))$, for (almost) all $s \in \mathcal{S}$, imply the same (Poincaré) inequality for g, that is $E_{\widetilde{\nu}}(\widetilde{g}) \leq \pi(E_{\widetilde{\mu}}(\widetilde{g}))$.

Remark. This can be seen in terms of a foliated space with a transversal measure ds, where the leaves are endowed with Poincaré structures. Here the integration of Poincaré structures (and thus, the Poincaré inequalities) is granted by the *definition* of a foliated Poincaré space (with the transversal measure playing the role of the Haar measure on Γ).

Let $\widetilde{X}$ be mapped into a Poincaré Γ-space $X = (X, \mu, \nu)$ by a (Borel) map $\varphi : \widetilde{X} \to X$ that is equivariant under a given homomorphism $\widetilde{\Gamma} \to \Gamma$ and denote by μ_* and ν_* the pushforwards of the measures $\widetilde{\mu}$ and $\widetilde{\nu}$ under the Cartesian square of φ.

If $\mu_* \leq \alpha\mu$ and $\nu \leq \beta\nu_*$ for *some* $\alpha, \beta \geq 0$, *then the Poincaré inequality* $E_{\widetilde{\nu}}(\widetilde{g}) \leq \pi(E_{\widetilde{\mu}}(\widetilde{g}))$ *on* $\widetilde{X}$ *descends to the following Poincaré inequality on* X,

$$E_\nu(g) \leq \beta\pi\big(\alpha E_\mu(g)\big)\,,$$

for $\widetilde{g}$ *being the* φ*-pull back of* g.

This is obvious but quite useful with judicial choices of $\widetilde{X} = \cup X'_s$ and φ. Here are the standard examples.

(i) Consider Riemannian manifolds of unit volumes X'_s parametrized by a finite measure space S and denote by $dsdx'_s$ and $dsd^2x'_s$ the product measures on $\widetilde{X} = \bigcup_{s\in S} X'_s$ and on $\widetilde{X}_2 = \bigcup_{s\in S} X'_s \times X'_s$ for the Riemannian dx'_s in X'_s. Let $\widetilde{\varphi}$ be a map from $\widetilde{X}$ to another riemannian manifold X such that

- $\widetilde{\varphi}$ is δ-Lipschitz on all fibers $X'_s, s \in S$, for some $\delta < \infty$.
- •• the pushforward of the measure from $\widetilde{X}$ to X satisfies

$$\varphi_*(dsdx'_s) \leq \gamma_1 dx \text{ on } X \qquad (*)$$

and

$$\gamma_2\varphi_*(dsd^2x'_s) \geq dxdx \text{ on } X \times X\,. \qquad (**)$$

Then the first eigenvalue of the Laplace operator on X, denoted $\lambda_1(X)$ is bounded as follows:

$$(\lambda_1(X))^{-1} \leq \delta^2\gamma_1\gamma_2\Big(\inf_{s\in S}\lambda_1(X'_s)\Big)^{-1}(\mathrm{Vol}\,X)^{-1} \qquad (\star)$$

(In fact, all eigenvalues of X satisfy this inequality.) For instance if S consists of a single unit atom, $\dim X = \dim(X' = \widetilde{X})$ and $\widetilde{\varphi}$ is injective, the $(*)$ amounts to the inequality $|\text{Jacobian } \widetilde{\varphi}| \geq \gamma_1^{-1}$, while $(**)$ is satisfied for *surjective* maps $\widetilde{\varphi}$ with $|\text{Jacobian }| \leq \gamma_2^{1/2}$. Thus we arrive at the classical (and obvious) monotonicity inequality for diffeomorphisms between equidimensional manifolds of finite volume

$$(\mathrm{Vol}\,X)\lambda_1^{-1}(X) \leq (\mathrm{Lip}\,\widetilde{\varphi})^2(\mathrm{Vol}\,X')\lambda_1^{-1}(X')\Big(\inf_{x'}|J\widetilde{\varphi}(x')|\Big)\sup_x|J\widetilde{\varphi}(x')|^2.$$

In particular, if the Jacobian $J\widetilde{\varphi}$ is *constant*, then

$$\lambda_1^{-1}(X) \leq (\mathrm{Lip}\,\widetilde{\varphi})^2\lambda_1^{-1}(X')\,.$$

(This, obviously, remains true for *non*-equidimensional manifolds if $\varphi_*(dx') = \mathrm{const}\,dx$).

(ii) The standard way to use $(\star)$ is to construct a family of submanifolds $X'_s \subset X$, where a lower bound on $\lambda_1(X'_s)$ is available, e.g. for X'_s being line segments in X, and where one seeks for smallest possible γ_1 and γ_2. (Typical choices are minimizing geodesic segments parametrized by the Liouville measure and "all" segments parametrized by the *Wiener measure*, where the latter is optimal in some sense).

3.10 Relation between Poincaré and Kazhdan constants. We have seen in 3.1 how a balanced measure on $X \times X$ gives rise to a random walk on X. Conversely, a diffusion $\mu(x \to x')$ on X with a stationary measure σ defines a measure on $X \times X$ denoted $\mu(x,x') = \mu(x \to x')\sigma(x)$ by

$$\int_{X\times X}\varphi(x,x')\mu(x,x')dxdx' = \int_X\sigma(x)dx\int_X\varphi(x,x')\mu(x\to x')dx'.$$

In fact, many useful Poincaré structures are of the form (X,μ,μ^n). If Y is Hilbertian with isometric action of Γ, then the Poincaré constants

$$\pi_n \overset{def}{=} \sup_f E_{\mu^n}(f)/E(f)$$

are related to Kazhdan's κ by the (obvious) relation

$$\pi_n = 1 + \kappa + \kappa^2 + \ldots + \kappa^n$$

Similarly, $\pi_{m,n} = \sup_f E_{\mu^n}(f)/E_{\mu^m}(f)$ equals $(1-\kappa^n)/(1-\kappa^m)$ for $n \geq m$ (and sometimes for $n < m$).

In general, given a bound $\kappa \le \kappa_0 < 1$ for maps $f : X \to Y$, one can (often) evaluate the L_2-distance of f from a constant map f_∞ terminating the heat flow, and thus obtain an inequality between different energies of f in terms of κ_0.

Conversely, one can bound κ in terms of π_n by confronting the Poincaré inequality with the (parabolic) growth inequality (whenever the latter is available). For example, for CAT(0)-spaces Y one has

$$\pi_n \ge 1 + \kappa + \kappa^2 + \ldots + \kappa^n ,$$

for all $n = 2, 3, \ldots$. (This is seen by looking at κ-extremal maps of X to CAT(0)-cones.) In particular, the inequality $\pi_n < n$, for some $n \ge 2$ implies the fixed point property for isometric Γ-actions on CAT(0)-spaces.

Remark. The essential difference between the constants π and κ is that the former reflects the geometry of Y on individual orbits (or, more generally, Γ-maps $f : X \to Y$) while κ's depend on the metric properties of Y on the *whole orbit* of the heat flow applied to a given f and so they need more structure (codiffusion) for their definition. Thus κ's are of more global nature compared to π's. However, when it comes to extremal (e.g. harmonic) maps $f : X \to Y$, the two (classes of) constants carry essentially the same information.

3.11 Garland lemma and generalizations. Let X^2 be a locally finite 2-dimensional polyhedron with the vertex set $X \subset X^2$ and let $L_x \subset X$ denote the link of a vertex $x \in X$. Consider the standard measure μ_x on the edge set $\overset{\leftrightarrow}{E}_x \subset L_x \times L_x$ assigning unit weight to each oriented edge and let $\underline{\nu}_x$ on L_x be the stationary measure associated to the random walk corresponding to μ_x, where each vertex $x' \in L_x$ is given the weight $d(x, x')$ equal the degree of L_x at x', i.e. the cardinality of the link of the edge $[x, x']$ in X^2. The collections of measures μ_x and $\nu_x = \underline{\nu}_x \times \underline{\nu}_x$ parametrized by the vertices $x \in X$ sum up over X to measures μ and ν on the space $X \times X$ for $X \subset X^2$, where the measures μ and ν are balanced and the corresponding random walks $\mu_\bullet$ and $\nu_\bullet$, clearly, satisfy the equality $\mu_\bullet^2 = \nu_\bullet$.

By the "integral geometry" estimate, the mutual Poincaré constant of (X, μ, ν) is bounded by the supremum over $x \in X$ of the corresponding constants for $(L_x, \mu_x, \underline{\nu}_x \times \underline{\nu}_x)$, for an arbitrary class $\mathcal{G}$ of metrics on X,

$$\pi(X) \le \sup_{x \in X} \pi(L_x) ,$$

where $\pi(X)$ refers to some (locally compact) group Γ simplicially acting on X^2 with bounded quotient space X^2/Γ.

If $\mathcal{G}$ consists of the Hilbertian metrics, then $\pi(L_x)$, for μ_x and ν_x normalized to *probability* measures, equals the reciprocal of the first eigenvalue $\lambda_1(X)$ of the graph L_x, where, observe, this λ_1 is related to Kazhdan's κ of the random walk associated to μ_x by the (obvious) equality $\lambda_1 = 1-\kappa$. Thus the Poincaré constant of X satisfies

$$\pi(X,\mu_\bullet,\mu_\bullet^2) \le \sup_{x\in X} \big(1-\kappa(L_x)\big)^{-1}.$$

and, therefore

$$\kappa(X,\mu_\bullet) = \big(\pi(X,\mu_\bullet,\mu_\bullet^2)\big)^{-1} - 1 \le \Big(\sup_{x\in X} \big(1-\kappa(L_x)\big)^{-1}\Big) - 1\,.$$

In particular (see [Z1])

($\star$) *if $\kappa(L_x) < 1/2$ for all $x \in X$, then $\kappa(X) < 1$ and the group Γ is Kazhdan T.*

REMARKS. (1) One is not obliged to start with the *standard* diffusion on L_x: the above argument works for arbitrary μ_x on L_x invariant under Γ, i.e. if $\gamma\mu_x = \mu_{\gamma(x)}$. But the standard diffusion, represented by the uniform (and, hence, entropy maximizing) measure μ on $\overset{\leftrightarrow}{E}$ is likely to have the smallest κ (among balanced measures supported on $\overset{\leftrightarrow}{E}$) for decent (all?) finite graphs.

(2) The above ($\star$) remains valid for every class of Γ-invariant metrics on X but it looses in precision due to the roughness of the growth inequality (for CAT(0)-spaces). If f is a *Hilbert* space valued function on a probability space $\sum$, then

$$\int \big|f(x)-f(x')\big|^2(\sigma\times\sigma)(x,x')dxdx' = 2\int \big|c-f(x)\big|^2\sigma(x)dx \qquad (*)$$

for c being the center of mass of the f-pushforward of the measure of $\sum$. But for CAT(0)-spaces one only has the inequality "$\ge$", but not, in general "$\le$". However, one can regain a sharp inequality by replacing the growth inequality for E_{μ^2} by a tautological equality automatically incorporating ($*$). Namely, for every metric (or any function for this purpose) $g: X\times X \to \mathbb{R}$, the energy over the measure μ on $X\times X$ supported on the edges of the polyhedron $X^2 \supset X$ and giving the weight equal the cardinality of the link of this edge in X^2, can be computed in two different ways:

I. Define $E_x^\bullet(g)$ for all $x \in X$ by

$$E_x^\bullet(g) = \sum_{x'} \big(g(x,x')\big)^2 \deg[x,x']\,,$$

where x' runs over all edges of X^2 issuing from x' and $\deg[x,x']$ stand for the cardinality of the edge $[x,x']$.

II. Let $e = e_x \subset L_x \times L_x/\{+1,-1\}$ denote the set of non-oriented edges in the link $L_x \subset X^2$ and set

$$E_x^\circ(g) = \sum_e g^2(x',x'')\,,$$

where the summation is taken over all non-ordered pairs $(x',x'') \in e$.

If the function $E_x^\bullet$ (and hence E_x°) is bounded on X, and if we are in a position to average bounded functions, then, clearly

$$\mathop{\mathrm{Av}}_{x\in X}(E_x^\bullet) = \mathop{\mathrm{Av}}_{x\in X}(E_x^\circ)\,. \qquad (\star)_A$$

(For example, one can average over amenable spaces X, e.g. those of subexponential growth and if g is Γ-invariant, one can average over X/Γ, in the case this quotient is finite, or even amenable). Therefore, if for all $x \in X$,

$$E_x^\bullet(g) \le \pi E_x^\circ(g) \quad \text{for} \quad \pi < 1\,,$$

then g is *on the average zero* on $X \times X$.

Next, consider maps f_x of the vertex set $L_x^\circ \subset L_x$ to a metric space Y and denote by $\mathcal{C}(f_x) \subset Y$ the subset where the weighted sum

$$E(y) = \sum_{x'\in L_x^\circ} \big|y - f_x(x')\big|_Y^2 \deg[x,x'] \qquad (\square)$$

assumes its minimum (which consists of a single point for CAT(0)-spaces Y). Suppose that all maps f_x, $x \in X$, satisfy the following Poincaré inequality for all $c_x \in \mathcal{C}(f_x)$,

$$\sum_{x'\in L_x^\circ} \big|c_x - f_x(x')\big|_Y^2 \deg[x,x'] \le \pi \sum_{e_x} \big|f(x') - f(x'')\big|_Y^2 \qquad (\Delta)$$

for $\pi < 1$. Then every harmonic map $f : X \to Y$ with $\sup_{x\in X} E_x(f) < \infty$ has $\mathrm{Av}_{x\in X}(f) = 0$, i.e. it is "constant on the average", where "harmonic" is *defined* by the inclusion

$$f(x) \in \mathcal{C}(f \mid L_x) \quad \text{for all } x \in X\,.$$

In order to have enough harmonic maps at our disposal we have to add to Y the limit spaces Y' of the form $Y' = \lim_{i\to\infty}(\lambda_i Y, y_i)$ where all $\lambda_i \ge \epsilon > 0$. Then for every group Γ acting on X^2 (simplicially and cocompactly), we depart from an arbitrary Γ-equivariant map $f_0 : X \to Y$ and, by minimizing the Γ-energy and applying the scaling limit argument, we conclude, in the case where Y is complete, that either f_∞ is a *constant* map to the *original* Y or $f_\infty : X \to Y'$ is a *non-constant energy minimizing* map for some of the above Y' (possibly, but not necessarily equal to Y). In a variety of cases, e.g. for CAT(0)-spaces, as we have seen earlier, this f_∞ is necessarily *harmonic* and then (Δ) *cannot* hold in Y' with $\pi < 1$. Therefore, *if* (Δ) *does hold for all* Y' *with* $\pi < 1$*, then every isometric action of* Γ *on* Y *has a fixed point*, where Γ is a group admitting a simplicial action on X^2, such that X admits a Γ-equivariant map to (Y, Γ), e.g. the action of Γ on X is free)

(b) One can relax the notion of harmonicity by defining $\mathcal{C}(f_x)$ via *local* minima of $E(y)$ (or suitable sets of *critical points* of $E(y)$).

Extra remarks. (a) The above "equal averaging" argument represents a (small) fragment of Garland's proof of the vanishing of $H^i(X, U(\Gamma))$ for $i < \dim X$, where X is a Bruhat–Tits building and $U(\Gamma)$ is a unitary representation. The emphasis in Garland's paper is laid on $i \geq 2$, since the case $i = 1$, corresponding to T, was covered by the original paper of Kazhdan (compare [Bo], [Z1]). This makes energy minimizing maps "harmonic" for a wide class of target spaces (e.g. for CAT($\kappa > \infty$)), but then one needs, accordingly, more general Poincaré inequalities.

(b) If Y is CAT(0), one needs the Poincaré inequality with $\pi < 1$ not for all Y', but only for $Y' \subset \mathcal{Y}_Y^{\uparrow}$; actually only for tangent cones of the spaces in $\mathcal{Y}_Y^{\uparrow}$. In particular, the inequality (Δ) for all $Y \in \overline{\mathcal{C}}_{\rm reg}$ (trivially) reduces to that for $\mathbb{R}$-valued maps f (as follows from the 2-convexity; see [W1,2] and [G3]).

(c) One can extend $(\star)_A$ to arbitrary pairs of measures on $X \times X$ with equal "averages", yet satisfying a non-trivial Poincaré inequality. A geometrically attractive case is where one has a "domain" $B \subset X$ and μ is supported on $\partial B \times \partial B \subset X \times X$ for the "boundary" ∂B of B and ν is supported on $B \times B$. The relevant Poincaré constant enters via the solution of the *Dirichlet* (*filling*) *problem*: given $f_0 : \partial B \to Y$, find $f : B \to Y$ extending f_0 and minimizing the ν-energy on B. Whenever one can guarantee f with $E_\nu(f) \leq \pi E_\mu(f)$ and $\pi < 1$, one rules out non-constant ν-harmonic maps $X \to Y$ compatible with the averaging.

About examples. The inequality $\kappa(L) < 1/2$ is rather restrictive and the main source of the X's with such links is provided by Euclidean buildings. Here is an attractive (I guess, known) family of 2-polyhedra with $\kappa(L) < 1/2$. Let Δ_d^1 be the full graph (clique) with $d+1$ vertices (the 1-skeleton of the d-simplex) and $\widetilde{\Delta}_d^1$ the 2-cover corresponding to the $\mathbb{Z}_2$-cocycle equal 1 on each edge in Δ_d^1. The automorphism group of $\widetilde{\Delta}_d^1$ is transitive on the vertices and the quotient by an isotropy subgroup is a

4-point graph with the vertices of degrees d, d^2, d^2, d. Hence, $\kappa(\widetilde{\Delta}_d^1)$, that is the maximal eigenvalue of the diffusion operator, equals the maximal root < 1 of the equation $1 - (d-1)\kappa = d\kappa^2$ and, obviously, $\kappa(\widetilde{\Delta}_d^1) = d^{-1} < 1/2$ for $d \geq 3$.

It remains to construct a polyhedron X_d^2 with all links isomorphic to $\widetilde{\Delta}_d^1$. Start with Δ_{d+1}^2, that is the 2-skeleton of the $(d+1)$-simplex with the links Δ_d^1 and then doubly ramify it at all vertices. This double cover $\widetilde{\Delta}_{d+1}^2$ does exist, since the corresponding cocycles extend from the links to $\Delta_{d+1}^2 \setminus \{\text{vertices}\}$ and it is CAT(0) since all cycles $\widetilde{\Delta}_d^1$ are ≥ 6. (If $d = 2$, this is the Weierstrass torus doubly covering S^2 with 4 ramification points). The universal cover of $\widetilde{\Delta}_{d+1}^2$ is our X_d^2; it is (freely) acted upon by $\Gamma_d = \pi_1(\widetilde{\Delta}_{d+1}^2)$, where the property T and the f.p. with respect to $\overline{\mathcal{C}}_{\text{reg}}$ start from $d = 3$. Thus every isometric action of Γ_d, $d \geq 3$, on a *smooth* CAT(0)-space, or on the Hausdorff limit of such spaces, has a fixed point.

3.12 Harmonic spread. Let us reformulate the "integral geometric" bound on κ in the present context in order to make use of growth and Poincaré inequalities relating the energies E_μ and E_{μ^n} for *all* $n \geq 2$. Given a class $\mathcal{Y}$ of metric (codiffusion) Γ-spaces Y and three invariant random walks μ, μ_1, μ_2 on a Γ-space X we define the upper and lower *μ-harmonic spreads of μ_2 relative* to μ_1 by considering the *non-constant* μ-harmonic Γ-equivariant maps $f : X \to Y$ for all $Y \subset \mathcal{Y}$ and by taking the supremum and infimum of the ratios of the corresponding energies over all these maps:

$$\overline{\text{h.spr}}_\mu(\mu_2/\mu_1) \overset{def}{=} \inf E_{\mu_2}(f)/E_{\mu_1}(f)\,,$$

$$\underline{\text{h.spr}}_\mu(\mu_2/\mu_1) \overset{def}{=} \inf E_{\mu_2}(f)/E_{\mu_1}(f)\,,$$

where the energies are measured with respect to a μ-stationary measure ν on X, where the basic example is $\mu_1 = \mu$ and $\mu_2 = \mu^n$, $n \geq 2$, and where in the presence of Γ the energy integrals are taken over X/Γ as earlier. We assume, for all μ_1 and μ_2 that they share a common Γ-invariant stationary measure ν with μ and, therefore, the energy relations can be understood as Poincaré's inequalities.

Remarks. (a) Clearly, $\overline{\text{h.spr}} \geq \underline{\text{h.spr}}$ unless every harmonic map $f : X \to Y$ is constant. In fact, the inequality $\overline{\text{h.spr}} < \underline{\text{h.spr}}$ is used below to rule out non-constant harmonic maps.

(b) Evaluations of the harmonic spreads seem interesting in situations unrelated to the present context, e.g. for ordinary harmonic functions (maps) on manifolds X with boundary where μ_1 is supported on the boundary, while μ_2 is some (possibly "infinitesimal") measure on all of X.

Next, let (L, μ'_1, μ'_2) be another space with two random walks, now without a Γ-action, yet sharing a common stationary measure ν' on L. Consider a map $\varphi : L \to X$ such that the pushforward measure $\varphi_*(\nu')$ on X sums up to ν over all Γ-translations of Γ and then take the sums of the Γ-translations of the pushforwards of μ'_1 and μ'_2, denoted by $\mu^*_1 = (\mu'_1)_*$ and $\mu^*_2 = (\mu'_2)_*$. (This is done by first replacing the random walks by measures $\mu'_1(\ell_1, \ell_2)$ and $\mu'_2(\ell_1, \ell_2)$ on L, pushing them down to $X \times X$, translating and summing up over Γ and finally, taking the corresponding random walks).

Now, the "integral geometric" inequality takes the following form:

if every map $f' : L \to Y$, for all $Y \in \mathcal{Y}$, satisfies the Poincaré inequality

$$E_{\mu'_2}(f') \le \pi' E_{\mu'_1}(f') \tag{E'}$$

then, for all μ,

$$\overline{\text{h.spr}}_\mu(\mu^*_2/\mu^*_1) \le \pi'. \tag{$\overline{\text{spr}}$}$$

Notice, that "μ" enters only via the μ-harmonicity condition on f that is not present in (E').

Next, suppose that μ^*_1 and μ^*_2 approximate μ_1 and μ_2 in the following way

$$\mu^*_1(x \to\ ) \le (1+\epsilon_1)\mu_1(x \to\ )$$

and

$$\mu^*_2(x \to\ ) \ge (1+\epsilon_2)^{-1}\mu_2(x \to\ )$$

for all $x \in X$ and some positive ϵ_1 and ϵ_2. Then, clearly

$$\overline{\text{h.spr}}_\mu(\mu_2/\mu_1) \le \pi'(1+\epsilon_1)(1+\epsilon_2)\,. \tag{$*$}$$

Thus, if we want to rule out harmonic maps $X \to Y$, it is sufficient to find (L, μ') with a small $\kappa' = \kappa(L)$ and a map $\varphi : L \to X$ such that μ^* sufficiently closely approximate μ and $\mu^*_n \overset{def}{=} ((\mu'^n)_*$ approximates μ^n (where, observe $((\mu'^n)_* \ne \mu'_*)^n$, in general). For example, if all Y's are CAT(0) it is sufficient to have

$$\big(1+\kappa'+\dots+(\kappa')^n\big)(1+\epsilon_1)(1+\epsilon_2) < n \tag{$\star$}$$

for a single $n \ge 2$.

Let us specialize the above to the case relevant to the present paper, where X equals the Cayley graph of a group Γ with k-generators and the standard random walk μ coming from that on the free group F_k. Suppose furthermore, that Γ contains a relation given by a map $\alpha : \overleftrightarrow{E} \to F_k$ for a graph (V, E) (where we switched from L to the notation V of §1). Since

α is a relation for Γ, it lifts to a map $\varphi : V \to X$, that is unique up to Γ-translations and if $\kappa' = \kappa(V) \le 1 - \epsilon'$, we shall satisfy ($\star$) for a *large* n, provided ϵ_1 and ϵ_2 are not too large.

It may be difficult to control these for an individual α, but if we average over all maps $\alpha : E \to F_k$ with respect to μ^E then the averaged measures (i.e. random walks), say $\overline{\mu}^*$ and $\overline{\mu}^*_n$, obviously satisfy

(i) $\overline{\mu}^* = \mu$;

(ii) The measure $\overline{\mu}^*_n$ is *radial*; this means it (i.e. $\overline{\mu}^*(\mathrm{id} \to\ ))$ is a push-forward of a *radial measure* $\widetilde{\mu}^*_n$ on F_k that is a convex combination of *spherical* probability measures on F_k, where "spherical" refers to a measure concentrated on some sphere in F_k and having equal weight at all points. (An explicit $\widetilde{\mu}^*_n$ is constructed below.)

In other words, the density function of $\overline{\mu}^*(\mathrm{id} \to\ )$ denoted $\overline{\mu}^*(\gamma)$, $\gamma \in \Gamma$, depends only on $r = |\gamma| = |\,\mathrm{id} - \gamma|$ and this function $\overline{\mu}^*(r)$ can be explicitly (and easily) computed in terms of (V, E) (see below). Notice, that the group Γ can vary as we take different samples of α, but $\widetilde{\mu}^*$ and $\overline{\mu}^*(r)$ make sense independently of Γ.

The measure $\widetilde{\mu}^*_n$ on F_k is constructed out of the standard diffusions μ' on V and $\mu = \mu_k$ on F_k as follows. Take the product $V \times V \times F_k$ with the measures $\mu^m(\mathrm{id} \to\ )$ in the fibers $(v, v) \times F_k$ with $|v - v'|_V = m$, $m = 0, 1, 2, \dots$, and the measure $(\mu')^n$ on $V \times V$ (with the usual identification between $\mu'(v \to v')$ and $\mu(v, v')$ via the μ'-stationary measure ν' on V). Denote by $\widetilde{\mu}_n$, the resulting "fiber product" measure on $V \times V \times F_k$ and let $\widetilde{\mu}^*_n$ be the pushforward of this measure to F_k under the projection $V \times V \times F_k \to F_k$. Clearly, $\widetilde{\mu}^*_n$ is radial and it goes to $\overline{\mu}^*_n$ under the epimorphism $F_k \to \Gamma$.

To get a feeling for the relation between the measures $\widetilde{\mu}^*_n$ and $\mu^n(\mathrm{id} \to\ )$ on F_k think of $\widetilde{\mu}^*_n$ as the composition of random walks in V and in F_k, where the picture is the clearest for random maps between trees. In fact, let $T' = T_i$ and $T = T_j$ be regular rooted trees with vertex degrees i and j correspondingly (where T' roughly corresponds to the universal covering of V, while $T_j = T_{2k}$ is the tree inhabited by F_k) and look at a random map $\widetilde{\alpha} : T' \to T$, where each *simple path* in T' starting at the root in T' turns into a *random rooted path* in T. For example, if $i = 1$, then $\widetilde{\alpha}$ reduces to the standard random walk in T.

The counterpart to the measure $\widetilde{\mu}^*_n$ is the composed random map $T_1 \to T' \to T$. This map contracts the distance to the root somewhat stronger than a random $T_1 \to T$ due to the contracting effect of randomness of the map $T' \to T$: the n-step random path in T typically has the distance $\approx n(i-2)/i$ for $i \geq 3$ from the root (observe that random maps to T_1 contract n to $\approx \sqrt{n}$; it may be amusing to look at iterated random selfmappings of trees and CAT(0)-spaces in general).

The harmonic spread of a radial measure on F_k can be explicitly computed since we know this spread for the spherical measures. In particular if some radial probability measure $\mu_\bullet$ on F_k has $\mu_\bullet(B(R)) \leq \epsilon$ for some R-ball in F_k, then $\underline{\text{h.spr}}(\mu_\bullet/\mu) \geq 0.1R(1-\epsilon)k/k-1$ for $k \geq 2$.

In order to apply this to $\widetilde{\mu}_n^*$, one needs to evaluate how much the measures $(\mu')^n(v \to\ )$ spread in V, i.e. one needs a bound on the $(\mu')^n(v \to\ )$-measures of the balls $B(v,R) \subset V$ for $R \ll n$. For example if these measures $\mu'_n(v,R) \overset{def}{=} (\mu')^n(v \to\ )B(v,R)$ satisfy the inequality

$$\int_V \mu'_n(v,R)\nu'(v)dv \leq \epsilon \tag{$*$}$$

for the stationary *probability* measure ν' on V, then, clearly,

$$\underline{\text{h.spr}}(\widetilde{\mu}_n^*/\mu) \geq 0.01R\big((1-\epsilon)k/k-1\big)^2,$$

(where, observe, the integral in $(*)$ equals the $\mu(v,v')$-measure of R-neighbourhood of the diagonal in $V \times V$).

As a basic example look at an i-regular graph V, where all vertices have degree $i \geq 3$. Then for all $R \leq \frac{1}{2}\text{girth}(V)$ and $n \geq g = \text{girth}(V)$ the above $(*)$ holds with $\epsilon \leq (0.9)^g$.

3.12.A From average to individual $\alpha : V \to F_k$. The measures μ_n^* on Γ, constructed for individual α are non-radial (unlike their average $\overline{\mu}_n^*$) but if $n \ll \log \operatorname{card} V$, then they are close to $\overline{\mu}_n^*$ for most α according to the *law of large numbers*. In fact the measure $\mu_n^*(\text{id} \to\ )$ on Γ is given by $N = N_n(\Gamma)$ real numbers for $N = \operatorname{card}(B(\text{id},n) \subset \Gamma)$ and should be thought of as a random $\mathbb{R}^N$-valued (actually $[0,1]^N$-valued) variable on the probability space $(\mathcal{A}, \mu^E)$ of symmetric maps $\alpha : \overset{\leftrightarrow}{E} \to (F_k, \mu = \mu(\text{id} \to\ ))$. If the space $\mathcal{A}$ is "large enough" compared to N, then $\mu_n^* = \mu_n^*(\alpha)$ is close to its average value (expectation) $\overline{\mu}_n^*$. Let us prove this in a rather special case that is sufficient for our applications. Look at the ball $B(v, 2n+2) \subset V$ around some vertex, consider the ratio of its cardinality, say $b_n(v)$, to that of V and let

$$\delta = \delta_n(V) = \sup_{v \in V} b_n(v)/\operatorname{card}(V).$$

Clearly, each (out of N) components w of μ_n^*, is δ-Lipschitz on $\mathcal{A}$ equipped with the *Hamming metric*. Indeed μ_n^* is obtained by summing pushforwards of $(\mu')^n$ over the translations in Γ, where the relevant translations are those moving the images of the vertices in V to $\mathrm{id} \in \Gamma$. If we change α at a single edge $e \in E$, this affects (at most) the δ-percentage of the terms in the sum, while each term changes at most by 1. It follows, by the (Levy–Milman) concentration of L_1-Lipschitz functions (see [L] and references therein) that w is close to its average $\overline{w}$ with overwhelming probability if $\delta\sqrt{\operatorname{card} E}$ is sufficiently small, say

$$\operatorname{prob}\bigl\{|w-\overline{w}| \geq C\delta\sqrt{\operatorname{card} E}\bigr\} \leq 100 \exp -0.01C$$

for all $C > 0$. Consequently

$$\operatorname{prob}\bigl(\mu_n^* - \overline{\mu}_n^* \geq C\delta\sqrt{\operatorname{card} E}\bigr) \leq 100N \exp -0.01C\,.$$

In our applications, the graphs (V, E) will have $\operatorname{card} E \leq 3 \operatorname{card} V$ and $b_n \leq (\operatorname{card} V)^{0.1}$ for relevant n (see below) thus allowing a bound on $\kappa(\Gamma)$ in terms of $\kappa(V)$ by the above discussion. Besides a uniform bound on $\kappa(V)$i, one must make certain that at least some groups Γ containing V in their Cayley graph, are *infinite*. (Finite groups are *trivially* T.) Such Γ's either come by an arithmetic route departing from $SL_n\mathbb{Z}$ or by means of small cancellation theory. (In the latter case, these groups are "generic" in any conceivable sense; on the contrary, the arithmetic groups appear very peculiar when seen from the point of view of the *combinatorial* group theory. If one enlists the essential group theoretic properties of $SL_3\mathbb{Z}$, for example, then a justifiable conjecture would be that such a group *does not* exist at all! In fact, add a single extra property to the list and most arithmetic groups will collapse to finite ones).

3.13 Useful expanders. Families of growing graphs V with a uniform positive lower bound on $\lambda_1(V) = 1 - \kappa(V)$ are produced either probabilistically, or number theoretically by taking the Cayley graphs of arithmetic groups modulo congruence subgroups. A bound on λ_1 for $SL_2\mathbb{Z}/SL_2p\mathbb{Z}$ was discovered by Selberg in (equivalent) terms of a uniform lower bound on the first eigenvalue ($\lambda_1 = 1 - \kappa$) of the Laplacian on the congruence covering of $H^2/SL_2\mathbb{Z}$. The relevance of this to the expanding property of graphs (defined purely combinatorially) was recognized by Margulis (see [M], [Lu], and references therein) who also proved the expander property for finite quotients of T-groups. Besides a uniform bound $\kappa(V) < 1$, we shall need a lower bound on the girth of V (the length of the shortest non-trivial simple cycle) that is available in the above expanders, as well as a bound on the cardinalities b_r of the n-balls in V. In combinatorics, one emphasizes *regular* expanders, where all vertices have the same degree, e.g. degree $= 3$. By subdividing each edge in such an i-regular V into j segments, one obtains V_j with $b_n(V_j) \leq b_{n'}(V)$ for $n'j \leq n$ and with $\lambda_1(V_j) \geq j^{-2}\lambda_1(V)$. Here is a summary of the relevant features of the resulting family of expanding graphs.

Selberg–Pinski–Margulis Theorem. *There exists a family of finite connected graphs $V_{ij} = (V_{ij}, E_{ij})$, $i, j = 1, 2, \dots$, with the following five properties,*

(i) $\mathrm{girth} V_{ij} = i$ *for all* $i, j = 1, 2 \dots$;
(ii) the degrees of the vertices in V_{ij} *are bounded by 3;*
(iii) $\lambda_1(V_{ij}) \geq 1/100j^2$;
(iv) $\mathrm{Diam}\, V_{ij} \leq 100i = 100\mathrm{girth} V_{ij}$ *(where the diameter of a graph is defined as the maximum of the lengths of the shortest paths of edges between the vertices);*
(v) the balls $B(n) = B(v, n) \subset V_{ij}$ *of radii* $n \leq i/2$ *satisfy*

$$2^{n/j} \leq \mathrm{card}\, B(v, n) \leq 10(2^{n/j})$$

for all $i, j = 1, \dots,$ *and all* $v \in V$.

3.13.A On bounds for $\kappa(\Gamma)$. If one wishes to obtain a specific bound on $\kappa(\Gamma)$, not just $\kappa(\Gamma) < 1$, one can use the *parabolic* growth inequality from 3.5. This needs to be applied only to κ-extremal maps (see 3.6) of X to *conical* spaces $Y \in \mathcal{Y}$ with an obvious generalization of the definition of harmonic spread (corresponding to $\kappa = 1$), to $\kappa < 1$.

3.13.B On f.p. property for non-isometric actions.[1] Consider the (simplest) case of Γ acting by affine transformations on a Hilbert space Y. If all transformations are $\mathcal{C}$-Lipschitz, i.e. the norms of the corresponding linear operators are bounded by $\mathcal{C}$, then either the action has a fixed point or there exists another such action of Γ on (another Hilbert space isomorphic to) Y possessing a *non-constant harmonic equivariant map* (orbit) $f : \Gamma \to Y$ (see 3.6). Then non-integrated μ^n-energy $E_{\mu^n}(\gamma) = \frac{1}{2}\int_\Gamma |f(\gamma) - f(\gamma')|_Y^2 \mu^n(\gamma \to \gamma') d\gamma'$ grows linearly in n,

$$E_{\mu^n}(\gamma) \geq \mathcal{C}^{-1}\, E_\mu(\gamma)$$

for the above $\mathcal{C}$ and all n and γ by the harmonic growth inequality, i.e.

$$\underline{\mathrm{h.spr}}_\mu(\mu^n/\mu) \geq \mathcal{C}^{-1},$$

where this spread is defined with the *infimum* of the ratios of the corresponding (non-integrated) energies over all $\gamma \in \Gamma$. On the other hand, the relation ($\overline{\mathrm{spr}}$) in 3.12 (obviously) remains valid in the form

$$\overline{\mathrm{h.spr}}_\mu(\mu_2^*/\mu_1^*) \leq \mathcal{C}\,\pi'.$$

[1]These were brought to my attention by V. Pestov (compare [P]).

This provides a lower bound on $\mathcal{C}$ in terms of $g = \text{girth}(V)$ and $\lambda_1 = \lambda_1(V)$ for groups Γ with an expander V (generically positioned in the Cayley graph of Γ),

$$\mathcal{C} \geq \mathcal{C}(g, \lambda_1) \underset{g\to\infty}{\longrightarrow} \infty \text{ for } \lambda_1 > 0\,.$$

In particular, if Γ contains above V_{ij} for a fixed j and a sequence $i_k \to \infty$, then *every affine uniformly Lipschitz action of* Γ *on a Hilbert space has a fixed point.*

(The existence of such *infinite* Γ follows from §2).

REMARKS. A careful look at the above argument shows that the bound on the norm of the linear parts of the γ-transformations for *all* $\gamma \in \Gamma$, can be relaxed to such a bound on γ's with $|\gamma| \leq \text{Diam}\,V$. Furthermore, one can extend the above to maps $f : \Gamma \to Y$ with bounded Laplacian Δf (or/and with $|\Delta^n f| \leq \mathcal{C}$ for some n and $\mathcal{C}$). In particular, one obtains the harmonic stability for such maps $f : \Gamma \to Y$ for an arbitrary affine action of Γ on a Hilbert space Y in the presence of arbitrary large expanders (in the Cayley graph of Γ). Yet it seems unclear if there are non-constant harmonic orbits in this case.

3.14 Random walk and recurrence. Every measure μ on a (discrete infinite) group Γ uniquely defines an equivariant random walk $\mu(\gamma_1 \to \gamma_2)$ by setting $\mu(\text{id} \to\ ) = \mu$, where we assume as earlier that

- • μ is symmetric; this can be equivalently expressed by $\mu(\gamma_1 \to \gamma_2) = \mu(\gamma_2 \to \gamma_1)$ or by $\mu(\gamma) = \mu(\gamma^{-1})$ for all $\gamma \in \Gamma$.
- •• the support of μ generates Γ.
- ••• the support of μ is finite (this is minor and not truly needed).

The basic example is seen in the Cayley graph X of Γ, where the standard random walk on (the vertices of) X equals that on Γ with μ assigning equal weight to the generators $g_i^{\pm 1} \in \Gamma$, $i = 1, \dots, k$.

Such a μ gives us the heat operator H_μ on functions $\Gamma \to \mathbb{R}$; here one may either forfeit the group action (with the definition extending to an arbitrary random walk) or, to interpret the H_μ as an operator on the L_2-space of function $\Gamma \to \mathbb{R}$, naturally acted upon by Γ, denoted $R = R_2(\Gamma)$ (also called the *regular representations*). Then one introduces $\kappa_{\text{reg}}(\Gamma) = \kappa(\Gamma, R)$ and the corresponding $\lambda_1 = \lambda_1(\Gamma) = 1 - \kappa_{\text{reg}}(\Gamma)$. Clearly

$$\kappa_{\text{reg}}(\Gamma) \leq \kappa(\Gamma)$$

and $\kappa_{\text{reg}} < 1$ if and only if the group Γ is *non-amenable* (actually, one can use this as a definition of amenability, where one has to check independence of the inequality $\kappa_{\text{reg}}(\Gamma) < 1$ of μ satisfying the above conditions.

If Γ is non-amenable, then the probability of recurrence $\mu_n(\mathrm{id} \to \mathrm{id})$ exponentially converges to zero. In fact, the limit $\lim = \lim_{n\to\infty}(\mu^n(\mathrm{id}\to\mathrm{id}))^{\frac{1}{n}}$ exists and equals κ_{reg}. Actually we shall need in sequel only the (obvious) inequality $\lim \leq \kappa_{\mathrm{reg}}$. Besides we shall use another inequality,

$$\kappa_{\mathrm{reg}}(\underline{\Gamma}) \leq \kappa(\Gamma)$$

for all factor groups $\underline{\Gamma}$ of Γ, that provides a simultaneous control on recurrence of factor groups of a given Kazhdan T group Γ, and that directly follows from the definitions of κ and κ_{reg}.

4 Scaling Limits and Entropy

Given a sequence of random events $\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_n, \dots)$ we define, following Gibbs, the entropy $\mathrm{ent}(\mathrm{ev})$ by

$$\mathrm{ent}(\mathrm{ev}) = \lim_{n\to\infty} n^{-1} \log \mathrm{prob}(\mathrm{ev}_n)\,, \qquad (*)$$

provided the limit exists. Otherwise we take the upper limit, denoted $\overline{\mathrm{ent}}(\mathrm{ev})$. We say that an event happens with *coentropy* $\leq \alpha \leq 0$, if the complementary event has $\overline{\mathrm{ent}}(\mathrm{ev}^\perp) \leq \alpha$.

4.1 Definition of $\mathbf{ent}_\Gamma(\epsilon)$. Let Γ be a group with left invariant metric $|\ |$ and diffusion μ. Since

$$\big(|\gamma_1| \leq r, |\gamma_2| \leq r\big) \Rightarrow |\gamma_1\gamma_2| \leq r_1 + r_2\,,$$

the entropy recurrence of the n-step random walk w (thought of as a word or path in the Cayley graph of Γ with the generating set $\mathrm{supp}\mu \subset \Gamma$) to the ball

$$B(n,\epsilon) = \big\{\gamma \in \Gamma \;\big|\; |\gamma| \leq n\epsilon\big\}$$

is well defined, i.e. the limit

$$\mathrm{ent}(\epsilon) = \mathrm{ent}\big(|w| \leq n\epsilon\big) \overset{def}{=} \lim_{n\to\infty} n^{-1} \log \mu^n(B(n\epsilon))$$

exists. Furthermore the function $\mathrm{ent}(\epsilon)$ is *concave* and increasing in ϵ.

Clearly $\mathrm{ent}(0) = \log \kappa_{\mathrm{reg}}(\Gamma)$ (defined in §3) and in the standard finitely generated case $\mathrm{ent}(\epsilon)$ is continuous at zero. In fact,

$$\mathrm{ent}(\epsilon) \leq \mathrm{ent}(0) + \epsilon \log(2k-1)\,, \qquad (+)$$

where k is the number of generators in Γ; more generally, (and obviously)

$$\mathrm{ent}(\epsilon) \leq \mathrm{ent}(0) + \epsilon\; \overline{\mathrm{ent}}|\Gamma|\,,$$

where

$$\overline{\mathrm{ent}}|\Gamma| \overset{def}{=} \limsup_{n\to\infty} n^{-1} \log \mathrm{card}(B(n))\,.$$

(This limit exists for word metrics on Γ but not in general).

It follows that *all infinite factor groups* $\underline{\Gamma}$ *of* Γ *have*

$$\mathrm{ent}_{\underline{\Gamma}}(\epsilon) \le \log\kappa(\Gamma) + \epsilon\,\overline{\mathrm{ent}}|\Gamma|\,. \qquad (\star)$$

4.2 Bound on $\overrightarrow{\mathrm{ent}}(\epsilon)$. Let $B_\gamma(n\epsilon) = \{\gamma' \in \Gamma \mid |\gamma - \gamma'| \le n\epsilon\}$ denote the $n\epsilon$-ball around γ and set

$$\overline{\mu}_n(\epsilon) = \sup_{\gamma\in\Gamma} \mu^n(B_\gamma(n\epsilon))\,,$$

$$\overrightarrow{\mathrm{ent}}(\epsilon) = \limsup_{n\to\infty} n^{-1}\log\overline{\mu}_n(\epsilon)\,.$$

It is clear that $\overrightarrow{\mathrm{ent}}(\epsilon) \ge \mathrm{ent}(\epsilon)$ and the two are equal in many (all?) cases. Furthermore, the above bound on $\mathrm{ent}(\epsilon)$ extends to $\overrightarrow{\mathrm{ent}}(\epsilon)$,

$$\overrightarrow{\mathrm{ent}}(\epsilon) \le \mathrm{ent}(0) + \epsilon\,\overline{\mathrm{ent}}|\Gamma|\,, \qquad (\star)$$

since, obviously,

$$\mu_n(\gamma) \le (\mu_{2n}(\mathrm{id}))^{1/2}$$

for all $\gamma \in \Gamma$. Consequently

$$\overrightarrow{\mathrm{ent}}_{\underline{\Gamma}}(\epsilon) \le \log\kappa(\Gamma) + \epsilon\,\overline{\mathrm{ent}}\Gamma \qquad \binom{\star}{\star}$$

for all infinite factor groups $\underline{\Gamma}$ of Γ.

4.3 Bound on $\overline{\mathrm{ent}}^{\square}(\epsilon)$. Given (words) $w_1, w_2 \in \Gamma$, set

$$|w_1 \underset{\gamma}{-} w_2| = |\gamma| + |\gamma w_1 - w_2|\,, \quad \gamma \in \Gamma$$

and

$$|w_1 \underset{\Gamma}{-} w_2| = \inf_{\gamma\in\Gamma} |w_1 \underset{\gamma}{-} w_2|\,.$$

We want to bound the entropy, and thus the probability, of the event $(|w_1 \underset{\Gamma}{-} w_2| \le \epsilon n)$ where w_1 and w_2 are μ-random (for given μ on Γ) *independent* words of length n_1 and n_2 with $n_1 + n_2 = n$. The probability of this event is bounded by

$$\sum_{|\gamma_1|\le\epsilon n_1}\ \sum_{|\gamma_2|\le\epsilon n_2}\ \sum_{w\in\Gamma} \mu_{n_1}(\gamma_1 w)\mu_{n_2}(\gamma_2 w)$$
$$\le \mathrm{card}\,B(\epsilon n_1)\cdot \mathrm{card}\,B(\epsilon n_2)\|\mu_{n_1}\|_{\ell_2}\cdot\|\mu_{n_2}\|_{\ell_2}\,.$$

Since

$$\|\mu_n\|^2_{\ell_2} \overset{def}{=} \sum_{\gamma\in\Gamma}\mu_n^2(\gamma) = \mu_{2n}(\mathrm{id})$$

by the symmetry of μ_n, we conclude that the corresponding entropy, denoted $\overline{\mathrm{ent}}^{\square}_{\Gamma}(\epsilon)$, satisfies

$$\overline{\mathrm{ent}}^{\square}(\epsilon) \le \mathrm{ent}(0) + \epsilon\,\overline{\mathrm{ent}}|\Gamma|\,, \qquad (+)$$

and, again, this entropy is bounded for all factor groups $\underline{\Gamma}$ of Γ by

$$\overline{\mathrm{ent}}^{\square}_{\underline{\Gamma}}(\epsilon) \leq \log(\kappa(\Gamma)) + \epsilon\, \overline{\mathrm{ent}}|\Gamma|\,. \qquad \binom{+}{+}$$

4.4 Entropy of displacements. Given an isometry of a metric space, $\gamma : X \to X$, set

$$di_\gamma = di_\gamma(x) = \big|\gamma(x) - x\big|_X\,.$$

If $X = \Gamma$ with the left (isometric) translation $\gamma : \Gamma \to \Gamma$, then

$$di_\gamma(\gamma_\bullet) = |\gamma_\bullet^{-1}\gamma\gamma_\bullet|_\Gamma = \big|[\gamma_\bullet^{-1},\gamma]\gamma^{-1}\big|_\Gamma$$

and this $di_\gamma(\gamma_\bullet)$ is related to the size of the commutator denoted

$$co_\gamma(\gamma_\bullet) \overset{def}{=} \big|[\gamma_\bullet^{-1}\gamma]\big|_\Gamma\,,$$

by

$$co_\gamma - |\gamma| \leq di_\gamma \leq co_\gamma + |\gamma|\,.$$

We want to evaluate the μ^n-measures of the subsets

$$\{di_\gamma \leq \rho\} = \big\{\gamma_\bullet \in \Gamma, di_\gamma(\gamma_\bullet) \leq \rho\big\} \subset \Gamma$$

and

$$\{di_\Delta \leq \rho\} = \bigcup_{\gamma\in\Delta}\{di_\gamma \leq \rho\} \subset \Gamma$$

for (large) $\rho \in \mathbb{R}_+$ and specific $\Delta \subset \Gamma$ (namely, for balls $B(n\epsilon) \subset \Gamma$) as well as the measures of the corresponding subsets $\{co_\Delta \leq \rho\} \subset \Gamma$. Clearly,

$$\{co_\Delta \leq \rho\} \subset \big\{di_\Delta \leq \rho + |\Delta|_\Gamma\big\}$$

and

$$\{di_\Delta \leq \rho\} \subset \big\{di_\Delta \leq \rho + |\Delta|_\Gamma\big\}$$

for

$$|\Delta|_\Gamma \overset{def}{=} \sup_{\gamma\in\Delta} |\gamma|_\Gamma\,.$$

We limit ourselves at this point to *finitely* generated *hyperbolic* groups Γ, where the metric is associated to the generators as the word metric. To avoid irrelevant complications we assume that Γ is *faithful*, i.e. the canonical action of Γ on the *ideal boundary* $\partial_\infty\Gamma$ is faithful.

4.4.A Density Lemma. *For each $\gamma \in \Gamma$, there exists a subgroup $C'_\gamma \subset \Gamma$ that centralizes an element $\gamma' \in \Gamma$ conjugate to γ, i.e. $\gamma' = \gamma_\bullet\gamma\gamma_\bullet^{-1}$ for some $\gamma_\bullet = \gamma_\bullet(\gamma) \in \Gamma$, such that the subset $\{di_\gamma \leq \rho\} \subset \Gamma$ is contained in the ρ'-neighbourhood of the corresponding left coset of C'_γ, namely,*

$$\{di_\gamma \leq \rho\} \subset \gamma_\bullet C'_\gamma + \rho'$$

where $\rho' \leq \frac{1}{2}\rho + \mathrm{const}_\Gamma$.

Proof. If Γ acts on a δ-hyperbolic geodesic space, then the subsets $\{di_\gamma \le \rho\} \subset X$ satisfy

$$\{di_\gamma \le \rho_0 + 2\rho - 10\delta\} \subset \{di_\gamma \le \rho_0\} + \rho$$

for all $\rho \ge 0$ and $\rho_0 \ge 20\delta$ (see [G5]) and thus the lemma is reduced to the case of $\rho_0 \le 20\delta$, where the required "density" of the centralizer in the sets $\{di_\gamma \le \rho_0\}$ is also established in [G5].

4.5 Definition and evaluation of $\overline{c_\bullet \mathrm{ent}}(\epsilon)$. Let Δ_n denote the ball $B(n\epsilon)$ in Γ minus $\{\mathrm{id}\}$ and set

$$\overline{c_\bullet \mathrm{ent}}(\epsilon) = \limsup_{n\to\infty} \tfrac{1}{n} \log \mu_n\{di_{\Delta_n} \le \epsilon n\}\,.$$

It follows from the above that

$$\mu_n\{di_{\Delta_n} \le \epsilon n\} \le \mathrm{const}_\Gamma \cdot \mathrm{card}(\Delta_n) \cdot \mathrm{card}\, B(\tfrac{\epsilon n}{2}) \overline{\mu}_n(\mathcal{C})$$

where $\mathcal{C}$ denotes the collection of the centralizers of all $\gamma \in \Gamma \backslash \{\mathrm{id}\}$. Consequently,

$$\overline{c_\bullet \mathrm{ent}}(\epsilon) \le \log \kappa + \tfrac{3}{2}\epsilon\, \overline{\mathrm{ent}}|\Gamma|\,. \qquad (\star)$$

Remark. If Γ has no torsion, or, more generally, if the centralizer of all torsion elements are virtually cyclic, then, clearly

$$\overline{c_\bullet \mathrm{ent}}(\epsilon) \le \log \kappa_{\mathrm{reg}} + \tfrac{3}{2}\epsilon\, \overline{\mathrm{ent}}|\Gamma|\,.$$

4.6 ϵ-geometry of random paths in Γ. Take a small, yet positive ϵ (specified later on) and let w be a random path of length n in Γ, thought of as a map of the segment $[0,n]$ into the Cayley graph $X^1 = X^1(\Gamma)$ of Γ. Let us bound the coentropy of the event expressed by the following properties of $w : [0,n] \to X^1$, where $n \to \infty$.

($\mathbf{pr_1}$) Quasiisometry: For every two points $t_1, t_2 \in [0,n]$ the distance between $w(t_1)$ and $w(t_2)$ in X^1 satisfies

$$\big|w(t_1) - w(t_2)\big| \ge \epsilon|t_1 - t_2| - 10\log n\,, \qquad (*)_\epsilon$$

where the metric in $X^1 \supset \Gamma = X^0$ equl27.35.9(sth)-9.r on Γ for *integer* t_1, t_2 (with $w(t_i)$ l ndi i the vertex set $X^0 = \Gamma$ of X^1) -31.2d i27.368.4(in)23.8(ter)-8.1(p)-42.2(olated)-383.2(f)-6.2(r
metric, th27.39027.400.9(-31.6)n ody (geodesic) g-31.6h metric assign ut le
to dg31.2in X^1.

($\mathbf{pr_2}$ $_2$) 27.335.4(c)0.5(on)23.7(tiguh)-9.3(ed)-350.3 ρ-neighbour of the corresponding im-ge $'$ ^{1} ^{1} of w,

$$w' \subset w\big([t_1,t_2]\big) + \rho \text{ for } \rho \le 100\delta\log n\,, \qquad (*)_n$$

and

$$w\big([t_1,t_2]\big) \subset w' + 100(1+\delta)\log n \qquad (*)^n$$

for all $t_1, t_2 \in [0, n]$ and for the hyperbolicity constant δ of the metric in Γ.

(pr$_3$) Separation of $\gamma(w)$ from w: For every $\gamma \in \Gamma \backslash \{\mathrm{id}\}$ the intersection of the ρ-neighbourhoods of the images of w and γw in X^1 satisfies

$$\mathrm{Diam}(w+\rho) \cap (\gamma w + \rho) \leq 10\rho + 1000(1+\delta)\log n \qquad (\star)_n$$

We claim, that if ϵ is sufficiently small depending on k and $\kappa_{\mathrm{reg}}(\Gamma)$, e.g.

$$\epsilon \leq (1 - \kappa_{\mathrm{reg}}(\Gamma))/100k$$

then the coentropy of the events pr_1, pr_2, pr_3 for a δ-hyperbolic Γ is negative and therefore all three properties are satisfied simultaneously with overwhelming probability for $n \to \infty$. In fact, pr_1 and pr_2 do not need the group structure (action), nor do they require the metric to be geodesic, as these directly follow from (+) in 4.1 and the standard (approximation by trees from [G5]) properties of hyperbolic spaces. On the other hand, pr_3 follows from $\binom{+}{+}$ in 4.3, and $(\star)$ in 4.5, since the bound $(\star)_n$ reduces (by the standard hyperbolic geometry and $\mathrm{pr}_1 \wedge \mathrm{pr}_2$) to excluding the following two (bad) possibilities.

1. The words w and γw are mutually close to each other along two subwords $w([t_1, t_2])$ and $w\gamma w([t'_1, t'_2])$, where the subsegment $[t_1, t_2]$ and $[t'_1, t'_2]$ are disjoint in $[0, n]$. This event has a small entropy by $\binom{+}{+}$.
2. The word $\gamma w([t_1 t_2])$ comes close to $w[t'_1, t'_2]$. This is ruled out (with negative coentropy) by $(\star)$ (where one should be aware of the possibility of γ switching the ends $w(t_1)$ and $w(t_2)$ in X^1 but this causes no problem as one can see readily). Notice, that only at this point do we need the full set of assumptions on Γ.

Remark. All one needs of "$\log n$" as far as the present applications go is the asymptotic relation $\log n = o(n)$.

4.7 Random trees and forests in Γ. Let us generalize the above to trees (or forests) T randomly mapped to Γ, or rather to the Cayley graph $X^1 = X^1(\Gamma)$. The basic examples come from graphs (V, E) with random maps $\alpha : \overleftrightarrow{E} \to \Gamma$, where the relevant T's are subtrees in V with $\alpha|T$ lifted to $X^1(\Gamma)$. Given a (finite) forest $\mathcal{T}$, that is a disjoint union of (finitely many finite) trees, denote by $b_m = b_m(\mathcal{T})$, $m = 1, 2, \dots$, the supremum of the (vertex) cardinalities of the m-balls in $\mathcal{T}$ at all vertices $v \in \mathcal{T}$ (where one uses the standard distance within each tree in $\mathcal{T}$) and for a sequence $\mathcal{T}_n$ set

$$\overline{b} = \overline{\mathrm{ent}}|\{\mathcal{T}_n\}| = \limsup_{\substack{m\to\infty \\ n\to\infty}} m^{-1} \log b_m\,.$$

Assume $\operatorname{diam} \mathcal{T}_n = n$, where the diameter of a forest is *defined* as the sum of diameters of the trees in it, and observe, obviously generalizing from the case $\mathcal{T}_n = [0,n]$, that if $\bar{b} > 0$ is small enough in comparison to $-\log \kappa(\Gamma) > 0$ and if $\epsilon > 0$ is sufficiently small, e.g.

$$\epsilon \leq \big(1 - \kappa(\Gamma)\big)/1000k$$

then, with overwhelming probability, random maps ("branched words") $w : \mathcal{T} \to X^1$ have the following properties:

$$\left.\begin{array}{ll} (1) & |w(t_1) - w(t_2)| \geq \epsilon |t_1 - t_2| - 10 \log n \\ (2) & w' \subset w([t_1,t_2]) + 100\delta \log n \\ (2') & w([t_1,t_2]) \subset w' + 100(1+\delta) \log n \\ (3) & \operatorname{Diam}(w+\rho) \cap (\gamma w + \rho) \leq 10\rho + 1000(1+\delta)\log n\,, \end{array}\right\} \quad (\star_*)$$

where $[t_1,t_2] \subset \mathcal{T}$ denotes the segment between t_1 and t_2 whenever they lie in the same tree with the rest of the notation and assumptions following those in 4.6.

4.8 Geometry of random relations. Let us return to random relations $W_\alpha \subset \Gamma$ associated to random $\alpha : \overset{\leftrightarrow}{E} \to \Gamma$ for graphs (V,E) where, specifically, we shall deal with the sequence of expanders $V_i = V_{ij_0} = V_{ij_0}, E_{ij_0})$ from 3.12 for a fixed large $j_0 \geq j_0(k, \kappa_{\mathrm{reg}}(\Gamma))$ and $i = \operatorname{girth} V_i \to \infty$. These V_i have small $\bar{b}$ and therefore every subtree (or subforest) $T \subset V_i$, when lifted to $X'(\Gamma)$ satisfy $(\star_*)$ with overwhelming probability for $i \to \infty$ (and playing essentially the same role as the above n). It follows, that each lift of α to the universal covering (tree), say $\tilde{\alpha} : \tilde{V} \to X^1$ also satisfies $(\star_*)$ with (essentially the same) overwhelming probability by the local $\Rightarrow$ global principle for δ-hyperbolic spaces (see [G5]), where it is shown, in particular that "local quasiconvexity" implies the global one with some loss in constants, but this is absorbed by our generous 10, 100, etc. in the inequalities $(\star_\star)$. It follows that:

Whenever we start with a faithful hyperbolic group Γ with k generators, then, generically, the corresponding quotient group $\Gamma_1 = \Gamma/[W_\alpha]$ is also faithful hyperbolic for all sufficiently large i_1, the quotient map $\Gamma \to \Gamma_1$ is injective on the ball of radius $\approx i$ in Γ and the map of $V_i = (V_i E_i)$ to the Cayley graph $X^1(\Gamma_1)$ is essentially one-to-one, as it is quasiisometric in the sense of (pr_1) in 4.6.

Then we take another value, say i_2 much larger than i_1 (actually $i_2 \approx \exp i_1$ is sufficient in the present discussion) and we can pass to Γ_2, provided the same j_0 serves Γ_2 as well as Γ_1. This j_0 depends on k, that is the number of generators of Γ and on the recurrence rate $\kappa_{\mathrm{reg}}(\Gamma)$. The latter may, a priori, approach 1, and then the j's go to infinity destroying the expander property of the graphs. However, if Γ is Kazhdan T, then all factor groups have $\kappa_{\text{reg}} \leq \kappa(\Gamma) < 1$ and this difficulty does not present itself. So one could *start* with a T-group Γ but, in truth, there is no need for this: if not Γ itself, then Γ_1 (and thus all following quotient groups) is T. Moreover, every isometric action of Γ_1 on a complete regular CAT(0)-space has a fixed point. This allows one to have the relations associated to a random map of (V_∞, E_∞) equal the disjoint union of V_{i_ν}, j_0 for an $i_\nu \underset{\nu\to\infty}{\longrightarrow} \infty$ and a *fixed* (large) j_0. Here is the list of the essential (generic) properties of the resulting factor group Γ_∞ one can obtain by this process:

(1) Γ_∞ is infinite, and if one starts with $\Gamma = F_k$, then the presentation with which Γ_∞ is (naturally) built is aspherical. This follows from the above and §2.
(2) Γ_∞ admits no uniform embedding into the Hilbert space, not even into any CAT(0)-space with bounded singularities. This follows from §3.
(3) Γ is Kazhdan T; moreover every isometric action of Γ_∞ on a complete CAT(0)-space with bounded singularities (e.g. on a regular one) has a fixed point. Furthermore, every affine action on the Hilbert space with bounded linear parts also has a fixed point. This follows from §3.
(4) Γ_∞ embeds into the fundamental group of a closed aspherical 4-manifold (where one should start with $\Gamma = F_2$ and replace "random" by "quasi-random" everywhere). This follows from (1) and an unpublished theorem by Ol'shanskiy and Sapir.
(5) The above properties are resilient under adding (extra) thin random relations.

4.9 Final remarks. (a) The above estimates on the constants involved are rather crude. Looking carefully through the proofs, one can see that everything works just as well if $i_{\nu+1} \geq \text{const}\, i_\nu$ (instead of $i_{\nu+1} = \exp i_\nu$ as we had above) for a *fixed* (possibly large) universal constant, where a specific evaluation of the critical value of this "const" has been done so far only for the simplest small cancellation scheme by counting Dehn diagrams (see [G2], where one should keep track of the diagrams containing several cells corresponding to the same relation; this was pointed out to me by Yann Ollivier).

(b) The passage from "probability" to "entropy", borrowed from Gibbs' formalism, is a particular instance of ("log" of) the (multiplicative scaling) limit for $n \to \infty$ of the semirings $R_n = (\mathbb{R}_n^\times, \times)$, where $a +_n b = (a^n + b^n)^{1/n}$

and where one obtains, by taking "log" in the limit, the idempotent algebra $(\mathbb{R}, \vee, +)$ for $a \vee b = \max(a, b)$.

(c) In many cases (e.g. for Gibbs ensembles with short range interaction) the events ev_n can be approximated by simultaneous occurrence of m (almost) independent events with m only slightly smaller than n, say $m = O(n)$; moreover, the corresponding probabilities $P(\mathrm{ev}_n)$ appear as total masses of certain measure spaces $\mathcal{M}_n$ (often subset in the powers of probability spaces). Then one is tempted to take a geometric (multiplicative) scaling limit $\lim_{n\to\inf}(\mathcal{M}_n)^{1/n}$, where the n-th root $(\mathcal{M}_n)^{1/n}$ is taken in a suitably extended category of measure spaces; for example, $\mathcal{M}_n$ could be "functorially approximated" by a Cartesian power of an actual measure space, say by $(\mathcal{M}'_n)^m$ for $m = O(n)$.

(d) When a probability measure μ_n is attached to a metric space X_n (e.g. a fixed space, like Γ, or, variable, e.g. the space of paths of length n) then one may also scale X_n by n^{-1} and go to the Hausdorff (ultra) limit X_∞ along with the Gibbs limit. Then the entropy emerges as a $\vee$-additive) function on certain (constructible) subsets in X_∞ defined with suitable observables on X and/or on X_∞. For example, the entropy $\mathrm{ent}(\epsilon)$ corresponds to observable $\gamma \mapsto |\gamma|$ on Γ, where we neglect its growth beyond its mean value (and do not use the Gibbs–Legendre transform. In fact, this works well for hyperbolic Γ). The problem in general is to identify (classes of) observables with good properties (e.g. some concavity in the Gibbs–Hausdorff limit.

(e) The properties (pr_1)–(pr_3) and their modifications suggest possible *definitions of mean hyperbolic* groups and/or spaces X, where the standard hyperbolic features appear on *generic* (not all!) paths and/or subtrees in X and where the ultimate theory, probably, needs to be set into the framework of Gibbs–Hausdorff (limit) spaces.

(f) Let us indicate an instance of implementations of the above (d) (and, in part (e)) by introducing "circular" entropy with true convergence and concavity unlike the "square" $\mathrm{ent}^{\square}$. For a word $\underline{w} = w_1 w_2, \ldots, w_n$, $w_i \in \Gamma$, consider all w' obtained from w by making $k \leq \kappa n$, for a fixed κ, insertions of $\gamma_1, \ldots \gamma_k$ into $\underline{w}$, i.e.

$$\underline{w}' = \gamma_1 w_1 \ldots w_{n_1} \gamma_2 w_{n_1} + 1, \ldots, w_n\,,$$

such that $\underline{w}' = \mathrm{id}$ in Γ. Denote by $|\underline{w}|_\kappa$ the minimal total "length" $\sum_{i=1}^{k} |\gamma_i|$ among all these $\underline{w}'$. Observe that this is a true norm, on the group $F(\Gamma)$ freely generated by all $w \in \Gamma$, i.e. symmetric and satisfying the triangle inequality $|\underline{w}_1\underline{w}_2|_\kappa \leq |\underline{w}_1|_\kappa + |\underline{w}_2|_\kappa$, where the zero ball contains the kernel of the tautological homomorphism $\tau : F(\Gamma) \to \Gamma$.

Now think of $\underline{w}$ as a sample of the n-step random walk in (Γ, μ), denote by $\underline{P}(n, \kappa, \epsilon)$ the μ_n-measure of the $\underline{w}'$s with $|\underline{w}|_\kappa \leq \epsilon n$ and set $\mathrm{ent}^\kappa(\epsilon) \overset{def}{=} \lim_{n\to\infty} n^{-1} \log P(n, \kappa, \epsilon)$. Observe that the limit exists and $\mathrm{ent}^\kappa(\epsilon)$ is concave in ϵ since $|\ |_\kappa$ is a norm. Furthermore, $\mathrm{ent}^\kappa(\epsilon)$ is monotone decreasing in κ and let

$$\mathrm{ent}^\circ(\epsilon) \overset{def}{=} \lim_{\kappa\to 0} \mathrm{ent}^\kappa(\epsilon)\,.$$

Clearly $\mathrm{ent}^\circ \geq \overline{\mathrm{ent}}^\square$ and one sees as earlier, that

$$\mathrm{ent}^\circ(\epsilon) \leq \mathrm{ent}(0) + \epsilon\ \mathrm{ent}|\Gamma|$$

for Γ's with word metrics and standard random walks.

(g) Another amusing entropy (depending on (Γ, μ) but not on the metric) is associated with the *commutor "norm"* $|w|^{[\ ]}$, that is the minimal number of commutors $[\gamma_1\gamma_2], [\gamma_3\gamma_4], \dots$ whose product equals γ. (If $H_1(\Gamma) = \Gamma/[\Gamma,\Gamma] \neq 0$, one should add some norm on H_1 to $|\gamma|^{[\ ]}$.) More generally, one may consider the minimal genus g of a surface bounding k random (loops represented by) random words $w_1, \dots, w_k$ of lengths $n_i\epsilon$, $\Sigma n_i = n$, and consider the corresponding $\mathrm{ent}(g \leq n\epsilon)$. This may admit an analytical expression for free groups and also for (fundamental groups of) hyperbolic manifolds with the Riemannian diffusion.

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