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Pinching constants for hyperbolic manifolds

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Summary. We show in this paper that for every $n \geqq 4$ there exists a closed n-dimensional manifold V which carries a Riemannian metric with negative sectional curvature K but which admits no metric with constant curvature $K \equiv -1$. We also estimate the (pinching) constants H for which our manifolds V admit metrics with $-1 \geqq K \geqq -H$.

0. Introduction

0.1. Basic construction. Start with an orientable manifold V with $K(V) \equiv -1$ and let V' be an oriented totally geodesic submanifold of codimension two in V, such that the homology class $[V'] \in H_{n-2}(V)$ is trivial. (See §1 for examples of such V and V'.) Then there exists a cyclic ramified (at V') covering $\tilde{V}^i \to V$ of a given order $i = 1, 2, \ldots$. It is not hard to construct a metric $\tilde{g}^i$ with $K \leqq -1$ on every manifold $\tilde{V}^i$ (see §2).

Let us estimate the volume of $(\tilde{V}^i, \tilde{g}^i)$. To do this we triangulate V in such a way that V' becomes a subcomplex, then lift this triangulation to $\tilde{V}^i$ and order the vertices of the lifted triangulation. Then, by induction on $k = 1, \ldots, n = \dim(V)$, we straighten all lifted k-simplices $\Delta^k \subset \tilde{V}^i$ as follows (compare [Th]). We lift Δ^k to the universal covering of $\tilde{V}^i$, replace it by the geodesic cone (for the metric $\tilde{g}^i$) from the first vetex (with respect to the ordering) over the opposite $(k-1)$-face (which has already been straightened out) and then project this cone back to $\tilde{V}^i$. Thus we obtain a covering of $\tilde{V}^i$ by iN straightened n-simplices, where $N = N(V, V')$ is the number of n-simplices of the original triangulation of V. Since $K(\tilde{g}^i) \leqq -1$ every straightened k-simplex has volume $\leqq \pi$ for $k \geqq 2$. (See [Th, Gr 1, I-Y].) Therefore volume grows at most in proportion to the degree of the covering:

$$\mathrm{Vol}(\tilde{V}^i, g^i) \leqq C i = N i \pi.$$

Suppose that there were a metric of constant curvature $K \equiv -1$ on $\tilde{V}^i$. Consider the action of the cyclic group $\mathbf{Z}_i$ on $\tilde{V}^i$ by deck transformations. If $n = \dim \tilde{V}^i \geqq 3$, then by the Mostow rigidity theorem there exists an isometric action of $\mathbf{Z}_i$ on $\tilde{V}^i$ whose fixed point set $V'' \subset \tilde{V}^i$ is diffeomorphic to V' and whose quotient $\overline{V}^i = \tilde{V}^i/\mathbf{Z}_i$ has a natural orbifold structure of constant curvature $K \equiv -1$. Observe that no two orbifolds $\overline{V}^i$ are isometric (though the topological spaces $\overline{V}^i$ may be homeomorphic to V) and that

$$\operatorname{Vol}(\overline{V}^i) = \frac{\operatorname{Vol} \tilde{V}^i}{i} \leqq C.$$

In the case that $n = \dim \overline{V}^i \geqq 4$, Wang's finiteness theorem (see [W]) for locally symmetric orbifolds can be applied. In particular, this theorem asserts for every $n \geqq 4$ that there are at most finitely many isometry classes of n-dimensional orbifolds $\overline{V}$ with $K(\overline{V}) \equiv -1$ and with $\operatorname{Vol} \overline{V} \leqq$ constant. (There are infinitely many three-dimensional manifolds V with $K(V) \equiv -1$ and $\operatorname{Vol}(V) \leqq 2.03$ - see [Th].) This obviously contradicts our assumption $K(\tilde{g}_i) \equiv -1$ for $i \geqq i_0$, so *the manifolds* $\tilde{V}^i$ *for* $i \geqq i_0$ *admit no metrics with* $K \equiv -1$.

0.2. Remark. It is likely that none of the $\tilde{V}^i$ $(i = 2, 3 \ldots)$ admit a metric with $K \equiv -1$. (Compare 3.6.)

0.3. Manifolds with curvature close to -1. Denote by $\operatorname{Rad}^{\perp} V'$ the *normal injectivity radius* of $V' \subset V$, that is, the greatest real number ρ for which the open ρ-neighborhood $U_\rho \subset V$ of V' is diffeomorphic to the normal bundle of V' in V. We shall see in §2 that every ramified covering $\tilde{V}^i$ of V ramified over V' admits a metric $\tilde{g}_i$ such that

$$-1 \geqq K(\tilde{g}_i) \geqq -1 - \varepsilon^2,$$

where $\varepsilon \leqq 10\rho^{-1} \log 2i$ for $\rho \geqq 10$. Using this we construct in §3 finite ramified coverings $\tilde{V}_i$ of certain manifolds V_i of a given dimension $n \geqq 4$, such that the manifolds $\tilde{V}_i$ admit metrics $\tilde{g}_i$ with $-1 \geqq K(\tilde{g}_i) \geqq -1 - \varepsilon_i^2$, where $\varepsilon_i \to 0$ for $i \to \infty$ and such that none of $\tilde{V}_i$ admits a metric with $K \equiv -1$.

0.4. Remark. The above example contrasts sharply with the pinching theorems for $K > 0$, where the pinching $1 \leqq K(V) < 4$ implies that V is homeomorphic to S^n if $\pi_1(V) = 0$ and where the stronger inequality $1 \leqq K(V) \leqq 1.4$ ensures the existence of a metric with $K \equiv 1$ on V for all V (see (G-K-R]).

0.5. Unpinchable manifolds. The Wang theorem can be generalized to n-dimensional orbifolds with $-1 \geqq K \geqq -x_0$. Namely, if $n \geqq 4$, there are at most finitely many diffeomorphism classes of n-dimensional orbifolds $\overline{V}$ with $-1 \geqq K(\overline{V}) \geqq -x_0$ and with $\operatorname{Vol} \overline{V} \leqq$ const, where the diffeomorphisms are understood in the category of orbifolds and where the implied bound on the number of non-diffeomorphic manifolds $\overline{V}$ depends on n, x_0 and the constant (compare [Gr2]). It follows that for every $x \geqq 1$ there exists an integer $i_0 = i_0(V, V')$ such that the ramified covering $\tilde{V}^i$ for $i \geqq i_0$ admits no $\mathbf{Z}_i$-equivariant metric with $-1 \leqq K \leqq -x$ (though it admits a $\mathbf{Z}_i$-invariant metric with $K < 0$ for all $i \geqq 1$). Moreover, we shall prove in §4, under some additional assumptions on V and V', that the equivariance condition is redundant, and thus exhibit for every $n \geqq 4$ and $x > 1$ a compact manifold $\tilde{V}$ with a metric of negative curvature which carries no metric with $-1 \geqq K \geqq -x$.

0.6. Remark. The first example of a compact manifold with $K<0$ which admits no locally symmetric metric is due to Mostow and Siu (See [M-S]). Namely, they produced a 4-dimensional manifold V with a Kähler metric of negative curvature such that the signature $\sigma(V)$ does not vanish and the Euler characteristic $\chi(V)$ satisfies $\chi(V) \neq 3|\sigma(V)|$.

§ 1. Examples of ramified coverings

1.1. Let $\Phi_n = \sum_{i=1}^{n} x_i^2 - \sqrt{2} x_{n+1}^2$ and denote by Γ_n the group of automorphisms of the form Φ_n over the ring of integers of the field $\mathbf{Q}(\sqrt{2})$. This group acts naturally on the hyperbolic space H^n and the action is discrete and cocompact. We consider subgroups $\Gamma \subset \Gamma_n$ of finite index and study the orbifolds $H^n \backslash \Gamma$, which are called *Φ_n-orbifolds*. If there are no torsion elements in Γ the orbifold $H^n \backslash \Gamma$ is actually a manifold of constant curvature -1.

1.2. Lemma. *Take a Φ_n-manifold V, for $n \geqq 2$, and fix a positive real ρ. Then there exists a finite covering $\tilde{V} \to V$ and an oriented totally geodesic hypersurface $\tilde{V}_0 \subset V$ such that:*

(i) $\operatorname{Rad}^{\perp} \tilde{V}_0 \geqq \rho$;
(ii) *the class $[\tilde{V}_0] \in H_{n-1}(\tilde{V})$ vanishes;*
(iii) *every connected component of $\tilde{V}_0$ is a Φ_{n-1}-manifold.*

Proof. Take a covering $\tilde{V}$ whose fundamental group is a *normal* subgroup in Γ_n. Then the reflection $x_1 \mapsto -x_1$ defines an involution $\tilde{I}\colon \tilde{V} \to \tilde{V}$ whose fixed point set $\tilde{V}_0$ covers $H_{n-1} \backslash \Gamma_{n-1} \subset H_n \backslash \Gamma_n$ under the obvious map $\tilde{V} \to H_n \backslash \Gamma_n$, where the hypersurface $H_{n-1} \subset H_n$ corresponds to the subspace $\{x_1 = 0\}$ in $\mathbf{R}_{n+1}$. Now choose $\tilde{V}$ with a sufficiently large injectivity radius (compared to ρ) at all $\tilde{v} \in \tilde{V}$, thus making $\operatorname{Rad}^{\perp} \tilde{V}_0 \geqq \rho$. Furthermore, we may assume (by passing to a double covering of $\tilde{V}$ if necessary) that the manifold $\tilde{V} \backslash \tilde{I}$ with boundary $\partial(\tilde{V} \backslash \tilde{I}) = \tilde{V}_0$ is orientable, and we take the induced orientation on $\tilde{V}_0$. Clearly $[\tilde{V}_0] = 0$ and the proof is concluded.

By applying the lemma twice, first to V and then to $\tilde{V}_0$, we obtain the following

1.3. Corollary. *For every $n = 2, 3, \ldots$ and every $\rho > 0$ there exists an orientable Φ_n-manifold V and a (possibly disconnected) totally geodesic oriented submanifold V' of codimension 2 in V such that*

(i) $\operatorname{Rad} V_0' \geqq \rho$;
(ii) $|V'| = 0$; *moreover, V' bounds a connected oriented totally geodesic hypersurface $V_0 \subset V$.*

1.4. Remark. The hypersurface V_0 lifts to an i-paged open book $\sum \tilde{V}_0^i$ in the $\mathbf{Z}_i$-ramified covering $\tilde{V}_i$ of V. This book, consisting of i copies of V_0 joined together at V' is in fact a *totally geodesic* (singular) subspace in $\tilde{V}^i$ with the singular Riemannian metric induced from V.

§ 2. Metrics of negative curvature on ramified coverings

2.1. Lemma. *Take real numbers* $i>1$ *and* $\rho>0$ *and set* $s(r)=\frac{1}{2}(e^r-e^{-r})$. *There exists a positive* C^∞ *function* $\sigma(r)=\sigma_{i,\rho}(r)$ *for* $r\geqq 0$ *such that*

(i) $\sigma'(r)>0$ *and* $\sigma''(r)>0$ *for* $r>0$;

(ii) $\sigma(r)=s(r)$ *for small* $r\geqq 0$ *and* $\sigma(r)=is(r)$ *for* $r\geqq\rho$;

(iii) the ratio σ''/σ *satisfies*

$$(C(i,\rho))^{-1}\leqq\frac{\sigma''}{\sigma}\leqq C(i,\rho),$$

where $C(i,\rho)\to 1$ *for* $\rho\to\infty$ *and* i *fixed. In fact, one can find* σ *for which*

$$|1-C^2|\leqq 10\,\rho^{-1}\log 2i$$

for all $\rho>10$;

(iv) the ratio $\sigma' s/\sigma s'$ *satisfies*

$$(C(i,\rho))^{-1}\leqq\frac{\sigma' s}{\sigma s'}\leqq C(i,\rho)$$

for the above $C(i,\rho)$.

The proof is an elementary exercise which is left to the reader.

2.2. Recall that the hyperbolic metric in H^2 can be written in polar coordinates as $dr^2+s^2(r)\,d\theta^2$, where $r\geqq 0$ denotes the distance to a fixed point $x\in H^2$ and $\theta\in[0,2\pi]$ is the length parameter on the unit circle. Consider the Z_i-covering $\tilde H_i^2\to H^2$ ramified at x. Then the induced metric (outside the lift of x to $\tilde H_i$) can be written as $dr^2+i^2s^2(r)\,d\theta^2$ for the same r and θ. This metric can be smoothed at x by the metric $dr^2+\sigma^2(r)\,d\theta^2$, for σ(r) as in the above lemma. The sectional curvature of this metric equals $-\sigma''(r)/\sigma(r)$. Hence, it is strictly negative and close to -1 for ρ large compared to i.

2.3. Take the hyperbolic space H^n for $n\geqq 2$ and a subspace $H^{n-2}\subset H^n$. The hyperbolic metric on H^n can be written as $dr^2+s^2(r)\,d\theta^2+c^2(r)\,dx^2$, for the above r and θ, where $c(r)=\frac{1}{2}(e^r+e^{-r})$ and dx^2 stands for the hyperbolic metric in H^{n-2}. Take the $\mathbf{Z}_i$-covering $\tilde H_i^n\in H^n$, ramified at H^{n-2}, and give it the metric

$$\tilde g_i=dr^2+s^2(r)\,d\theta^2+c^2(r)\,dx^2.$$

This $\tilde g_i$ equals the induced metric (with $K\equiv -1$) outside the ρ-neighborhood of (the lift of) H^{n-2}.

2.4. Lemma. *The sectional curvature of* $\tilde g_i$ *is strictly negative. Moreover,*

$$-(C(i,\rho))^{-1}\geqq K(\tilde g_i)\leqq -C(i,\rho)$$

for the constant C *in* 2.1.

Proof. The submanifolds $\theta=$constant and $x=$constant clearly are totally geodesic for $\tilde g_i$. Hence the curvature tensor $R=R_{\tilde g_i}$ is diagonal in the basis

$$\left\{\partial r=\frac{\partial}{\partial r},\partial\theta=\frac{\frac{\partial}{\partial\theta}}{\left\|\frac{\partial}{\partial\theta}\right\|},X_1,\ldots,X_{n-2}\right\},$$

where $X_1,\ldots,X_{n-2}$ are orthonormal fields in H^{n-2}. Namely, for every four vectors from this basis, say A, B, C, D, we have

$$R(A,B,C,D)=0$$

unless $A\neq B$ and either $(A,B)=(C,D)$ or $(A,B)=(D,C)$. Furthermore, by a straightforward computation, we have

$$R(\partial r,\partial\theta,\partial r,\partial\theta)=-\frac{\sigma''(r)}{\sigma(r)},$$
$$R(X_i,X_j,X_i,X_j)=-1 \quad \text{for } i\neq j,$$
$$R(X_i,\partial r,X_i,\partial r)=-1 \quad \text{for } i=1,\ldots,n-2,$$
$$R(\partial\theta,X_i,\partial\theta,X_i)=\frac{\sigma' c'}{\sigma c}.$$

Since R is diagonal, the curvature at every two-plane is a convex combination of the above numbers -1, $-\sigma''/\sigma$ and $\sigma' c'/\sigma c$, and the lemma follows from 2.1.

2.5. Let V be a complete manifold with $K\equiv -1$ and let $V'\subset V$ be a totally geodesic submanifold with $\mathrm{Rad}^{\perp}(V')\geqq\rho$. Since the metric $\tilde{g}_i$ in 2.3 is invariant under the isometries of the ramified covering $\tilde{H}^n_i$, it "descends" to the ramified covering $\tilde{V}\to V$. Thus we get metrics with $K<0$ on $\tilde{V}$ and also metrics with K pinched between -1 and $-1-(10\rho^{-1}\log 2i)^2$ for $\rho\geqq 10$.

§ 3. Manifolds with pinched negative curvature

3.1. Consider a compact totally geodesic submanifold V' in a complete manifold V with $K<0$. Every non-trivial (i.e., non-contractible to V') homotopy class p of paths in V with enpoints in V' is homotopic to a unique geodesic segment γ in V normal to V' at the endpoints. We set length p = length γ and observe that

$$\mathrm{Rad}^{\perp}(V')=\tfrac{1}{2}\inf_p \text{length}\, p,$$

where p runs over the non-trivial classes. (If p is trivial length $p=0$.) We also have the following trivial

3.2. Lemma. *Let γ_1 and γ_2 be geodesic segments issuing from a point $v\in V\setminus V'$ and ending in V'. If γ_1 and γ_2 are normal to V' and if the (class of) the composite path $\gamma_1\gamma_2^{-1}$ is trivial, then $\gamma_1=\gamma_2$.*

Now consider distinct geodesic segments γ_1 and γ_2 from v to V' having equal length, say length γ_1 = length $\gamma_2 = l$, and let α denote the angle between γ_1 and γ_2 at v.

3.3. Lemma. *If* $K(V) \geqq -1$ *the class* $[\gamma_1 \gamma_2^{-1}]$ *satisfies*

$$\operatorname{length}[\gamma_1 \gamma_2^1] < \alpha \left(\frac{e^l - e^{-l}}{2}\right). \tag{$*$}$$

Proof. If $K \equiv -1$ the base of a triangle in H^2 having sides of length l and angles α satisfies

$$\operatorname{length}(\text{base}) \leqq \alpha \left(\frac{e^l - e^{-l}}{2}\right),$$

which proves the lemma for $K \equiv -1$. The general case (which is not very important for this paper) follows from the Toponogov comparison theorem.

3.4. Lemma. *Let* V' *be the fixed point set of a faithful isometric action of the group* Z_i *on* V. *If* $K(V) \geqq -1$ *and* codim $V' = 2$ *we have*

$$\operatorname{Rad}^\perp V' > 0.1 \log i,$$

provided that $i > 10$.

Proof. Take the minimal non-trivial geodesic segment γ of length $l = 2 \operatorname{Rad}^\perp V'$ contained in V and with endpoints in V'. There exists some isometry in Z_i which moves γ to another such segment, say γ', making a angle $\alpha = 2\pi/i$ with γ. Since γ is minimal, the path $\gamma' \gamma^{-1}$, which is non-trivial by 3.2, has length $\geqq l$. Hence, by ($*$), the number l satisfies the inequality

$$l \leqq \frac{\pi}{i}(e^l - e^{-l}),$$

which, for $i \geqq 10$, implies

$$l \geqq 0.2 \log i.$$

3.5. Consider a compact manifold V with $K \equiv -1$ and let $V' \subset V$ be a totally geodesic submanifold of codimension 2 with $\operatorname{Rad}^\perp V' = \rho$ for ρ large (eventually we'll make $\rho \to \infty$). Assume further that $[V'] = 0$ and that $\dim V \geqq 4$. Then, by Wang's theorem (see 0.1), there exists a least integer j such that the ramified $\mathbf{Z}_i$-covering $\tilde{V}^i \to V$, for $i = 2^{j+1}$, admits no metric with $K \equiv -1$. Let us show that this $\tilde{V}^i$ has a metric with K close to -1. Indeed, by 2.5 there is a metric with

$$-1 \geqq K \geqq -1 - 100\,\rho^{-1}(j+2). \tag{1}$$

On the other hand, the manifold $\tilde{V}^{i'}$, for $i' = 2^j$, does admit a metric with $K \equiv 1$, and the axial manifold $\tilde{V}'$ in $\tilde{V}^{i'}$ is the fixed point set of the action of $Z_{i'}$. Hence $\rho' = \operatorname{Rad}^\perp \tilde{V}' \geqq j/20$ by 3.4. Since $\tilde{V}^{i'}$ is a double cover of $\tilde{V}^i$, we have, for some metric on $\tilde{V}^i$, the inequality

$$-1 \geqq K \geqq -1 - (10 \log 4)(\rho')^{-1} \geqq -1 - 400 j^{-1}. \tag{2}$$

Now we use (1) if j is small compared with ρ^{-2}, and (2) if j is large. In any case we get a metric on $\tilde{V}^i$ as $\rho\to\infty$, and this metric satisfies

$$-1 \geqq K \geqq -1-\varepsilon^2(\rho)$$

for $\varepsilon\to 0$ as $\rho\to\infty$.

3.6. *Remark.* Let us assume there exist isometric involutions I_1 and I_2 of V with the following two properties:

(1) the involution I_α for $\alpha=1,2$ fixes a totally geodesic hypersurface $V_\alpha\subset V$ and the quotient V/I_α is orientable for $\alpha=1,2$.

(2) the hypersurfaces V_1 and V_2 meet at a totally geodesic submanifold $V'\subset V$ with the angle $\pi/2$.

Notice that many Φ_n *manifolds* do admit such involutions.

Let us show, for this $V'=V_1\cap V_2$, that *no ramified (at V') Z_i-covering $\tilde{V}\to V$ admits a metric with $K\equiv -1$ provided that* $\dim V\geqq 4$.

Proof. Observe that the hypersurfaces V_1 and V_2 divide V into 4 isometric sectors $S_\beta\subset V$ with the dihedral angles $\pi/2$ at V'. If $\tilde{V}$ admits a metric $\tilde{g}$ with $K(\tilde{g})\equiv -1$ then, by Mostow's theorem (compare 0.1), the involutions I_α lift to $2i$ isometric involutions on $\tilde{V}$. The hypersurfaces fixed by these divide $\tilde{V}$ into $4i$ isometric sectors $\tilde{S}^j_\beta\subset\tilde{V}$ where $j=1,\dots,i$ and $\beta=1,2,3,4$, with dihedral angles $\pi/2i$ at $\tilde{V}'\subset\tilde{V}$. The boundary $\partial\tilde{S}^j_\beta$ is an $(n-1)$-dimensional manifold (for $n=\dim V$) with $K\equiv -1$ which is isometric (using Mostow's theorem again) to $\partial\tilde{S}_\beta$ for all $j=1,\dots,i$ and $\beta=1,2,3,4$. Therefore one can isometrically attach $2i$ sectors $\tilde{S}^j_\beta$ to two sectors S_β. Since the total dihedral angle of the attached sectors equals $\pi+\frac{2i\pi}{2i}=2\pi$, the resulting manifold, say W, is smooth with $K\equiv -1$. Now we apply Mostow's theorem to W (compare 0.1) and obtain $i+1$ *isometric* involutions of W which permute sectors $\tilde{S}^j_\beta$ and $\tilde{S}_\beta$. Therefore $\tilde{S}^i_\beta$ is isometric to S_β and so $i=1$.

3.7. *Conformally flat manifolds with pinched metrics.* Assume that the manifold V admits i-isometric involutions fixing some hypersurfaces in V which divide V into $2i$ isometric sectors meeting at V'. (To construct such a V, take a totally real algebraic number s all of whose conjugates but one are positive and such that the field $Q(s)$ contains $\sin\frac{\pi}{i}$ and $\cos\frac{\pi}{i}$. For example, take a generic elements $s'\in Q\left(\sin\frac{\pi}{i},\ \cos\frac{\pi}{i}\right)$ and let $s=s'-r$, where r is a rational number between the minimal conjugate of s' and the following one. The desired manifold V' can now be obtained by dividing the hyperbolic space by an appropriate discrete group of $Q(s)$ automorphisms of the form $x_1^2+x_2^2+\dots+x_n^2-s\,x_{n+1}^2$) Glue together $2\tilde{i}$ such sectors and obtains a new manifold, say $\tilde{V}$, with the total angle $\frac{2\pi\tilde{i}}{i}$ at the "corner" V'. By applying the argument in 3.6 one can easily show that most manifolds $\tilde{V}$ admit no metric with $K\equiv -1$. On the other hand, if i is large and $i/\tilde{i}$ is close to one, then there is an embedding of $\tilde{V}$ into some $(n+1)$-dimensional manifold $\tilde{V}_+$ (where $n=\dim V=\dim \tilde{V}$) with $K\equiv -1$, such that each sector in $\tilde{V}$ isometrically goes to a totally geodesic sector (of codimension one) in $\tilde{V}_+$ and such that these sectors meet at (dihedral) angles close to π in $\tilde{V}_+$. Then one can easily construct a *conformally flat* metric on $\tilde{V}$ whose curvature is close to -1. Thus, one produces a *sequence of conformally flat manifolds* $(\tilde{V}_i, \tilde{g}_i)$ of a given dimension $n\geqq 4$, such that

$$-1\geqq K(\tilde{g}_i)\geqq -1-\varepsilon_i^2$$

for $\varepsilon_i\to 0$, and such that the $\tilde{V}_i$ admit no metrics of curvature $K\equiv -1$.

§4. Manifolds with large pinching constants

4.1. We start with an elementary study of integral m-dimensional *cycles* in a complete manifold V with $\dim V\geqq m$. A cycle in V is, by definition, a compact oriented *m-dimensional pseudomanifold* Z mapped into V by a Lipschitz map. Similarly, a *chain* in V is an oriented pseudomanifold C with boundary and a Lipschitz map $C\to V$. To avoid notational complications we treat our chains (and cycles) as *embedded* subpseudomanifolds $C\subset V$ (and $Z\subset V$) even if the implied Lipschitz maps are not one-to-one. However, the volume $\mathrm{Vol}_m\, C$ of a chain C, for $m=\dim C$, is always counted with the *geometric multiplicity* of the implied map $C\to V$.

Definitions. The (flat norm) *distance* $\overline{Z_1 Z_2}$ between two m-cycles Z_1 and Z_2 in V is the lower bound of the $(m+1)$-volumes of the *$(m+1)$-chains* $C\subset V$ for which $\partial C=Z_1-Z_2$.

We denote by $\mathrm{Rad}(v)$ the *convexity radius* of V at $v\in V$, that is, the radius of the largest ball $B\subset V$ around v such that any two points in B can be joined by a unique geodesic in B. If $K(V)\leqq 0$, the convexity radius $\mathrm{Rad}(v)$ equals the injectivity radius.

4.2. Lemma. *Fix an m-cycle $Z_0\subset V$ and real numbers $A\geqq 1$ and $\delta>0$. There exists a cycle $Z\subset V$ such that:*

(i) $\overline{ZZ_0}\leqq A\exp\left(\dfrac{\mathrm{Vol}\,Z_0}{\varepsilon_m\rho^m}\right)$, *where* $\varepsilon_m=(4m)^{-4m}$;

(ii) every cycle $Z'\subset V$ with $\mathrm{Vol}\,Z'\leqq \mathrm{Vol}\,Z-\varepsilon_m\rho^m$ satisfies $\overline{ZZ'}\geqq \overline{ZZ_0}+A$;

(iii) if $v\in Z\subset V$ is a point for which $\mathrm{Rad}(v)\geqq\rho$, the intersection $Z_\rho=Z\cap B_\rho$, where $B_\rho\subset V$ is the ball of radius ρ around v, satisfies

$$\mathrm{Vol}_m\, Z_\rho\geqq 3\varepsilon_m\rho^m.$$

Proof. We may assume (by scaling the metric in V by ρ^{-2}) that $\rho=1$. Introduce the following function W on m-cycles:

$$W(Z)=\mathrm{Vol}\,Z+\varepsilon_m\log(1+\overline{ZZ_0}/A).$$

If a cycle Z minimizes W, conditions (i) and (ii) are satisfied. In fact, it suffices to take a δ-minimizing cycle $Z \subset V$ for a sufficiently small $\delta>0$, where "δ-minimizing" means that

$$W(Z_r) \leqq \inf W(Z) + \delta.$$

To prove (i), intersect Z with the balls B_r around $v \in V$, for $0 \leqq r \leqq \rho = 1$, and denote by ∂_r the boundary of $Z_r = Z \cap B_r$. Each cycle ∂_r bounds a chain C_r in B_r such that

$$\mathrm{Vol}_m C_r \leqq \alpha_m (\mathrm{Vol}\, \delta_r)^{m/(m-1)},$$

for $\alpha_m = \mathrm{m}^{2m}$ (see §3.4 in [Gr3]). Furthermore, the cycle $C_r - Z_r$ clearly bounds a chain D_r such that

$$\mathrm{Vol}\, D_r \leqq r(\mathrm{Vol}\, C_r + \mathrm{Vol}\, Z_r).$$

Hence the cycle $Z' = Z - Z_r + C_r$ satisfies

$$\mathrm{Vol}\, Z' - \mathrm{Vol}\, Z \leqq \alpha_m (\mathrm{Vol}\, \partial_r)^{m/(m-1)} - \mathrm{Vol}\, Z_r$$

and

$$\overline{Z' Z_0} \leqq \overline{Z Z_0} + r(\mathrm{Vol}\, Z_r + \alpha_m (\mathrm{Vol}\, \partial_r)^{m/(m-1)}).$$

Now, if Z minimizes the functional W, we have

$$\alpha_m (\mathrm{Vol}\, \partial_r)^{m/(m-1)} - \mathrm{Vol}\, Z_r + \varepsilon_m r(\alpha_m (\mathrm{Vol}\, \partial_r)^{m/(m-1)} + \mathrm{Vol}\, Z_r) \geqq 0.$$

Hence $(\mathrm{Vol}\, \partial_r)^{m/(m-1)} \geqq \mathrm{Vol}\, Z_r/(2\alpha_m)$, and (iii) follows from the coarea inequality

$$\mathrm{Vol}\, Z_r \geqq \int_0^r \mathrm{Vol}\, \partial_r \, \mathrm{dr}$$

(compare §2.4 in [Gr3]). Finally, the formalism in §6.4 of [Gr3] shows that every δ-minimizing cycle Z' can be *regularized* to a cycle Z such that $|\mathrm{Vol}\, Z - \mathrm{Vol}\, Z'| + Z Z' \leqq \delta'$, where $\delta' \to 0$ for $\delta \to 0$, and such that Z satisfies

$$(\mathrm{Vol}\, \partial_r)^{m/(m-1)} \geqq \mathrm{Vol}\, Z_r |(2\alpha_m + \delta').$$

Now (iii) follows by the previous argument.

4.3. Consider a covering map $p\colon \tilde{V} \to V$, take a cycle $\tilde{Z}_0 \subset \tilde{V}$ and let the projection $Z_0 = p(\tilde{Z}_0) \subset V$ bound a chain $C_0 \subset V$. Fix a positive $\rho \leqq 1$, take $A = \max(1, \mathrm{Vol}\, C_0)$ and let $\tilde{Z} \subset V$ be a cycle satisfying (i)–(iii) in 4.2.

Lemma. *If the manifold $\tilde{V}$ has* Rad $\tilde{v} \geqq \rho$ *at some point $\tilde{v} \in \tilde{Z}$, there exists a loop γ in V based at $v = p(\tilde{v})$ such that*

(i) the homotopy class $[\gamma] \in \pi_1(V, v)$ is not contained in the subgroup $\pi_1(V, \tilde{v}) \subset \pi_1(V, v)$;

(ii) length $\gamma \leqq 2l$, where

$$l = \frac{2A}{\varepsilon_m \rho^m} \left(1 + \exp\left(\frac{\mathrm{Vol}\, \tilde{Z}_0}{\varepsilon_m \rho^m}\right)\right).$$

Proof. The cycle $Z=p(\tilde{Z})\subset V$ bounds a chain $C\subset V$ such that

$$\operatorname{Vol} C \leqq \operatorname{Vol} C_0 + \overline{Z_0 Z} \leqq A\left(1+\exp\left(\frac{\operatorname{Vol}\tilde{Z}_0}{\varepsilon_m \rho^m}\right)\right).$$

By the coarea formula, the intersection of C with the ball $B_R \subset V$ around v satisfies

$$\operatorname{Vol} \partial(B_r \cap C) \leqq l^{-1} \operatorname{Vol} C$$

for some $R \leqq l$. Then the cycle $Z'=Z-B_R\cap Z+\partial(B_r\cap C)$ satisfies

$$\overline{ZZ'} \leqq \operatorname{Vol} C$$

and

$$\operatorname{Vol} Z - \operatorname{Vol} Z' \geqq 3\varepsilon_m \rho^m - l^{-1} \operatorname{Vol} C \geqq \varepsilon_m \rho^m.$$

Now, if every loop in V of length $\leqq 2l$ and based at v lifts to $\tilde{V}$, the cycle Z' also lifts to a cycle $\tilde{Z}' \subset \tilde{V}$ such that

$$\tilde{Z}\tilde{Z}' \leqq \operatorname{Vol} C \leqq \tilde{Z}_0 \tilde{Z} + A$$

and

$$\operatorname{Vol} \tilde{Z}' \leqq \operatorname{Vol} \tilde{Z} - \varepsilon_m \rho^m.$$

Hence, by (ii) of 4.2, some loop γ does not lift, and we have $[\gamma] \neq \pi_1(\tilde{V}, \tilde{v})$.

4.4. Let $\tilde{V}$ be an arbitrary $K(\Pi, 1)$-space. Consider all virtually nilpotent subgroups $N \subset \Pi$ (virtually nilpotent means that N contains a nilpotent subgroup of finite index), and call a homology class $h \in H_*(\tilde{V}) = H_*(\Pi)$ *cuspidal* if it lies in the span of the images of the inclusion homomorphisms $H_*(N) \to H*(\Pi)$. Observe that all $h \in H_1(\Pi)$ are cuspidal. On the other hand, if Π is the fundamental group of a compact manifold of negative curvature, no non-zero element $h \in H_m(\Pi)$ is cuspidal for $m \geqq 2$.

Now let $\tilde{V}$ be a complete manifold with pinched negative curvature,

$$-1 \geqq K(\tilde{V}) \geqq -x.$$

Lemma. *Let $\tilde{Z}$ be a non-cuspidal m-cycle in $\tilde{V}$ (i.e., $[\tilde{Z}] \in H_m(\tilde{V})$ is non-cuspidal). Then there exists a point $\tilde{v} \in Z$ such that the manifold $\tilde{V}$ satisfies*

$$\operatorname{Rad}(\tilde{v}) \geqq \mu = \mu(x, \dim V) > 0.$$

Proof. Take μ to be the Margulis constant (see [Gr2, B-K]). By Margulis' lemma, for each connected component U of the region $\{\tilde{v} \in \tilde{V} \mid |\operatorname{Rad} \tilde{v} \leqq \mu\}$ the image of the inclusion homomorphism $\pi_1(U) \to \pi_1(\tilde{V})$ is virtually nilpotent.

4.5. Let Γ be an arbitrary group, let $\Pi \subset \Gamma$ be a subgroup and let $\tilde{h} \in H_*(\Pi)$ be a non-cuspidal element which goes to zero under the inclusion homomorphism $H_*(\Pi) \to H_*(\Gamma)$. Denote by $\Gamma(i)$ the free product of i copies of Γ amalgamated at Π.

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