NON-ARITHMETIC GROUPS IN LOBACHEVSKY SPACES

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0. Introduction

In this paper we construct *non-arithmetic* lattices Γ (both cocompact and non-cocompact: see 1.3 for the definition) in the projective orthogonal group $PO(n, 1) = O(n, 1)/\{+1, -1\}$ for all $n = 2, 3, \ldots$. We obtain our Γ by "interbreeding" two *arithmetic* subgroups Γ_1 and Γ_2 in $PO(n, 1)$ as follows. Recall that $PO(n, 1)$ is the isometry group of the *Lobachevsky space* L^n and assume the subgroups $\Gamma_i \subset PO(n, 1)$, for $i = 1, 2$, have no torsion. Then the quotient spaces $V_i = \Gamma_i \backslash L^n$ are *hyperbolic manifolds* (i.e. complete Riemannian of constant curvature) and Γ_i is the fundamental group of V_i for $i = 1, 2$. Next, to make the interbreeding possible, we assume there exist connected submanifolds $V_1^+ \subset V_1$ and $V_2^+ \subset V_2$ of dimension n with boundaries $\partial V_1^+ \subset V_1$ and $\partial V_2^+ \subset V_2$, such that

a) The hypersurface $\partial V_i^+ \subset V_i$ for $i = 1, 2$ is totally geodesic in V_i. That is, the universal covering of ∂V_i^+ is a hyperplane in the universal covering L^n of V_i. In particular, ∂V_i^+ is an $(n-1)$-dimensional hyperbolic manifold.

b) The manifolds ∂V_1^+ and ∂V_2^+ are isometric.

Now we produce *the hybrid manifold* V by gluing together V_1^+ and V_2^+ according to an isometry between ∂V_1^+ and ∂V_2^+. This V carries a natural metric of constant negative curvature coming from those on V_1^+ and V_2^+ and this metric is complete apart from a few irrelevant exceptional cases (see 2.10). Then the universal covering of V equals L^n and the fundamental group Γ of V is a lattice in $PO(n, 1) = \mathrm{Is}\, L^n$. Note that if the subgroups Γ_1 and Γ_2 are *cocompact* (i.e. if V_1 and V_2 are compact) then also Γ is cocompact.

Also note that the fundamental group Γ_i^+ of V_i^+ *injects* into Γ_i for $i = 1, 2$ (see 2.10) and that in the relevant cases Γ_i^+ satisfies the following.

0.1. Density property (see 1.7). — *The subgroup* $\Gamma_i^+ \subset PO(n, 1)$ *is Zariski dense in* $PO(n, 1)^0$ for $i = 1, 2$, where ^{0} stands for "the identity component of".

This density for $i = 1$ implies (see 1.2 and 1.6) the following

0.2. Commensurability property. — *If the group* Γ *(as well as* Γ_1*) is arithmetic then* Γ *and* Γ_1 *are commensurable. That is there exists a hyperbolic manifold admitting locally isometric finite covering maps onto* V *and onto* V_1.

Similarly, arithmeticity of Γ implies commensurability between Γ and Γ_2 and hence, commensurability between Γ_1 and Γ_2. Therefore, *one obtains a non-arithmetic* Γ *by taking* Γ_1 *and* Γ_2 *non-commensurable* (compare 2.6, 2.7 and 2.8).

0.3. *Historical remarks.* — *a)* Examples of non-arithmetic lattices Γ in L^3 (the existence of non-arithmetic lattices in L^2 is trivial) were first found by Makarov (see [M]) among *reflection groups* that are groups generated by reflections in some hyperplanes. Then non-arithmetic reflection lattices were constructed in L^4 and L^5. It is yet unknown for which n there exists a non-arithmetic reflection lattice in L^n, but one does know this n cannot be too large. In fact, no reflection lattice exists in L^n for $n \geqslant 995$ (see [V], [N] and references therein).

b) A famous theorem by Margulis asserts that every lattice in a simple Lie group G with $\operatorname{rank}_{\mathbf{R}} G \geqslant 2$ is arithmetic. The remaining non-compact groups (groups with $\operatorname{rank}_{\mathbf{R}} = 1$) are (up to local isomorphism): $O(n, 1)$, $U(n, 1)$, and their quaternion and Cayley analogues. Apart from $O(n, 1)$ where our interbreeding provides non-arithmetic lattices for all n, the existence of non-arithmetic lattices is only known for $SU(2, 1)$ and $SU(3, 1)$. Non-arithmetic lattices in these two groups were constructed by Mostow (see [Mo]) by using reflections in complex hyperplanes.

0.4. *Questions.* — Call a discrete subgroup $\Gamma_0 \subset PO(n, 1)$ *subarithmetic* if Γ_0 is Zariski dense and if there exists an arithmetic subgroup $\Gamma_1 \subset PO(n, 1)$ such that $\Gamma_0 \cap \Gamma_1$ has finite index in Γ_0. Does every lattice Γ in $PO(n, 1)$ (maybe for large n) contain a subarithmetic subgroup? Is Γ generated by (finitely many) such subgroups? If so, does $V = \Gamma \backslash L^n$ admit a "nice" partition into "subarithmetic pieces"?

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1. Rudiments of arithmetic groups

1.1. *Integral points in linear reductive groups.* — A connected Lie group G is called *reductive* if the center of G is compact and G/Center is semisimple. Such a G obviously contains a unique maximal compact normal subgroup $K \subset G$. The quotient group $G' = G/K$, clearly is of *adjoint type*. That is the adjoint representation $\mathrm{ad} : G' \to \mathrm{Aut}\, L'$ is injective, where L' denotes the Lie algebra of G' and Aut is the group of linear automorphisms of L'. Our basic example is $G' = PO(n, 1)^0$.

Sufficiently dense subgroups. — Call $\Gamma \subset G$ *sufficiently dense* if the image of Γ in $G' \subset \operatorname{Aut} L'$ is Zariski dense in G'.

Let $G \subset GL_N \mathbf{R}$ be a reductive subgroup and let $\Gamma \subset G$ be the subgroup of integral matrices in G with $\det = \pm 1$. That is

$$\Gamma = G \cap GL_N \mathbf{Z}.$$

Property A. — We say that G satisfies A if Γ is sufficiently dense in G.

1.2. Basic Theorem. — *A reductive subgroup* $G \subset GL_N \mathbf{R}$ *satisfies* A *if and only if* Γ *is a lattice in* G, *that is,* $\operatorname{Vol} G/\Gamma < \infty$.

Proof. — The implication

$$\operatorname{Vol} G/\Gamma < \infty \Rightarrow \text{Zariski density of } \Gamma' \text{ in } G'$$

holds true for all discrete subgroups $\Gamma \subset G$ and is called *Borel density theorem.* A short proof of this can be found in [Z] and $[G]_2$.

Let us indicate the (well-known, see $[B]_1$) proof of the implication $\operatorname{Vol} G/\Gamma < \infty \Leftarrow A$.

Step 1. — By elementary properties of reductive groups (see $[B]_2$), G equals the identity component of the Zariski closure $\overline{G} \subset GL_N \mathbf{R}$. Therefore, G contains the identity component $\overline{\Gamma}_0$ of the Zariski closure $\overline{\Gamma} \subset GL_N \mathbf{R}$.

Note that the inclusion $\overline{\Gamma}_0 \subset G$ is automatic in all our cases and so Step 1 can be omitted.

Step 2. — Property A immediately implies that the homomorphism $G \to G'$ maps $\overline{\Gamma}_0$ *onto* G'. It follows that $\overline{\Gamma}_0$ is reductive.

Step 3. — The Zariski density of integral points in $\overline{\Gamma}$ implies that $\overline{\Gamma}$ is defined over $\mathbf{Q}$. In fact one only needs Zariski density of *rational* points in $\overline{\Gamma}$. This easily follows from the very definition of the Zariski closure.

Step 4. — Since $\overline{\Gamma}$ is reductive, there exists a polynomial map $P : (\mathbf{R}^N)^k \to \mathbf{R}^\ell$ for some k and ℓ, such that

a) The set of linear transformations of $\mathbf{R}^N$ fixing P equals $\overline{\Gamma}$.

Furthermore, since $\overline{\Gamma}$ is defined over $\mathbf{Q}$ one can choose the above P integral. That is

b) $$P((\mathbf{Z}^N)^k) \subset \mathbf{Z}^\ell.$$

The existence of P is easy (see $[B]_1$) and follows directly from Step 2. (We included Step 3 only to bring our discussion nearer to the standard language.)

Step 5. — The orbit $\overline{\Gamma}(\mathbf{Z}^N)$ is *closed* in $GL_N \mathbf{R}/GL_N \mathbf{Z}$, where the quotient space $GL_N \mathbf{R}/GL_N \mathbf{Z}$ is identified in a natural way with the space of lattices in $\mathbf{R}^N$. (Note that this step brings us from algebra to geometry.)

Proof of step 5. — Observe that for each lattice $L \subset \mathbf{R}^N$ there exists a finite subset $F \subset L$, such that the values of P on F^k uniquely determine P among the polynomials of the same degree on $(\mathbf{R}^N)^k$. Thus the inequality $P \circ g = P$ on F^k implies $g \in \overline{\Gamma}$ for all $g \in GL_N \mathbf{R}$ and the diagonal action of $GL_N \mathbf{R}$ on $(\mathbf{R}^N)^k$.

If L lies in the closure of the orbit $\overline{\Gamma}(\mathbf{Z}^N)$, then there exists a sequence g_i converging to 1 in $GL_N \mathbf{R}$ and a sequence γ_i in $\overline{\Gamma}$ such that $g_i L = \gamma_i \mathbf{Z}^N$ for all $i = 1, 2, \ldots$. This follows from the very definition of the topology in the space of lattices, that is $GL_N \mathbf{R}/GL_N \mathbf{Z}$.

Since P is integer valued (i.e. $\mathbf{Z}^l$-valued) on $(\mathbf{Z}^N)^k$ and $\overline{\Gamma}$-invariant, the equality $g_i L = \gamma_i \mathbf{Z}^N$ shows that $P \circ g_i$ is integer valued on F^k.

Since $P \circ g$ is continuous in g and F is finite, we have $P \circ g_i = P$ on F^k for almost all i. This implies $P \circ g_i = P$ on all of $(\mathbf{R}^N)^k$ by our choice of F. Therefore, $g_i \in \overline{\Gamma}$ and $L = g_i^{-1} \gamma_i(\mathbf{Z}^N) \in \overline{\Gamma}(\mathbf{Z}^N)$. Q.E.D.

Step 6. — If the orbit $G(\mathbf{Z}^N)$ is *precompact* in $GL_N \mathbf{R}/GL_N \mathbf{Z}$, then by the previous step $G/\Gamma = G(\mathbf{Z}^N)$ is compact. That is, Γ is a *cocompact* lattice in G. Note that this case is sufficient for our examples of *compact* hybrids V.

If $G(\mathbf{Z}^N)$ is not precompact the proof of the lattice property

$$\operatorname{Vol} G(\mathbf{Z}^N) < \infty$$

is more complicated (see § 16 in [B]$_1$ and § 10 in [R]). Yet, in the cases needed for our purpose the proof is relatively simple (see § 2).

1.3. *Arithmetic groups.* — A discrete subgroup Γ in a reductive group G is called *arithmetic* if there exists a reductive subgroup $\overline{G} \subset GL_N \mathbf{R}$ for some $N = 1, 2, \ldots$ satisfying A and a continuous surjective homomorphism $\rho : \overline{G} \to G$ such that

(i) the kernel of ρ is a *compact* subgroup in $\overline{G}$;

(ii) the ρ-image of $\overline{G} \cap GL_N \mathbf{Z}$ is *commensurable* with Γ. That is, the intersection

$$\Gamma \cap \rho(\overline{G} \cap GL_N \mathbf{Z})$$

has finite index in Γ as well as in $\rho(\overline{G} \cap GL_N \mathbf{Z})$.

Remarks. — *a)* Since G is reductive and Ker ρ is compact, the group $\overline{G}$ is *necessarily* reductive.

b) Since $\overline{G} \cap GL_n \mathbf{Z} \subset \overline{G}$ is a lattice by 1.2, the subgroup $\Gamma \cap \rho(\overline{G} \cap GL_N \mathbf{Z})$ has finite index in Γ. Thus, it is enough to assume in (ii) that this subgroup has finite index in $\rho(\overline{G} \cap GL_N \mathbf{Z})$.

c) For our applications, we only need $G = PO(n, 1)$ and $PO(n, 1) \times PO(n, 1)$.

1.4. Criterion for non-arithmeticity. — *Let* $H \subset G$ *be a reductive subgroup. Then the intersection of* H *with an arithmetic subgroup* $\Gamma \subset G$ *is arithmetic in* H *if and only if this intersection* $H \cap \Gamma$ *is sufficiently dense in* H.

Proof. — Use $\overline{H} = \rho^{-1}(G) \subset \overline{G} \subset GL_N \mathbf{R}$ and 1.2.

1.4.A. Corollary. — *If* $\Gamma \subset G$ *is arithmetic and* $H \cap \Gamma$ *is sufficiently dense in* H *then* $H \cap \Gamma$ *is a lattice in* H. *That is*, $\mathrm{Vol}\ H/H \cap \Gamma < \infty$.

Proof. — Apply 1.2 again.

1.5. *Remarks.* — *a)* If Γ is cocompact in G, then 1.4.A obviously implies that $\Gamma \cap H$ is cocompact in H, provided Γ is arithmetic.

b) The above corollary can be used as a criterion of non-arithmeticity for Γ. For example, let H be isomorphic to $SL_2\,\mathbf{R}$ or $PSL_2\,\mathbf{R}$. Then an elementary argument shows that a discrete subgroup $\Gamma' \subset H$ is either sufficiently dense (here it is equivalent to Zariski dense) or virtually cyclic (i.e. contains a cyclic subgroup of finite index). Therefore, the intersection of an *arithmetic* subgroup $\Gamma \subset G$ with every H isomorphic to $SL_2\,\mathbf{R}$ or $PSL_2\,\mathbf{R}$ is either a lattice in H or a virtually cyclic group. (This observation is due to D. Toledo.)

1.6. Commensurability criterion. — *Let* Γ *and* Γ_1 *be arithmetic subgroups in* G *such that* $\Gamma \cap \Gamma_1$ *is sufficiently dense in* G. *Then* $\Gamma \cap \Gamma_1$ *has finite index in* Γ *as well as in* Γ_1.

Proof. — Observe that $\Gamma \times \Gamma_1$ is an arithmetic subgroup in $G \times G$ and that $\Gamma \cap \Gamma_1 \subset G$ equals $G \cap (\Gamma \times \Gamma_1)$ for the diagonal embedding $G \subset G \times G$. Hence, $\Gamma \cap \Gamma_1$ is a *lattice* in G by 1.4.A which implies the desired commensurability.

1.6.A. *Example* : *Commensurability of hyperbolic manifolds* (compare 0.2). — Let V and V_1 be n-dimensional hyperbolic manifolds whose fundamental groups Γ and Γ_1 are *arithmetic* subgroups in $PO(n, 1)$. Let $V^+ \subset V$ and $V_1^+ \subset V$ be connected mutually isometric submanifolds with sufficiently dense fundamental groups Γ^+ and Γ_1^+. That is, the images of Γ^+ and Γ_1^+ in Γ and Γ_1 respectively are Zariski dense in the ambient group $PO(n, 1)$. *Then there exists a hyperbolic manifold* V′ *which admits a finite locally isometric covering map onto* V *and onto* V_1.

Proof. — Since V^+ is isometric to V_1^+ the image of Γ^+ in $PO(n, 1)$ is conjugate to that of Γ_1^+. Therefore, we way assume that the intersection $\Gamma' = \Gamma \cap \Gamma_1$ in $PO(n, 1)$ contains the image of Γ^+. According to 1.6 this Γ' has finite index in Γ as well as in Γ_1. Hence, the manifold $V' = \Gamma' \backslash L^n$ *finitely* covers V and V_1.

1.7. *Density criterion for hyperbolic manifolds with boundary.* — Let V^+ be a connected n-dimensional manifold of constant negative curvature with non-empty totally geodesic boundary ∂V^+ having finitely many connected components. Assume V^+ is complete as a metric space and $\mathrm{Vol}\ V^+ < \infty$.

1.7.A. Lemma. — *Let the (image of the) fundamental group of every component of* ∂V^+ *have finite index in the fundamental group of* V^+. *Then* $n = 2$ *and* V^+ *is simply connected. It follows that* V^+ *is isometric to a* k*-gon in* L^2 *with vertices at infinity.*

Proof. — The finite index condition shows that the universal covering $\widetilde{V}^+$ also has finitely many boundary components. Then one may assume without loss of generality that the deck transformation group Γ maps every component into itself. Let ∂_0 be one of the components of $\partial\widetilde{V}^+$ and let $\bar{\partial}_i \subset \partial_0$ be the normal projections of the remaining components ∂_i, $i = 1, \ldots, k$, to ∂_0. The condition $\mathrm{Vol}\, V^+ < \infty$ implies that $\bigcup_{i=1}^{k} \bar{\partial}_i \subset \partial_0$ is a subset of full measure. Hence, $n = 2$, and the action of deck transformations is trivial. Q.E.D.

1.7.B. Corollary (compare 0.1). — *If* $\mathrm{Vol}\, \partial V^+ < \infty$, *then the fundamental group* Γ^+ *of* V^+ *is Zariski dense in* $\mathrm{PO}(n, 1)^0$.

Proof. — Since $\mathrm{Vol}\, \partial V^+ < \infty$ the Zariski closure $\bar{\Gamma}^+ \subset \mathrm{PO}(n, 1)$ of Γ^+ contains $\mathrm{PO}(n-1, 1)$ by Borel density theorem (see 1.2), where $\mathrm{PO}(n-1, 1) \subset \mathrm{PO}(n, 1)$ is identified with the isometry group of the space L^{n-1} serving as the universal covering of each component of ∂V^+. By the above lemma, $\dim \bar{\Gamma}^+ > \dim \mathrm{PO}(n, 1)$ because the (algebraic!) group $\bar{\Gamma}^+$ has at most finitely many connected components. It follows that $\bar{\Gamma} = \mathrm{SO}(n, 1)$, since $\mathrm{O}(n-1, 1)^0$ is a *maximal* connected subgroup in $\mathrm{SO}(n, 1)$.

2. Arithmetic subgroups in $\mathrm{O}(n, 1)$.

2.1. *Orthogonal groups.* — Let $K \subset \mathbf{R}$ be a number field and F be a non-singular quadratic form in $n+1$ variable with coefficient in K. Denote by $\Gamma(F) \subset \mathrm{GL}_{n+1}\mathbf{R}$ the group of K-*integral* automorphisms of F. That is the group of F-orthogonal matrices with entries from the ring of integers in K. If the form F has real type (p, q), then $\Gamma(F)$ is contained in (some conjugate of) the orthogonal group $\mathrm{O}(p, q)$. We are mainly interested in the case $p = n$ and $q = 1$.

Suppose K is totally real of degree $d+1$ and let $\mathbf{I}_i : K \subset \mathbf{R}$, $i = 0, \ldots, d$ be the various embeddings where $\mathbf{I}_0$ is the original embedding $K \subset \mathbf{R}$. For our applications we shall only need the fields $\mathbf{Q}$ and $\mathbf{Q}(\sqrt{2})$. Note that the embedding $\mathbf{I}_1 : \mathbf{Q}(\sqrt{2}) \subset \mathbf{R}$ is obtained from I_0 by applying the automorphism $\mathbf{I} : \alpha + \beta\sqrt{2} \mapsto \alpha - \beta\sqrt{2}$ to $\mathbf{Q}(\sqrt{2})$.

The following classical theorem (see $[B]_1$, for example) provides a variety of arithmetic subgroups in $\mathrm{O}(n, 1)$.

2.2. Arithmeticity of $\Gamma(F)$. — *If the forms* $\mathrm{I}_i F$ *are positive definite for* $i = 1, \ldots, d$, *then the subgroup* $\Gamma(F) \subset \mathrm{O}(p, q)$ *is arithmetic. In particular,* $\Gamma(F)$ *is discrete and* $\mathrm{Vol}\, \mathrm{O}(p, q)/\Gamma(F) < \infty$.

Proof. — The pertinent group $\bar{G}$ here (compare 1.3) for $G = \mathrm{O}(p, q)$ is the Cartesian product of the real orthogonal groups $\mathrm{O}(\mathrm{I}_i F)$, $i = 0, 1, \ldots, d$ (where $\mathrm{O}(\mathrm{I}_0 F) = \mathrm{O}(F) = \mathrm{O}(p, q)$). Thus $\bar{G} \subset \mathrm{GL}_N \mathbf{R}$ for $N = (n+1)(d+1)$, where $\mathbf{R}^N$ is given a K-*rational* basis, that is, a basic of vectors whose projections to the copies of

$\mathbf{R}^{d+1}$ lie in $K \subset \mathbf{R}^{d+1}$, where K embeds into $\mathbf{R}^{d+1}$ by $x \mapsto (I_0(x), \ldots, I_d(x))$ for all $x \in K$. Then the verification of the A-property of $\overline{G}$ and arithmeticity of $\Gamma(F)$ is straightforward (see $[B]_1$).

2.3. Cocompactness of $\Gamma(F)$. — *The above arithmetic group* $\Gamma(F)$ *is cocompact in* $O(p, q)$ *if and only if* F *has no non-trivial zero in* K.

This is a simple corollary of Mahler compactness theorem for lattices in $\mathbf{R}^N$ (see $[B]_1$, [R]).

2.3.A. *If* $d + 1 \geqslant 2$, *then* $\Gamma(F)$ *is cocompact.*

Proof. — If $F(x, x) = 0$, then also $I_i F(I_i(x), I_i(x)) = 0$ for $i > 0$, as I_i is an isomorphism. Since $I_i F$ is positive definite for $i > 0$, we have $I_i(x) = 0$ and thus $x = 0$.

2.4. *Remark.* — If $K = \mathbf{Q}$, then $\Gamma(F)$ may be both cocompact and non-cocompact for $n = 2, 3, 4$. But $\Gamma(F)$ is not cocompact for $n \geqslant 5$ as every indefinite rational quadratic form in five variables has a non-trivial rational zero by the Minkovski-Hasse theorem.

2.5. *Action of* $\Gamma(F)$ *on* L^n. — Let F be of signature $(n, 1)$ and consider the (pseudo)-sphere $S = S_F = \{x \in \mathbf{R}^{n+1} \mid F(x, x) = -1\} \subset \mathbf{R}^{n+1}$. This S has two connected components isometric to L^n for the metric induced from the pseudo-Euclidean metric F on $\mathbf{R}^{n+1}$. Thus $S/\{+1, -1\} = L^n$ and $PO(n, 1) = PO(F)$ acts isometrically on L^n. If $\Gamma \subset \Gamma(F)$ is a subgroup of finite index without torsion, then $\Gamma/\{+1, -1\}$ acts *freely* on L^n and the quotient space $\Gamma \backslash L^n$ is a hyperbolic manifold such that, according to 1.2,

$$\mathrm{Vol}(\Gamma \backslash L^n) < \infty.$$

2.5.A. *Congruence subgroups in* $\Gamma(F)$. — Take a prime ideal p in the ring of integers of K and define the *congruence subgroup* $\Gamma_p(F) \subset \Gamma(F)$ by

$$\Gamma_p(F) = \{\gamma \in \Gamma(F) \mid \gamma \equiv \mathrm{Id} \ (\mathrm{mod}\ p)\}.$$

If $|p|$ is sufficiently large, then $\Gamma_p(F)$ has no torsion and the action of $\Gamma_p(F)$ on L^n is free (see $[B]_1$, [R]).

2.6. Commensurable manifolds. — *Let* F_1 *and* F_2 *be two forms over* K *of type* $(n, 1)$ *for* $n \geqslant 2$, *such that the corresponding groups* $\Gamma(F_1)$ *and* $\Gamma(F_2)$ *are commensurable (we stick to the assumptions in* 2.2 *so that these groups are arithmetic) in the following sense. There exists an isometry* α *of the (Lobachevsky) space* $L_1 = S_{F_1}/\{+1, -1\}$ *onto* $L_2 = S_{F_2}/\{+1, -1\}$ *which sends some subgroup of finite index* $\Gamma_1 \subset \Gamma(F_1)/\{+1, -1\}$ *(acting on* L_1*) into* $\Gamma(F_2)/\{+1, -1\}$ *(acting on* L_2*). Then the forms* F_1 *and* F_2 *are similar over* K. *That is there exists a linear* K*-isomorphism* $\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ *sending* F_1 *to* λF_2 *for some* $\lambda \in K$.

Proof. — There obviously exists a unique (up to $\{+1, -1\}$) linear map $\bar{\alpha} : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ sending F_1 to F_2 such that the induced map $L_1 \to L_2$ is α. Denote by $\bar{\Gamma}_1 \subset \Gamma(F_1)$ the $\{+1, -1\}$-extension of Γ_1. Since $\bar{\Gamma}_1$ is Zariski dense in $O(n, 1)$ and the action of $O(n, 1)$ on $\mathbf{R}^{n+1}$ is $\mathbf{C}$-irreducible for $n \geqslant 2$, the K-linear span of $\bar{\Gamma}_1$ in $\operatorname{End} \mathbf{R}^{n+1}$ equals $\operatorname{End} K^{n+1} \subset \operatorname{End} \mathbf{R}^{n+1}$. Since $\bar{\alpha}$ sends $\bar{\Gamma}_1$ in $\Gamma(F_2)$, the K-span of $\bar{\Gamma}_1$ goes to that of $\Gamma(F_2)$ and then the equality $\operatorname{Span}_K \bar{\Gamma}_1 = \operatorname{End} K^{n+1}$ implies that $\bar{\alpha} = \mu\bar{\alpha}_0$, where $\bar{\alpha}_0$ is defined over K and $\mu \in \mathbf{R}^{\times}$. Now, α_0 sends F_1 to $\mu^{-2} F_2$ and since $F_1 \neq 0$, the factor μ^{-2} lies in K. Q.E.D.

2.7. Corollary. — *Let* F_1 *and* F_2 *be diagonal,*

$$F_1 = \sum_{i=1}^{n+1} a_i x_i^2 \quad \textit{and} \quad F_2 = \sum_{i=1}^{n+1} b_i x_i^2$$

for a_i *and* b_i *in* K. *Then for* $n+1$ *even the ratio of the discriminants*

$$\prod_{i=1}^{n+1} a_i \,\Big|\, \prod_{i=1}^{n+1} b_i \quad \textit{lies in } (K^{\times})^2.$$

Proof. — A linear transformation over K with determinant D multiplies discriminants by D^2 and similarity $F \mapsto \lambda F$ multiplies the discriminant of F by λ^{n+1}.

2.7.A. *Example.* — *a)* Let $K = \mathbf{Q}$ and

$$F_1 = x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2$$
$$F_2 = 2x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2.$$

Then for $n+1$ even the groups $\Gamma(F_1)$ and $\Gamma(F_2)$ are not commensurable as 2 is not a square in $\mathbf{Q}$. Also note that these groups are not cocompact as $F_i(x, x) = 0$ for $x = (0, 0, \dots, 0, 1, 1)$ and $i = 1, 2$ (compare 2.4).

b) Let $K = \mathbf{Q}(\sqrt{2})$ and

$$F_1 = x_0^2 + x_1^2 + \dots + x_{n-1}^2 - \sqrt{2}\, x_n^2$$
$$F_2 = 3x_0^2 + x_1^2 + \dots + x_{n-1}^2 - \sqrt{2}\, x_n^2.$$

Here again the corresponding groups are not commensurable for $n+1$ even, but now these groups are cocompact (see 2.3.A).

2.8. *Totally geodesic submanifolds in hyperbolic manifolds.* Take a $(k+1)$-dimensional linear subspace $R_0 \subset \mathbf{R}^{n+1}$ which meets the sphere $S = S(F) \subset \mathbf{R}^{n+1}$. Then the intersection $S_0 = S \cap R_0$ is a totally geodesic submanifold in S of dimension k. For a subgroup $\Gamma \subset \Gamma(F)$ denote by $\Gamma_0 \subset \Gamma$ the subgroup stabilizing R_0. If the subspace R_0 is K-rational and Γ_0 has finite index in Γ, then Γ_0 is arithmetic. That is, the image of Γ_0 in the full isometry group $\operatorname{Is} S_0 = O(k, 1)$ gives a *proper immersion* $\Gamma_0 \backslash S_0 \to \Gamma \backslash S$ (by Step 5 in 1.2).

2.8.A. *Embedding criterion.* — Denote by $I_0 \in O(n, 1)$ the orthogonal reflection of $\mathbf{R}^{n+1}$ in R_0.

If I_0 normalizes Γ, then the canonical map $\Gamma_0 \backslash S_0 \to \Gamma \backslash S$ is a proper embedding, provided Γ has no torsion.

Proof. — Suppose two distinct points s and s' from S_0 go to the same point in $\Gamma \backslash S$. That is $s' = \gamma(s)$ for some $\gamma \in \Gamma$. Since s and s' are fixed by I_0, the commutator $\delta = \gamma^{-1} I_0 \gamma I_0^{-1}$ fixes s. Since I_0 normalizes Γ this δ is contained in Γ and as Γ has no torsion and acts freely on S_0, we obtain $\delta = \mathrm{Id}$. Since S_0 equals the fixed point set of I_0, the equality $[\gamma, I_0] = \mathrm{Id}$ implies that $\gamma \in \Gamma_0$. Q.E.D.

2.8.B. *Remark.* — If $\Gamma_0 \backslash S_0 \to \Gamma \backslash S$ is an embedding, then, obviously, the corresponding map $\Gamma'_0 \backslash S_0 \to \Gamma' \backslash S$ also is an embedding for every subgroup $\Gamma' \subset \Gamma$.

Corollary. — *If the group generated by Γ and $I_0 \Gamma I_0^{-1}$ is discrete without torsion, then the map $\Gamma_0 \backslash S_0 \to \Gamma \backslash S$ is an embedding.*

2.8.C. *Example.* — Let F_0 be a quadratic form in variables $x_1, \ldots, x_n$ over $K \subset \mathbf{R}$ of type $(n-1, 1)$ and $F = a x_0^2 + F_0$ for $a > 0$ in K. Then the reflection I_0 in the hyperplane $R_0 = \{ x_0 = 0 \} \subset \mathbf{R}^{n+1}$,

$$I_0 : (x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$$

lies in $\Gamma(F)$ and the previous discussion applies to the congruence subgroups $\Gamma_p(F) \subset \Gamma(F)$ with $|p|$ sufficiently large. Therefore the hyperbolic manifold

$$V(F_0, p) = \Gamma_p(F_0) \backslash L^{n-1}$$

(where we identify L^{n-1} with $S_0/\{+1, -1\}$) isometrically embeds into $V(F, p) = \Gamma_p(F) \backslash L^n$.

Note that for p prime to 2 both manifolds $V(F, p)$ and $V(F_0, p)$ are orientable. In fact, if $-1 \neq 1 \pmod p$, then $\Gamma_p(F) \subset SO(n, 1)$ and $\Gamma_p(F_0) \subset SO(n-1, 1)$.

The hypersurface $V(F_0, p)$ does not necessarily bound in $V(F, p)$. (In fact for large $|p|$ it does not bound). However, there exists an obvious double covering $\widetilde{V}(F, p)$ of $V(F, p)$, such that the lift of $V(F_0, p)$ to $\widetilde{V}(F, p)$ consists of two disjoint copies of $V(F_0, p)$ which do *bound* some connected submanifold $V^+ \subset \widetilde{V}(F, p)$. That is the boundary ∂V^+ is the union of two copies of $V(F_0, p)$.

2.9. *Interbreeding hyperbolic manifolds.* — Take the forms $F_i = a_i x_0^2 + F_0$ as in the previous example for $i = 1, 2$, and assume for the uniformity of notation that $V(F_0, p)$ does not bound in either of the two manifolds $V(F_i, p)$. (As we mentioned earlier, this is the case for large $|p|$.) Then we take the corresponding manifolds $V_i^+ \subset \widetilde{V}(F_i, p)$ for $i = 1, 2$ and recall that V_1^+ and V_2^+ have isometric boundaries equal to $2V(F_0, p)$.

If $n + 1$ is even and a_1/a_2 is not a square in K then the forms F_1 and F_2 are not similar over K (compare 2.7) and the groups $\Gamma(F_1)$ and $\Gamma(F_2)$ are not commensurable (see 2.6). In this case the manifold V obtained by gluing V_1^+ to V_2^+ along the boundary is non-arithmetic (i.e. the fundamental group is not arithmetic: compare 0.2, 1.6.A).

If $(n+1)$ is odd, we consider a K-rational hyperplane $R' \subset \mathbf{R}^{n+1}$ normal to R_0. For example, let $F_0 = \sum_{i=1}^{n} b_i x_i^2$, where $b_1 > 0$ and take

$$R' = \{ x_1 = 0 \} \subset \mathbf{R}^{n+1}.$$

Then the corresponding hypersurfaces $V'_i \subset V(F_i, p)$ are normal to $V(F_0, p)$. Therefore, their "halfs" $V'_1 \cap V_1^+$ and $V'_2 \cap V_2^+$ glue together to a *totally geodesic* hypersurface $V' \subset V$. If V is arithmetic, then so is V' (see 1.4). But V' is non-arithmetic for $n - 1 = \dim V' \geqslant 2$ by the previous argument and thus the non-arithmeticity of V (i.e. of the fundamental group Γ of V) is established for all $n \geqslant 3$. We leave the (trivial) case where $n = 2$ to the reader.

2.10. *Final hyperbolic remarks.* — To complete our discussion we need two simple facts from hyperbolic geometry.

2.10.A. *The fundamental group of* V^+ *injects into that of* V.

Proof. — The submanifold $V^+ \subset V$ has convex (in fact, totally geodesic) boundary and so every class in $\pi_1(V^+)$ is represented by a *geodesic* loop in V^+. Such a loop is not contractible in V, as V is complete of negative curvature. Q.E.D.

2.10.B. *The manifold* V *obtained by gluing* V_1^+ *and* V_2^+ (see § 0) *is complete provided these manifolds as well as their (totally geodesic) boundaries have finite volumes.*

Proof. — The claim is obvious if $V_1^+ = V_2^+$ is compact.

If V_1^+ is non-compact then the geometry at infinity is described with the following notion.

2.10.C. *Cusps.* — An n-dimensional *cusp with boundary* is a Riemannian manifold $C^+ = F^+ \times \mathbf{R}_+$, where F^+ is a compact flat manifold with totally geodesic boundary and where the metric in C^+ is $dt^2 + e^{-t} g$, where $t \in \mathbf{R}_+$ and g is the flat metric on F^+.

Observe that a compact connected flat manifold F^+ with a non-empty boundary either is isometric to a product $F_0 \times [-a, a]$ for some compact flat manifold F_0 without boundary, or has a double covering isometric to $F_0 \times [-a, a]$. In both cases the connected components of the levels of the distance function $\mathrm{dist}(x, \partial F^+)$ foliate F^+ into closed connected totally geodesic submanifolds F_θ for $\theta \in [0, a]$. It follows that a connected cusp with non-empty boundary is canonically foliated into leaves $C_\theta = F_\theta \times \mathbf{R}_+$. Note that this splitting of C_θ is unique. In fact, for each $x \in C_\theta$, there exists a unique closed connected $(n-2)$-dimensional hypersurface $F(x) \subset C_\theta$ passing through x, such that

a) the induced metric in $F(x)$ is flat;

b) also the induced metrics in the *parallel* hypersurfaces (which are defined as the level of the distance function to $F(x)$ in C_θ) are flat.

Since the hypersurfaces $F_\theta \times t \subset C_\theta$ have these properties, the hypersurface $F(x)$ for $x = (f, t)$ equals $F_\theta \times t$.

The $(n-2)$-dimensional volume of $F_\theta \times t$ is obviously const $\exp(n-2)\,t$. Hence, if $n \geqslant 3$, the parameter $t = t(x)$ for $x = (f, t)$ can be recaptured (up to an additive constant) by taking $\log \operatorname{Vol} F(x)$, for those x, for which the hypersurface is normally orientable and $\log 2 \operatorname{Vol} F(x)$ for the others.

Now it is clear that a manifold C, obtained by gluing together two cusps $C_i^+ = F_i^+ \times \mathbf{R}_+$ by isometries along their boundary cusps $\partial F_i^+ \times \mathbf{R}_+$, is again a cusp. In fact, the foliations on C_i^+ define a geodesic foliation of C into $(n-1)$-dimensional cusps C_θ without boundary and the cusp structure in C is seen with $t = \log \operatorname{Vol} F(x)$.

Finally, we conclude the proof of 2.10.B by invoking the following.

2.10.D. Proposition. — *Let* V^+ *be a complete hyperbolic manifold with totally geodesic boundary. If* $\operatorname{Vol} V^+ < \infty$, *then the complement to a compact subset in* V^+ *is isometric to a (possibly disconnected) cusp.*

Proof. — If V^+ has no boundary, this is standard (see $[B]_1$, [R], $[G]_1$), and the case with boundary follows by taking the double of V^+.

This proposition and the above discussion show that the glued manifold V is cuspidal at infinity. Since cusps are complete, V is complete. Q.E.D.

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