Hilbert Volume in Metric Spaces, Part 1.

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Contents

Abstract

We introduce a notion of *Hilbertian n-volume* in metric spaces with Besicovitch-type inequalities built-in into the definitions. The present Part 1 of the article is, for the most part, dedicated to reformulation of known results in our terms with proofs being reduced to (almost) pure tautologies. If there is any novelty in the paper, this is in forging certain terminology which, ultimately, may turn useful in Alexandrov kind of approach to singular spaces with positive scalar curvature [17].

1 Partitions of Unity $\mu = d \ \ p$ in Banach Spaces, $\tilde{L}$ -Dilation $||min.\widetilde{dil}^{*}||\ _q$ and Hilbertian $\widetilde{Hilb}_{/}$.

We recall in this section a few (more or less) standard definitions and introduce notations used throughout the paper.

The dilation of a map between metric spaces, $\ :\ \ \to\ $, is a function in two variables $\ _1 \neq \ _2 \in\ $, that is

$$_f(\ _1\ \ _2) = \ \ _Y(\ (\ _1)\ \ (\ _2)/\ \ \ _X(\ _1\ \ _2)\ \ \ _1 \neq \ _2 \in$$

The Lipschitz constant of is

$$(\ ) = \sup_{x_1 \neq x_2 \in X} \ \ _f(\ _1\ \ _2)$$

Equivalently,

$$(\ ) = \sup_{B \subset X} \ \ _Y(\ (\ ))/\ \ \ _X(\ )$$

where the supremum is taken over all bounded subsets in .

Let $\mathcal{L} = \mathcal{L}(\ )$ be the space of (usually Lipschitz) functions $\ :\ \ \to \mathbb{R}$, let $\mathcal{L}/$ be the space of functions modulo additive constants, and let $\ = \ \ _\mu$ be a Borel measure on $\mathcal{L}/$. Define the $\ _q$-*dilation* of a map $\ :\ \ \to\ $, $1 \leq\ \ \leq \infty$, as

$$||\ \ ^*\ ||_{L_q(\mu)} = \left(\int_{\mathcal{L}/const} \ \ ^q(\ \circ\ )\ \ _\mu \right)^{1/q}$$

Example. Let be the Euclidean space $\mathbb{R}^n$ and let be supported on the orthogonal projections (modulo constants) of $\mathbb{R}^n$ onto the coordinate axes with equal weights 1 assigned to all projections. Then every isometric map $: \mathbb{R}^m \to \mathbb{R}^n$ satisfies

$$||\ \ ^*\ ||_{L_2(\mu)} = \sqrt{\ \ }$$

In general, *an axis* or a *(parametrized) straight line*

Observe that is uniquely determined by up to ±-sign; it will be sometimes denoted $_p$ or suppressed from the notation.

Denote by $\mathcal{P} = \mathcal{P}( )$ the space of axial projectors ( ) in and, given a Borel measure $= _\mu$ on $\mathcal{P}$, define *the* $_q$*-dilation* of a map : → , where is a metric space, by

$$|| \ ^* ||_{L_q(\mu)} = \left(\int_{\mathcal{P}} \ ^q(\ \circ \ ) \ _\mu \right)^{1/q} \quad 1 \leq \ \leq \infty$$

Identity Example. The $_q$-dilation of the identity map $_Y$: → is obviously expressed by the total mass of ,

$$|| \ ^* ||_{L_q(\mu)} = (\mathcal{P})^{1/q}$$

Maps , $\mathcal{I}_\mu$ *and the Norm* $||\widetilde{\ }^* ||_{L_q}$. The space tautologically embeds into the space $\Phi(\mathcal{P})$ of functions $\mathcal{P} \to \mathbb{R}$ for : →

The map $\mathcal{I}_\mu : Ł_1(\mathcal{P} \quad) \to$ extends to $\mathcal{I}_\mu : Ł_2(\mathcal{P} \quad) \to$ that is, in the Hilbertian case, equals the adjoint to , since the operators $: \mathbb{R} \to$ and $: \quad \to \mathbb{R}_a = \mathbb{R}$ are mutually adjoint. Explicitly,

$$\langle \mathcal{I}_\mu(\ (\ )) \quad \rangle_Y = \left\langle \int_{\mathcal{P}} \ (\ (\ )) \ \mu \quad \right\rangle_Y =$$

$$= \int_{\mathcal{P}} \ (\ (\ )) \cdot \ (\ ) \ \mu \ = \int_{\mathcal{P}} \ (\ ) \cdot \ (\ ) \ \mu \ = \langle \ (\ ) \quad (\ ) \rangle_{L_2(\mathcal{P},\mu)}$$

In other words, $\mathcal{I}_\mu$ equals the orthogonal projection of $\ _2(\mathcal{P} \quad)$ onto $=$ $(\ ) \subset \ _2(\mathcal{P} \quad)$. Consequently,

$$(\mathcal{I}_\mu) = 1 \textit{ for all partitions of unity } \quad \textit{in Hilbert spaces}$$

It follows, that every Lipschitz map $\tilde{}$ from a metric space to $\ _2(\mathcal{P} \quad)$, that is a $\mathcal{P}$-family of $\mathbb{R}$-valued functions $\tilde{}_p : \quad \to \mathbb{R} = \mathbb{R}_a \subset \ , (\quad) \in \mathcal{P}$, satisfies

$$|| \quad ^*(\mathcal{I}_\mu \circ \tilde{})||_{L_2(\mu)} \leq \left(\int \quad ^2(\tilde{}_p) \ \mu \right)^{1/2}$$

Partitions of Hilbertian Forms into Squares ^{2}. Let be the Hilbertian quadratic form in . Then the partition of unity condition on can be equivalently (and obviously) expressed in term of the integral of the squares of linear functions (forms):

$$\int_{\mathcal{P}} \quad _p = \ _Y \Leftrightarrow \int_{\mathcal{L}} \ ^2 \quad _l =$$

where $= \ _p : \quad \to \mathbb{R} = \mathbb{R}_p$ are linear functions corresponing to the projectors $: \quad \to \mathbb{R}_p$ and $\ _l$ is the pushforward of from $\mathcal{P}$ to to $\mathcal{L}$ under the map $\mapsto \ _p$.

If and are Hilbert spaces and $: \quad \to$ is a linear map, then $|| \quad ^* \ ||_{L_2(\mu)}$ *does not depend* on the partition of unity in : it equals the trace of the induced quadratic form on

$$|| \quad ^* \ ||_{L_2(\mu)} = \int_{\mathcal{L}} \quad _Y(\ \circ \ )^2 \quad _l = \quad _X \int_{\mathcal{L}} (\ \circ \ )^2 \quad _l = \quad _X \ ^*(\ )$$

and it is called the *Hilbert -Schmidt* norm $|| \ ||_{L_2} = || \quad ^* \ ||_{L_2(\mu)}$ of .

Furthermore, since the map $\mathcal{I}_\mu : \ _2(\mathcal{P} \quad) \to$ is 1-Lipschitz,

$$||\widetilde{\quad}^* \ ||_{L_2(\mu)} = || \quad ^* \ ||_{L_2(\mu)}$$

For instance, isometric linear maps $: \quad \to$ have

$$||\widetilde{\quad}^* \ ||_{L_2(\mu)} = || \quad ^* \ ||_{L_2(\mu)} = || \ ||_{L_2} = \sqrt{\quad (\ )}$$

for all partitions of unity ; in particular

$$|| \quad \widetilde{\quad}^* \ _Y||_{L_2} = || \quad ^* \ _Y||_{L_2} = || \ _Y||_{L_2} = \sqrt{\quad (\ )}$$

In general, Lipschitz maps from metric spaces to Hilbertian satisfy

$$(\ ) \leq || \quad \widetilde{\quad}^* \ ||_{L_2} \leq || \quad ^* \ ||_{L_2} \leq \quad (\ ) \cdot \sqrt{\quad (\ )}$$

This suggests the notation

$$(\ ) = \ _{/id}(\ ) = \ _{/id_Y}(\ ) = || \quad ^{*} \ ||_{L_2}/|| \quad ^{*} \ _Y||_{L_2}$$

and

$$\widetilde{\quad}(\ ) = \widetilde{\quad}_{/id}(\ ) = \widetilde{\quad}_{/id_Y}(\ ) = || \quad \widetilde{\ }^{*} \ ||_{L_2}/|| \quad \widetilde{\ }^{*} \ _Y||_{L_2}$$

These definitions make sense for all arbitrary metric spaces and Banach spaces , where, if is Hilbertian,

$$\widetilde{\quad}(\ ) \leq \quad (\ ) \leq \quad (\ ) \leq \sqrt{\quad (\ )} \cdot \widetilde{\quad}(\ )$$

Up to some point, we formulate and prove obvious general properties of the "norms" $|| \quad ^{*} \ ||_{L_q}$ and $|| \quad \widetilde{\ }^{*} \ ||_{L_q}$ for all Banach spaces and all – this does not cost us anything, but only maps into Hilbert spaces and $= 2$ will be essential for the present day geometric applications.

Local Dilations $|| \ ^{*} (\ )||$, $(\ (\ ))$ *etc. in Finsler Spaces.* All of the above dilation invariants depends only on the metric but not on the linear structure in . This allows the following local versions of these dilations for maps into Finsler, e.g Riemannian, manifolds .

Namely, let $= (\quad)$ be a metric space, where each point $\in$ admits a local Banach metric in a neighbourhood $_y \subset$ of , say $_y$ for all $\in$, such that

$$_y(\ _1 \ _2)/ \quad (\ _1 \ _2) \to 1 \text{ for } _1 \ _2 \to$$

Define

$$|| \ ^{*} (\ )||_{L_q(\mu)}, || \quad ^{*} (\ )||_{L_q}, ||\widetilde{\ }^{*} \ (\ )||_{L_q(\mu)}, \text{Hilb(f(x)), etc.}$$

for maps $: \ \to$ by restricting to small neighbourhoods $_x \subset$ for all $\in$, such that $(\ _x) \subset \ _{f(x)} \subset$, evaluating the corresponding dilations with respect to the local Banach metrics $_y$ and then taking the infimum over all neighbourhoods $_x$ of a point $\in$ for all $\in$.

Equivalently, in the case is a Riemannian manifold, one can, by Nash' theorem, isometrically immerse $\subset \mathbb{R}^N$ and define the local dilations via axial partitions of unity in $\mathbb{R}^N$.

Notice, that unlike the Lipschitz constant, where $(\ ) = \sup_{x\in X} \ (\ (\ ))$ for maps from *length* spaces , there is no(?) apparent passage from local to global $_q$-dilations.

2 Hadamard's $Jac^{[\ ]} \leq \widetilde{Hilb}$ and Inverse Lipschitz Maps.

Hadamard's Inequality for $^{[n]}$. Given a map between Hilbert spaces, say $: \ \to$, denote by $^{[n]}(\ )$ the norm of the -th exterior power of , that is the supremum of the *absolute values* of the Jacobians of on all -dimensional subspaces $' \subset$,

$$^{[n]}(\ ) = \sup_{dim(A')=n} \ (\ |\ ') \text{ for } |\ ' : \ ' \to \ _{A'} \subset$$

If : → is a locally Lipschitz, e.g. ^{1}-smooth, map between Riemannian manifolds then the differential $( ) : _x( ) \to _{f(x)}( )$ exists for almost all $\in$ by *Rademacher-Stepanov theorem* and the Jacobian of is defined as

$$^{[n]}(( )) = \quad ^{[n]}(\quad ( )) \text{ and } \quad ^{[n]}( ) = \sup_x \quad ^{[n]}(\quad ( ))$$

where $\sup_x$ refers to *almost all* $\in$.

Hadamard's Inequality. Let and be Riemannian manifolds and : → be a Lipschitz map.

Then

$$^{[n]}(( )) \leq || \quad ^{-1/2} \quad \widetilde{\quad}^{*} \quad ( )||^{n}_{L_2} \textit{ for } = \quad ( ) \textit{ and almost all } \in$$

In particular, Lipschitz maps from -manifolds to $\mathbb{R}^n$ *satisfy*

$$^{[n]}( ) \leq \widetilde{\quad\quad}^{n}( )$$

Proof. In fact, if

$$= \mathcal{I}_\mu \circ \tilde{\ } = \int_{\mathcal{P}} \ _p(\tilde{\ }) \ _\mu \quad \text{for } \tilde{\ } : \quad \to \ _2(\mathcal{P} \quad)$$

then

$$|| \quad ( )||_{L_2} \leq || \quad \tilde{\ }( )||_{L_2} \leq || \quad \widetilde{\quad}^{*} \quad ( )||_{L_2} \text{ for almost all } \in$$

while

$$^{[n]}(\quad) \leq \ ^{-1/2}|| \quad ||^{n}_{L_2}$$

since the discriminants of quadratic forms (induced by from the Hilbert form in target Hilbert spaces) are bounded by their traces via the arithmetic/geometric mean inequality for the eigenvalues of these forms.

However trivial, the inequality $^{[n]}( ) \leq \ ^{n}( )$ is significantly sharper than mere $| \ ^{[n]}|( ) \leq \ ^{n}( )$.

For instance, let a smooth map from a Riemannian -manifold to $= \mathbb{R}^n$ be given by functions $_1 \quad _n$. Then $( )$ may be as big as $(\sum_i \ ^2(_i))^{1/2}$, while the Hadamard's inequality, applied to the Hilbert form $\sum_{i=1}^{n} \ _i \ ^2_i$ for $_i = (\quad (_i))^{-1/2}$ on , says that

$$| \quad ^{[n]}( )| \leq \prod_{i=1,\dots,n} \quad (_i)$$

(This is, of course, obvious anyway.)

Sharpness of Hadamard. Hadamard's inequality bounds the volume of the image of a Lipschitz map between -dimensional Riemannian manifolds, : → , by

$$_n(\ (\ )) \leq \int_X \quad ^{n}(\quad ( )) \quad \leq \quad _n(\ ) \cdot \sup_x \quad ^{n}(\quad ( ))$$

since

$$_n(\ (\ )) \leq \int_X \quad ^{[n]}(\ ( )) \quad \leq \quad _n(\ ) \cdot \sup_x \quad ^{[n]}(\quad ( ))$$

where the two suprema are taken over *almost all* $\in$.

Since the algebraic inequality $^{[n]}(\ ) \leq \ ^{n}(\ )$ is sharp for linear maps $: \mathbb{R}^n \to \mathbb{R}^n$, i.e. the equality $^{[n]}(\ ) = \ ^{n}(\ )$ implies that is a homothety, the corresponding integral inequalities are also sharp:

A. *If*

$$_n(\ (\ )) = \int_X \ ^n(\ (\ )) \ < \infty \ \textit{and} \ (\ ) = \ (\ ) \geq 2$$

then the map is a conformal diffeomorphism on its image.

(This is not used in the sequel.)

B. *If*

$$\sup_x \ ^n(\ (\ )) = 1 \ \textit{and} \ _n(\ (\ )) = \ _n(\ ) < \infty$$

then is an isometric diffeomorphism onto its image.

Proof. The equality $_n(\ (\ )) = \int_X \ ^{[n]}(\ (\ ))$ implies that the local topological degree of , either equals +1 at almost all $\in$ or it is almost everywhere −1. If such an is conformal *almost everywhere* and ≥ 2, then it is conformal *everywhere*. For example, if $= 2$ this is seen with the Cauchy integral formula, while the case ≥ 3 is (essentially) trivial because of the Liouville theorem.

The remaining case of **B** for $= 1$ is, of course, obvious but it points out to the following extension to more general Lipschitz maps.

C. *If a Lipschitz map* $: \ \to$ *between -dimensional Riemannian manifolds without boundaries satisfies*

$$_n(\ (\ )) = \int_X \ ^{[n]}(\ (\ )) \ < \infty$$

and

$$_n(\ ^{-1}(\ )) \leq \ \cdot \ (\ ) \ \textit{for all open} \ \subset \ \textit{and a constant} \ < \infty.$$

Then is a homeomorphism onto its image $_f = (\ ) \subset$ *and the inverse map* $^{-1}: \ _f \to$ *satisfies*

$$\sup_{y \in im_f} \ _y(\ ^{-1}) \leq \ \cdot \ ^{[n-1]}(\ ) \leq \ \cdot \ ^{n-1}(\ )$$

This is well known with the proof, probably, buried somewhere in [8]. Here is how it goes.

Since $_n(\ ^{-1}(\ )) \leq \ \cdot \ (\ )$, the pullbacks $^{-1}(\ ) \subset$ are *zero dimensional* subsets in for all $\in \ _f$; in fact, this follows from the inequality $_n(\ ^{-1}(\ _y(\ ))) = (\ ^{n-1})$ for small -balls in .

Since $(\ ^{-1}(\ )) \leq 0$, every point $\in \ ^{-1}(\ )$ admits an arbitrarily small neighbourhood, say $_x \subset$, such that the the boundary $_x \subset$ does not intersect $^{-1}(\ )$; hence, the local topological degree of is defined at .

This degree must be *non-zero*, otherwise, all points $'$ close to would have "partners" $''$ such that $(\ ') = (\ )$ that is incompatible with $_n(\ (\ )) = \int_X \ ^{[n]}(\ (\ ))$.

Therefore, the image of every neighbourhood of $\in$ contains a neighbourhood of $= (\ ) \in$ for all $\in$. Thus, is what is called an *open map.*

Besides, again because of $_n(\ (\ )) = \int_X \ ^{[n]}(\ (\ ))$, the map is one-to-one on a dense subset (of full measure) in , and, obviously,

densely one to one + open ⇒ one-to-one.

Granted one-to-one, take an -narrow cylinder $_\varepsilon \subset \ _f$ around a distance minimizing geodesic segment $[\ _1\ _2] \subset \ _f$, denote by $_1\ _\varepsilon \ni \ _1$ and $_2\ _\varepsilon \ni \ _2$. the top and the bottom of this cylinder and let $_{lat}\ _\varepsilon$ be the remaining (lateral) part of its bounadry,

$$_{lat}\ _\varepsilon = \ _\varepsilon \setminus (\ _1\ _\varepsilon \bigcup \ _2\ _\varepsilon)$$

Observe that the relative homology group $_{n-1}(\ _\varepsilon\ \ _{lat}\ _\varepsilon)$ is free cyclic and let $_{n-1}\ ^{n-1}$ denote the minimum of the $(\ -1)$-volumes of relative $(\ -1)$-cycles representing *non-zero* classes in $_{n-1}(\ _\varepsilon\ \ _{lat}\ _\varepsilon)$.

Observe that $_{n-1}$ is close to the volume of the Euclidean $(\ -1)$-ball and

$$_n(\ _\varepsilon) = \ _{n-1}\ ^{n-1} \cdot \ (\ _1\ _2) + (\ ^{n-1})$$

Take the pull-backs $\tilde{}_\varepsilon = \ ^{-1}(\ _\varepsilon) \subset$ and $_{lat}\ \tilde{}_\varepsilon = \ ^{-1}(\ _{lat}\ _\varepsilon)$ and let $\tilde{}_{n-1}\ ^{n-1}$ be the minimum of the $(\ -1)$-volumes of relative $(\ -1)$-cycles representing *non-zero* classes in $_{n-1}(\tilde{}_\varepsilon\ \ _{lat}\ \tilde{}_\varepsilon)$.

Clearly,

$$_{n-1} \leq \ ^{[n-1]}(\ ) \cdot \tilde{}_{n-1}$$

On the other hand,

$$_n(\tilde{}_\varepsilon)) \geq \ _X(\ _1\ \tilde{}_\varepsilon\ \ _2\ \tilde{}_\varepsilon) \cdot \tilde{}_{n-1}\ ^{n-1}$$

for $_1\ \tilde{}_\varepsilon\ \ _2\ \tilde{}_\varepsilon \subset$ being the pullbacks of the top and the bottom of the cylinder $_\varepsilon \subset$.

The proof of this is standard: map $\tilde{}_\varepsilon$ to $\mathbb{R}_+$ by $:\ \tilde{} \mapsto \ _X(\tilde{}\ \ _1\ \tilde{}_\varepsilon)$, observe that the pullbacks $^{-1}(\ ) \subset \tilde{}_\varepsilon$ support non non-zero classes in $_n(\tilde{}_\varepsilon\ \ _{lat}\ \tilde{}_\varepsilon)$ for $0 \leq\ \leq\ \ _X(\ _1\ \tilde{}_\varepsilon\ \ _2\ \tilde{}_\varepsilon)$ and apply the classical (and obvious) *coarea inequality* (that happens to be equality in the present case and that is extended in section 8 to more general setting of Hilbert volumes),

$$_n(\tilde{}_\varepsilon) \geq \int_{\mathbb{R}_+} \ _{n-1}(\ ^{-1}(\ ))$$

Then the proof follows with $\to 0$.

D. *Remark.* The statement of **C** can be obviously reformulated without any reference to differentiability of and the above argument still applies as it does not use the Rademacher-Stepanov theorem.

Besides, one does not truly need to be Lipschitz: what is essential is a bound on the $(\ -1)$-Jacobian $^{[n-1]}(\ )$ that can be defined as

$$^{[n-1]}(\ ) = \sup_H \ _{n-1}(\ (\ ))/\ \ _{n-1}(\ )$$

for all rectifiable hypersurfaces $\subset$.

Moreover, and do not have to carry any metrics, what is needed are measures on and and the notion of "$(\ -1)$-volume" for hypersurfaces. In fact, the distance can be *derived* from these data via the *Almgren-Besicovitch-Derrick inequality* (see section 9).

Warnings. (a) A Lipschitz map : → where ( ) : $_x($ $) \to$ $_{f(x)}($ $)$ is an isometry almost everywhere does not have to be one-to-one. In fact, paradoxically, every Riemannian -manifold admits a 1-Lipschitz map into $\mathbb{R}^n$, that preserves the lengths of *all* rectifiable curves in , see 2.4.11 in [16].

(b) There are lots of ª -smooth maps : → such that $_n($ $($ $))$ $\int_X$ $^{[n]}($ $($ $))$, where $^{[n]}($ $)$ x 0 almost everywhere; yet, where the unions of those pullbacks $^1($ $)$ ` that are diffeomorphic to the (− 1)-ball, are *dense* in .

3 Controlled $\tilde{L}$ -Dilation Extensions, Banach Straightening and Hilbertian Volume Domination.

If a $_0 \overset{f_0}{\to}$ is a partially defined -Lipschitz map between metric spaces, then it admits an extension to a -Lipschitz map : → only for rather exceptional spaces . For instance, this is, obviously, possible if is a tree, e.g. $\mathbb{R}$, a (finite or infinite measurable) Cartesian product of trees with the -metric, e.g. the a -space over a measure space, such as a $(\mathcal{P}$), or else, an *Uryson's universal space.*

But such extensions do not exist, in general, not even for $\mathbb{R}^n$ if C 2, except for particular domains , such as *Alexandrov's spaces* with curvature C 0 [21], while for general , only an extension : → $\mathbb{R}^n$ with ( ) B $^{1/2}$ ($_0$) is (obviously) possible.

However, Lipschitz extensions of maps $\tilde{}_0$: $_0$ → a $(\mathcal{P}$) to $\tilde{}$: → a $(\mathcal{P}$) trivially yield a control over $\tilde{}_q$-dilation of maps $\mathcal{I}_\mu \circ \tilde{}$: → as follows.

Let be a metric space and be a Banach space with an axial partition of unity .

Then every partial map, a $_0 \overset{f_0}{\to}$ *, extends to a map* : → *, such that*

$$||\widetilde{}^{\ddagger} ||_{L_q(\mu)} \text{ B} ||\widetilde{}^{\ddagger}{}_0||_{L_q(\mu)} \text{ B} || {}^{\ddagger}{}_0||_{L_q(\mu)} \text{ B}$$

$$\text{B} || {}^{\ddagger}{}_0||_{L_\infty(\mu)} \cdot ((\mathcal{P}))^{1/q} \text{ B} ({}_0) \cdot ((\mathcal{P}))^{1/q}$$

for all 1 B B ∞

Remark. The only point here which needs a minor attentions (and which is unneeded for our applications) is to make sure that the same extension $\tilde{}$ of $\tilde{}_0$ serves the spaces $_q(\mathcal{P}$) simultaneously for all .

This is achieved by making Lipschitz extensions $\tilde{}_p$: → $\mathbb{R}$ $\mathbb{R}_p$ ` , $>\mathcal{P}$, of the corresponding functions $(\tilde{}_0)_p$ : $_0 \to \mathbb{R}$ $\mathbb{R}_p$ ` depend *measurably* on , e.g. by taking, for each $>\mathcal{P}$, a *minimal* such $\tilde{}_p$. Namely, in general, given a partial defined function a $_0 \overset{\phi_0}{\to} \mathbb{R}$, its minimal Lipschitz extension is

$$( ) \sup_{x_0 \in X_0} ({}_0({}_0) - {}_X({}_0))$$

For example, if $_0$ is isometric to an axis $\mathbb{R} \subset$ and $_0 \to \mathbb{R} = {}_0$ is the identity map, this minimal extension equals the Busemann function $\to \mathbb{R}$.

Banach Straightening at Infinity. Define the *Lipschitz constant at infinity* of a map $:\ \to$ between metric spaces by

$$_\infty(\ ) = \limsup_{dist_X(x_1,x_2)\to\infty} {}_f(\ _1\ _2)$$

Equivalently, this can be defined with restrictions of to *-separated nets* $_D \subset$, that are subsets in with distances between all pairs of points $\geq$, as follows,

$$_\infty(\ ) = \limsup_{X_D\to\infty} (\ _{|X_D})$$

Similarly, define the $\tilde{\ }_q$-dilation at infinity, denoted $||\widetilde{\ }^*_\infty||_{L_q(\mu)}$, by restricting to -separated nets and taking of $||\widetilde{\ }^*_{|X_D}||_{L_q(\mu)}$ over $_D$ with $\to \infty$.

Then the above extension of maps $|_D :\ _D \to$ from $_D \subset$ to all of implies the following

Banach -Straightening. Let $:\ \to$ be a map from a metric space to a Banach space with a partition of unity in .

Then, for every > 0*, there exists a map* $_\varepsilon :\ \to$ *within finite distance from , i.e.*

$$||\ _\varepsilon(\ ) - (\ )||_Y \leq\ =\ _\varepsilon < \infty \textit{ for all } \in$$

and such that

$$||\widetilde{\ }^*\ ||_{L_q(\mu)} \leq (||\ ^*_\infty\ ||_{L_\infty(\mu)} +\ ) \cdot\ (\mathcal{P})^{1/q} \textit{ for all } \in [1\ \infty]$$

On $\to 0$. One can, in many cases, pass to the limit map $_{\varepsilon\to 0} :\ \to$ but this $_{\varepsilon\to 0}$ may be far away from , e.g. it may be constant for an *isometric at infinity*, i.e. where $\lim_{D\to\infty}\ _{f|X_D} = 1$.

Hilbert-Hadamard Volume Domination. Let be an -dimensional Riemannian manifold, let be a partition of unity in the Euclidean/Hilbertian space $= \mathbb{R}^n$ and let $:\ \to$ a proper (pullbacks of compact sets are compact) continuous map.

Let $\Omega_R \subset \mathbb{R}^n$, $\to \infty$, be bounded domains, such that $_n(\Omega_R) = \ \cdot\ ^n$ and such that for every $_0$ the volumes of the $_0$-neighbourhoods of their boundaries satisfy

$$_n(\ _{r_0}(\ \Omega_R)) = (\ ^n) \text{ for } \to \infty$$

e.g. Ω_R are Euclidean -balls.

If has non-zero topological degree, then the Euclidean volumes of the domains $\Omega_R \subset \mathbb{R}^n$ *are "asymptotically sharply bounded" by the Riemannian volumes of their -pullbacks as follows*

$$_n\big(\ ^{-1}(\Omega_R)\big) \geq ||\ ^*_\infty||^n_{L_\infty(\mu)} \cdot\ \ _n(\Omega_R) - (\ ^n) \textit{ for } \to \infty$$

Proof. Since $(\ ) \neq 0$ and $(\quad _{\varepsilon}) < \ _{\varepsilon}$, the images $_{\varepsilon}(\ ^{-1}(\Omega_R)) \subset \mathbb{R}^n$ contain $[\Omega_R - \ _{\varepsilon}] =_{def} \Omega_R \smallsetminus \ _{r^{n}}(\ \Omega_R)$; hence,

$$_{n}(\ ^{-1}(\Omega_R)) \geq \quad _{n}(\ _{\varepsilon}^{-1}[\Omega_R - \ _{\varepsilon}])$$

On the other hand, by the above,

$$\widetilde{\quad\quad}(\ _{\varepsilon}) = \ ^{-1/2}\|\widetilde{\quad}^{*} \ _{\varepsilon}\|_{L_2(\mu)} \leq \| \ _{\infty}^{*} \ \|_{L_\infty(\mu)} +$$

while the Jacobian of $_{\varepsilon}$ is bounded by Hadanmard's inequality $\ ^{[n]}(\ _{\varepsilon}) \leq \widetilde{\quad\quad}^{n}(\ _{\varepsilon})$; hence,

$$_{n}(\ _{\varepsilon}^{-1}[\Omega_R - \ _{\varepsilon}]) \geq \widetilde{\quad\quad}^{-n}(\ _{\varepsilon}) \cdot \quad _{n}[\Omega_R - \ _{\varepsilon}]$$

$$\geq \| \ _{\infty}^{*} \ \|_{L_\infty(\mu)}^{-n} \cdot \quad _{n}(\Omega_R) - \ \cdot \qquad \cdot \ ^{n} - \ (\ ^{n}) \text{ for all } \ > 0$$

QED.

Local $\tilde{\ }_{q}$*-Extensions.* Let be a smooth manifold with a continuous Riemannian metric and $\supset \ _{0} \overset{f_0}{\to}$ a partial Lipschitz map. Suppose that that

$$\| \quad \widetilde{\ }^{*} \ _{0}(\ )\|_{L_q} < \ (\ ) \text{ for all } \ \in \ _{0}$$

where is a continuous function on .

Then $_{0}$ *extends to a neighbourhood* $_{1} \supset \ _{0}$ *by a map* $\supset \ _{1} \overset{f}{\to}$ *, such that*

$$\| \quad \widetilde{\ }^{*} \ (\ )\|_{L_q} < \ (\ ) \textit{ for all } \ \in \ _{0}$$

Proof. Combine the above extension with the ordinary Lipschitz partition of unity in by the following trivial argument.

First, let $= \mathbb{R}^N$ and let $_{i} \subset \ , \ \in \ $, be a locally finite covering of such that

$$\| \quad \widetilde{\ }^{*} \ _{0|U_i \cap X_0}\|_{L_q} < \ (\ ) \text{ for all } \ \in \ _{i}$$

Extend the maps $_{0|U_i \cap X_0} : \ _{i} \cap \ _{0} \to \mathbb{R}^N$ to $_{i} : \ _{i} \to \mathbb{R}^N$, such that

$$\| \quad \widetilde{\ }^{*} \ _{i}\|_{L_q} \leq \| \quad \widetilde{\ }^{*} \ _{0|U_i \cap X_0}\|_{L_q} < \ (\ ) \text{ for all } \ \in \ _{i}$$

Let $_{i} : \ _{i} \to \mathbb{R}_{+}$ be Lipschitz functions with supports strictly inside $_{i}$ and such that $\sum_{i \in I}$=1. Then the map

$$= \sum_{i \in I} \ _{i} \cdot \ _{i} \text{ satisfies } \| \quad \widetilde{\ }^{*} \ (\ )\|_{L_q} < \ (\ )$$

in some neighbourhood $_{1} \subset$ of $_{0}$ by an obvious computation.

Finally, if is ^{0}-Riemannian, isometrically ^{1}-embed $\subset \mathbb{R}^N$ and observe that some neighbourhood $_{1} \subset \mathbb{R}^N$ of admits a Lipschitz projection $: \ _{1} \to$ such that $(\ (\ )) = 1$ for all $\in$. (If $\subset \mathbb{R}^N$ is ^{2}-smooth, one can use the normal projection to , that does not always exist for ^{1}-submanifolds.)

Compose the above with this and thus, obtain the required extension $= \ \circ \ : \ \to$ of $_{0}$ from $_{0}$ to $_{1} \supset \ _{0}$. QED.

Question on $_q$-Dilation at Infinity. Is there a counterpart to the above with $|| \quad ^* \quad ||_{L_q}$ instead of $||\widetilde{\quad}^* \quad ||_{L_q}$?

Namely, what is the minimal (or, rather, infimal) constant $\quad = \quad (\quad \quad ')$, such that every $\quad : \quad \to \quad$ admits an $\quad ' : \quad \to \quad$ within bounded distance from $\quad$, such that $|| \quad ^* \quad '||_{L_{q'}(\mu)} \leq \quad \cdot || \quad ^*_\infty \quad ||_{L_q(\mu)}$?

4 John's $h_\diamond$ and John's Ellipsoid in Banach Spaces with a Riemannian Corollary.

So far, everything was compiled of definitions + trivial generalities. Now comes something deceptively simple but more substantial.

Fritz John's Ellipsoid Theorem. *An -dimensional Banach space $\quad = (\quad || \quad ||_Y)$ admits a unique Hilbert quadratic form $_\diamond$, such that*

$$|| \quad ||_{h_\diamond} \geq || \quad ||_Y$$

and such that the identity map $\quad_\diamond = \quad : \quad = (\quad || \quad ||_Y) \to \quad = \quad_\diamond = (\quad || \quad ||_{h_\diamond})$ *satisfies*

$$_{h_\diamond} = \quad _{/Y_\diamond}(\quad _\diamond) \leq 1$$

for $\quad _{/Y_\diamond}(\quad _\diamond) = \quad ^{-1/2}|| \quad \quad ^* \quad _\diamond||_{L_2}$.

Proof. Let $_+(1)$ be the subset in the space of positive semidefinite (Hilbertian) quadratic forms on such that $\quad _h \leq 1$, i.e such that the identity map $\quad = \quad _h : (\quad || \quad ||_Y) \to (\quad \quad)$. satisfies $\quad ^{-1/2}|| \quad \quad ^* \quad _h||_{L_2} \leq 1$

Observe that this $_+(1)$ is a *convex* susbset in the linear space of all quadratic forms on by the definition of $|| \quad \quad ^*||_{L_2}$ via partitions of Hilbertian forms into squares ^{2} (see section 1).

($\diamond$) Let $\quad_\diamond \in \quad _+(1)$ *maximize* the Hilbertian (Euclidean) Haar measure of associated to it among all $\quad \in \quad _+(1)$, i.e. the Jacobian of the identity map $: (\quad _\diamond) \to (\quad \quad)$, denoted $\quad ^{[n]}_{h_\diamond \to h}$, satisfies $\quad ^{[n]}_{h_\diamond \to h} \leq 1$ for all $\quad \in \quad _+(1)$. Then

$$|| \quad ||_{h_\diamond} \geq || \quad ||_Y$$

Indeed, assume otherwise, let be a linear form on such that $|| \quad ||^2_Y = \quad = \quad (\quad)$ and $|| \quad ||^2_{h_\diamond} = \quad / \quad$ for $\quad > 1$ and let $\quad _\varepsilon = (1 - \quad) \quad _\diamond + \quad ^2$, $0 \leq \quad \leq 1$.

Clearly,

$$^{[n]}_{h_\diamond \to h_"} = \left((1 - \quad)^{(n-1)} \cdot ((1 - \quad) + \quad \quad))\right)^{1/2}$$

and

$$\log \quad ^{[n]}_{h_\diamond \to h_"} = \frac{1}{2}(-(\quad - 1) \quad + \quad (\quad - 1) \quad) - \quad (\quad) > 0 \text{ for } \quad \to 0;$$

hence, the form $_\varepsilon$ has greater Haar measure than $_\diamond$ for small $\quad > 0$.

Since $\quad \cdot \quad ^2 \in \quad _+(1)$, the form $_\varepsilon$ also lies in $_+(1)$ by convexity of $_+(1)$; this contradicts the extremality of $_\diamond$ and uniqueness of $_\diamond$ follows by the same argument. QED.

John's $_\diamond$. The partition of unity $\quad _\diamond = \quad _{min}$ in the Hilbert space $(\quad || \quad ||_{h_\diamond})$ for which $|| \quad ^* \quad _\diamond||_{L_2} = \sqrt{\quad}$ is not, in general unique. However, the inequalities $|| \quad ||_{h_\diamond} \geq || \quad ||_Y$ and $\quad _{h_\diamond} \leq 1$ trivially imply that *every projector* $\in \mathcal{P}(\ \ \|\ \ \|_{h_\diamond})$ *from the support of some* $_\diamond$ *lies in* $\mathcal{P}(\ \ \|\ \ \|_Y)$, *i.e.* $\|\ \ \|_Y = 1$, *and the norm* $\|\ \ \|_{h_\diamond}$ *equals* $\|\ \ \|_Y$ *on the axes* $\mathbb{R}_p = \ \ _p \subset$ *of all* $\in \quad (\ _\diamond)$.

Remark. The above shows that the total masses of John's partitions of unity $_\diamond$ in satisfy $_\diamond(\mathcal{P}) = \ =\dim$. In fact, the existence of a partition of unity in with the total mass $\leq$ is equivalent (by a trivial argument) to the full John's theorem.

Riemannian Lower Volume Bound. Let be an -dimensional Banach space, let $\Omega_R \subset$, $\to \infty$, be bounded domains, as in the Hilbert-Hadamard volume domination from section 3, i.e. such that their -volumes with respect to John's Hilbert/Euclidean metric $_\diamond$ satisfy $_\diamond(\Omega_R) = \quad \cdot \ ^n$ and such that for every $_0$ the volumes of the $_0$-neighbourhoods of their boundaries satisfy

$$_\diamond(\ _{r_0}(\ \Omega_R)) = \ (\ ^n) \text{ for } \ \to \infty$$

e.g. Ω_R are the -balls $\{\|\ \ \| \leq \ \} \subset$.

Asymptotic Banach-John Volume Inequality of Burago-Ivanov. Let be an -dimensional Riemannian manifold, let $:\ \to$ be a proper (pullbacks of compact set are compact) map of non-zero degree and let $_\infty(\ )$ be the Lipschitz constant of at infinity with respect to the metric $\|\ _1 - \ _2\|_Y$ in .

Then the (John's Euclidean) $_\diamond$*-volumes of* $\Omega_R \subset$ *are "asymptotically sharply bounded" by the Riemannian volumes of their -pullbacks by*

$$_n(\ ^{-1}(\Omega_R)) \geq \ _\infty^{-n}(\ ) \cdot \ _\diamond(\Omega_R) - \ (\ ^n) \text{ for } \to \infty$$

Consequently, since $\|\ \ \|_{h_\diamond} \geq \|\ \ \|_Y$, *the asymptotic volume growth of -balls in around any given point* $_0 \in$ *is asymptotically sharply bounded by the growth of the Euclidean balls,*

$$\liminf_{R\to\infty} \ (\ _X(\ ))/\ (\ _{\mathbb{R}^n}(\ )) \geq 1$$

Proof. Since John's $_\diamond$ serves as partition of unity in $(\ \ \|\ \ \|_Y)$ as well as in $(\ \ \|\ \ \|_{h_\diamond})$, the map $:\ \to \mathbb{R}^n = (\ \ \|\ \ \|_{h_\diamond})$ satisfies

$$\|\ \ _{h_\diamond}^{*}\ \|_{L_\infty(\mu)} \leq \ _\infty(\ )_{\|\ldots\|_Y}$$

where – this is the main point – the dilation $\|\ \ ^*\ \|$ is measured with the (Euclidean) $_\diamond$-metric in , while $_\infty(\ )$ is evaluated with the original Banach metric. Hence the Hilbert-Hadamard volume domination applies and the proof follows.

5 Co-Lipschitz at Infinity, Federer-Whitney Metric Descendants and Abelian Volume Growth Inequality of Burago-Ivanov.

Federer-Whitney Theorem. Define the *co-Lipschitz constant at infinity* of a map between metric spaces, $:\ \to$, by

$$_\infty(\ ) = \limsup_{diam_Y(B)\to\infty} \ _X(\ ^{-1}(\ ))/\ _Y(\ )$$

where $\subset$ run over all *bounded* subsets in .

Observe that if

$$\sup_{y\in Y} \quad {}_X(\ ^{-1}(\ )) < \infty$$

(which is weaker than $\ {}_\infty(\ ) < \infty$) then

$${}_\infty(\ ) = \Big(\liminf_{dist(x_1,x_2)\to\infty} \quad {}_f(\ {}_1\ \ {}_2)\Big)^{-1}$$

A map between metric spaces, $\ :\ \to\ $, is called an *isometry at infinity* if the image of intersect all -balls in for $\geq\ {}_0 < \infty$ and

$${}_\infty(\ ) = \quad {}_\infty(\ ) = 1;$$

this is equivalent to

$${}_f(\ {}_1\ \ {}_2) = \frac{{}_X(\ {}_1\ \ {}_2)}{{}_Y(\ (\ {}_1)\ \ (\ {}_2))} \to 1 \text{ for } \quad {}_X(\ {}_1\ \ {}_2) \to \infty$$

Let be a locally compact metric space, an $\mathbb{R}$-linear -space, where both spaces are acted upon by a locally compact group Γ, such that the action of Γ on is *isometric* and the action on is *affine.*

Let this affine action be *co-compact quasi-translational*, i.e. $/\Gamma$ is compact and Γ admits a *co-compact* subgroup that acts on by *parallel translations.*

Federer-Whitney -Descendant Metric. Let $\ :\ \to\ $ be a proper continuous Γ-equivariant map.

Then there exists a unique Banach distance on , denoted $\ {}_Y(\ {}_1\ \ {}_2) = \|\ {}_1 - \ {}_2\|_Y$ *and called Federer-Whitney -Descendant of the metric* $\ {}_X$*, such that the map is isometric at infinity with respect to* $\ {}_X$ *and* $\ {}_Y$*.*

Proof. Clearly, there exists a unique *maximal* Banach distance $\ {}_Y$ in with respect to which $\ {}_\infty(\ ) \leq 1$.

To see that $\ {}_\infty(\ ) = 1$ as well, observe that every two disjoint -balls in satisfy

$${}_Y(\ {}_{y_1}(\ )\ \ {}_{y_2}(\ )) = \lim_{\lambda\to\infty}\ ^{-1}\cdot \quad {}_X\big(\ ^{-1}(\ {}_{\lambda y_1}(\ ))\ \ ^{-1}(\ {}_{\lambda y_2}(\ ))\big)$$

where the scaling $\to$ is understood with some point in taken for zero and where the existence of the limit, as well the convexity and homogeneity of the limit distance function, trivially follow from

• the triangle inequality for $\ {}_X$,

• elementary properties of *asymptotically sub-additive* positive functions $(\ )$, > 0, i.e. such that $(\ {}_1 + \ {}_2) \leq (\ {}_1) + (\ {}_2) + (\ {}_1)$; the essential property of such functions is the existence of the limit $\lim_{\rho\to\infty}(\ )/\ $.

Remarks. (a) The above is the special (trivial) case of the full *Federer-Whitney duality theorem* between the *stable volume norm on homology* $\ {}_k(\ )$ and the *comass norm* on cohomology $\ ^k(\ ;\mathbb{R})$. (See sections 4.C and 4.D in [10] and 7.4 in [11] for the statement and further applications of this theorem.)

(b) The Federer-Whitney theorem generalizes [22] to maps $\to$, where is a simply connected *nilpotent* Lie group with an *expanding* automorphism and where the existence of the limit

$$\lim_{N\to\infty}\ ^{-N}\cdot \quad {}_X\big(\ ^{-1}(\ ^N(\ {}_{y_1}))\ \ ^{-1}(\ ^N(\ {}_{y_2}))\big)$$

is not so obvious.

(c) The property of being isometric at infinity for equivariant maps some-times implies a much stronger (and non-obvious) one:

$$| \quad _X(\ ^{-1}(\ _1) \quad ^{-1}(\ _1)) - \quad _Y(\ _1 \quad _2)| \leq \qquad < \infty$$

This was proven by D. Burago [5] for surjective maps from length metric spaces (where the distance is given by infima of length of curves between points) into linear spaces and, later on, for maps into some nilpotent Lie groups and all hyperbolic Γ-spaces by S. Krat [20], but also there are counterexamples [4]. Yet, the full geometry of isometric at infinity equivariant maps has not been fully clarified at the present day.

Abelian Volume Growth Inequality of Burago-Ivanov . Let $\tilde{\ }$ be the universal covering of a compact Riemannian -manifold that admits a continuous map of non-zero degree onto the -torus, $\to \mathbb{T}^n$, e.g. is homeomorphic to $\mathbb{T}^n$.

Then the asymptotic volume growth of -balls ( ) in $\tilde{\ }$ for $\to \infty$ is minorized by that in $\mathbb{R}^n$,

$$\liminf_{R\to\infty} \quad (\ (\ ))/ \quad (\ _{\mathbb{R}^n}(\ )) \geq 1$$

Proof. Let be the universal covering of the torus $\mathbb{T}^n$, let be the corre-sponding $\mathbb{Z}^n$-covering of and : $\to$ the $\mathbb{Z}^n$-equivariant map induced by our $\to \mathbb{T}^n$.

Let $\| \ \|_Y$ be the Federer-Whitney norm on and $_\diamond$ John's quadratic Hilbertian (Euclidean) form.

Since is isometric at infinity with respect to the Federer-Whitney metric $\| \ _1 - \ _2\|_Y$, the -pullbacks of the (Euclidean) $_\diamond$-balls $_\diamond(\ - \ (\ )) \subset \mathbb{R}^n =$ $(\quad _\diamond)$ of radii $- \ (\ ^n)$ are contained in the Riemannian -balls in $= \tilde{\ }$, and since $_\infty(\ ) = 1$, the asymptotic Banach-John volume inequality from the previous section applies.

Remarks. (a) This proof:

Federer-Whitney + John's $_\diamond$ + Hadamard's inequality

is similar to the original one in [7], except that the authors of [7] use at some point a rather subtle Burago's theorem (stated in the above Remark (c)) instead of the (almost obvious) Federer-Whitney theorem.

(b) *Volume Rigidity.* The above argument combined with "sharpness of Hadamard" (see **B** in section 2) implies by a simple argument another theorem from [7].

Let be a Riemannian Γ-manifold of dimension , where the asymptotic volume growth of balls equals that in $\mathbb{R}^n$,

$$\liminf_{R\to\infty} \quad (\ _X(\ ))/ \quad (\ _{\mathbb{R}^n}(\ )) = 1$$

If admits a Γ-equivariant map $_0 : \ \to \mathbb{R}^n$ of a non-zero degree, where the action of Γ on $\mathbb{R}^n$ is affine co-compact quasi-translational, then is isometric to $\mathbb{R}^n$.

This, however, leaves open the following

Question. What is the minimal volume growth of an with the correspond-ing Federer-Whitney Banach space being in a given isometry class?

(We shall address this in Part 2 of this paper.)

(c) An attractive feature of the volume growth theorem, is that a general convexity argument (John's e theorem) yields, rather unexpectedly, a *sharp purely Riemannian* inequality.

(A non-sharp lower bound on asymptotic volumes of balls follows by the above argument, see [3] and 4.C in [10], from Federer-Whitney and the inequality $|| \ ||_A \geq \sqrt{\ } \cdot || \ ||_{h_\diamond}$ that is the standard corollary to John's $\ _{h_\diamond} \leq 1$.)

But we shall see with our definition of *Hilbertian volume* in section 9 that, in truth, there is not so much "Riemannian" in this inequality after all: the Abelian volume growth inequality and the volume rigidity hold for (almost) arbitrary metric Γ-spaces .

6 Γ-Spaces, Averaging and Hilbert's Straightening Map $f_\diamond$.

The Federer-Whitney Banach metric is, obviously, invariant under actions of a group Γ acting on our spaces. Let us explain how to make other constructions Γ-invariant as well.

Recall, that *a* Γ*-space* in a given (topological) category is a space with an action of a group Γ, denoted $\ _X : \ \to \ , \ \in \Gamma$, where these maps $\ _X$ are morphisms in our category. We are mainly concerned with *metric* Γ*-spaces* where the maps $\ _X$ are isometries.

Equivariant Hilbert Straightening Theorem. Let and be topological Γ-spaces as in the Federer-Whitney theorem, i.e. is a locally compact metric Γ-space, and an ℝ-affine -space isomorphic to $\mathbb{R}^n$, where the action of Γ is co-compact quasi-translational. We denote by $\ _\bullet$ the linear space associated to that is the space of parallel translations of and that can be (non-functorially) thought of as with a point in it distinguished for 0.

Let $\ _0 : \ \to \ $ be a proper continuous Γ-equivariant map.

Then there exist a unique Γ*-invariant (Federer-Whitney-John's) Euclidean (Hilbertian) metric* $\ _\diamond = || \ ||_{h_\diamond}$ *on , and a (non-unique)* Γ*-equivariant Lipschitz map* $\ _\diamond : \ \to \ $*, such that*

$$\ _{h_\diamond}(\ _\diamond \ \ _0) \leq \infty$$

the Hilbert constant of $\ _\diamond$ *with respect to the Hilbertian form* $\ _\diamond$ *on satisfies*

$$\widetilde{\quad}(\ _\diamond) = 1$$

and, at the same time,

$$\ _\infty(\ _\diamond) \leq 1$$

Proof. The distance $|| \ ||_{h_\diamond}$ in for John's $\ _\diamond$ associated to the Federer-Whitney Banach metric $\ _Y$ is Γ-invariant due to uniqueness of $\ _\diamond$ but the underlying measure $\ _\diamond$ – partition of unity ($\ _{\mu_\diamond}$) or equivalently, partition of $\ _\diamond$ (that is $\ _{\mu_\diamond}$), is not necessarily invariant.

But since the linear isometry group of ($\ _\bullet \ || \ ||_Y$), where $\ _\bullet$ is the linear space associated to , is compact, one can average partitions of unity $\ _\diamond$ (or rather partitons of $_\diamond$ into squares of linear functions) over and have John's partition of unity $_\diamond$ invariant under the action of Γ.

Let us make the -straightened maps $_\varepsilon$: → from section 3 equivariant by the following standard averaging over Γ.

Observe that Γ acts on the space of maps : → by $_F$: ↦ $_Y^{-1} \circ \; \circ \; _X$ with fixed points of this action corresponding to equivariant maps .

If we replace maps ∈ by $_f$ = − $_0$ for our (equivariant!) $_0$: → , this actions becomes the obvious shift action on the space of maps : → • for : ( ) ↦ ($_X$( )).

Since the full isometry group ( ) ⊃ Γ is *amenable* one can average *bounded* functions to Γ-invariant ones, where "bounded for $_f$" corresponds to the "finite distance from $_0$."

$$_Y(\quad _0) = \sup_{x \in X} \quad _\diamond(\; (\;) \quad _0(\;)) < \infty$$

Since the straightening maps $_\varepsilon$: → from section 3 do lie within finite distance from $_0$, they can be averaged to equivaruant one, say $\bar{}_\varepsilon$: → , and since $\|\widetilde{\;}^* \;_\varepsilon\|_{L_2(\mu)} \le 1 +$ and the function(al) ↦ $\|\widetilde{\;}^*\|^2_{L_2(\mu)}$ is (obviously) *convex*, these averaged maps also have their $\tilde{}_2$-dilations bounded by

$$\|\widetilde{\;}^* \bar{}_\varepsilon\|_{L_2(\mu)} \le 1 +$$

Finally, since the maps $\bar{}_\varepsilon$ are uniformly Lipschitz, some sequence of them converges with → 0 to the desired equivariant map $_\diamond$: → . QED.

Remark. The Hilbert straightening map $_\diamond$ can be regarded as a metric counterpart to the *Abel-Jacobi-Albanese maps* from Kähler manifolds to their Jacobians. The latter maps, being holomorphic are (pluri)harmonic; they minimize Dirichlet's $\int_{X/\Gamma} \| \quad (\;)\|^2_{L_2}$ over all Γ-equivariant maps , while the map $_\diamond$ minimizes, in a way, the norm $\sup_x \| \quad (\;)\|_{L_2}$.

Besides $_\diamond$, there is also a *dynamical* counterpart to the Abel-Jacobi-Albanese, called *the Shub-Franks map* that is associated with *hyperbolic endomorphisms* of tori and infra-nil-manifolds in general. The three maps and their generalizations intricately intertwine in their applications to the geometric rigidity theory (See [15] for a related discussion on A-J-A – S-F connection.)

6.1 On Infinite Dimensional Γ-Spaces and Codiffusion Spaces.

The above suggests the following Γ-version of the notion of the $_q$-dilation from section 1.

Let be a Banach space with an affine isometric action of a locally compact (e.g. discrete) group Γ. Let Π ⊂ Γ be the subgroup of parallel translations in Γ and let $\Gamma_0 = \Gamma/\Pi$.

Thus, Γ_0 and, hence, Γ itself, act on by *linear* isometric transformations.

A proper axial Γ*-partition of unity in* is a Γ_0*-invariant* measure $= \; _\mu$ on the spaces $\mathcal{P}$ of axial projectors : ↦ $\mathbb{R}_p = \; _p \subset$, such that the action of Γ_0 on the measure space ($\mathcal{P}$) is *proper*:

there exists a measurable subset ⊂ $\mathcal{P}$ such that the Γ_0-orbit of equals almost all $\mathcal{P}$ and such that the subset of those ∈ Γ_0 for which (( ) ∩) > 0 is precompact in Γ_0.

If the action is proper, the quotient space $\underline{\mathcal{P}} = \mathcal{P}/\Gamma_0$ carry a Lebesgue-Rochlin measure, say $\underline{\ }$, and one can integrate invariant functions on $\mathcal{P}$ over $(\underline{\mathcal{P}}\ \underline{\ })$

Thus, one can define the $_q$-dilation of a Γ-equivariant map from a metric Γ-space into as

$$|| \quad ^* ||_{L_q(\underline{\mu}=d\_p)} = \left(\int_{\underline{\mathcal{P}}} \quad ^q(\ ) \ \underline{\mu}\right)^{1/q}$$

Several constructions we met earlier, namely those which do not involve integration apart from that over $\mathcal{P}$ generalize to this setting: that are

• Equivariant minimal "norms" $|| \quad ^* ||_{L_q}$ and $|| \quad \widetilde{\ }^* ||_{L_q}$,

• Banach straightening at infinity (section 3) and equivariant Hilbert straightening,

• John's $_\diamond$, where one needs to(?), regretfully, assume that the Banach space is "essentially Hilbertian" to start with: it admits a Γ_0-invariant Hilbert form , such that $|| \ ||_Y \leq || \ ||_h \leq \quad \cdot || \ ||_Y$.

(This assumption rules out, for example, the only natural candidate for John's $_\diamond$ on the spaces $_q(\Gamma)$, that is Hilbert's $= || \ ||^2_{l_2(\Gamma)}$ for countable groups Γ.)

Where are we to go from this point? Are there unexplored domains populated by interestingly structured infinite dimensional Γ-spaces ?

One may start a search for them among *symbolic* Γ*-spaces*, [12], [14], and/or *concentrated spaces* [13]).

Axes, Non-Linear Partitions of Unity and Codiffusion. A distance minimizing geodesic in a metric space that is the image of an isometric embedding $\mathbb{R} \mapsto \mathbb{R}_\bullet = \ _p \subset$ admits a 1-Lipschitz projection onto itself. Thus "many" spaces admit "many" axial projectors $: \ \to \mathbb{R}_p \subset$.

Notice that an axis $\mathbb{R}_\bullet \subset$ may have several projections $\to \mathbb{R}_\bullet$. Two, apparently, most natural ones correspond to the two *Busemann functions* $_\pm$ on associated to this axes with two possible orientations: $\mapsto \ _\pm(\ ) \in \mathbb{R} = \mathbb{R}_\bullet \mapsto$.

There are two alternative ways to define the $_2$-dilation of a map $: \ \to$.

(1) If is a Riemannian/Hilbertian manifold with the Riemannian quadratic differential form , one may forfeit projectors and use measures $= \ _\mu$ on the space $\mathcal{B}$ of 1-Lipschitz functions $: \ \to \mathbb{R}$.

The normalization condition of being a partition of unity may be formulated by requiring the integral of the squares of the differentials of our functions to be equal to ,

$$\int_{\mathcal{B}} \ ^2 \ _\mu =$$

Here it seems reasonable to limit the support of relevant measures to the space of Busemann functions or, more generally of all *horofunctions.* Besides, if is a proper Γ-space, it may be to one's advantage to restrict to Γ invariant measures on $\mathcal{B}$ and integrate over $\mathcal{B}/\Gamma$ rather than over $\mathcal{B}$.

(2) In order to sum/integrate projectors with a measure on the space of projectors one may use an additional structure in , called

Codiffusion. Let $\mathcal{N} = \mathcal{N}(\ )$ be the space of probability measures on where is embedded to $\mathcal{N}$ by assigning Dirac's $_y$ to all $\in$.

A codiffusion in is a projection $: \mathcal{N} \to \subset \mathcal{N}$, where "projection" means $^2 =$.

For example, every complete convex (locally) affine topological space, e.g. a closed convex subset in a Banach space, comes with a natural codiffusion that sends every measure on to its *center of mass*

The center of mass construction generalizes to those geodesic spaces, in particular to Riemannian manifolds, where the square distance functions $' \mapsto {}^2(\ ')$ are strictly convex on all geodesics: the center of mass of a is defined as the minimum point $'_{min} \in$ of the (strictly convex!) function $' \to \int_Y {}^2(\ ')\ _\nu$ (see [18], [19] and also [2] for the history of this concept and new applications.)

However, all this does not seem to help in answering the following

Questions (a) What are "interesting" instances of equivariant maps $: \to$ between -dimensional Riemannian (Finsler?) Γ-spaces such that

$$_\infty(\ ) \cdot \widetilde{\quad}(\ (\ )) = 1 \text{ for all } \in$$

where the group Γ is *not* virtually Abelian?

(b) Let be a metric Γ-space, $= (\quad)$ a Finsler (e.g. Riemannian) Γ-manifold and $: \to$ be a proper equivariant map. Suppose that all geodesics in are distance minimizing.

What is the minimal $= (\ ) > 0$ for which there exists a Γ-equivariant map $_\bullet : \to$ within bounded distance from such that $\widetilde{\quad}(\ _\bullet(\ )) \leq \cdot {}_\infty(\ )$ for all $\in$?

(c) Let be a contractible metric Γ-space. Consider all, possibly singular, (oriented?) Riemannian Γ-spaces of dimension that admit equivariant maps $: \to$ that have $_\infty(\ ) \leq 1$ and such that the induced homomorphisms on their invariant -dimensional real cohomologies do not vanish, e.g. $(\ ) = (\ ) =$ and $(\ ) \neq 0$.

When does there exist an $_{min}$ among them that minimizes $_n(\ /\Gamma)$?

How does $_{min}$ change (if at all) with passing to subgroups of finite indices in Γ?

Can this $_{min}$ be explicitly described for particular "simple" (possibly, infinite dimensional, e.g. isometric to Hilbert's $\mathbb{R}^\infty$) spaces ?

Given subgroups $\Gamma_i \subset \Gamma$ of finite indices $_i = (\Gamma/\Gamma_i)$, let $_i : {}_i \to$ be an $_i^{-1}\ _n(\ /\Gamma_i)$-minimizing sequence of Γ_i-equivariant maps $_i : {}_i \to$ with $_\infty(\ _i) \leq 1$. When and how does such a sequence converge?

7 Burago-Ivanov's Solution of Hopf Conjecture.

Let us show (essentially) following the original argument in [6] how Hilbert's straightening implies *Hopf conjecture*:

Let $= (\quad)$ be a compact Riemannian manifold without conjugate points and $\underline{\ } : \to \mathbb{T}^{\bar{n}}$ be a continous map such that the induced homomorphism of the fundamental groups, $_1(\ ) \to {}_1(\mathbb{T}^n) = \mathbb{Z}^n$, is an isomorphism. Then the universal covering $= (\quad)$ of is isometric to $\mathbb{R}^n$.

In fact, if all geodesic segments in are distance minimizing, then

Hilbert's straightening map $_\diamond : (\quad) \to (\quad _\diamond) = \mathbb{R}^n$ is an isometry,

where is the universal covering of the torus $\mathbb{T}^n$ and $_\diamond$ is John's Hilbertian form associated to the Federer-Whitney -descendant $|| \ ||_Y$ of the metric $_g$ on for : → being the lift of $\_$ to .
This is immediate with the following simple classical formulas **(LS1)** and **(LS1)**.

Liouville-Santalo Integral Identities. Let = ( ) be a complete Riemannian manifold.

Given a measure = $_\lambda$ on the tangent bundle ( ), denote by $\_$ = $\underline{\lambda}$ the push-forward of under the projection : ( ) → , and assume that $\_$( ) < ∞ for all compact subsets ⊂ .

Call a measure *balanced* if, for every continuous quadratic form on ( ),

$$\int_X \quad {}_{g_x}(\ {}_x) \ \underline{\lambda} \ = \ (\ ) \int_{T(X)} (\ ) \ \lambda$$

For example, the *Liouville measure*, that is supported on the unit tangent bundle ( ) ⊂ ( ), is balanced, since the trace of a quadratic form on $\mathbb{R}^n$ equals -times the average values of on the unit sphere $^{n-1} \subset \mathbb{R}^n$.

Let Σ( ) = Σ(;) be the space of geodesic segments of length in that are locally isometric (geodesic) maps : [0] → and denote by $_\sigma$ ∈ $_{\sigma(0)}$( ) ⊂ ( ) the unit tangent vector to our geodesic at the point (0) ∈ .

The correspondence ↔ = $_\sigma$ identifies Σ( ) with the unit tangent bundle ( ); accordingly, the measures on Σ( ) corresponding to = $_\lambda$ on ( ) are denoted $_\lambda$.

Let be acted upon discretely and isometrically by a group Γ, and let be a measure on ( ) that is *invariant under the action of* Γ and that is *normal* in the sense the total -mass of ( )/Γ equals one,

$$(\ (\ )/\Gamma) = \int_{S(X)/\Gamma} \ \lambda \ = 1$$

Let = ( ) be another complete Riemannian Γ-manifold and let : → be a proper Γ-equivariant Lipschitz map.

Average/integrate the dilation of at the two ends of our geodesic segments, or, rather, the dilation of the composed maps ∘ : [0] → ,

$$_{f\circ\sigma}(0 \ \ ) = \ ^{-1} \quad _Y(\ \circ \ (0) \quad \circ \ (\ ))$$

over (Σ( ) $_\lambda$) and denote this averaged dilation by

$$_{av}(\ \circ \quad) = \int_{S(X)/\Gamma=\Sigma(R)/\Gamma} \quad _{f\circ\sigma}(0 \quad) \ \lambda$$

Clearly, this $_{av}$ is bounded by the *average length* of the parametized curves ∘ : [0] → divided by ,

$$_{av}(\ \circ \quad) \leq \ ^{-1} \qquad _{av}(\ \circ \ [0 \quad]) = \ ^{-1} \int_{\Sigma(R)/\Gamma} \qquad (\ \circ \ [0 \quad]) \ \lambda$$

If the measure is *invariant under the geodesic flow*, then, by Fubini's theorem, this integral equals the integrated norm of $_\sigma$ ∈ $_{f(x)}$( ), for = (0) = ($_\sigma$), that is the derivative (differential) of in the direction of the unit tangent vector $_\sigma$ ∈ $_x$ ⊂ ( ),

(LS1) $\quad {}^{-1} \quad {}_{av}(\circ [0 \ ]) = \int_{\Sigma(R)/\Gamma} \| \ {}_{\sigma} \ \| \ \lambda$

where, by *Schwartz inequality,*

$$\int_{\Sigma(R)/\Gamma} \| \ (0)\| \ \lambda \ \leq \left(\int_{\Sigma(R)/\Gamma} \| \ (0)\|^2 \ \lambda \right)^{1/2}$$

If is *balanced*, then

(LS2) $\quad \int_{\Sigma(R)/\Gamma} \| \ (0)\|^2 \ \lambda \ = \ {}^{2}_{av}(\ )$

where ${}_{av}(\ )$ is the normalized *Dirichlet-Hilbert* energy of that is

$$ {}^{2}_{av}(\ ) =_{def} \ {}^{-1} \int_{X/\Gamma} \| \ \|^2_{L_2} \ \underline{\lambda} \ = \ {}^{-1} \int_{X/\Gamma} \ {}_{g(x)}(\ (\ ))^{*}(\ ) \ \underline{\lambda}$$

for the pullback $(\ (\ ))^{*}(\ )$ of the Riemannian form from $(\ )$ to $(\ )$ by the differential of .

In sum, **(LS1)**+**(LS2)**+(Schwartz inequality) imply:

if is a normal Γ*-invariant measure on* $(\ )$ *that is balanced and invariant under the geodesic flow and* $: \ \to$ *is a* Γ*-equivariant map, then the -average dilation of at the ends of geodesic segments in of length satisfies*

$$ {}_{av}(\ \circ \ ) \leq \ {}_{av}(\ ) = \ {}^{-1/2} \left(\int_{X/\Gamma} \| \ \|^2_{L_2} \ \underline{\lambda} \right)^{1/2}$$

This integral inequality is sharp: the equality is possible only if all quantities involved are equal point-wise and this sharpness is independent of , since the integral $\int_{\Sigma(R)/\Gamma} \| \ {}_{\sigma} \ \| \ \lambda \ = \ {}^{-1} \ {}_{av}(\ \circ \ [0 \ ])$ does not depend on .

It follows, in particular, that

if

$$\limsup_{R\to\infty} \ {}_{av}(\ \circ \ ) \geq 1 \ \textit{and} \ \ {}_{av}(\ ) \leq 1$$

then the map is isometric on every geodesic $: (-\infty \ +\infty) \to$ *that is contained in the support of .*

The Hopf conjecture follows by applying the above to the universal covering of a torus without conjugate points with the Liouville measure on $(\ )$ and to the Hilbert straightening map ${}_{\diamond} : \ \to \mathbb{R}^n = (\ {}_{\diamond})$.

In fact, "no conjugate points" says that all geodesic segments in , that are $: [0 \ ] \to$, have ${}_{X}(\ (0) \ (\ )) = \ $. Therefore, the average dilation of ${}_{\diamond}$ satisfies

$$\left(\limsup_{R\to\infty} \ {}_{av}(\ {}_{\diamond} \circ \ ) \right)^{-1} \leq \ {}_{\infty}(\ {}_{\diamond})$$

where, recall,

$$ {}_{\infty}(\ ) = \liminf_{dist(x_1,x_2)\to\infty} \ (\ (\ {}_{1}) \ (\ {}_{2}))/ \ (\ {}_{1} \ \ {}_{2})$$

for proper maps .

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It is immediate that

$$\widetilde{\quad}\ {}^{[n]}(\ ) \leq \ {}^{n}(\ )$$

and that composed maps $\underset{f}{\to} \ \underset{g}{\to}$ satisfy

$$\widetilde{\quad}\ {}^{[n]} \circ (\ ) \leq \widetilde{\quad}\ {}^{[n]}(\ ) \cdot \widetilde{\quad}\ {}^{[n]}(\ ) \text{ for } = (\ )$$

Also observe that $\widetilde{\quad}\ {}^{[n]}(\ )$ equals the (absolure value of the) ordinary Jacobian ${}^{[n]}(\ )$ for smoth maps between Riemannian manifolds.

Homologically Stable Hilbertian Volume. We want to define -volumes in metric spaces with a usual behaviour under (Jacobians of) Lipschitz maps $: \to = (\ {}^{n}\ )$, e.g. decreasing under maps with $\widetilde{\quad}\ {}^{[n]}(\ (\ )) \leq 1$, where we are concerned with *stability of the images* under small continuous perturbations maps that is defined as follows.

Fix a coefficient group (e.g. a field) $\mathbb{F}$ and call a point $\in$ *cohomologically* ($^{n}_{\mathbb{F}}$*-image) stable* for a continuous map $: \to$ if for all sufficiently small neighbourhoods $= {}_{y} \subset$ of the induced homomorphism on the relative cohomology

$${}^{*n}: \ {}^{n}(\quad ;\mathbb{F}) \to \ {}^{n}(\ {}^{-1}(\ )\ {}^{-1}(\ );\mathbb{F})$$

does not vanish. The set of these stable points is denoted by ${}_{\mathbb{F}}\ {}_{f}(\ ) \subset$.

As we shall be dealing with locally compact spaces, the cohomology will be *Čech;* to be specific, we stick to $\mathbb{F} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

The hyper-Hilbertian global -volume ${}_{\mathbb{F}}\ {}^{[n]}(\mathbb{R}^n \backslash\ )$ *of over* $\mathbb{R}^n$ is the supremum of the numbers such that admits a Lipschitz map $: \to \mathbb{R}^n$ with $\widetilde{\quad}\ {}^{[n]}_{x}(\ ) \leq 1$, $\in$, and such that the Lebesgue-Haar measure of the set ${}_{\mathbb{F}}\ {}_{f}(\ ) \subset \mathbb{R}^n$ is $\geq$.

For example, ${}_{\mathbb{F}}\ {}^{[n]}(\mathbb{R}^n \backslash\ ) = \ {}_{n}(\ )$ for smooth open Riemannian manifolds , while closed manifolds satisfy ${}_{\mathbb{F}}\ {}^{[n]}(\mathbb{R}^n \backslash\ ) = \ {}_{n}(\ )/2$ (To get rid of this silly 1/2 one has to map to the -spheres ^{n} rather than to $\mathbb{R}^n$.)

A more interesting example is provided by closed minimal hypersurfaces $=$ ^{n} in compact Riemannian manifolds ${}^{n+1} \supset$ representing homology classes in ${}_{n}(\ {}^{n+1})$: such an admits a -Lipschitz map, $: \to \ {}^{n}$ of non-zero degree, where $\leq \ {}_{0}(\ {}^{n-1}\ {}_{n}(\ )) < \infty$ as it follows from the *compactness* of the space of minimal subvarieties in ${}^{n+1}$ with bounded volumes and the *maximum principle* for minimal hypersurfaces.

Then one sees, by looking at tangent cones, that every point in such a minimal $\subset \ {}^{n+1}$ admits an arbitrarily small neighbourhood ${}_{x} \subset$ such that

$${}_{\mathbb{F}}\ {}^{[n]}(\mathbb{R}^n \backslash\ {}_{x}) \geq \ \cdot \ {}_{n}(\ {}_{x}) \text{ for some } \geq (\ ) > 0$$

Homology Domination and Semicontinuity of ${}_{\mathbb{F}}\ {}^{[n]}$. Let be a metric space, $\subset$ a locally compact locally closed (open $\cap$ closed) subset, and let ${}_{i}$, i=1,2,..., be a sequence of locally closed subsets that converges to in the sense that for every neighbourhood ${}_{X} \subset$ of all but finitely many of ${}_{i}$ are contained in ${}_{X}$.

Say that ${}_{i}$ *(co)homologically dominate in dimension with a given coefficient domain* $\mathbb{F}$ if for every pair of open subsets ${}_{1} \subset \ {}_{2} \subset$ every relative cohomology class in $^n(\ _2 \ \ _1;\mathbb{F})$ that restricts to a *non-zero* class in $^n(\ _2\cap$ $_1\cap \ ;\mathbb{F})$ also restricts to *non-zero* classes in $^n(\ _2\cap \ _i \ \ _1\cap \ _i;\mathbb{F})$ *for all but finitely many* .

By invoking the local $\tilde{\ }_q$-extension with $= 2$ from section 1, one obtains:

If subsets $_i \subset$ *homologically dominate a subset* *in a metric space* *then*

$$\liminf_{i\to\infty} \ _{\mathbb{F}} \ ^{[n]}(\ _i) \geq \ _{\mathbb{F}} \ ^{[n]}(\ )$$

(Semi)Local Coarea Inequality. Let $= \ ^m, \ \leq$, be a topological manifold and $: \ \to$ a continuous map. Call the map *(co)mologically (co)stable over a point* *in dimension* *with coefficients* $\mathbb{F}$ if the fiber $_y = \ ^{-1}(\ ) \subset$ and near $_y$ have the following property.

[STB] Let $_y : \ \to \mathbb{R}^{n-m}$ be a continuous map and $\subset \mathbb{R}^{n-m}$ be an open subset such that the induced cohomology homomorphism

$$^{n-m}(\ ;\mathbb{F}) \to \ ^{n-m}(\ _y^{-1}(\ )\cap \ _y \ _y^{-1}(\ )\cap \ _y;\mathbb{F})$$

does not vanish. Then for all sufficiently small closed neighbourhoods $_y \subset$ of the map

$$= (\ _y) : \ ^{-1}(\ _y) \to \ _y \times \mathbb{R}^{n-m}$$

induces *non-zero* homomorphism

$$^* : \ ^n(\ _y \times \ (\ _y \times \ );\mathbb{F}) \to \ ^n(\ ^{-1}(\ _y \times \ ) \ ^{-1}(\ (\ _y \times \ );\mathbb{F}))$$

For example an $\mathbb{R}$-valued function on a topological manifold is stable over a point $\in \mathbb{R}$ if the level $^{-1}(\ )$ contains no local maxima and/or minima points of .

Now let be smooth -manifold with a smooth measure $^{[m]}$ on it, let $: \ \to$ be a Lipschitz map and let $_y(\ ) \subset$ be -balls around for some metric in .

If *satisfies* **[STB]** *then*

$$\liminf_{\varepsilon\to 0} \ _{\mathbb{F}} \ ^{[n]}(\ ^{-1}(\ _y(\ )) \geq \frac{^{[m]}(\ _y(\ )) \cdot \ _{\mathbb{F}} \ ^{[n-m]}(\ ^{-1}(\ ))}{\widetilde{\quad} \ ^{[m]} \ /y}$$

Proof. The condition **[STB]**, albeit unpleasantly restrictive, matches the local $\tilde{\ }_q$-extension property and the coarea inequality trivially follows.

Let us partially localize the above "volume" with countable systems $\mathcal{U}$ of open subsets $_i \in \ , \ \in$ that have bounded intersection multiplicity, denoted $(\mathcal{U}) < \infty$, and let

$$^{[n]}_{\mathbb{F}}(\mathbb{R}^n \setminus \ ) = \sup_{\mathcal{U}}(\ (\mathcal{U}))^{-1} \sum_{i\in I} \ _{\mathbb{F}} \ ^{[n]}(\mathbb{R}^n \setminus \ _i)$$

where this will be often abbreviated as follows

$$\widetilde{\quad} \ ^{[n]}(\ ) \text{ instead of } \widetilde{\quad} \ ^{[n]}_{\mathbb{F}}(\mathbb{R}^n \setminus \ )$$

with $\mathbb{F} = \mathbb{F}_2$, unless otherwise indicated.

The homological volume $\widetilde{\quad}^{[n]}(\ \ )$ of smooth and piece-wise smooth Riemannian spaces equals the ordinary volume $_n(\ \ )$.

But, in general, $\mapsto \widetilde{\quad}^{[n]}(\ \ )$, $\subset$, is not, a priori, an additive set function. This can be amended but using countable systems $\mathcal{U}_j$ of open sets $_{ij} \subset$ where $(\ \ _{ij}) \to 0$ for $\to \infty$ and localizing with

$$(\ \ ) = \limsup_{j\to\infty} \mathcal{U}_j(\qquad (\mathcal{U}_j))^{-1} \sum_{i\in I} \quad {}_{\mathbb{F}} \quad {}^{[n]}(\mathbb{R}^n \setminus \ _{ij})$$

This "local volume" equals $\widetilde{\quad}^{[n]}(\ \ )$ if small -balls in have volumes $\geq \ _\varepsilon$ where $_\varepsilon / \ ^{n+1} \underset{\varepsilon\to 0}{\to} \infty$; in general, however, the local volume, probably, may be (I have not worked out a convincing example) smaller than $\widetilde{\quad}^{[n]}$ that is defined with unrestricted $_i$. To compensate for this, we shall introduce another "volume" in Part 2 of the paper by means of maps $\to \mathbb{R}^\infty$ with controlled filling volumes of the boundaries of images of subsets (or rather of -cycles) similarly in certain respects to [1].

Jonn's-Hilbert Volume of Banach Balls. The obvious corollary to John's Ellipsoid Theorem (pointed out in slightly different terms in [**?**]) reads:

the Hilbertian volume of the unit ball $_X(1)$ *in an -dimensional Banach space equals its Euclidean volume with respect to John's Euclidean (Hilbertian) metric* $_\diamond$,

$$^{[n]}(\ \ _X(1)) = \quad _{h_\diamond}(\ \ _X(1))$$

Consequently, since $_{h_\diamond}(1) \subset \ _X(1)$,

the Hilbertian volume $_{h_\diamond}(\ \ _X(1))$ *is greater than the (ordinary) volume of the unit Euclidean ball, where the inequality is strict unless is isometric to* $\mathbb{R}^n$.

9 Lower Volume Bounds.

Let us reformulate and reprove a few standard geometric inequalities in the Hilbert volume setting where we abbreviate: $^{[n]}(\ \ ) = \qquad ^{[n]}(\ \ )$ for general metric spaces .

$\square$-*Spaces and their Faces.* A topological space with a given collection of pairs of disjoint closed subsets, $\pm \ _i = \pm \ _i(\ \ ) \subset$, $= 1$, is called an -*cubical space* where $\pm \ _i$ are called *faces* and their union $= \bigcup_i \pm \ _i$ is regarded as *the boundary* of .

Every such , assuming it is a normal space, admits a continuous map into the -cube, $: \ \to \square = [0\ 1]^n$, such that the faces $\pm \ _i(\ \ )$ go to the $(\ \ - 1)$-faces $\pm \ _i(\square)$ of the cube, where such a map is given by -functions $_i : \ \to [0\ 1]$ where $- \ _i$ goes to 0 and $+ \ _i$ goes to 1. Clearly, such a map is unique up-to homotopy.

$\square$-*Straightening Lemma.* Let be an -cubical metric space, and $: \ \to \square = [0\ 1]^n$ be a continuous map that sends faces of to the corresponding faces of the cube $\square$.

If

$$_P(\ \ _i - \ _i) \geq \ _i \qquad = 1$$

then *is homotopic to a Lipschitz map* $: \to \square = [0\ 1]^n$ *given by* *-functions* $_1 \quad _n : \to [0\ 1]$, *such that*

$$(_i) \leq \ _i^{-1}$$

Proof. Take $_i( ) = \ ^{-1} \quad (\quad _i)$ for all $\in$,where $(\quad _i) \leq \ _i$ and let $_i( ) = 1$ for $(\quad _i) \leq \ _i$.

Corollary: Besikovich-Derrick-Almgren $\square$*-Inequality.* If the map $: \to \square = [0\ 1]^n$ has non-zero degree, i.e. the induced cohomology homomorphism $^* : \ ^n(\square \quad \square; \mathbb{F}) \to \ ^n(\quad ; \mathbb{F})$ doe not vanish for some coefficient domain $\mathbb{F}$, then

the $\mathbb{F}$*-homological Hilbertian volume of* *is bounded from below by the product of the distances between opposite faces in* ,

$$^{[n]}(\ ) \geq \prod_{i=1,\ldots,n} \quad _P(\ _i - \ _i)$$

Proof. Since is homotopic to and $(\ ) \neq 0$, the homomorphism $^* : \ ^n(\square \quad \square; \mathbb{F}) \to \ ^n(\quad ; \mathbb{F})$ does not vanish; hence, all points $\in$ are cohomologically stable and

$$1 = \ _n(\square) \leq || \quad ^{[n]}(\ )||_{sup} \cdot \quad ^{[n]}(\ ) \leq \quad ^{[n]}(\ ) \cdot \prod_i \ _i^{-1}$$

Example. Let a Riemannian -manifold and $\pm \ _i \subset$, $= 1\ 2$, be smooth closed domains that are -submanifolds with smooth boundaries $(\pm \ _i) \subset \ _i$, e.g. $= \mathbb{R}^n$ and $\pm \ _i \subset \mathbb{R}^n$ are half-spaces.

Let equal the intersection of $\pm \ _i$,

$$= \bigcap_i \pm \ _i \text{ and } \pm \ _i = \pm \ _i(\ ) = \quad \cap \quad (\pm \ _i)$$

If is compact, if the hypersurfaces $(+ \ _i)$, $= 1\ 2$, intersect *transversally* and if the intersection

$$\bigcap_i + \ _i = \quad \cap \quad (+ \ _1) \cap \quad (+ \ _2) \cap \quad \cap \quad (+ \ _n)$$

(that is necessarily finite) consists of *odd* number of points, then

$$_n(\ ) \geq \prod_i \quad (+ \ _i - \ _i)$$

Indeed, our map $: \to \square$ has non-zero $\mathbb{F}_2$-degree because the -pullback of every point $\in \square$ close to the corner $\bigcap_i + \ _i(\square)$ of $\square$ is a finite set of odd cardinality.

Recall that the map $: \to \square$, for every -cubical space sends $\to \square$ and the essential property of used in the construction of , say for $_P(- \ _i + \ _i) \geq 1$, is that is 1-*Lipschitz on the boundary* with respect to the *-norm* on $\mathbb{R}^n \supset [0\ 1]^n = \square$, that is $||(\ _1 \quad _i \quad _n)||_{sup} = \quad _i| \ _i|$.

Seen from this angle, the $\square$-inequality appears as a special case of the following corollary (stated slightly differently in [7]) of Jonn's theorem.

Filling Extremality of Banach spaces. Let $= (\ \| \ \|_A)$ be an -dimensional Banach space and let $\subset$ be a bounded open subset with connected boundary .

Let be a compact metric space, let $\subset$ be a closed subset and let $: \ \to \ \subset$ be a 1-Lipschitz map.

If the inclusion/restriction homomorphism $^{n-1}(\ ;\mathbb{F}) \to \ ^{n-1}(\ ;\mathbb{F})$ *vanishes for some coefficient group* $\mathbb{F}$*, while the cohomology homomorphism* $^* :$ $^{n-1}(\ ;\mathbb{F}) \to \ ^{n-1}(\ ;\mathbb{F})$ *does not vanish, then*

$$^{[n]}(\ ) \geq \ ^{[n]}(\ ) = \ _{h_\diamond}(\ )$$

for John's Euclidean (Hilbertian) quadratic form $_\diamond$ *on* .

Proof. Extend to a Lipschitz map $: \ \to$ with $_{h_\diamond}(\ ) \leq 1$ (see section 1) and observe that the cohomology assumptions imply that this map is homologically image stable at all points $\in$.

Since the Hilbertian Jacobian of $: \ \to (\ \ _\diamond)$ is bounded by

$$^{[n]}(\ ) \leq \ ^n(\ ) \leq 1$$

the proof follows from John's volume identity in section 8.

$\triangle$*-Inequalities.* There is another generalization of the $\square$-nequality, besides the above Banach filling extremality, where the cube $[0 \ 1]^n$ is replaced by the Cartesian product of regular Euclidean $_j$-simplices, denoted $\times_j \triangle_{n_j}$, $= 1$.

Start with the case $= 1$ and define a $\triangle_n$*-space* as a topological space with a distinguished collection of subsets, called *faces* $_i = \ _i(\ )$, $= 0 \ 1$, indexed by the $+ 1$ codimension one faces in $\triangle_n$, such that the intersection $\bigcap_i \ _i$ is empty and where the union $\bigcup_i \ _i \subset$ is regarded as the boundary $\subset$

If is a normal space, it admits a continuous map $: \ \to \triangle_n$, such that every face of goes to the corresponding face of the simplex $\triangle_n$, where this is unique up to homotopy.

Let me a metric space and denote by $\Sigma = \Sigma_P : \ \to \mathbb{R}_+$ the sum of the distance functions to the faces,

$$\Sigma(\ ) = \sum_i \ _P(\ \ _i)$$

let

$$\Sigma_\partial = \Sigma_\partial(\ ) = \inf_{p \in \partial P} \Sigma(\ )$$

and let

$$= \Sigma_\partial(\ )/\Sigma_\partial(\triangle_n)$$

If the map $: \ \to \triangle_n$ *has non-zero degree, then*

$$^{[n]}(\ ) \geq \ ^n \ \ _n(\triangle_n)$$

Proof. There obviously exits an axial partition of unity $\mathcal{P} = \{ \ _i\}$ $= 0 \ 1$ in $\mathbb{R}^n \supset \triangle_n$ with $+ 1$ axes $\mathbb{R}_i \subset \mathbb{R}^n$ that are normal to the $(\ - 1)$-faces of $\triangle_n$

and such that the quaisi-projectores $_i$ on $\triangle_n$, that are $_i : \triangle_n : \mathbb{R}_i = \mathbb{R}$, are given by the functions $\mapsto \cdot \ (\ )$ for $= \sqrt{\ /(\ + 1)}$

Let $_i : \ \to \mathbb{R} = \mathbb{R}_i$ be given by $_i \mapsto \ ^{-1} \cdot \ (\ _i)$ and observe that the map

$$= \sum_i \ _i : \ \to \mathbb{R}^n \supset \triangle_n$$

sends the boundary of *outside the interior of* $\triangle_n$ where, moreover, this map

$$: \ \to \mathbb{R}^n \setminus \ (\triangle_n)$$

is *homotopic to*

$$: \ \to \ \triangle_n \subset \mathbb{R}^n \setminus \ (\triangle_n)$$

Since $(\ ) = \ (\ ) \neq 0$, the image of homologically stably covers $\triangle_n$ and the proof follows for

$$^{[n]}(\ ) \leq \ ^n(\ ) \leq \ ^{-n}$$

Now we turn to metric $\times_j \triangle_{n_j}$*-spaces* , where $= 1 \ 2$, and where a $\times_j \triangle_{n_j}$-structure is given by a collection of "faces" $_{ji} \subset$, $= \ _j = 0 \ _j$ and where $_{ji}$ correspond to those faces of codimension 1 in the Cartesian product that equal the pullbacks of the faces of $\triangle_{n_j}$ under the coordinate projection $\times_j \triangle_{n_j} \to \triangle_{n_j}$.

The condition we impose on the set of faces $_{ji}$ reads: a subset of faces has a common point in only if the corresponding faces in $\times_j \triangle_{n_j}$ have a common point.

This is equivalent to the existence of a continuous map $: \ \to \times_j \triangle_{n_j}$ that sends faces to faces, where such a is unique up-to a homotopy.

Let

$$\Sigma_j(\ ) = \sum_{i=1,\dots,n_j} \ _P(\ _{ji}) \ \Sigma_{\partial_j} = \Sigma_{\partial_j}(\ ) = \inf_{p \in \partial P} \Sigma_j(\ )$$

$$_j = \Sigma_{\partial_j}(\ )/\Sigma_\partial(\triangle_{n_j})$$

If the map $: \ \to \triangle_n$ *has non-zero degree, then*

$$^{[n_1+\dots+n_j]}(\ ) \geq \prod_{j=1,\dots,k} \ ^{n_j}_j \ _{n_j}(\triangle_{n_j})$$

Proof. Take a partition of unity in $\mathbb{R}^{n_1+\dots+n_k} \supset \times_j \triangle_{n_j}$ with the axes normal to the faces of $\times_j \triangle_{n_j}$ and argue as earlier.

9.1 Digression: Acute Polyhedra.

What should be a generalization of the above for *non-regular* simplices $\triangle_{n_j}$? Namely, what is the sharp lower bound on the distance function $_P(\ _1 \ _2)$ for $_1 \ _2 \in$ that would imply $^{[n]}(\ ) \geq \times_j \triangle_{n_j}$?

Notice in this respect, if is a convex *acute* Euclidean polyhedron, i.e. all dihedral angles are $\leq \ /2$ then the distances to its codimension 1 faces $_i : \ \mapsto (\ _i(\ ))$ satisfy

$$\sum_i \ _i \ _i = \ = \ (\ ) \text{ for } _i = \ ^{-1} \ _{n-1}(\ _i(\ ))$$

but the mere $\sum_i \ _i \ _i \geq$ does not non-sufficient even for (homologically) over (i.e. mapped with non-zero degree to) *non-regular* Euclidean triangles $\triangle_2$.

What is the minimal set of (preferably linear and/or log-linear) inequalities between $_i$ that would imply $^{[n]}(\ ) \geq \times_j \triangle_{n_j}$?

This question make sense for all acute Euclidean polyhedra besides $\times_j \triangle_{n_j}$ (some "non-acute" lower volume bounds can be derived from "acute" ones , see 7.3 in [11]) but, in fact, there aren't any other by *Steinitz*(?) *theorem*:

every acute polyhedron $\subset \mathbb{R}^n$ *is a Cartesian product of simplices.*

Proof. Start by observing that an acute convex *spherical polyhedron* (that is an intersection of hemispheres in ^{n}) is a either a *simplex*, or, in the degenerate case, the *spherical suspension* over a simplex in $^{n-i} \subset \ ^n$.

Indeed, the dual polyhedron, say $^{\perp}_p \subset \ ^n$, has all its edges longer than $/2$. Consequently, the distance between *every two* vertices in $^{\perp}_p$ is $\geq \ /2$; hence, there are at most $+1$ vertices in $^{\perp}_p$.

Also note that (non-strictly) acute spherical triangles $\triangle \subset \ ^2$ have all their edges bounded in length by $/2$. It follows that all -faces, $= 2 \ 3 \quad - 1$, of acute spherical -simplices are acute.

Now, let be *Euclidean* acute take an (-1)-face $\subset$, move its supporting hyperplane $^{n-1} = \ _Q \supset$, inward parallel to itself until it hits a vertex, say $\in$ and denote by $_p \subset \mathbb{R}^n$ the so moved hyperplane.

Since the normal projections from onto the hyperplanes supporting (-1)-faces adjacent to send *into* (hence, onto) these faces, all of is contained in the "band" $[\ _Q \ \ _p] \subset \mathbb{R}^n$ between the parallel hyperplanes $_Q \subset \mathbb{R}^n$ and $_p \subset \mathbb{R}^n$. Moreover,

• if the opposite face $- \ = \ _p \cap$ has $(- \ ) = \ -1$ then is orthogonally splits into the Cartesian product of with a segment; it is seen with the two orthogonal projections $\to \ - \ .$

•• If is an (-1)-simplex and $(- \ ) < \ -1$ then, obviously, is an -simplex.

Since the spherical polyhedra underlying the tangent cones at all vertices of are acute, • and •• apply to the faces of .

Namely, let $\Delta^m \subset$ be a *simplex*-face of *maximal* dimension . Then every ($+1$)-face containing this Δ^m orthogonally splits according to • and ••. It follows by induction on that all of splits, $= \Delta^m \times \ '$, and the proof is concluded by induction on .

Conclude by noticing that the number of (-1)-faces of a *convex* polyhedron $^n \subset \mathbb{R}^n$ with all dihedral angles $\leq \ -$, is (obviously) universally bounded, say by $100^{n/\varepsilon}$; a classical problem in convexity is to " effectively enumerate" these .

10 Volume Stability In The Riemannian Category.

The Burago-Ivanov Volume rigidity theorem (see section 5), like any other sharp inequality between two geometric invariants,

$$_1(\ )/ \ \ _2(\ ) \geq 1$$

where the equality implies that is isometric to a particular space $_0$ (or a member of a "small explicit" class of manifolds) raises the following

Stability Problems. (a) Describe spaces where $_1(\ )/\ _2 \leq 1 +$;

(b) find a metric in the space $\mathcal{X}$ of all , such that the subspace $\mathcal{X}_\varepsilon \subset \mathcal{X}$ of those where $_1(\ )/\ _2 \leq 1 +$ is compact.

Prior to engaging into general discussion on relevant metrics in the space of metric spaces (we shall come to this in Part 2 of the paper and in [17]) it is instructive to look at a "metric" in the space of Riemannian manifolds, which (almost) adequately reflects the volume rigidity picture.

10.1 *Directed Lipschitz Metric Normalized by Volume.*

To simplify/normalize, let and $'$ be closed connected oriented Riemannian -manifolds with $(\ ) = \ (\ ') = \ .$

Define $\overrightarrow{\ }_{Lip/vol}(\ '\ )$ as the infimum of ≥ 0, such that $'$ admits an $^\varepsilon$-Lipschitz map $: \ ' \to$ of degree ±1, where, observe, $^\varepsilon = 1 + \ + (\ )$.

This "metric" is *non-symmetric*, it may be equal $+\infty$, but it satisfies the *triangle inequality* and it equals zero if and only if and $'$ are isometric.

There is nothing wrong with being non-symmetric. Limits make perfect sense for such "metrics".

For example, the "true", in the category theoretic sense, Hausdorff metric on subsets $\subset$, call it $\overrightarrow{\ }_{Hau}(\ _1\ \ _2)$ – *the minimal such that* $_1$ *is contained in the -neighbourhood of* $_2$, is also non-symmetric.

Let us show that suitable bounds on the diameters and curvatures guaranty an "almost isometric" diffeomorphism between $/$ -close manifolds.

Denote the minimium of the diameters of and $'$ by

$$_{min} = \ (\ \ (\ )\ \ (\ '))$$

and the maximum of the absolute values of their sectional curvatures at all tangent 2-planes by

$$|\ |_{max} = \ \big(\sup_{\tau_2 \subset T(X)} |\ \ _{\tau_2}|(\ )\ \sup_{\tau_2' \subset T(X')} |\ \ _{\tau_2'}|(\ ')\big)$$

$\overrightarrow{\mathbf{D}}\ \leq\ \Rightarrow \mathbf{D}$. *Given numbers* $= 1\ 2$ *, and* ≥ 0*, there exists an* $= (\ \ ^2) > 0$*, such that the Riemannian -manifolds* $'$ *and satisfying the inequalities*

$$_{min} \leq \ |\ |_{max} \leq \ \textit{and}\ \overrightarrow{\ }_{Lip/vol}(\ '\ ) \leq$$

are diffeomorphic.

Moreover, the implied diffeomorphism $' \to$ *is* $(1 +\ )$*-bi-Lipschitz with* $\to 0$ *for* $\to 0$.

To show this, rescale the metrics in and $'$ in order to have $|\ |_{max} = 1$ and then approximate the implied $^\varepsilon$-Lipschitz map $: \ ' \to$ by a $(1 + \ )$-bi-Lipschitz diffeomorphism $\bar{\ } : \ ' \to$ as follows.

Denote by $\tilde{\ }_x(\ ) \underset{\exp}{\to}$ the ball in the tangent space $_x(\ )$ with the metric induced from by the exponential map and observe that if is sufficiently small, $\leq\ _0(\ ) > 0$, and is much smaller than , then, for every $' \in$,

the map from the ball $\tilde{\ }_{x'}(\ ) \subset\ '$ *to* $_x(\ ^\varepsilon) \subset\ ,\ =\ (\ ')$, *uniquely lifts to a map* $\tilde{\ }_{x'} : \tilde{\ }_{x'}(\ ) \to \tilde{\ }_x(\ ^\varepsilon)$.

Indeed, if otherwise, the map would send some short geodesic loops from $'$ to loops contractible in their immediate vicinity in ; this, in turn, would imply that $(\ ) = 0$.

Now we turn into a diffeomorphism as follow. Take the center of mass $\tilde{\ } \in \tilde{\ }_x(\ ^\varepsilon) \subset$ of the Riemannian measure of $\tilde{\ }_{x'}(\ )$ mapped to $\tilde{\ }_x(\ ^\varepsilon)$ by $\tilde{\ }_{x'}$ and let $\bar{\ }(\ ') = \exp_x(\tilde{\ }) \in$. It is easy to see, as in [?], [?], that for small > 0 and $<<$,

the so defined map $\bar{\ } :\ ' \to$ *is a* $(1 + \ )$*-Lipschitz diffeomorphism, where* $\to 0$ *for* $\to 0$ *and* $/\ \to 0$.

It is significant that the above does not work if you replace the bound on the diameters of and $'$ by volume bounds, i.e. if you bound $|\ |_{max}\ ^{2/n}$ while allowing $|\ |_{max} \cdot\ ^2_{min} \to \infty$.

(Counter)example. Let $_0$ and $'_0$ be complete Riemannian $(\ -1)$ manifolds that are cylindrical at infinity and have their sectional curvatures bounded by $|\ | \leq 1/2$. Let $_0 :\ '_0 \to\ _0$ be a 1-Lipschitz map that is isometric at infinity.

Thus, both manifold equal $\times [0\ \infty)$ on their common cylindrical end for a closed $(\ - 2)$-manifold , where we denote by $(\ ) =\ (\ ') \in [0\ \infty)$, $\in\ _0$, $' \in\ '_0$, the axial parameter on this end.

(All this can be arranged, for instance, if is diffeomorphic to $\mathbb{R}^{n-1}$ and $'$ is diffeomorphic to the complement of a ball in an *arbitrary* closed $(\ - 1)$-manifold, where $_0$ and $'_0$ can be made isometric to $^{n-2} \times [0\ \infty)$ at infinity.)

Let be a smooth positive function on $_0$ and let $'(\ ') = (1 - \ )\ (\ _0(\ '))$, $' \in\ '_0$. (One may take an which depends only on on the cylindrical end of $_0$, i.e. $(\ ) =\ (\ (\ ))$, and which is constant away from this end.)

Multiply $_1$ and $'_1$ by the unit circle ^{1} and modify the product metrics in $_1 =\ _0 \times\ ^1$ and $'_1 =\ '_0 \times\ ^1$ by multiplying the circles over $_0 \in\ _0$ and $'_0 \in\ '_0$, that are $_0 \times\ ^1$ and $'_0 \times\ ^1$, by the functions and $'$ respectively. Denote by $_f$ and $'_{f'}$ the resulting complete Riemannain -manifolds, where we organize the matter with a suitable , such that both manifolds $_f$ and $'_{f'}$ have $|\ | \leq 1$.

Observe that the map $_1 :\ '_{f'} =\ '_0 \times\ ^1 \to\ _f =\ _0 \times\ ^1$ for $(\ '\ ) \mapsto (\ _0(\ ')\ )$ is $(1 + \ )$-Lipschitz for these metrics.

There obviously exists an axial value $_\varepsilon \in [0\ \infty)$, such that the integrals of the two functions over the cut-off manifolds $_0(\ _\varepsilon) \subset\ _0$ and $'_0(\ _\varepsilon) \subset\ '_0$, defined by $(\ )\ (\ ') \leq\ _\varepsilon$, are equal. Then the corresponding compact -manifolds $_f(\ _\varepsilon) \subset\ _f$ and $'_{f'}(\ _\varepsilon) \subset\ '_{f'}$, with the boundaries $_f(\ _\varepsilon) =\ '_{f'}(\ _\varepsilon) = \times\ ^1 \times\ _\varepsilon$, have equal -volumes.

If we replace $\mapsto$ and $' \mapsto\ '$ with an arbitrary > 0, we do not change the curvatures of the manifolds; thus we can make the volumes of $_f(\ _\varepsilon)$ and $'_{f'}(\ _\varepsilon)$ arbitrarily small, say both equal $/2$ (to avoid more letters in the notation) still keeping $|\ | \leq 1$.

Finally, we take the doubles of these manifolds and obtain, for all ≥ 3 and arbitrarily small > 0, lots of

closed Riemannian -manifolds $= ( )$ *and* $' = '( )$ *both with* $| \quad | \leq 1$ *and with the volumes* $=$ *, where admits a* $(1+)$*-Lipschitz map* $: ' \to$ *of degree* 1 *that is not a homotopy equivalence.* (The diameters of these manifolds are $\approx \ _\varepsilon \approx \ ^{-1}$.)

On can recapture, however, the implication $\overrightarrow{\mathbf{D}} \ \leq \ \Rightarrow \mathbf{D}$ for complete Riemannian manifolds with $| \quad | \leq 1$ if, for a given $= 1\ 2$, the condition $(') = ( )$ is replaced by

$[\ ' \lesssim_\varepsilon \ \cdot \ ]$ $(\ ^{-1}(\ )) \leq (\ + \ ) \ (\ )$ for all unit balls $\subset$

Then, the argument used for $\overrightarrow{\mathbf{D}} \ \leq \ \Rightarrow \mathbf{D}$ also delivers

an approximation of every proper $(1+)$*-Lipschitz map* $: \ \to \ '$ *of degree by a locally diffeomorphic locally* $(1+)$*-bi-Lipschitz map* $' \to$ *for all sufficiently small* $\leq \ _0(\ ) > 0$ *and* $\leq (\ ) \underset{\varepsilon\to 0}{\to} 0$. (Such an approximating map $' \to$ necessarily is a -sheeted covering map.)

Questions. Let and $'$ be Riemannian -manifolds (or possibly singular Alexandrov spaces for this matter) with their sectional curvatures bounded from below by -1 and let $: ' \to$ be a $(1+)$-Lipschitz map of degree that satisfy $[\ ' \lesssim_\varepsilon \ \cdot \ ]$.

Does then, for small > 0, the map lift to a *homotopy equivalence* between $'$ and a -sheeted covering of ? (This is easy in the non-collapsed case.)

Is, moreover, homotopic to a locally homeomorphic map?

Is there anything meaningful here with the lower bound on the sectional curvatures relaxed to a lower bound on the Ricci curvatures of and/or $'$?

What happens for $(') - ( ) = \ > 0$?

Namely, let $: ' \to$ be an $(1+)$-Lipschitz map such that $(') \leq || \ _k(\ )||_{vol_k} \cdot \ (\ )$, where $|| \ (\ )||_{vol_k}$ is the $\mathbb{R}$-*mass* of the homology class $_k(\ ) \in \ _k(')$ that is the class represented by a generic pullback $^{-1}(\ ) \subset \ '$, $\in$.

Are there particular topological/homological constrains on such an for small in the case where and $'$ satisfy specified bounds on their sizes and the local geometries, say an upper bound on the diameters and on the absolute values of the sectional curvatures?

10.2 Mean Curvature Stability.

Below is another kind of situation where Lipschitz and volume confront one another.

Let and $'$ be smooth Riemannian -manifolds, $= \ _\varepsilon : ' \to$ be a $(1+)$-bi-Lipschitz homeomorphism and let $: \ \to \mathbb{R}$ be a continuous function.

Let $\subset$ be a smooth closed cooriented hypersurface with mean curvature $_y(\ ) = (\ )$ and let positive numbers > 0 be given.

If $\leq \ _0 = \ _0(\quad) > 0$, then there exists a closed hypersurface $_{min'} \subset$ such that the following conditions $[\mathbf{U}_\rho]$, **[mn.curv±]** and **[diff]** are satisfied.

$[\mathbf{U}_\rho]$ $_{min'}$ *is contained in the -neighbourhood* $_\rho(\ ) \subset$ *of where it is homologous to* .

[mn.curv±] *The hypersurface* $'_{min} = {}^{-1}({}_{min'}) \subset {}'$ *is* ^{2}*-smooth and its mean curvature with respect to the Riemannian metric in* $'$ *satisfies*

$$| \quad {}_{y'}({}'_{min}) - {}'({}')| \leq \quad \text{for} \quad {}'({}') = \circ ({}') \text{ and all } {}' \in {}'_{min}$$

Notice that even if is smooth, the hypersurface ${}_{min'} \subset$ is *not, in general,* ^{1}*-close to* . Moreover, even for $( ) = 1$, the normal projection ${}_{min'} \to$ is *not,* typically, one-to-one. However,

[diff] *the manifold* ${}'_{min}$ *is diffeomorphic to* . *In fact the composition of* $: {}'_{min} \to {}_{\rho}( )$ *with the normal projection* ${}_{\rho}( ) \to$ *can be approximated by a diffeomorphism* ${}'_{min} \to$.

Proof. We may assume that is connected and $\supset$ equals a small normal neighbourhood of . Thus, divides into two halves, call them *inside and outsides* of , written $( ) \subset$ and $( ) \subset$, with common boundary $(( )) = (( )) =$.

-Area and -Bubbles. Given a measure on , let

$$ {}_{-\mu}( ) =_{def} \quad {}_{n-1}( ) - (( ))$$

and call a hypersurface $\subset$ a *stable -bubble* if it *locally minimizes* the function $\mapsto \; {}_{-\mu}( )$ among all hypersurfaces in homologous to .

If is given by a continous density function $( )$, $\in$, i.e. $= \cdot \; {}_n$ for the Riemannian -volume (measure) ${}_n$, then these are called *-bubles.* Clearly, the mean curvature of a -bubble $\subset$ satisfies $( ) = ( )$.

In particular, *-bubbles*, where equals the Riemannian -volume ${}_n$ in times $\in \mathbb{R}$, have constant mean curvature

Local Traps. Let $\subset$ be a closed smooth cooriented hypersurface and $( )$ be a ^{1}-smooth function such that $( ) = \; {}_y( )$ for all $\in$.

If *the inward normal derivative* $\frac{d\phi(y)}{d\nu^{in}}$ *on* , *is sufficiently large*, namely

$$\frac{( )}{{}^{in}_{y}} > \quad {}^2_y( ) + \quad {}_X({}^{in}_y \; {}^{in}_y) \text{ for all } \in$$

where ^{2} denotes the sum of squares of the principal curvatures of and where, observe, $({}^{in} \; {}^{in}) = ({}^{out} \; {}^{out})$, then

is a stable -bubble; moreover, there is a (small) neighbouthood ${}_0 \subset$ *of* , *such that every hypersurface* $' \neq \subset {}_0$ *homologous to* *has strictly greater -area than* .

This trivially follows from the *second variation formula* for ${}_{n-1}( )$.

Now, given a smooth hypesurface $\subset$, let $: \to \mathbb{R}$ be equal the mean curvature of on and have large inward normal derivative. Then (almost) obviously, the domain ${}^{-1}_{\varepsilon}({}_0) \subset {}'$ contains a locally minimal $'$-bubble, say ${}'_{min} \subset {}^{-1}_{\varepsilon}({}_0)$, homologous to ${}^{-1}_{\varepsilon}( ) \subset {}'$ for all $\leq ( ) > 0$

The hypersurface ${}'_{min}$ can be, a priori, singular. However, if is sufficient215 to the volume of the Eucliden ball $\ ^{n-1}_{Eucl}(\ )$ for all $\ \in\ ^{\prime}_{min}$ Since $\ ^{\prime}$ on the - scale has almost the same filling inequalities as $\mathbb{R}^{n-1}$, it follows, by the standard monotonicity argument, that the ratio

$$\frac{_{n-1}(\ ^{\prime}_{y}(\ )\cap\ ^{\prime}_{min})}{_{n-1}(\ ^{n-1}_{Eucl}(\ ))}$$

is close to one for all *arbitrarily small* balls.

Hence, by Almgren-Allard regularity theory (that is the only non-elementary ingredient of our argument) the hypersurface $\ ^{\prime}_{min}$ is $\ ^{2}$-smooth with the mean curvature equal $\ ^{\prime}_{\epsilon}$ on it.

It remains to show that $\ ^{\prime}_{min}$ is diffeomorphic to .

You may assume is smooth, this does not cost you anything for small , but the geometric measure theory tells you nothing, a priori, about the topology and geometry of $\ ^{\prime}_{min}\subset\ ^{\prime}$ and of the corresponding $\ _{min^{\prime}} = (\ ^{\prime}_{min})\subset$.

Yet, since $\ ^{\prime}_{\mathrm{min}}$ is a *minimal* bubble, and is $(1+\ )$-bi-Lipshitz, the hypersurface is $\ _{min^{\prime}}$ is $\ ^{\prime}$*-quasi-minimal* for $\ _{n}$ and $\ ^{\prime}\approx$ in the sense of [?] on *all sufficiently small scales.* Then the standard blow-up rescaling/limit argument yields the following.

Weak Distortion Property. If the distance function from a point $\in$ to $\ _{min^{\prime}}$ has two minima $\ _{1}\ \ _{2}\in\ _{min^{\prime}}$,

$$(\ \ _{1}) = (\ \ _{2}) = = (\ \ _{min^{\prime}})$$

then the angle between the corresponding minimal geodesic segments $[\ \ _{1}]$ and $[\ \ _{2}]$ at must be small, roughly of order , for all not very large .

It follows that the distance function $\rightarrow (\ \ _{min^{\prime}})$, say outward $\ _{min^{\prime}}$, can be "bi-Lipschitz" approximated in the -neighbourhood $\ _{\delta}(\ _{min})$ for, say $= 100^{n}$ for a sufficiently small and . by a smooth function $(\ )$ *without critical points* which vanishes on $\ _{min^{\prime}}$. Similarly one sees that the normal projection of the /2-level of $(\ )$ to is a diffeomorphism. QED.

Concluding Remarks. The $(1+\ )$-bi-Lipschitz assumption, as well as the $\ ^{\varepsilon}$-Lipschitz constrain on maps in the previous section, are unduly restrictive. These will be relaxed in the Hilbert volume framework, in the spirit of Stephan Wenger's metric in the space of manifolds, [23] [24], in Part 2 of our paper.

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