# Metric Inequalities with Scalar Curvature.

Misha Gromov

August 8, 2017

## Contents


1 **Formulation of Key Inequalities.** **2**

2 **Bounds on Widths of Over-torical, Over-cubical and Related Riemannian Bands.** **5**

3 **Toric Bands in Spheres and Lower Bounds on Lipschitz Constants of Map $X \to S^n$ in terms of $Sc(X)$.** **8**

4 **$\frac{4\pi}{n}$- and $\frac{2\pi}{n}$-Inequalities for Iso-Enlargeable Bands.** **11**

5 **Schoen-Yau-Schick Manifolds and $\mathcal{SYS}$-Bands.** **17**

6 **$\mathcal{SYS}$-Enlargeable Manifolds and Codimension Two Depth Inequalities.** **19**

7 **External Curvature, Focal Radius and Depth in Codimension$>$ 2** **22**

8 **Symmetrization by Reflections with Point-wise Control of the Scalar Curvature.** **24**

9 **Application of Symmetrization to Manifolds with Positive and with Negative Scalar Curvatures.** **29**

10 **Comparison with Results Obtained with Twisted Dirac Operators.** **34**

11 **References.** **44**



**Abstract**

We establish several inequalities for manifolds with *positive scalar curvature* and, more generally, for the scalar curvature bounded from below, in the spirit of the classical bound on the distances between conjugates points in surfaces with *positive sectional curvature.*

# 1 Formulation of Key Inequalities.

Our point of departure is the following inequality for *torical bands* which are smooth manifolds homeomorphic to *tori times intervals.*

$[\mathbf{O}_\pm]$ **Torical $\frac{2\pi}{n}$-Inequality.** Let $V$ be an $n$-dimensional torical band, $V = \mathbb{T}^{n-1}\times[-1,+1]$, where the boundary is

$$\partial(V) = \partial_- \cup \partial_+ = \partial_-(V)\cup\partial_+(V) = (\mathbb{T}^{n-1}\times\{-1\})\cup(\mathbb{T}^{n-1}\times\{+1\}).$$

*Let $g$ be a smooth Riemannian metric on $V$, where the scalar curvature is bounded from below by a positive constant $\sigma > 0$,*

$$Sc(g) \geq \sigma > 0.$$

*Then the distance between the two boundary components of $V$ satisfies*

$$\left[\mathbf{O}_\pm \leq 2\pi\sqrt{\tfrac{n-1}{\sigma n}}\right] \qquad dist_\pm = dist_g(\partial_-(V),\partial_+(V)) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}\ \left(<\frac{2\pi}{\sqrt{\sigma}}\right).$$

On Normalisation of $Sc$. We use the customary normalisation of the scalar curvature, where the unit spheres satisfy

$$Sc(S^n) = n(n-1).$$

Thus, by scaling, the inequality $\left[dist_\pm \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}\right]$ for a non specified $\sigma > 0$ reduces to that for $Sc(V) \geq n(n-1)$, where it reads

$$\left[\mathbf{O}_\pm \leq \tfrac{2\pi}{n}\right] \qquad dist_\pm = dist_g(\partial_-(V),\partial_+(V)) \leq \frac{2\pi}{n}$$

In particular,

*all torical bands in the unit sphere, satisfy*

$$dist(\partial_-,\partial+) \leq \frac{2\pi}{n}.$$

This is obvious for $n = 2$, where $[\mathbf{O}_\pm \leq \frac{2\pi}{2}]$ is sharp as well as obvious. One also expects a two line proof of a stronger inequality for all $n$, but to my surprise, I was unable to directly prove even the corresponding inequality for *principle curvatures* of $(n-1)$-tori embedded to $S^n$ , where this inequality is formulated below in terms of *focal coradii* as follows.

Normal Tubes, Normal Bands and $rad^{\odot}(Y)$. The *normal focal radius* of a smooth submanifold $Y$ in a Riemannin manifold $X$, denoted $rad^{\odot}(Y) = rad^{\odot}(Y\subset X)$ is the maximal $r$ such that the normal exponential map

$$\exp : T_\perp(Y) = T(X)|Y \ominus T(Y) \to X$$

is one-to-one[1] on the subset of vectors $\nu \in T_\perp(Y)$, such that $||\nu|| < r$.

[1]It would be more in the spirit of "focal" to require the normal exponential map to be *locally* one-to-one, but this, probably, makes no difference in the present context for $X = S^n$.

In other words, this is the maximal $r$ such that *the normal $r$-tube* around $Y$, called *normal $r$-band* if $codim(Y) = 1$, that is the open $r$-neighbourhood $U_r(Y) \subset X$ for $r = rad^{\odot}$, normally projects[2] to $Y$ and fibers $U_r(Y)$ into $r$-balls of dimension $dim(X) - dim(Y)$.

*Examples.* (a) The normal focal radii and the geodesic curvatures of subspheres

$$S^m(\rho) = S^m_s(\rho) \subset S^n = S^n(1) \subset \mathbb{R}^{n+1},\ \rho \le 1,$$

centred at ponts $s \in S^n$ are

$$r = rad^{\odot}_{S^n}(S^m(\rho)) = \arcsin\rho \text{ and } curv_{S^n}(S^m(\rho)) = \frac{\sqrt{1-\rho^2}}{\rho} = \tan r.$$

(b) The Clifford torus $\mathbb{T}^n_{Cl} \subset S^{2n-1} \subset (\mathbb{R}^2)^n$, that is the product of $n$ circles of radii $\frac{1}{\sqrt{n}}$ in the plane, satisfies:

$$rad^{\odot}(\mathbb{T}^n_{Cl}) = \arcsin\frac{1}{\sqrt{n}}.$$

Conjecturally, $\mathbb{T}^n_{Cl}$ has maximal $rad^{\odot}$ among all $n$-tori smoothly embedded to $S^{2n-1}$.

**Normal Radius Inequality for $\mathbb{T}^{n-1} \subset S^n$.** *If a smooth hypersurface $Y$ in the unit $n$-sphere is homeomorphic to the $(n-1)$-torus, then*

$[\odot \le \frac{\pi}{n}]$. $$rad^{\odot}(Y) \le \frac{\pi}{n}.$$

This inequality– this will become clear later on – is non-sharp.

Conjecturally, the sharp constant must be asymptotic for $n \to \infty$ to

$$\frac{const}{n^\alpha} \text{ for some } \alpha > 1 .$$

$\blacksquare_\pm$ **Sub-Rectangular $[\square_{\frac{2\pi}{n}}]$-Inequality**. Let $X$ be a Riemannian $n$-manifold, let $Q \subset X$ be a domain diffeomorphic to the $n$-cube $[-1,+1]^n$ and let $Q_i^\pm \subset \partial Q \subset Q$, $i = 1, ..., n$, denote the pairs of opposite codimension 1 faces in $Q$ which correspond to such pairs in the cube.

*Let*

*(i) the faces $Q_i^\pm$ for $i = 1, ..., n-1$, are mean curvature convex, i.e.*

$$mean.curv(Q_i^\pm) \ge 0,$$

*(ii) the dihedral angles $\angle_{\pm i,\pm j} = \angle(Q_i^\pm, Q_j^\pm)$ between these faces are non-obtuse at all points in the $(n-2)$-"edges" where these faces meet,*

$$\angle_{\pm i,\pm j} \le \pi/2, \text{ for all } i,j = 1,...,n-1, i \ne j,$$

*(iii) the scalar curvature of $X$ satisfies $Sc(X) > n(n-1)$.*

*Then the distance between the two remaining opposite faces satisfy*

$\left[\square_\pm < \frac{2\pi}{n}\right]$ $$dist_\pm = dist_X(Q_n^+, Q_n^-) < \frac{2\pi}{n}.$$

[2] This projection sends each $x \in U_r(Y)$ to the unique(!) nearest point in $Y$.

Sketch of the Proof. Start by recalling that manifolds with $Sc > 0$

*can't contain mean curvature convex (e.g. convex) cubical domains $Q$, where all dihedral angles, including $\angle_{\pm i, \pm n}$, are non-obtuse.*

This is shown in [Gr 2014] by reflecting such $Q$ in the faces, smoothing the resulting metric with $Sc > 0$ and invoking non-existence theorem for metrics with Sc>0 on the tori [SY-Str 1979], [GL 1980].

Here we also reflect $Q$, but only in the faces $Q_i^\pm$ with $i < n$. Thus we construct a torical band with $Sc > \sigma$ and apply $[\circledcirc_{\frac{2\pi}{\sqrt{\sigma}}}]$ to this band.

*On Sharpness of* $\left[\mathbf{O}_\pm \leq \frac{2\pi}{n}\right]$ *and* $[\square_\pm \leq \frac{2\pi}{n}]$. These inequalities agree with the obvious ones in the 2-sphere (where the conventionally defined scalar curvature equals twice the sectional curvature) where the widths of the bands between concentric circles as well as the distances between opposite sides of (all) quadrilaterals are bounded by $\frac{2\pi}{2} = \pi = diam(S^2)$ and where these inequalities become sharp for doubly punctured spheres (in the case $\mathbf{O}$) and for quadrilaterals (in the case $\square$) which degenerate to *geodesic digons* joining opposite points in $S^2$.

And if $n \geq 2$, we shall see in the next section that the *extremal* $\mathbf{O}$ and $\square$, where $dist_\pm = \frac{2\pi}{n}$, also have *constant scalar curvatures* and their opposite sides *collapse to points*, but they *do not have constant sectional curvatures* for $n > 2$ anymore.

**Quadratic Decay Theorem.** Let $X$ be a complete Riemannian manifold, and let

$$\min_{B(R)} Sc(X)$$

denote the minimum of the scalar curvature (function) of $X$ on the ball $B(R) = B_{x_0}(R) \subset X$ for some centre point $x_0 \in X$.

*If $X$ is homeomorphic to $\mathbb{T}^{n-2} \times \mathbb{R}^2$, then there exists a constant $R_0 = R_0(X, x_0)$, such that*

$$[\asymp \tfrac{4\pi^2}{R^2}] \qquad \min_{B(R)} Sc(X) \leq \frac{4\pi^2}{(R - R_0)^2} \text{ for all } R \geq R_0.$$

*Outline of the Proof.* Let $X_0 \subset X$ corresponds to the torus $\mathbb{T}^{n-2} \times \{0\} \subset \mathbb{T}^{n-2} \times \mathbb{R}^2$ under the homeomorphism $\mathbb{T}^{n-2} \times R^2 \leftrightarrow X$ and let $R_0 = diam_X(X_0)$. Then the $(R - R_0)$-neighbourhood $U_{R-R_0}(X_0) \subset X$ is contained in the ball $B_{x_0}(R)$ for $x_0 \in X_0$.

If $U_{R-R_0}(X_0)$ is homeomorphic to $\mathbb{T}^{n-1} \times (-1, +1)$, then $[\asymp \frac{4\pi^2}{R^2}]$ follows from the torical $\frac{2\pi}{n}$-inequality and if the topology of $U_{R-R_0}(X_0)$ is more complicated, then we apply a generalisation of the $\frac{2\pi}{n}$-inequality from section 4.

On Uniformly Positive Scalar Curvature. The obvious corollary to $[\asymp \frac{4\pi^2}{R^2}]$ is

*non-existence of complete metrics with* $Sc \geq \sigma > 0$ on $\mathbb{T}^{n-2} \times \mathbb{R}^2$.

Notice that there are similar results for other manifolds $X$ proven with *Dirac operators twisted with suitable "almost flat" bundles over $X$* [GL 1983], [HaPS 2015].

However, for all I know, one can't rule out metrics with uniformly positive scalar curvature on $\mathbb{T}^{n-2} \times \mathbb{R}^2$ with the present day Dirac operator methods.

Examples of Metrics on $\mathbb{T}^{n-2}\times\mathbb{R}^2$ with Quadratic Decay of Scalar Curvature. Let $g = dt^2+\varphi(t)^2d\theta^2$, $t\in[0,\infty),\theta\in[0,2\pi]$, be a radial (rotationally symmetric) metrics on $\mathbb{R}^2$. Then

$$-\frac{1}{2}Sc(g)(t) = -\frac{\varphi''(t)}{\varphi(t)};$$

thus, the metrics

$$g_0 + dt^2 + t^{2\alpha}d\theta^2$$

on $\mathbb{R}^2\times\mathbb{T}^{n-2}$, where $g_{fl}$ are flat on $\mathbb{T}^{n-2}$ and $0<\alpha<1$ do the job.

On Proofs and Generalisations. Simple generalisations of everything we stated so far is proven section 2.

Then, in the following sections we formulate and prove further *generalisations and refinements* of these. Also we indicate additional *applications* and articulate several *conjectures.*

Our approach is based on the Schoen-Yau *dimension descent argument* [SY-Str 1979], [SY 2017] accompanied by *torical symmetrisation* [GL 1983] and/or *symmetrizarion by reflection* [Gr 2014].

About Singularities. Applications of minimal hypersurfaces $Y\subset X$ to $Sc\geq 0$ depends on the regularity of these $Y$ which is known to hold for all $Y$ if $n = dim(X)\leq 7$ and for generic ones for $n=8$ by a Nathan Smale theorem [NS 1993)]. More recently, Lokhamp [Loh 2016] and Schoen and Yau [SY 2017] suggested ways of bypassing the singularity problem.

As far as I understand, the regularity results by Schoen and Yau in [SY 2017], such as theorem 4.6, suffice for the needs of the present paper and this is, probably, true about the corresponding results by Lokhamp. But since I have not studied these papers in depth, I can vouch for the validity of our proofs only for $n\leq 8$, where the singularity problem does not exist.

# 2 Bounds on Widths of Over-torical, Over-cubical and Related Riemannian Bands.

A *band* is a manifold $V$ with two distinguished disjoint non-empty subsets in the boundary $\partial(V)$, denoted

$$\partial_- = \partial_-(V)\subset\partial V \text{ and } \partial_+ = \partial_+(V)\subset\partial V.$$

A band is called *proper* if $\partial_\pm$ are unions of connected components of $\partial V$ and

$$\partial_-\cup\partial_+ = \partial V.$$

*Band maps* $V\to\underline{V}$ are those continuous ones which respect these $\pm$-boundaries, $\partial_\pm\to\underline{\partial}_\pm$.

If $V$ is endowed with a Riemannin metric then the *width of a band* is the distance between $\partial_-$ and $\partial_+$, that is the infimum of length of curves in $V$ between $\partial_-$ and $\partial_+$.

A compact proper orientable band is called *over-torical* if it admits a band map to the toric band,

$$f : V\to\underline{V} = \mathbb{T}^{n-1}\times[-1,+1],\ n = dim(V),$$

with *non-zero degree.*

Another way to put it is by saying that the relative fundamental class $[V] \in H^n(V, \partial V; \mathbb{Q})$ decomposes to the product

$$[V] = h_1 \smile ... \smile h_{n-1} \smile h_n$$

where $h_i$, $i = 1, ..., n-1$, are (absolute) 1-dimensional cohomology classes, $h_i \in H^1(V; \mathbb{Q})$, and $h_n \subset H^1(V, \partial V; \mathbb{Q})$ is the (relative) class, of the differential of a function $V \to [-1, +1]$ such that $\partial_\pm \mapsto \pm 1$.

If $V$ is non-orientable, then *overtorical* means that an orientable finite cover of $V$ is overtorical.

**Torical Symmetrisation.** *There exists a quasi-functorial symmetrizartion "operator" from Riemannian over-torical bands to torical ones*

$$\mathsf{Sym} : V \rightsquigarrow \underline{V}$$

*where $\underline{V}$ admits a free isometric action of the torus $\mathbb{T}^{n-1}$ and such that*

$$width(\underline{V}) \geq width(V)$$

*and*

$$Sc(V) > \sigma \Rightarrow Sc(\underline{V}) > \sigma.$$

*Proof.* This is proven in a slightly different form in [GL 1983] for $n \leq 7$ by induction as it is explained below.

(Earlier, such symmetrization for $n = 3$ was used by Fisher-Colbrie and Schoen [FCS 1980], while the proof for $n = 8$ is essentially the same as for $n \leq 7$ due to Nathan's Smale generic regularity theorem.)

Induction Step. Let $V_k$ be a $\mathbb{T}^k$ invariant Riemannin band, $k = 0, ..., n-2$, which admits a $\mathbb{T}^k$-equivariant band map to the torical band

$$f_k : V_k \to \mathbb{T}^{n-1} \times [-1, +1]$$

where $\mathbb{T}^k$ acts on $\mathbb{T}^{n-1} \times [-1, +1]$ via the standard (coordinate) embedding $\mathbb{T}^k \subset T^{n-1}$ and such that $deg(f_k) \neq 0$.

Let $Y_k \subset V_k$ be a volume minimising hypersurface which is homologous to the $f_k$-pullback of

$$\mathbb{T}^{n-2} \times [-1, +1] \subset \mathbb{T}^{n-1} \times [-1, +1]$$

for the torus $\mathbb{T}^{k+1} \supset \mathbb{T}^k$, where "homologous" refers to the relative group $H_{n-1}(V_k; \partial V_k = \partial_- \cup \partial_+)$.

It is easy to see that this $Y_k$ is $T^k$-invariant and that the lowest eigenfunction $\phi(y)$ of the second variation operator $L$ on $Y_k$,

$$L = -\Delta + \frac{1}{2}(Sc(Y_k) - Sc(V_k|Y_k) - ||curv_{V_k}(Y_k)||^2),$$

is also $T^k$-invariant. (Here, $\Delta$ is the Laplacian on $Y_k$, that is $\sum_i \frac{\partial^2}{dy_i^2}$ and $curv_X(Y)$ denotes the second fundamental form of $Y_k \subset V_k$.)

Then we let $V_{k+1} = Y_k \times \mathbb{T}^1$ with the metric $dy^2 + \phi^2 dt^2$, where a simple computation shows that if the scalar curvature of $V_k$ restricted to $Y_k$ is $\geq \sigma$, then the scalar curvature of $V_{k+1}$ is also bounded from below by $\sigma$.

It is also clear that $V_{k+1}$ admits a $\mathbb{T}^{k+1}$-equivariant band map of degree $= deg(f_k)$ to the torical band and that $width(V_{k+1}) \geq width(V_k)$.

Thus, the inductive step is completed and the existence of torical symmetrisation follows. (See [GL 1983] for details).

*Remark on Singularities.* $\mathbb{T}^k$-invariant minimal hypersurfaces in $V_k$ correspond to hypersurfaces in the quotient manifolds $V_k/\mathbb{T}^k$, which are minimal with respect to the quotient metrics with obvious conformal weights. Then theorem 4.6 in [SY 2017] says, in effect, that even if some hypersurfaces $Y_k$ were singular, say for $n-k \geq 8$, the final $\mathbb{T}^{n-1}$-symmetric $V_{n-1} = \underline{V}$ are non-singular.

(Schoen and Yau formulate their theorem for closed manifolds but the needed regularity for manifolds $V$ with boundaries trivially reduces to that for doubles of $V$.)

$\frac{2\pi}{n}$**-Inequality for Over-Torical Bands.** *Overtorical bands with scalar curvatures* $\geq n(n-1)(= Sc(S^n))$ *satisfy*

$$\left[dist_{\pm} \leq \tfrac{2\pi}{n}\right] \qquad\qquad width(V) \leq \frac{2\pi}{n}.$$

*Proof.* Torical symmetrization reduces the general case to that of $\mathbb{T}^{n-1}$-invariant metrics $g$, on torical bands, where

$$g = dt^2 + \sum_i \varphi_i(t)^2 d\tau_i^2,\ i = 2,3,...,n.$$

Then one easily computes

$$Sc(g)(t,\tau_2,...,\tau_n) = -2\sum_i \frac{\varphi''(t)}{\varphi_i(t)} - 2\sum_{i<j} \frac{\varphi_i'(t)}{\varphi_i(t)}\frac{\varphi_j'(t)}{\varphi_j(t)}$$

and shows that the the longest $t$-interval where this function remains defined for $Sc(g) \geq \sigma > 0$ is achieved with $\varphi_2 = ... = \varphi_n = \varphi$, where the proof follows by simple computation on p.401 in [GL 1983] which is reproduced below in the description of optimal (maximal) torical bands with $Sc \geq \sigma$.

*Proof of Propositions from Section 1.* The inequality $\left[dist_{\pm} \leq \frac{2\pi}{n}\right]$ implies everything we have stated so far, where in the case of the quadratic decay theorem one needs to observe that the domains $U_{R-R_0}(X_0) \subset X$ (defined following the statement of this theorem) are, in an obvious sense, *open overtorical bands* to which the above $\frac{2\pi}{n}$-Inequality applies.

Notice at this point that this argument automatically delivers the following

Generalisation of The Quadratic Decay Theorem. *If a complete orientable Riemannin $n$-manifold $X$ admits a proper continuous map $X \to \mathbb{T}^{n-2} \times \mathbb{R}^2$ of non-zero degree, then the minima of the scalar curvature of $X$ over concentric $R$-balls in $X$ satisfy*

$$[\asymp \tfrac{4\pi^2}{R^2}]^* \qquad \min_{B(R)} Sc(X) \leq \frac{4\pi^2}{(R-R_0)^2} \text{ for some } R_0 \geq 0 \text{ and all } R \geq R_0.$$

Optimality of $\frac{2\pi}{n}$. *Every smooth manifold $V = Y \times [-1,1]$ admits a Riemannian metric $g = g_\varepsilon$ with $Sc(g) \geq n(n-1)$ and the $g$-distance between the two boundary components $Y \times \{-1\}$ and $Y \times \{1\}$ in $V$ equal $2\pi/n - \varepsilon$ for a given $\varepsilon > 0$.*

For instance, if $Y = S^1$, then the *spherical suspension* $V = \ominus(Y)$ serves this purpose for all $\varepsilon > 0$.

More generally, given a Riemannian metric $g_0$ on $Y$ and a real function $\varphi(t)$, let $g = dt^2 + \varphi(t)^2 g_0$ be the metric on $Y \times [-l, l]$. If $g_0$ is flat then

$$◑ \qquad \sigma = Sc(g) = -2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\frac{\varphi'^2}{\varphi^2},$$

or

$$\frac{\sigma}{n-1} = -2\left(\frac{\varphi''}{\varphi} + \frac{\varphi'^2}{\varphi^2}\right) - n\frac{\varphi'^2}{\varphi^2},$$

that is

$$-2f' - nf^2 = \frac{\sigma}{n-1} \text{ for } f = \frac{\varphi'}{\varphi}.$$

Now let $\sigma = Sc(S^n) = n(n-1)$ and rewrite the above as

$$\frac{f'}{1+f^2} = (\arctan f)' = \frac{n}{2}$$

and

$$f = f(t) = \tan\frac{n}{2}t$$

which is a function defined on the $\frac{2\pi}{n}$-interval $(-\frac{\pi}{n}, +\frac{\pi}{n})$.

This settles the matter for flat manifolds $Y$ and the general case follows by rescaling general metrics in $Y$ with a large constants.

# 3 Toric Bands in Spheres and Lower Bounds on Lipschitz Constants of Map $X \to S^n$ in terms of $Sc(X)$.

Suppose, there is a *toric band of width* $d$ in the unit $n$-sphere $S^n$ that is a domain $\underline{V} \subset S^n$ which is homeomorphic to $\mathbb{T}^{n-1} \times [-1,1]$ and such that the distance between the two boundary components $\partial_\pm(\underline{V})$ of $V$ is equal to $d$ and let $f$ be a continuos map of non-zero degree from an oriented Riemannin $n$-manifold $X$ to $S^n$.

Recall that saying "degree" presupposes that $f$ is locally constant at infinity, i.e. constant on each boundary component of $X$ and, if $X$ is non-compact, on every component of some (large) compact subset in $X$, and let us additionally assume that the (finite) $f$-image of the so defined infinity *does not intersect* $V$. (This is relevant only if $\partial X$ is disconnected and/or if $X$ is disconnected at infinity.)

Then the pullback $V = f^{-1}(\underline{V}) \subset X$ is a Riemannin over-torical band, such that the distance between the two parts $\partial_\pm(V)$ of its boundary is $\geq \lambda^{-1}d$, and the inequality

$$d = width(V) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}$$

(this is $[dist_\pm \leq \frac{2\pi d}{n}]$ formulated for $\sigma = n(n-1)$ in the previous section) shows that

$$\bigcirc^n_{Lip} \qquad [Sc(X) \geq \sigma] \Rightarrow \left[Lip(f) \geq \frac{d}{2\pi}\sqrt{\frac{\sigma n}{n-1}}\right],$$

where, recall,

$$Lip(f) = \sup_{x_1 \neq x_2} \frac{dist_{S^n}(f(x_1), f(x_2))}{dist_X(x_1, x_2)}.$$

Notice that the $\frac{2\pi}{n}$-inequality $[dist_\pm \leq \frac{2\pi}{n}]$, that is (essentially) $\bigcirc^n_{Lip}$ applied to the identity map, shows that torical bands in $S^n$ have widths $d \leq \frac{2\pi}{n}$.

Conjecturally, the maximal widths $d$ for large $n \to \infty$ must be asymptotic to $\frac{1}{n^{1+\alpha}}$ for some $\alpha > 0$.

*Round Tori and in $S^n$ in $\mathbb{R}^n$.* Let us show that this $\alpha$ must be $\leq \frac{1}{2}$ by exhibiting embedded tori $\mathbb{T}^{n-1}$ with bands of width $\approx \frac{1}{n^{\frac{3}{2}}}$ around them, where we use the following terminology.

**Over-Torical Width** $width_{\hat{\mathcal{T}}}(X)$**.** This is defined for Riemannian manifolds $X$ as the supremum of numbers $d$, such that $X$ admits an equidimensional locally isometric (not necessarily globally one-to-one) immersion from an overtorical Riemannian band of width $d$.

For instance, it is obvious that

$$width_{\hat{\mathcal{T}}}(S^2) = \pi.$$

More significantly, since the Clifford torus in $S^3$ has $rad^{\odot} = \pi/4$, (see section 1)

$$width_{\hat{\mathcal{T}}}(S^3) \geq \pi/2$$

and consequently,

*all continuous maps $f$ from Riemannian (possibly incomplete) 3-manifold $X$ with $Sc(X) \geq 6 = Sc(S^3)$ to $S^3$ which are constant at infinity and have $deg(f) \neq 0$, satisfy*

$$Lip(f) \geq \frac{3}{4}.$$

This improves the inequality $Lip(f) \geq \frac{3}{8\pi}$ from [GL 1983] but falls short of the conjectural bound $Lip(f) \geq 1$.

Another natural conjecture is the *equality*

$$width_{\hat{\mathcal{T}}}(S^3) = \pi/2.$$

Moreover, one expects that

*all (possibly non-complete) 3-manifolds $X$ with sectional curvatures $\geq 1$ satisfy*

$$width_{\hat{\mathcal{T}}}(X) \leq \pi/2.$$

Starting from $n = 4$, codimension one tori in $S^n$ can't be rotationally invariant any more; we construct certain "roundish" ones with relatively large focal coradii $r = rad^{\odot}$, i.e. with the normal exponential maps of these tori is one-to-one within distance $\leq r$ from them.

We construct these tori in the unit Euclidean $n$-balls (rather than in the unit spheres) by induction as follows.

Given codimension one tori $Y_1 \subset B^{n_1} \subset \mathbb{R}^{n_1}$, and $Y_2 \subset B^{n_2} \subset \mathbb{R}^{n_2}$ with focal coradii $r_1$ and $r_2$, take $c_1, c_2 > 0$, such that

$$c_1^2 + c_2^2 = 1 \text{ and } c_1 r_1 = c_2 r_2,$$

observe that the product of the $c_i$-scaled $Y_i$ in $\mathbb{R}^{n_i}$ is contained in the unit ball

$$Y_\times = c_1Y_1 \times c_2Y_2 \subset B^{n_1+n_2} \subset \mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

and

$$rad^{\odot}(Y_\times) = r_\times = c_1r_1 = c_2r_2.$$

Then let

$$Y_{\times_+} = \frac{1}{1+\delta}(Y_\times)_{+\delta} \subset B^{n_1+n_2}$$

be the $\frac{1}{1+\delta}$-scaled boundary of the $\delta$-neighbourhood of $Y_\times$ in $\mathbb{R}^{n_1+n_2}$ with $\delta = \frac{1}{2}r_\times$ and observe that

$$rad^{\odot}(Y_{\times_+}) \geq \frac{1}{2+r_\times} rad^{\odot}(Y_\times).$$

In particular, if a torus $Y_1 = Y_2 = Y(n) \subset B^n$ has normal focal radius $r = r(n)$, the resulting $Y(2n) = (Y \times Y)_+ \subset B^{2n}$ satisfies

$$r(2n) = rad^{\odot}(Y \times Y)_+ = \frac{r(n)}{2\sqrt{2} + r/2}$$

and the normal focal radius of

$$Y(2n+1) = \left((c_1Y(n)) \times (c_2Y(n+1))\right)_+ \subset B^{2n+1}.$$

satisfies a similar inequality.

Then, starting from $Y(2) = S^1 \subset \mathbb{R}^2$ with $r(2) = 1$ one obtains $Y(n) \subset B^n$, such that

$$rad^{\odot}(Y(n)) \geq cn^{-\frac{3}{2}} \text{ for some } c > \frac{1}{4}.$$

(If $n = 2^i$, then $c > \frac{1}{2}$.)

Eventually, since the normal bands around these tori $Y(n)$, can be transported from $B^n$ to $S^n$ by the obvious expanding map $B^n \to S^n$, we conclude that

$$width_{\hat{\mathcal{T}}}(S^n) \geq n^{-\frac{3}{2}}$$

which combined with $\bigcirc^n_{Lip}$ implies the following.

**Spherical Lipschitz Bound Theorem.** *If the scalar curvature of a (possibly incomplete) Riemannian $n$-manifold is bounded from below by $n(n-1) = Sc(S^n)$, then all continous maps $f$ from $X$ to the sphere $S^n$ (and also to the hemisphere to $S^n_+$) of non-zero degrees*[3] *satisfy*

$$Lip(f) \geq \frac{1}{2\pi\sqrt{n}}.$$

(If $n = 2^i$ then $Lip(f) \geq \frac{1}{\pi\sqrt{n}}$, but since this inequality is unlikely to be qualitatively sharp anyway there is no point fiddling with constants.)

*Remarks.* (a) If $X$ is a *complete spin*[4] manifold, then the *sharp* spherical Lipschitz bound $Lip(f) \geq 1$ is known to hold for these maps $f : X \to S^n$ by the

[3]Here such a map $X \to S^n$ is supposed to be constant at infinity, including $\partial X$ and to be proper from the interior of $X$ to that of $S^n_+$.

[4]In fact, it suffices to have the universal covering of $X$ spin – we return to this later on'; here we recall that an orientable smooth manifold $X$ is *spin* if the restrictions of the tangent bundle $T(X)$ to all surfaces $S \subset X$ are trivial bundles.

work of Llarull [Ll 1998]. This is accomplished by carefully analysing the *algebraic Schroedinger-Lichnerowicz-Weitzenboeck formula* for *the Dirac operator on $X$ twisted with the spin bundle $\mathbb{S}^+(S^n)$ pulled back to $X$* and applying the index theorem.

In fact, this Dirac operator proof rules out smooth proper maps $f : X \to U \subset S^n$ of non-zero degrees, which *strictly decrease areas of surfaces $S \subset X$* (such $f$ may have $Lip(f) >> 1$) and where the complements to the (open) subsets $U \subset S^n$ are *zero dimensional*, or, more generally, where *all connected subsets $A \subset S^n \setminus U$ are trees and/or closed curves with trivial (i.e. identity) Levi-Civita monodromy transformations around them* (see section 10).

(b) It remains unknown:

• *if the spin condition is essential for ruling out maps $f$ for which $area(f(S)) < area(S)$,*

• *if the completeness condition is essential for $Lip(f) \geq 1$,*

• *if one may allow closed curves in $S^n \setminus U$ with nontrivial Levi-Civita monodromies even if $X$ complete and spin.* (See section 10 for further questions of this kind.)

(c) The above inequality $Lip(f) \geq \frac{1}{2\pi\sqrt{n}}$ (which applies to non-complete non-spin manifolds) improves upon $Lip(f) \geq \frac{n}{2^n\pi}$ in [GL 1983]. This gains in significance as $n \to \infty$, where the proof for $n \geq 9$ depends on the controlled singularity results by Lohkamp and Schoen-Yau, which the present author has not studied in detail.

(d) The above estimates of torical width of $S^n$ and of focal radii of tori in $S^n$ raise a multitude of questions concerning $width_{\mathcal{T}}(X)$, $rad^{\odot}(Y \subset X)$ and their generalisations for various $X$ and $Y$. These will be briefly discussed in section 7.

## 4 $\frac{4\pi}{n}$- and $\frac{2\pi}{n}$-Inequalities for Iso-Enlargeable Bands.

Hypersphericity and Iso-Enlargeability. An oriented Riemannin manifold $X$ is called *hyperspherical* if it admits continuous maps $f$ to $S^n$, $n = dim(X)$ with arbitrarily small $Lip(f) > 0$, which are *constant at infinity* which have *non-zero degrees.*

A Riemannian manifold $X$ is called *iso-enlargeable* if there exists a sequence of Riemannin manifolds $X_i$ of dimension $n = dim(X)$ and of locally isometric maps $X_i \to X$, such that $X_i$ admit continuous maps constant at infinity

$$f_i : \tilde{X}_i \to S^n,$$

such that

$$deg(f)_i \neq 0 \text{ and } Lip(f_i) \to 0 \text{ for } i \to \infty.$$

*Examples.* (a) The archetypical hyperspherical manifolds are the Euclidean spaces $\mathbb{R}^n$.

(b) Complete simply connected manifolds $X$ with *non-positive sectional curvatures* $\kappa$ are also hyperspherical.

This follows from (1), since the the inverse exponential maps $\exp^{-1} : X \to \mathbb{R}^n = T_{x_0}(X)$ satisfy $Lip(\exp^{-1} \leq 1$ for $\kappa(X) \leq 0$.

(c) If a compact manifold $X$ is fibered over an $\underline{X}$, where $\kappa(\underline{X}) \leq 0$ and where the fibers also admit metrics with $\kappa \leq 0$ then the universal covering of $X$ is hyperspherical by an easy argument.

(d) Compact locally symmetric spaces $Y$ that have no (local) factors isometric to real and/or complex hyperbolic spaces are enlargeable but *not overtorical*, since the homology groups $H_1(Y)$ are *finite* for these $Y$.

Instances of such $Y$ are compact quotients $H^n_{\mathbb{H}}/\Gamma$ of *quaternion hyperbolic spaces* (here the sectional curvature $\kappa(Y) < 0$) and compact quotients $SO(n)\backslash SL(n)/\Gamma$ , $n \geq 3$ (here $\kappa(Y) \leq 0$).

*Remark/Question.* If the, locally isometric maps $X_i \to X$ in the definition of isoenlargeability are required to be *covering maps*, which is equivalent to *completeness* of $X_i$ in the case where $X$ itself is complete (e.g. compact), then $X$ is called *enlargeable*, see [GL 1983], [Dr 2000], [DFW 2003], [HaSch 2006], [BH 2009], [Han 2011].

It is obvious that

$$enlargeable \Rightarrow iso\text{-}enlargeable$$

and if $X$ is compact, the inverse implication also seems plausible.

Indeed, sequences $X_i$ (sub)converge in a natural way to some $\tilde{X}$, where the maps $X_i \to X$ (sub)converge to a covering map $\tilde{X} \to X$ and where properly scaled maps $X_i \to S^n$ (sub)converge to Lipschitz maps $\tilde{f}_i : X \to S^n$.

But, in general, these $\tilde{f}_i$ are neither constant at infinity nor do they have non-zero degree, at least not in the ordinary sense (even if $f_i : X_1 \to X$ were covering maps to start with). Thus

enlargeablility of compact iso-enlargeable manifolds remains problematic even for compact *aspherical*[5] manifolds $X$.

(Examples of enlargeable manifolds with non-hyperspherical it universal coverings exhibited in [BH 2009] tilts one toward accepting a possibility of iso-enlargeable but non-enlargeable compact manifolds $X$.)

On the other hand, there is the following relation between iso-enlargeability and the overtorical width $width_{\tilde{\mathcal{T}}}(X)$ which was defined in the previous section.

*If $X$ is compact, then*

$$[width_{\tilde{\mathcal{T}}}(X) = \infty] \Leftrightarrow [X \text{is iso-enlargeable}].$$

In fact, the (quantitative form of the obvious) implication "$\Leftarrow$" has been already established the previous section.

Now, to prove "$\Rightarrow$", we observe that the maps $f : V \to \mathbb{T}^{n-1} \times [-1, 1]$ used in the definition of "over-torical" can be assumed Lipschitz, where, moreover, the corresponding maps (coordinate projections) $X \to [-1, 1]$ can be arranged to have their Lipschitz constants equal to

$$\frac{2}{width(V)}.$$

These $f$, by passing to the $\mathbb{Z}^{n-1}$-coverings $\tilde{V} \to V$, become Lipschitz maps $\tilde{f} : \tilde{V} \to \mathbb{R}^{n-1} \times [-1, 1]$. which, by scaling $\varepsilon : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, turn to maps

$$\tilde{f}_\varepsilon : \tilde{V} \to \mathbb{R}^{n-1} \times [-1, 1]$$

[5] A manifold is called *aspherical* if its universal covering is contractible.

with Lipschitz constants arbitrarily close to $\frac{2}{width(V)}$ and which remain proper with degrees $\neq 0$.

Finally, we compose these $\tilde{f}_\varepsilon$ with the obvious map $\mathbb{R}^{n-1}\times[-1,1]\to S^n$ of degree one and $Lip=\pi$ and obtain maps

$$\tilde{F}_\varepsilon:\tilde{V}\to S^n, \text{ where } deg(\tilde{F})\neq 0 \text{ and } Lip(\tilde{F})\leq \tfrac{2\pi}{width(V)}+\varepsilon'$$

with arbitrarily small $\varepsilon'$, and the implication

$$[width_{\hat{\mathcal{T}}}(X)=\infty]\Rightarrow[X\text{is iso-enlargeable}].$$

is thus established.

*Definition of $\mathcal{V}$-width and $\mathcal{IE}$-Width.* Given a class $\mathcal{V}$ of Riemannian bands $V$ define $width_{\mathcal{V}}(X)$ of a Riemannian manifold $X$ as we did it for $width_{\hat{\mathcal{T}}}$, namely, as

the supremum of numbers $d$, such that $X$ admits an equidimensional locally isometric (not necessarily globally one-to-one) immersion from a band $V\in\mathcal{V}$ with $width(V)=d$.

Here, we are concerned with the class of *iso-enlargeable orientable bands* $V$ which admits proper maps (i.e. boundary to boundary) $f:V\to Y\times[-1,1]$, where $Y$ must be compact orientable iso-enlargeable manifolds without boundaries and where $deg(f)\neq 0$.

**Iso-enlargeable $\frac{4\pi}{n}$-Inequality.** *The iso-enlargeable widths of $n$-dimensional Riemannian manifolds $X$ are bounded by the over-torical widths as follows.*

$$width_{\hat{\mathcal{T}}}(X)\leq width_{\mathcal{IE}}(X)\leq 2width_{\hat{\mathcal{T}}}(X).$$

*Consequently, if $Sc(X)\geq\sigma>0$, then*

$$width_{\mathcal{IE}}(X)\leq 4\pi\sqrt{\frac{n-1}{\sigma n}}.$$

*Proof.* The inequality $width_{\hat{\mathcal{T}}}\leq width_{\mathcal{IE}}$ is obvious.

To prove $width_{\mathcal{IE}}\leq 2width_{\hat{\mathcal{T}}}$ let us show that iso-enlargeable bands $V$ with width $d$ contain over-torical ones with width $d/2$.

In fact, since the above $Y$ is iso-enlargeable, there exist locally isometric immersions of $(n-1)$-dimensional over-torical bands $Y_D$ to $Y$ with $width(Y_D)\geq D$ for all $D>0$. Then the pullbacks[6] of $Y_D\times[-1,1]$ under the maps $f:V\to Y\times[-1,1]$ come with natural maps

$$f^{-1}(Y_D\times[-1,1])\to[0,D] \text{ and } f^{-1}(Y_D\times[-1,1])\to[0,d],$$

both with $Lip\leq 1$.

Then the pull back of the circle of radius $d/2$ in $[0,D]\times[0,d]$ under the pair of these maps (which may be assumed smooth and transversal to this circle) serves as the required overtorical band of width $\geq d/2$.

[6]Even if $g:A\to B$ is a non-injective map, we speak of the $f$-pullback of $A$ for a map $f:C\to B$, where $f^{-1}(A)$ is understood as the set of pairs $\{a,c\}_{g(a)=f(c)}\in A\times C$ which comes with the map $f^{-1}(A)\to C$, $(a,c)\mapsto c$ which has the same kind of (non)-injectivity as $g:A\to B$.

Finally, we recall the $\frac{2\pi}{n}$-inequality for over-torical bands in section 2 and obtain our $\frac{4\pi}{n}$-inequality for iso-enlargeable bands.

Iso-enlargeable $[\asymp \frac{8\pi^2}{R^2}]$ -Decay Theorem. Let a manifold $X$ admits a proper map of non-zero degree to the total space $\underline{X}$ of a two dimensional vector bundle $\underline{X} \to Y$ where $Y$ is a compact iso-enlargeable (e.g. admitting a metric with non-positive curvature) manifold.

*If the bundle $\underline{X} \to Y$ is trivial then the scalar curvatures of all complete Riemannin metrics $g$ in $X$ restricted to concentric balls $B(R) = B_{x_0}(R) \subset X$ satisfy*

$$[\asymp \tfrac{8\pi^2}{R^2}] \quad \min_{B(R)} Sc(X) \leq \frac{8\pi^2}{(R-R_0)^2} \text{ for some } R_0 = R_0(X,g,x_0) \text{ and all } R \geq R_0.$$

*Proof.* This follow word for word the argument for the quadratic decay theorem in section 1 and its generalisation in section 2 with "iso-enlargeable " for "over-torical".

What happens to nontrivial bundles $\underline{X} \to Y$? The above argument applies to non-trivial bundles, where the (total spaces of the) corresponding *circle bundles are iso-enlargeable*, which is so, for instance by the above (c) for $Y$ which admit metrics with non-positive sectional curvatures.

In general, the examples in [BH 2009] indicate a possibility of non-enlargeable circle bundles over enlargeable $Y$; yet, it seems hard(er) to find such examples, where the corresponding $\underline{X}$ would admit complete metrics with $Sc \geq \sigma > 0$.

❒ **From $\frac{4\pi}{n}$ to $\frac{2\pi}{n}$.** The $\frac{4\pi}{n}$-inequality for *compact iso-enlargeable bands* $V$ can be improved to the following *sharp* one:

$$\left[\mathbf{O}^{\mathcal{IE}}_{\pm} \leq 2\pi\sqrt{\tfrac{n-1}{\sigma n}}\right] \qquad width_{\mathcal{IE}}(V) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}.$$

But our argument presented below is more demanding on the geometry of minimal hypersurfaces than that for $\frac{4\pi}{n}$ and it needs extra analysis for $n \geq 8$.

We claim $\frac{2\pi}{n}$ in the $\mathcal{IE}$-inequality only for $dim(V) \leq 7$. [7]

*Proof of* $\left[\mathbf{O}^{\mathcal{IE}}_{\pm} \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}\right]$. Start by observing that this inequality trivially reduces to the inequality $\left[d_0(❒) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}\right]$ stated below as follows.

**$\frac{2\pi}{n}$-Inequality for Wide Over-Cubical Domains.** Let $V$ be a compact $n$-dimensional orientable Riemannian manifold with a boundary and let $\phi_i : V \to [0, d_i]$ be continuous functions, such that the following conditions be satisfied.

- The functions $\phi_i$ are 1-Lipschitz,

$$Lip(\phi_i) \leq 1, \; i = 0, 1, ...n-1.$$

- The map $\Phi = (\phi_0, \phi_1, ..., \phi_{n-1})$ from $V$ to the solid ❒ = ❒$(d_i) = \times_i [0, d_i]$,

$$\Phi : V \to ❒ = [0, d_0] \times [0, d_1] \times ... \times [0, d_{n-1}],$$

[7]It seems to work for $n = 8$ but since I have not written down the proof – this may take, as I see it, 5-10 pages – I can't be 100% certain.

has non-zero degree.

$$deg(\Phi) \neq 0.$$

$\star$ The sectional curvatures of $V$ are bounded in absolute values by

$$|\kappa(V)| \leq \kappa_0 \text{ for some constant } \kappa_0 \geq 0.$$

*If*

$$d_0 << d_1 << d_2 << ... << d_{n-1},$$

*say if,*

$$d_i \geq C^{1+|\kappa|d_{i-1}}$$

*then height of* ❒*, that is the size of smallest edge is bounded in terms of the minimum* $\sigma = \min_{v \in V} Sc(V)(v)$ *of the scalar curvature of* $V$*,*

*by*

$\left[d_0(❒) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}\right]$

$$d_0 = d_0(❒) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}} + \varepsilon,$$

where

$$\varepsilon = \varepsilon_n(C) \to 0 \text{ for } C \to \infty.$$

*Proof.* Let $V_i \subset V$ denote the $\Phi$-pullbacks of the 0-faces of ❒$(d_i)$, i.e.

$$V_i = \phi_i^{-1}(0),$$

let

$$f_i : V \to \mathbb{R}_+$$

be the distance functions to $V_i$

$$f_i(v) = dist_V(v, V_i)$$

and let $F$ be the resulting map

$$F = (f_0, f_1, ..., f_{n-1}) : V \to \mathbb{R}^n_+ = [0, \infty)^n.$$

Denote by $W \subset V$ the $F$-pullback of our solid,

$$W = F^{-1}(❒) \text{ for } ❒ = \times_i[0, d_i] \subset [0, \infty)^n$$

and observe that the map $F$ restricted to $W$,

$$F_{|W} : W \to ❒$$

sends $\partial W \to \partial$❒ and it has the same (non-zero!) degree as $\Phi$,

$$deg(F_{|W}) = \deg \Phi \neq 0.$$

Now, let us symmetrise $V$ similarly to how it was done in section 2.

To perform the first step of such symmetrization we need a minimising hypersurface $Y \subset V$ which would separate the faces $f_{n-1}(0)$ and $f_{n-1}(d_{n-1})$ in $W$.

In general, such a $Y$ may intersect these faces, but since for large $C$

$$d_{n-1} > const \cdot \sum_{i<n-1} d_i$$

where, besides being large in the absolute terms, this *const* is also large compared to the sectional curvature of $W$, one can slightly perturb the metric in $W$ such that the minimising hypersurface $Y_1 \subset W$ with respect the perturbed metric would stay away from $f_{n-1}(0)$ and $f_{n-1}(d_{n-1})$.

Then we can symmetrize the perturbed $W$ and arrive at a $\mathbb{T}^1$-symmetric Riemannin manifold $W_1$, which comes with a $\mathbb{T}^1$-invariant 1-Lipschitz map

$$W_1 \to \bigtimes_{i=0,\dots,n-2} [0, d_i].$$

Now, in order to proceed further with $Y_2 \subset W_1$, etc, one must show that $W_1$ satisfies the same kind of conditions as $W$, where the only non-trivial one is bound $\star$ on the sectional curvature of $W_1$.

To establish such a bound for $Y$, first, let $W$ have *bounded geometry*, which may be understood as the existence of controlled diffeomorphisms from the balls $B_w(r) \subset W$, to $\mathbb{R}^n$, where

$$r \le r_0 = \kappa_0^{-1/2} \text{ and } dist(w, \partial W) \ge 2r_0$$

and where "controlled" means that

*the differentials* of these diffeomorphisms and their inverses are bounded by a constant, say by $const = 2$ and also

*the covariant derivatives* of these diffeomorphisms up-to order $2n$ (in fact, 2 will do) are bounded by a constant, say by $\kappa_0$.

Since the $(n-1)$-volumes of the intersections of *minimising* submanifolds $Y \subset W$ of *codimension one* with $B(r)$ are bounded, by the $(n-1)$-volumes of the spheres $\partial B(r_0)$, the bound on the curvature of $Y$ follows by the standard compactness theorem for minimal varieties with bounded volumes $+$ regularity of minimal hypersurfaces for $n \le 7$.

Because of this, the symmetrisation argument proceeds without hitches and delivers a $\mathbb{T}^{n-1}$ invariant manifold $W_{n-1}$ with a (proper surjective) $(1+\varepsilon)$-Lipschitz map to $[0, d_0]$.

Then the proof in the bounded geometry case is concluded as in section 2, while the needed estimates for the general case where $|\kappa(W)| \le \kappa_0$ follow by passing to the $r$-balls $\tilde{B}_w(r) \subset T_w(W)$ with the metrics induced by the exponential maps $\exp_w : \tilde{B}_w(r) \to W$.

*Remarks and Questions.* $(\star)$ It is unclear if the above $\star$ – the bound on the curvature of $V$ – is needed. In fact, it is unneeded if we allow $4\pi$ instead of $2\pi$, where it follows from $\left[\bigcirc^{\mathcal{IE}}_{\pm} \le 4\pi\sqrt{\frac{n-1}{\sigma n}}\right]$.

$(\varepsilon = 0?)$ It is also unclear if our $\varepsilon = \varepsilon_n(C)$ is, actually, zero for large $C$.

$(\square)$ Let $V$ be an $n$-dimensional manifold with sectional curvatures $\kappa(V) \ge 1$ which admits a proper map $\Phi : V \to [0, d]^n$ given by $n$ functions $\phi_i(v)$ with $Lip(\phi_i) \le 1$ and such that $deg(\Phi) \ne 0$.

Does the maximal $d$ for these $V$ is achieved by the regular cube $\square$ in the hemisphere $S^n_+$ with the boundary $\partial\square \subset S^{n-1} = \partial S^n_+$?

$(\triangle)$ The same question for spherical simplices $\triangle \subset S^n_+$ with $\partial\triangle \subset S^{n-1}$:

*do these simplices have maximal distances between opposite faces among all simplices with $\kappa \geq 1$?*

$[\mathcal{V} \rightsquigarrow \mathcal{VE}]$: From $\mathcal{V}$-manifolds to to $\mathcal{V}$-Enlargeable ones. Given a "natural" class $\mathcal{V}$ of manifolds one defines an, a priori larger, class $\mathcal{VE}$ of $\mathcal{V}$-enlargeable manifolds $X$ by the condition

$$width_{\mathcal{V}} = \infty.$$

Thus, for instance the class $\hat{\mathcal{T}}$ of over-torical manifolds leads to the class $\hat{\mathcal{T}}\mathcal{E} \supsetneqq \hat{\mathcal{T}}$ of $\hat{\mathcal{T}}$-enlargeable manifolds, which, as we know, is equal to the class $\mathcal{IE}$ of iso-enlargeable manifolds,

On the other hand, if we depart from the class $\mathcal{IE} = \hat{\mathcal{T}}\mathcal{E}$, then the new class $\mathcal{IEE}$ defined by $width_{\mathcal{IE}} = \infty$ will coincide with $\mathcal{IE}$.

In the following section, following Schoen Yau and Schick, we define class $\mathcal{SYS} \supsetneqq \hat{\mathcal{T}}$, where the corresponding class $\mathcal{SYSE}$ of $\mathcal{SYS}$-enlargeable manifolds is *strictly greater than the class of iso-enlargeable ones.*

# 5 Schoen-Yau-Schick Manifolds and $\mathcal{SYS}$-Bands.

*Schoen-Yau Definition.* [SY-Str 1979], [SY 2017]. A compact orientable $n$-manifold is *SYS*, if there exist $n-2$ integer homology classes $h_1, h_2, ..., h_{n-2} \in H_1(X)$, such that their intersection consecutive

$$h_1 \frown h_2 \frown ... \frown h_{n-2} \in H_2(X)$$

is *non-spherical*, i.e. it is *not contained* in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$.

*Schick Definition.* [Sch 1998] A homology class $h \in H_n(K)$, where $K = K(\Pi, 1)$ is the Eilenberg-MacLane space for an Abelian group $\Pi$, is called $SYS$, if its consecutive *cap-producs* with some cohomology classes $h_1, h_2, ..., h_{n-2} \in H^1(K, \mathbb{Z})$ are non-zero,

$$(...((h \cap h_1) \cap h_2) \cap .... \cap h_{n-2}) = h \cap (h_1 \smile, ..., \smile h_{n-2}) \neq 0 \in H_2(K).$$

(Geometrically speaking, generic 2-dimensional intersections of the $n$-cycles $C \subset K$ representing $h$ with $(n-2)$-codimensional pullbacks of generic points of, some, say piecewise linear, maps $K \to \mathbb{T}^{n-2}$ are non-homologous to zero.)

Then a manifold $X$ is SYS if the Abel classifying map $X \to K(\Pi, 1)$ for $\Pi = H_1(X)$ sends the fundamental class $[X] \in H_n(X)$ to a SYS class in this $K(\Pi, 1)$.

(Recall that, by definition, the spaces $K(\Pi, 1)$ have *contractible* universal coverings and fundamental groups *isomorphic* to $\Pi$. The standard finite dimensional approximations to these $K$ are products of tori and *lens spaces* $L_i = S^N/\mathbb{Z}_{l_i}$, where the latter, observe, carry natural metrics with $Sc > 0$.

Abel's $X \to K$ maps are uniquely up-to homotopy, are characterised by inducing isomorphisms on the 1-dimensional homology groups.)

*Historical Remark.* In 1979 Schoen and Yau proved that SYS manifolds (defined slightly differently in [SY 1979] with incorporation of some spin manifolds) of dimensions $n \leq 7$ *carry no metrics with* $Sc > 0$. Then, in the recent paper [SY 2017], they published the proof for all $n$.

Meanwhile, Schick [Sch 1998] has shown that *no available Dirac operator methods can rule out $Sc > 0$ on these manifolds.*

## Examples.

$\bullet_1$ Overtorical manifolds are $SYS$.

$\bullet_2$ Let $X$ be obtained by a 2D-surgery applied to a closed curve $C$ in the $n$-torus.

If $n \geq 4$, then $X$ is SYS if and only if $C$ represents a *divisible* homology class in $H_1(\mathbb{T}^n)$.

(Such an $X$ is over-torical if and only if $C$ is *homologous to zero.*)

$\bullet_3$ If a compact orientable manifold $X$ admits a map $f$ of degree one to a SYS manifold that $X$ is SYS.

But if $deg(f) > 1$ then $X$ is not necessarily SYS, unlike the case of the overtorical and iso-enlargeable manifolds. For instance if the curve $C$ in $\bullet_2$ is $m$-divisible, than the some $m$-sheeted covering of $X$ is non-SYS.

Probably, these non-SYS coverings carry metrics with $Sc > 0$.

$\bullet_4$ Products of SYS manifolds by overtorical ones are SYS.

But products SYS $\times$ SYS and SYS $\times$ [iso-enlargeable] are, in general, not SYS.

*SYS-Bands.* A band $V$ is called $SYS$ if it admits a band map ($\partial_\pm \to \partial_\pm$) of degree $\pm 1$ to $Y \times [-1, 1]$ where $Y$ is a compact SYS manifold.

Accordingly, define the $\mathcal{SYS}$-width of $width_{\mathcal{SYS}}(X)$ of Riemannian manifolds $X$ based on the class $\mathcal{SYS}$ as we did it for $\mathcal{IE}$ in the previous section.

**$\frac{4\pi}{n}$-Inequality for SYS-Bands.** *All Riemannin manifolds $X$ with $Sc(X) \geq \sigma > 0$ satisfy*

$$width_{\mathcal{SYS}}(X) \leq 4\pi\sqrt{\frac{n-1}{\sigma n}}.$$

*Consequently, compact manifolds without boundaries, which have*

$$width_{\mathcal{SYS}}(X) = \infty$$

*admit no metrics with positive scalar curvatures.*

Proof. By symmetrising a $SYS$-band $V \to Y \times [-1, 1]$ as in the proof of the overtorical $\frac{2\pi}{n}$-Inequality in section 2 (now $Y$ plays the role of the torus $\mathbb{T}^{n-1}$ in section 2) we arrive at $V_{\circ^{n-3}}$ with $\mathbb{T}^{n-3}$-invariant metric with $Sc \geq \sigma$, such that

*the quotient space $\underline{V}^3 = V_{\circ^{n-1}}/\mathbb{T}^{n-3}$ is an orientable 3-manifold with the boundary decomposed into two (possibly disconnected) disjoint parts say*

$$\partial \underline{V}^3 = S_- \cup S_+,$$

*where*

$$dist_{\underline{V}^3}(S_-, S_+) \geq d$$

*for $d$ equal to the distance between the two boundary components in $V$,*

and where the Schoen-Yau-Schick property of $Y$ implies that

*if a closed surface $S \subset \underline{V}^3$ separates $S_-$ from $S_+$, then the homomorphism*

$$\pi_1(S) \to \pi_1(\underline{V}^3)$$

*has infinite image.*

Therefore the $d/2$-equidistance surface to $S_-$ (or to $S_+$) contains a circle $C$ which has infinite order in $\pi_1(\underline{V}^3)$ and, by the Poincaré duality, the covering $\tilde{\underline{V}}^3$ of $\underline{V}^3$ with the cyclic $\pi_1(\tilde{\underline{V}}^3)$ generated by the (homotopy class of) $C$ contains a relative 2-cycle $\tilde{C}^\perp$ [8] with non-zero intersection index with the lift $\tilde{C}$ of $C$ to $\tilde{\underline{V}}^3$.

Take the pull back of the cycle $\tilde{C}^\perp$ to the corresponding covering $\tilde{V}_{\circ^{n-3}}$ of

$$V_{\circ^{n-3}} = (V_{\circ^{n-3}}/\mathbb{T}^{n-3}) \times \mathbb{T}^{n-3},$$

write this pullback cycle as

$$\tilde{C}^\perp \times \mathbb{T}^{n-3} \subset \tilde{V}_{\circ^{n-3}},$$

and symmetrize the minimal cycle in the $(n-1)$-homology class of $\tilde{C}^\perp \times \mathbb{T}^{n-3}$.

Since $dist(\tilde{C}, \partial\tilde{\underline{V}}^3) = dist(C, \partial\underline{V}^3) \geq d/2$, the quotient surface of the resulting $\tilde{V}_{\circ^{n-2}}$ contains a point within distance $\geq d/2$ from its boundary, which implies (compare p. 310 in [GL 1983]) that

$$d/2 \leq 2\pi\sqrt{(n-1)/\sigma n}.$$

QED.

**Question.** What should be done to replace the above $4\pi\sqrt{\frac{n-1}{\sigma n}}$ by $2\pi\sqrt{\frac{n-1}{\sigma n}}$?

# 6 $\mathcal{SYS}$-Enlargeable Manifolds and Codimension Two Depth Inequalities.

A Riemannin manifold $X$ is called $\mathcal{SYS}$*-Enlargeable* if it has *infinite* $\mathcal{SYS}$-width.

For instance, SYS manifolds and iso-enlargeable manifolds are $\mathcal{SYS}$*-Enlargeable.*

What is more interesting is that the

*if an $n$-manifold $X$ admits a proper Lipschitz map $\phi$ (Lipschitz* means $Lip(\phi) < \infty$*) to an iso-enlargeable manifold of dimension $n-2$, say $\phi : X \to \underline{X}$, such that the homological pullback $\phi^![\underline{x}] \in H_2(X)$, $[\underline{x}] = 1 \in H_0(\underline{X}) = \mathbb{Z}$, is non-spherical (as in the first definition of SYS in the previous section), then $X$ is* $\mathcal{SYS}$*-enlargeable.*

Therefore, by the above $\frac{4\pi}{n}$-inequality,

*If such an $X$ is compact, then it admits no metric with $Sc > 0$.*

Thus, for example,

*products $X$ of SYS manifolds by compact iso-enlargeable ones (e.g. those which admit metrics with $\kappa(X_2) \leq 0$) admit no metrics with positive scalar curvatures.*

(These $X$, in general, are neither iso-enlargeable nor SYS.)

$\frac{8\pi}{n}$**-Inequality for SYSE-Bands.** Denote by $\mathcal{SYSE}$ the class of SYS-enlargeable manifolds, say that a compact band $V$ is SYSE if it admits a map of degree

[8] Relative means relative to $\partial\tilde{\underline{V}}^3 + \partial_\infty\tilde{\underline{V}}^3$ where $\partial_\infty$ stands for the complement of a large ball in $\tilde{\underline{V}}^3$.

$\pm 1$ to $Y \times [-1,1]$, where $Y$ is SYSE and accordingly define $width_{SYSE}(X)$ for Riemannian manifold $X$ (see $[\mathcal{V} \rightsquigarrow \mathcal{VE}]$ in section 4).

Then by arguing as for iso-enlargeable $\frac{4\pi}{n}$-inequality in section we conclude that

$$width_{\mathcal{SYS}}(X) \leq width_{\mathcal{SYSE}}(X) \leq 2width_{\mathcal{SYS}}(X)$$

for all Riemannin manifolds $X$.

Consequently,

*if* $Sc(X) \geq \sigma > 0$ *then*

$$width_{\mathcal{SYSE}}(X) \leq 8\pi\sqrt{\frac{n-1}{\sigma n}}.$$

*Remark/Question.* Probably, arguing as in ❐ of section 4 one can improve $8\pi$ to $4\pi$, but getting $2\pi$ is less apparent.

*Depth Inequalities.* Define *the depth of a homology class* $h$ in a Riemannin manifold $X$ with boundary as the supremum of $d \geq 0$ such that $h$ can be represented by a cycle positioned within distance $\geq d$ from the boundary of $X$ . (If $X$ is non-complete, we include the points obtained by completion of $X$ in the boundary of $X$.)

Let $Y$ be a closed $(n-2)$dimensional manifold and $p : \underline{X} \to Y$ be a the disc bundle, e.g. e.g. the trivial one $\underline{X} = Y \times B^2$.

Let $X$ be a compact $n$-manifold with boundary and $f : X \to \underline{X}$ be a proper continuous map where *proper*, means *boundary* $\to$ *boundary*. Let $h = f^!([Y] \in H_{n-2}(X)$ be the homology pull-back of the homology class of the zero section $Y = Y_{\mathbf{0}} \subset \underline{X}$.

Let $\underline{X}_{-\varepsilon} \subset \underline{X}$ be the complement of the open $\varepsilon$-neighbourhood of $Y_{\mathbf{0}}$ in $\underline{X}$ and observe that the boundary of $\underline{X}_{-\varepsilon}$ consists of two components, call them $\underline{\partial}_{\pm}$ which are canonically homeomorphic to the total space of the circle bundle associated to $\underline{X} \to Y$, denoted $p_\circ : Y_\circ \to Y$.

Let $\partial_{\pm} = \partial_{\pm}(X) \subset \partial X$ be the two parts of the boundary of $X$ which are sent by the map $f : X \to \underline{X}$ to $\underline{\partial}_{+}$ and to $\underline{\partial}_{-}$ correspondingly.

Observe that

- $\underline{X} = Y_\circ \times [\varepsilon, 1]$;
- If $Y$ is iso-enlargeable then $Y_\circ$ is also iso-enlargeable.
- if the fibration $p : \underline{X} = Y$ is trivial, $\underline{X} = Y \times B^2$, and

if $Y$ is over-toric then also $Y_\circ$ is over-toric,

if $Y$ is SYS then $Y_\circ$ is also SYS,

if $Y$ is SYSE then $Y_\circ$ is also SYSE.

Now

*let the fibration* $p : \underline{X} \to Y$ *be trivial* and let

$$Sc(X) \geq n(n-1) = Sc(S^n).$$

Observe that the band-width $\frac{k\pi}{n}$-inequalities, $k = 2, 4, 8$, (see sections ???) imply the following bounds on the depths of $h \in H_{n-2}(X)$ by the argument that we have already used several times, e.g. in the proofs of the quadratic decay inequalities (see sections ???,///).

$[\hat{\mathcal{T}}]_\circ$ *If* $Y$ *is over-torical, i.e. if it admits a map to the torus* $\mathbb{T}^{n-2}$ *with degree* $\neq 0$, *then*

$$depth(h) \leq \frac{2\pi}{n}.$$

This is the only case where our inequality is (known to be) sharp,

$[\mathcal{IE}]_\circ$ *If $Y$ is iso-enlargeable, e.g. if it admits a metric with non-positive sectional curvature, then*

$$depth(h) \leq \frac{4\pi}{n}.$$

(Here the fibration $p$ need not be trivial.)

$[\mathcal{SYS}]_\circ$ *If $Y$ is SYS and if the map $f : X \to \underline{X}$ has $deg(f) = \pm 1$, then*

$$depth(h) \leq \frac{4\pi}{n}.$$

(The simplest example of a non-overtoric SYS manifold $Y$ for $n-2 \geq 4$ is obtained from the $(n-2)$-torus by attaching a 2-handle based on a $k$-multiple of closed curve in this torus where $k \neq \pm 1$. In this case one only need $deg(f)$ to be non-divisible by $k$.)

$[\mathcal{SYSE}]_\circ$ *If $Y$ is SYSE and if the map $f : X \to \underline{X}$ has $deg(f) = \pm 1$, then*

$$depth(h) \leq \frac{8\pi}{n}.$$

(Recall, this was stated earlier, here as everywhere in this paper the above inequalities are established unconditionally for $n \leq 8$, while the case $n \geq 9$ relies on the recent partial regularity results by Lohkamp and by Schoen and Yau which the present author have not studied in detail.)

*On nontrivial bundles $p : \underline{X} \to Y$.* Here, similarly to where we addressed this issue in section 4, one may drop the triviality of $p$ assumption, if, for instance, $Y$ admits a metric with $\kappa \leq 0$.

No reasonable assumption of this kind, however, seems in view for SYS and SYSE manifolds.

In fact, circle bundles over many SYS manifolds, say on those obtained by 2D-surgery from $\mathbb{T}^n$ (see section 5) are very likely to carry metrics with $Sc > 0$ and so the above inequality can't hold with any constant for non-trivial fibrations $p : \underline{X} \to Y$.

*On Complete Manifolds and Dirac Operators.* The inequality $depth(h) < \infty$ implies that the interiors of the manifolds $X$ in $[\mathcal{IE}]_\circ$ and $[\mathcal{SYSE}]_\circ$

*admit no complete metrics $g$ with $Sc(g) \geq \sigma > 0$.*

(The inequality $depth(h) < \infty$ in the remaining cases follow from these two.)

Strangely enough, even if $X$ is spin, this was proven by the Dirac operator methods for *enlargeable and related* manifolds $Y$ [GL1983], [HPS2015] *only under additional geometric assumptions* on $X$ in spirit of "bounded geometry".

(To be honest, I am not 100% certain this is the case for [HPS2015] . The main result is stated in this paper for closed manifolds and I have not followed the proofs in sufficient details to understand what is actually proven there for complete non-compact manifolds.)

*Question. Do all products manifolds $Y \times \mathbb{R}^2$, and, more generally, the total spaces of all $\mathbb{R}^2$-bundles admit complete metrics $g$ with $Sc(g) \geq 0$?*

*Do, for example, such metrics $g$ exist for compact manifolds $Y$ which admit metrics with strictly negative sectional curvatures?*

If there are no such $g$ among rotationally symmetric warped product metrics,[9] then, probably, no complete metric $g$ on $Y \times \mathbb{R}^2$ has $Sc(g) \geq 0$, where the best candidates of this kind of manifolds with no complete metrics on them with $Sc \geq 0$ are non-trivial $\mathbb{R}^2$-bundles over surfaces of genera $\geq 2$.

# 7 External Curvature, Focal Radius and Depth in Codimension> 2

Observe that by Gauss theorema ergergium the scalar curvature of hypersurfaces $Y \subset S^n$, $n \geq 2$, with principal curvatures $c_i = c_i(y)$, $y \in Y$, $i = 1, ..., n-1$, satisfies

$$Sc(Y) = Sc(S^{n-1}) + \left(\sum_i c_i\right)^2 - \sum_i c_i^2 \geq (n-1)(n-2) - \sum_i c_i^2.$$

It follows that if an $(n-1)$-dimensional manifold manifold $Y$ *admits no metric with* $Sc > 0$, that the suprema of the principal curvatures of all smooth immersions from $Y$ to the unit sphere $S^n$ satisfy

$$\sup_{i,y} |c_i(y)| \geq \sqrt{n-2}$$

This is significantly weaker then the $\frac{\pi}{n}$-inequality for the normal radius of $\mathbb{T}^{n-1} \subset S^n$, which implies that $\sup_{i,y} c_i(y) \geq \frac{(1+\varepsilon_n)n}{\pi}$. But it applies to such manifolds, for instance, as certain exotic spheres $Y$ of dimensions $8m+1$ and $8m+2$ which carry no metrics with $Sc > 0$ by a theorem of Hitchin [H1974], yet are immersible (but not embeddable!)[10] to $S^n$ by Smale-Hirsch theorem.

Besides, this $\sup c_i$ inequality obviously generalises to $Y$ in $S^n$ of all codimensions $k$ where it reads

$$\sup_{i,j,y} |c_{i,j}(y)| \geq \frac{\sqrt{n-k-1}}{k}, \ i = 1, ..., n-1, \ j = 1, ..., k, \ y \in Y,$$

for all $Y$ which admit no metric with $Sc > 0$.

Then, obviously, the same holds true for Riemannin manifolds $X \supset Y$ with *sectional curvatures* $\kappa \geq 1$.

More interestingly, a similar inequality holds for immersions to unit Euclidean balls $B(1) \subset \mathbb{R}^n$ Namely,

*if an $(n-k)$ dimensional $Y$ admits no metric with $Sc > 0$, then the principal curvatures of all smooth immersions $Y \to B(1) \subset \mathbb{R}^n$ are bounded from below by*

$$\sup_{i,j,y} |c_{ij}(y)| \geq \frac{1}{const} \frac{\sqrt{n-k-1}}{k}$$

*for some universal positive constant $const \leq 100$.*

[9] Figuring this out does not seem hard, but I have not tried doing this.

[10] According to textbooks' terminology, a smooth map $A \to B$ is an *immersion* if it is *locally* one-to-one and the inverse map is smooth, while *embeddings* are immersions which are globally one-two-one and, if $Y$ is non-compact, are additionally required to be homeomorphisms from $Y$ to their (possibly, non-closed in B) images.

*Proof.* The Euclidean case reduces to the spherical one, since the standard projective map $\mathbb{R}^n \supset B(1) \to S^n$ distorts curvatures of the curves in $B(1)$ by a bounded amount. (Compare Lemma (C′) in 3.2.3 in [Gr 1986]).

*Remark.* This $\sup c_{i,j}$-inequality also holds in the balls in the hyperbolic spaces with sectional curvature $\kappa = -1$.

Also, the following weaker form of this inequality holds for the unit balls in all $n$-dimensional Riemannin manifolds $X$ with $-1 \le \kappa(X) \le 1$.

$$\sup_{i,j,y} |c_{ij}(y)| \ge \frac{1}{const} \frac{\sqrt{n-k-1}}{k} - const'.$$

In fact – this is obvious by the to-day's standards – the exponential maps $exp : T_x(X) \subset B(1) \to X$ in these $X$ can be approximated by maps with controlled distortion of curvatures of the curves in $B(1)$.

*Discussion.* There is a huge gap between the above lower bounds on the curvatures of submanifolds in $S^n$ (and/or in $B(1) \subset \mathbb{R}^n$) and the observed curvatures in the available examples $Y \subset S^n$.

Probably, certain homogeneous submanifolds $Y \subset S^n$, such as

• real and complex projective spaces *Veronese* represented by symmetric/Hermitian forms of rank one,

• Grassmannians *Plücker* embedded to exterior powers of linear spaces,

• the same Grassmannians represented by projectors in spaces of operators,

give a fair idea of embeddings with economical $c_{ij}$.

For instance, the curvature of the obvious embedding of the product of spheres

$$Y = S^{n_1} \times S^{n_2} \times ... \times S^{n_k} \subset S^{n_1+n_2+...+n_j+j-1} = \partial B(1) \subset \mathbb{R}^{n_1+n_2+...+n_j+j}$$

has $\max c_{ij} = \sqrt{k}$ and it is plausible (?) that

*no embedding/immersion of this $Y$ to $S^n$ may have a (significantly) smaller curvatures $c_{ij}$.*

Notice that above local bound $\max c_{ij} \gtrsim \sqrt{n}/k$ is non-vacuous only if all spheres are one dimensional, while the only known improvement of this bound is the inequality $\max c_{ij} \gtrsim n$ which was established in the previous section only for codimensions 1 and 2 and only for $\mathcal{SYSE}$-manifolds $Y$ (e.g. for $Y$ which admits metrics with $\kappa \le 0$.)

This, for instance, leaves the following questions open.

(a) *Does the torus*

$$\underbrace{S^1 \times S^1 \times ... \times S^1}_{n-3}$$

*embed to $S^n$ with the principal curvature $c_{ij} \le 100/n$?*

(b) *Does the product*

$$\underbrace{S^1 \times S^1 \times ... \times S^1}_{n-3} \times S^2$$

*embed to $S^n$ with the principal curvature $c_i \le 10$?*

In fact, we are more interested in *depth of homology and cohomology classes* in Riemannin manifolds $V$ rather than in their curvatures, where, *by definition*, *depth*$(h) \geq d$ for an $h \in H^*(V)$ if *the restriction of* $h$ *to the subset* $V_{-d} \subset V$ of the points within distance $\geq d$ from the boundary of $V$ (including the infinity for non-compact $V$, as in the previous section) *does not vanish.*

Problem. *Bound "complexity" of an* $h$ *in terms of* $d = depth(h)$.

For instance, let the sectional curvature of $V$ be bounded from below by $\kappa(V) \geq 1$ and let $h$ be induced by a continuous map from the fundamental cohomology class of a product of spheres,

$$h = f^*[Y] \text{ for } f : V \to Y = S^{n_1} \times S^{n_2} \times ... \times S^{n_k}.$$

*Does the depth of* $h$ *necessarily tend to zero for* $k \to \infty$*?*

# 8 Symmetrization by Reflections with Point-wise Control of the Scalar Curvature.

Torical symmetrization of bands $V$ from section 2 can be performed with point-wise control of the scalar curvatures of $V$ along with the mean curvatures of their boundaries $\partial V$ as follows.

**Definition:** $\mathbb{R}^{n-1} \rtimes O(n-1)$**-Symmetrisable Bands.** Let $V$ be an *n-dimensional Riemannian band*, i.e. a Riemannian manifold with two non-empty disjoint subsets in its boundary,

$$\partial_-(V) \sqcup \partial_+(V) \subset \partial(V).$$

For instance, cylinders $B \times [-1, +1]$ are bands for $\partial_\pm = B \times \{\pm 1\}$.

Write the full isometry group of $\mathbb{R}^{n-1}$ as the semidirect product $\mathbb{R}^{n-1} \rtimes O(n-1)$ and say that a band $V$ is $\mathbb{R}^{n-1} \rtimes O(n-1)$*-symmetrisable* if there exist a manifold (open band) $\hat{V}$ homeomorphic to $\mathbb{R}^{n-1} \times [-1, 1]$ with $\mathbb{R}^{n-1} \rtimes O(n-1)$-invariant Riemannian metric $\hat{g}$ and $\mathbb{R}^{n-1}$-invariant map $\Upsilon : \hat{V} \to V$ with the following properties.

(i) $\Upsilon$ sends $\mathbb{R}^{n-1} \times \{-1\}$ to $\partial_-(V)$ and $\mathbb{R}^{n-1} \times \{1\}$ to $\partial_+(V)$.

(ii) $\Upsilon$ is distance non-increasing,

$$Lip(\Upsilon) = \sup_{\underline{v}_1 \neq \underline{v}_2} \frac{dist_V(\Upsilon(\hat{v}_1)\Upsilon(\hat{v}_2))}{dist_g(\hat{v}_1, \hat{v}_2)} \leq 1.$$

(iii) $\Upsilon$ is scalar curvature non-increasing

$$Sc(V)(\Upsilon(\hat{v})) \leq Sc(\underline{g})(\hat{v}) \text{ for all } \hat{v} \in \hat{V}.$$

(iv) $\Upsilon$ is mean curvature non-increasing,

$$mean.curv(\partial V)(\Upsilon(\hat{v})) \leq mean.curv_g(\partial\hat{V})(\underline{v}) \text{ for all } \hat{v} \in \partial\hat{V} = \{-1\}\times\mathbb{T}^{n-1}\cup\{1\}\times\mathbb{T}^{n-1},$$

where, our sign convention is such that *convexity* of domains, say in $\mathbb{R}^n$, corresponds to *positivity* of the mean curvatures of their boundaries.

**Symmetrization of Overtorical, Iso-Enlargeable and Subrectangular Bands.** Recall the following.

(1) A band is called *proper* if $\partial_-(V)$ and $\partial_+(V) = \partial(V)$ are unions of connected components of $\partial V$ and

$$\partial V = \partial_-(V) \cup \partial_+(V).$$

(2) A compact orientable proper band $V$ is called *overtorical* if $V$ admits a map $f : \mathbb{T}^{n-1} \times [-1, 1]$ such that

$$\partial_\pm(V) \to \mathbb{T}^{n-1} \times \{\pm 1\} \text{ and } deg(f) \neq 0.$$

(3) A proper band $V$ is called *isoenlargeable* if there exist orientable coverings $\tilde{V}_i \to V$ and continuous maps $\phi_i : \tilde{V}_i \to [-1, 1]$ and $f_i : \tilde{V}_i \to S^{n-1}$, $n = dim(V)$, such that

• $\phi_i : \partial_\pm \tilde{V}_i \to \{\pm 1\}$;

• the maps $f_i$ are constant at infinity, i.e. constant outside compact subsets $W_i \subset \tilde{V}_i$;

• the maps

$$F_i = (f_i, \phi_i) : \tilde{V}_i \to S^{n-1} \times [-1, 1]$$

have

$$deg(F_i) \neq 0;$$

• the Lipschitz constants of $f_i$ satisfy

$$Lip(f_i) \to 0 \text{ for } i \to \infty.$$

(4) A (non-proper)compact orientable band $V$ is called *subrectangular* if

• it is carries an $n$-cubical corner structure on the boundary a continuous map

$$F = (f, \phi) : [0, 1]^{n-1} \times [-1, 1],$$

such that

$\bullet_1$ the map $F$ sends the boundary $\partial V$ to $\partial([0, 1]^{n-1} \times [-1, 1])$ and

$$deg(F) \neq 0;$$

$\bullet_2$ the function $\phi$ sends

$$\partial_\pm(V) \to \pm 1 = \partial_\pm[-1, 1];$$

$\bullet_3$ the map $F$ is smooth and it is transversal to the $(n-1)$-faces of the $n$-cube $[0, 1]^{n-1} \times [-1, 1]$ at the interior points in these faces (i.e. away from the $(n-2)$-faces).

Besides,

$\star_1$ the $(n-1)$-faces

$$V_i^0, V_i^1 \subset \partial V,\ i = 1, ..., n-1,$$

which are the $f$-pullbacks of the the $(n-2)$-faces $C_i^0, C_i^1 \subset [0, 1]^{n-1}$, (corresponding to 0 and 1 in the $i$-th $[0, 1]$ factor of $[0, 1]^{n-1}$) have non-negative mean curvature.

$\star_2$ the faces $V_i^{0,1}$ = and $V_j^{0,1}$, $i, j = 1..., n-1$, are mutually transversal at all points where they meet and the dihedral angles between them everywhere satisfy

$$\angle(V_i^{0,1}, V_j^{0,1}) \leq \pi/2.$$

**Symmetrization Theorem.** *Compact $n$-dimensional overtorical, isoenlargeable and subrectangular bands are $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetrisable.*

*Caveat.* The argument presented below is 100% unconditional in the overtorical and subrectangular cases for $n \leq 8$ and in the isoenlargeable case for $n \leq 7$.

The needed regularity of the symmetrised overtorical and subrectangular bands for $n \geq 9$ relies on the technical theorem ??? in [SY 2017] the proof and the range of applicability of which I haven't fully thought trough, while the overtorical symmetrisation needs "uniformity of regularity" which, may be (or may be not) implicitly present in the Schoen-Yau proof of their theorem???.

On the positive side, the symmetrization argument presented below increases some "weak regularity" of manifolds applies to, similarly how the *Steiner* symmetrization does, which suggest a direct approach to the proof of our theorem for all dimensions $n$.

**Main Step in Symmetrization by Reflections.** Let $X$ be a compact Riemannian manifold, with a (possibly empty) boundary, and let $(Y_0, \partial Y) \subset (X, \partial X)$ be a cooriented hypersurface which is *strictly locally volume minimising*, i.e. all sufficiently close to $Y$ hypersurfaces $(Y_0, \partial Y_0) \subset (X, \partial X)$ different from $Y_0$ satisfy

$$vol_{n-1}(Y) > vol_{n-1}(Y_0).$$

Let $U \subset X$ be a (small) neighbourhood of $Y_0$ in $X$ which is divided by $Y_0$ into two "halves", denoted $U_\pm \subset U$, and let $Y_{\pm\varepsilon} \subset U_\pm$ be hypersurfaces homologous to $Y_0$ in $U_\pm$ which minimise the functionals

$$Y \mapsto vol_{n-1}(Y) - \varepsilon \cdot vol_n(U_{\pm\varepsilon})$$

where

$$U_{\pm\varepsilon} = U(Y_{\pm\varepsilon}) \subset U_\pm$$

denote the regions bounded by $Y_{\pm\varepsilon}$ and $Y_0$.

By the basic regularity theorems of Simons-Federer-Almgran-Allard these $Y_{\pm\varepsilon}$ do exist for small $\varepsilon \geq 0$ and they are smooth away from closed subsets of Hausdorff codimension $\geq 7$.

Moreover, if $n = dim(X) = 8$, these $Y_\pm(\varepsilon)$ everywhere smooth for an open dense set of $\varepsilon > 0$ [NS 1993].

The mean curvatures of all these $Y_{\pm\varepsilon}$ at the regular points satisfies

$$mean.curv(Y_{\pm\varepsilon}) = \varepsilon$$

and $Y_{\pm\varepsilon}$ is *normal to* $\partial X$ at all regular points of $Y_{\pm\varepsilon}$ on the boundary $\partial Y_{\pm\varepsilon}$.

Let

$$U_{[-\varepsilon,\varepsilon]} = U_{-\varepsilon} \cup U_{+\varepsilon}$$

and let $\tilde{U}_{[-\varepsilon,\varepsilon]}$ be obtained by reflecting $U_{[-\varepsilon,\varepsilon]}$ in the two parts $Y_{\pm\varepsilon}$ of its boundary

$$\partial U_{[-\varepsilon,\varepsilon]} = Y_{-\varepsilon} \cup Y_{+\varepsilon}.$$

Namely, $\tilde{U}_{[-\varepsilon,\varepsilon]}$ is a space, which is acted by the semidirect product group $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}_2$, such that

- $$\tilde{U}_{[-\varepsilon,\varepsilon]}/\Gamma = U_{[-\varepsilon,\varepsilon]},$$

• there is an embedding $E : U_{[} - \varepsilon] \hookrightarrow \tilde{U}_{[-\varepsilon,\varepsilon]}$ which is inverse to the quotient map $Q : \tilde{U}_{[-\varepsilon,\varepsilon]} \to U_{[} - \varepsilon]$,

$$Q \circ E = Id : U_{[-\varepsilon,\varepsilon]} \to U_{[-\varepsilon,\varepsilon]},$$

• the group $\Gamma$ is generated by two involutions (reflections) of $\tilde{U}_{[-\varepsilon,\varepsilon]}$ which fix

$$Y_{\pm} \subset \partial U_{[-\varepsilon,\varepsilon]} \subset \tilde{U}_{[-\varepsilon,\varepsilon]}.$$

Thus, the action of our $\Gamma = \Gamma_{\varepsilon}$ on $\tilde{U}_{[-\varepsilon,\varepsilon]}$ mimics the action of the same group on the line $(\infty, \infty)$, where $\Gamma_{\varepsilon}$ is generated by the transformations

$$t \mapsto \pm\varepsilon - t$$

and where $\tilde{U}_{[-\varepsilon,\varepsilon]}$ admits a $\Gamma$-equivariant map to $(\infty, \infty)$, such that the pullback of $[-\varepsilon, \varepsilon] \subset (-\infty, \infty)$ is equal to $U_{[-\varepsilon,\varepsilon]} \subset \tilde{U}_{[-\varepsilon,\varepsilon]}$.

In particular, the action of the group $\mathbb{Z} = 2\varepsilon\mathbb{Z} \subset \Gamma$ on $\tilde{U}_{[-\varepsilon,\varepsilon]}$ is free and the quotient space is equal to the double of $U_{[-\varepsilon,\varepsilon]}$,

$$\tilde{U}_{[-\varepsilon,\varepsilon]}/\mathbb{Z} = U_{[-\varepsilon,\varepsilon]} \bigcup_{\partial U_{[-\varepsilon,\varepsilon]}} U_{[-\varepsilon,\varepsilon]}.$$

(Here the boundary $\partial U_{[-\varepsilon,\varepsilon]}$ is understood as $Y_{-\varepsilon} \cup Y_{-\varepsilon}$, while the part of the boundary coming from $\partial X$is excluded.)

If $n \leq 7$ and the manifolds $Y_{\pm\varepsilon}$ are non-singular, then the natural metric on $\tilde{U}_{[-\varepsilon,\varepsilon]}$ is a continuous piecewise smooth Riemannin one, call it$\tilde{g}_{\varepsilon}$, the derivatives of which jump at the "edges" $\Gamma Y_{\pm\varepsilon} (= \mathbb{Z} Y_{\pm\varepsilon})$ that are the $\Gamma$-translates of $Y_{\pm\varepsilon}$ in $\tilde{U}_{[-\varepsilon,\varepsilon]}$.

Since

$$mean.curv(Y_{\pm\varepsilon}) \geq 0,$$

$[\star]$ *the metric $\tilde{g}$ can be $\Gamma$-invariantly smoothed eventually without diminishing its scalar curvatures*

Namely, there are smooth metrics $\tilde{g}_{\varepsilon,\delta}$ which uniformly, i.e. $C^0$, converge to $\tilde{g}_{\varepsilon}$

$$||\tilde{g}_{\varepsilon,\delta} - \tilde{g}_{\varepsilon}|| \to 0 \text{ for } \delta \to 0$$

and such that

• the metrics $\tilde{g}_{\varepsilon,\delta}$ are equal to $\tilde{g}_{\varepsilon}$ away from the $\delta$-neighbourhood of $Y_{\pm\varepsilon}$;

• the scalar curvatures of $\tilde{g}_{\varepsilon,\delta}$ at all points $\tilde{u} \in \tilde{U}_{[-\varepsilon,\varepsilon]}$ are bounded from below, up to $\delta$, by the scalar curvatures of $\tilde{g}$ are these points,

$$Sc(\tilde{g}_{\varepsilon,\delta})(\tilde{u}) \geq Sc(\tilde{g}_{\varepsilon})(\tilde{u}) - \delta \text{ for all } \tilde{u} \in \tilde{U}_{[-\varepsilon,\varepsilon]}$$

where $Sc(\tilde{g}_{\varepsilon,\delta})(\tilde{u})$ is understood as the scalar curvature at $\tilde{u}$ of a $\Gamma$-translate of $U_{[-\varepsilon,\varepsilon]}$ which contains $\tilde{u}$.

In fact, these $\tilde{g}_{\varepsilon,\delta}$ can be obtained by suitably stretching $\tilde{g}_{\varepsilon}$ in the directions normal to the hypersurface $\Gamma Y_{\pm\varepsilon} \subset \tilde{U}_{[-\varepsilon,\varepsilon]}$.

(See [GL 1980], [Al 1985], [Gr 2014] for several versions of the geometric proof of this and see [Mi 2002] for a computation in local coordinates. Alternatively – and in some respects more satisfactory – a similar smoothing can be achieved by applying the Ricci flow to $\tilde{g}_{\varepsilon}$ [McFSzk 2011], [Bam 2015].)

Besides smoothing $Y_{\pm\varepsilon}$ at the interior points of $X \supset U_{[-\varepsilon,\varepsilon]}$ one needs to take care of the break of the second derivatives at the points in $\partial Y_{\pm\varepsilon} \subset \partial Y$, which is achieved by a simple smoothing of the boundary of $\tilde{U}_{[-\varepsilon,\varepsilon]}$, compare section 4.3 in [Gr 2014].

Let

$$\tilde{U}_{[0,0]} = \lim_{\varepsilon\to 0} \tilde{U}_{[-\varepsilon,\varepsilon]}$$

be obvious limit space, which, observe, is naturally acted by

$$\mathbb{R} = \lim_{\varepsilon\to 0} 2\varepsilon\mathbb{Z}$$

and which, if the original Riemannin manifold $X$ is $C^\infty$-smooth and if $Y_0 \subset X$ is nonsingular, is also $C^\infty$-Riemannian, despite edge singularities present in $\tilde{U}_{[-\varepsilon,\varepsilon]}$.

In fact, the hypersurfaces $Y_{\pm\varepsilon} \subset X$ for small $\varepsilon$ are graphs of smooth functions $\phi_{\pm\varepsilon}(y)$, $y \in Y_0$, where $\phi_{\pm\varepsilon}(y)$ is the length of the geodesic segment normal to $Y_0$ at $y \in Y_0$ and having its two ends on the hypersurfaces $Y_{\pm\varepsilon}$.

Then it is easy to see that suitably normalised function $\phi_{-\varepsilon} + \phi_{+\varepsilon}$ converge for $\varepsilon \to 0$ to a function $\phi > 0$ on $Y_0$,

$$\phi(y) = \lim_{\varepsilon\to 0} \frac{\phi_{-\varepsilon}(y) + \phi_{+\varepsilon}(y)}{C(\varepsilon)},$$

where one may take, for instance, $C(\varepsilon) = vol(U_{[-\varepsilon,\varepsilon]})$ and and where $\phi$ serves as a solution of a smooth linear elliptic partial differential equation with the principal term $L$ equal the second variation operator (see section 2).

Therefore $\phi$ is smooth and since $Y_0 \times \mathbb{R}$ with the metric $dy^2 + \phi(y)^2 dt^2$ is isometric to $\tilde{U}_{[0,0]}$ (this is easy to see), the limit space $\tilde{U}_{[0,0]}$ is smooth Riemannian as well.

Finally – this is the raison d'être of the symmetrization – the natural $\mathbb{R}$-invariant map

$$f : \tilde{U}_{[0,0]} \to X,$$

which is the composition of the quotient map

$$\tilde{U}_{[0,0]} \to Y_0 = \tilde{U}_{[0,0]}/\mathbb{R}$$

and the embedding

$$Y_0 \hookrightarrow X$$

is

$[\star\star]$ *scalar curvature non-decreasing,*

$$Sc(X)(f(\tilde{u})) \leq Sc(\tilde{U}_{[0,0]})(\tilde{u}).$$

In fact, this follows from the above $[\star]$ and semi-continuity of the scalar curvature under $C^0$-limits of Riemannin metrics[11] [Gr 2014], [Bam 2015].

*Proof of The Symmetrization Theorem.* If $V$ is an *overtorical* then we proceed as in section 2 with the replacement of each induction step in there by the above main step, where, if you want to keep the symmetrised bands compact, you pass from $\tilde{U}_{[0,0]}$ to $\tilde{U}_{[0,0]}/\mathbb{Z}$.

[11]The inequality $Sc(X)(f(\tilde{u})) \leq Sc(\tilde{U}_{[0,0]})(\tilde{u})$ can be also derived by an elementary computation with the PDE-equation satisfied by the function $\phi(y)$.

Thus, we obtain $\mathbb{T}^{n-1}$-invariant metrics $\underline{g}$ on $\mathbb{T}^{n-1}\times[-1,1]$ along with 1-Lipschitz maps $\Upsilon:\mathbb{T}^{n-1}\times[-1,1]\to V$ which don't increase the scalar corvettes and the mean curvatures.

Similarly, to $\mathbb{T}^{n-1}$-symmetrise *isoenlargeable bands* $V$ we consecutively apply the above main step of the symmetrization by reflections instead of the earlier used induction steps in the the proof of he inequality $\left[\bigcirc^{\mathcal{IE}}_{\pm}\leq 2\pi\sqrt{\frac{n-1}{\sigma n}}\right]$ in ❒ in section 4.

And the case of *subrectangular* $V$ reduces to that of it overtorical ones exactly as in sections 1 and 2.

Finally, we upgrade $\mathbb{T}^n$-symmetry to $\mathbb{R}^{n-1}\rtimes O(n-1)$ by $O(n)$-symmetrising $\mathbb{T}^{n-1}$-invariant bands $(\mathbb{T}^{n-1}\times[-1,1],\underline{g})$ as follows

Apply infinitely many "main steps" to the relative homology classes of sub-bands $\mathbb{T}^{n-2}\times[-1,1]\subset\mathbb{T}^{n-1}\times[-1,1]$ for all, not only the coordinate ones, subtori $\mathbb{T}^{n-2}\subset\mathbb{T}^{n-1}$ and observe that the natural action of the group $\mathbb{R}^{n-1}\rtimes O(n-1)$ on the universal covering of the limit Riemannin manifold which results from these symmetrisations is isometric. QED.

On Symmetries and Singularities. Sequential symmetrization based on all subtori $\mathbb{T}^{n-2}\subset\mathbb{T}^{n-1}$ can be performed on the original bands with *non-zero* $\varepsilon_i\to 0$, thus, delivering a sequence of smooth Riemannian bands $V_i$ which are defined even if $n\geq 8$, where the corresponding minimal hypersurfaces can be smoothed as in section 4.3 of [Gr 2014].

If $n\leq 7$ the universal covering of the resulting limit spaces are smooth $\mathbb{R}^{n-1}\rtimes O(n-1)$-symmetric Riemannian manifolds.

Conjecturally, this remains true for all $n$, where the main issue is the existence of this limit.

Possibly this can be proved by techniques of [SY 2017] and/or of [Loh 2006], [Loh 2008], [Loh 2016] or, which would be more satisfactory, by a more direct argument.

# 9 Application of Symmetrization to Manifolds with Positive and with Negative Scalar Curvatures.

Let $V$ be a Riemannian band and let $Z_0\subset V$ be a closed hypersurface which separates $\partial_-V$ from $\partial_+V$ and, thus, divides $V$ into two halves $V_\pm\supset\partial_\pm(V)$. Let

• the mean curvatures of $\partial_\pm V$ are bounded from below by some constants $M_\pm$;

• the scalar curvature of $V$ is bounded by a given function $\sigma=\sigma(d)$ of the signed distance $d=d(v)$ from $v$ to $Z_0$, that is

$$Sc(V)(v)\geq\sigma(dist_\pm(v,Z_0)),$$

where

$$dist_\pm(v,Z_0)=dist(v,Z_0)\text{ for }v\in V_+\text{ and }dist_\pm(v,Z_0)=-dist(v,Z_0)\text{ for }v\in V_-.$$

Since $\mathbb{R}^{n-1}\rtimes O(n-1)$-symmetric metrics on $\mathbb{R}^{n-1}\times[-l,l]$ cane be written as

$$\hat{g}=\hat{\varphi}^2 g_{Eu}+dt^2$$

where $g_{Eu}$ is the flat Euclidean metric on $\mathbb{R}^{n-1}$ and where the scalar curvature of $\hat{g}$, which depends only on $t \in [-l, l]$, satisfies

$$◑ \qquad Sc(\hat{g}) = -2(n-1)\frac{\hat{\varphi}''}{\hat{\varphi}} - (n-1)(n-2)\frac{(\hat{\varphi}')^2}{\hat{\varphi}^2},$$

the net effect of $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetrisation of $V$ can be stated in concrete terms as follows.

Reformulation of Symmetrization. *If the band $V$ is $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetrisable, then there exists a smooth function $\hat{\varphi}(t) = \hat{\varphi}_\sigma(t)$, on the segment $[-l, +l]$ such that*

$$\pm l \geq dist(Z, \partial_\pm V),$$

$$\frac{\hat{\varphi}'(-l)}{\hat{\varphi}(l)} \leq \frac{-M_-}{n-1}, \ \frac{\hat{\varphi}'(l)}{\hat{\varphi}(l)} \geq \frac{M_+}{n-1}$$

*and*

$$-2(n-1)\frac{\hat{\varphi}''(t)}{\hat{\varphi}} - (n-1)(n-2)\frac{\hat{\varphi}'(t)^2}{\hat{\varphi}(t)^2} \geq \sigma(t),$$

*that is*

$$\hat{◑}_l \qquad -2f'(t) - nf(t)^2 \geq \frac{\sigma(t)}{n-1} \text{ for } f(t) = \frac{\hat{\varphi}'(t)}{\hat{\varphi}(t)} \text{ and all } t \in [-l, l].$$

**Symmetrization Corollary for** $Sc \geq 0$. Let $V$ be an isoenlargeable band and $Z_0 \subset V$ be a hypersurface which separates $\partial_-(V)$ from $\partial_+(V)$. Let $Sc(V) \geq 0$ and let

$$Sc(V) \geq \sigma_0 > 0 \text{ on the } \delta_0\text{-neighbourhood of } Z_0.$$

Then

*the distance from $Z$ to the boundary $\partial V$ is bounded by a constant which depends only on the dimension of $V$, on $\sigma_0 > 0$ and on $\delta_0 > 0$,*

$$dist(Z, \partial V) \leq C = C_n(\sigma_0, \delta_0).$$

Moreover, this remains true

*if the inequality $Sc(V) \geq 0$ is replaced by $Sc(V) \geq -\varepsilon$ for a small positive $\varepsilon \leq \varepsilon_n(\sigma_0, \delta_0) > 0$.*

*Proof.* If $\sigma(t) \geq \sigma_0$ for $t \in [-\delta_0, \delta_0] \subset [-l, l]$ and $\sigma(t) \geq -\varepsilon$ for all $t \in [-l, l]$, where $\varepsilon << \delta_0, \sigma_0$, then the inequality $\hat{◑}_l$ implies that

$$l \leq C = C_n(\sigma_0), \delta_0$$

and the proof follows.

Notice that no condition $mean.curv(\partial_\pm(V) \geq M_\pm$ has been used at this point.

**Sub-corollary for Complete Manifolds with $Sc \geq 0$.** *Open isoenlargeable bands carry no complete metrics with scalar curvatures $Sc > 0$.*

Moreover,

*complete metrics with $Sc \geq 0$ on such bands are Riemannian flat.*

In fact, a deformation theorem by Kazdan and Warner together with the Cheeger-Gromoll splitting theorem imply that if such a band admits no complete metric with $Sc > 0$ then every complete metric with $Sc \geq 0$ is flat.

**Representative Example.** *If a compact manifold $Z$ admits a metric with negative sectional curvature, then there is no complete metrics with $Sc \geq 0$ on connected sums*

$$X = (Z \times \mathbb{R}) \#_i P_i$$

*for compact manifolds $P_i$.*

(A similar result is proven in 6.12 and 6.13 of [GL 1983] for spin manifolds $X$.)

**Symmetrization Corollary for** $Sc \geq \sigma < 0$. To get a perspective look at the following

*Model Example.* Let $V_{[-l,l]}$ be the band of width $2l$ between concentric horospheres in the hyperbolic space $H^n$ of constant curvature $-1$, which is the product

$$V_{[-l,l]} = \mathbb{R}^{n-1} \times [-l,l] \subset \mathbb{R}^{n-1} \times (-\infty + \infty) = H^n,$$

where the hyperbolic metric in these coordinates is $g_{hyp} = e^2 t g_{Eu} + dt^2$.

The scalar curvature of $g_{hyp}$ in these coordinates, , in agreement with ◑, is $n(n-1)$, while the mean curvatures of the boundaries $\partial_\pm V_{V_{[-l,l]}} = \mathbb{R}^{n-1} \times \{\pm l\}$ are

$$mean.curv(\partial_\pm V_{V_{[-l,l]}}) = \pm(n-1).$$

Such a band becomes compact if divided by the action of $\mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$ for

$$\mathbb{R}^{n-1} \times [a,b]/\mathbb{Z}^{n-1} = \mathbb{T}^{n-1} \times [a,b].$$

Now,

*let $V$ be a compact band, where the scalar curvature and the mean curvatures of the boundaries satisfy*

$$Sc(V) \geq -n(n-1),$$

$$mean, curv(\partial_-(V)) \geq -(n-1),$$

$$mean.curv(\partial_+(V)) \geq (n-1).$$

*then, in fact,*

$$Sc(V) = -n(n-1),$$

$$mean, curv(\partial_-(V)) = -(n-1),$$

$$mean.curv(\partial_+(V)) = (n-1).$$

*Proof.* If either of the above three inequalities is *strict at some point*, then, by slightly conformally perturbing the metric of $V$, one can make *all three non-strict at all points.*

This, by symmetrization, would result in a function $\hat{f} = \frac{\varphi'}{\varphi}$ on some segment $[-l,l]$, $l > 0$, such that

$$f(-l) < 1, \ f(l) > 1$$

and

$$2f' < -n(f^2 - 1)$$

which is, obviously, impossible. QED.

**Sub-corollary: Weak Rigidity of $H^n/\mathbb{Z}^{n-1}$.** Let $X = H^n/\mathbb{Z}^{n-1}$, where $H^n$ is the hyperbolic space with the sectional curvature $\kappa(g_{hyp}) = -1$, and the group $\mathbb{Z}^{n-1}$ discretely and isometrically acts on $H^n$ by parabolic transformations, i.e. preserving a horosphere in $H^n$.[12]

*If a Riemannin metric on $X$, which coincides with the hyperbolic one (descended from $H^n$ to $X$) outside a compact subset in $X$, satisfies*

$$Sc(g) \geq Sc(H^n) = -n(n-1)$$

*then*

$$Sc(g) = -n(n-1)$$

*everywhere on $X$.*

Soap Bubbles and Rigidity of Bands. A sharper version of the above sub-corollary, namely the implication

$$[\mathbf{O}_{-1}] \qquad Sc(g) \geq -n(n-1) \Rightarrow \kappa(g) = -1$$

follows from the existence of *stable minimal bubbles* in $X$, which are closed hypersurfaces $Y$ which separate the two ends in $X$ and which minimise the functional

$$Y \mapsto vol_{n-1}(Y) - (n-1)vol(X_{<Y}),$$

where $X_{<Y} \subset X$ is the part of $X$ which is bounded by $Y$ and which has $vol < \infty$.

*Proof.* $[\mathbf{O}_{-1}]$ for $n \leq 8$ follows from $\mathbf{O}[M]$ below by taking $M = n-1$.

$\mathbf{O}[M]$. Let a compact Riemannin band $V$ admits a function $M = M(v)$ such that

$$M_{|\partial_- V} \leq -mean.curv(\partial_- V) \text{ and } M_{|\partial_+ V} \geq mean.curv(\partial_+ V)$$

and

$$\frac{n}{n-1}M^2 - 2||dM|| + Sc(V) \geq 0.$$

If $n = dim(V) \leq 8$, then either

*there is closed hypersurface $Y_\circ \subset V$, which separates $\partial_- V$ from $\partial_+ V$ and which admits a metric with* $Sc > 0$,

or

*$V$ decomposes into the (warped) product, $V = Y \times [-l, l]$ with the metric*

$$\varphi(t)^2 g_Y + dt^2,$$

*where the Riemannin metric $g_Y$ on $Y$ has zero Ricci curvature.*

This is shown for $n \leq 7$ in section $5\frac{5}{6}$ in [Gr 1996]) and the case $n = 8$ can be taken care of with a help of ideas from [NS 1993].

But it seems that the regularisation techniques of [Loh 2016] and/or of [SY 2017] do not apply, at least not directly, to this case and the validity of the above statement for $n \geq 9$ remains quite problematic.

[12] If $n \geq 3$ then all discrete isometric actions of $\mathbb{Z}^{n-1}$ on $H^n$ are parabolic.

On the other hand [$\mathbf{O}_{-1}$], where the implied $Y_\circ$ is the $n-1$ torus, must follow from these techniques which are, in principle, applicable whenever torical symmetrization works.

On Min-Oo Rigidity Theorem. By adapting an idea of Witten to a "hyperbolically modified" Dirac operator, Min-Oo [Mi 1989] proved a version of the positive mass theorem for $H^n$. In particular he has shown the following.

**[MRT]** *If a complete spin manifold $X$ is isometric to $H^n$ outside a compact subset and if $Sc(X) \geq -n(n-1)$ then $X$ is isometric to $H^n$.*

Since compact perturbations of $H^n$ can be periodically extended by discrete actions of isometry groups $\Gamma$ on $H^n$, e.g. for the above parabolic $\mathbb{Z}^n$,

**[MRT]** *follows from* [$\mathbf{O}_{-1}$].

Thus, **[MRT]** *remains valid without assuming $X$ is spin* (but with some reservations for $n \geq 9$).

Moreover, this is shown in [AndMinGal 2007], that [MRT] combined with an argument from [Loh 1999] implies the positive mass theorem for $H^n$ and the spin condition can be disposed of in the context of the full Min-Oo(-Wang Chruściel-Herzlich) theorem (unconditionally for $n \leq 8$).

Proving Rigidity by Symmetrization. The rigidity of bands $V$ in the symmetrization context says that the universal coverings of

*the extremal bands, where our $\frac{2\pi}{n}$-inequality becomes equalities, inequality must be $\mathbb{R}^{n-1} \rtimes O(n-1)$-invariant.*

We shall indicate below the proof of this for $n \leq 7$, where the dimension $n = 8$ needs a little effort, and where the regularisation as developed in [Loh 2016] and in [SY 2017] for $n \geq 9$ may need an additional refinement to yield rigidity.

Now, assuming minimal varieties are non-singular, we observe that

the symmetrization process *strictly enlarges* the scalar curvature of $V$, unless the minimal hypersurfaces $Y \subset V$ used for this process are *totally geodesic*.

In fact, by the second variation formula in the form given to it in [SY 1979-Inc], the corresponding operator $L$ from section 2 is *strictly positive*, which implies increase of the scalar curvature under symmetrization. And this also work for symmetrization by reflection in section 2 if one replaces the smoothing of edges argument by an appeal to the corresponding operator $L$.

Thus, one represent all our homology classes in $H_{n-1}(V, \partial V)$ by totally geodesic submanifolds. This strongly restricts the geometry of $V$ but does not, at least not obviously, imply the required $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetry of $V$.

However, by applying the same argument to the soap bubbles $Y_{\pm\varepsilon} \subset X$ which lie close to minimal $Y$ and minimise the functional $Y \to vol_{n-1}(Y) - \varepsilon \cdot vol_n U_{\pm\varepsilon}$ as in section 8 one sees that no minimal $Y$ can be locally strictly minimising in either of the two halves it divides $V$ into.

This shows, that minimal $Y$ in all homology classes, besides being totally geodesic, are "freely movable" in $V$, namely, they serve as fibers of a fibrations of $V$ over the circle.

Then the required $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetry of $V$ easily follows.[13]

[13]Since I have not written this down in detail, I might have missed some hidden difficulty in this apparently quite innocuous argument.

# 10 Comparison with Results Obtained with Twisted Dirac Operators.

Besides the metod of of minimal hypersurfaces, a non-trivial information on geometry (and topology) of Riemannin manifolds $X$ with $Sc(X) \geq \sigma$, $(-\infty, \infty)$, can be obtained by confronting

I: Atiyah-Singer type *index theorems for Dirac operators* which yield *non-zero harmonic spinors* on $X$

with

II: the *twisted Schroedinger-Lichnerowicz-Weitzenboeck formula* for manifolds with *lower bounds on their scalar curvatures* which rules out, or significantly restrict, such spinors.

Comparison of (partly overlapping) results obtainable with minimal hypersurfaces and with Dirac operators exposes *limitations of both methods* and exhibits wide gaps in our understanding of scalar curvatures; this beg for a new approach.

Let us briefly demonstrate this on a few simple examples.[14]

(1) Spin, Spinor Bundles and Dirac Operators. Since the fundamental group of the special (i.e. orientation preserving) orthogonal group $SO(n)$ for $n \geq 3$ is $\mathbb{Z}/2\mathbb{Z}$, there are exactly *two* different orientable bundles of rank $n \geq 3$ over closed connected surfaces. The trivial bundle is, by definition, *spin* and the non-trivial one is *non-spin.*

An orientable manifold $X$ is called *spin* if the restrictions of the tangent bundle $T(X)$ to all surfaces $S \subset X$ are spin (i.e trivial).[15]

For instance, all orientable hypersurfaces $X^n \subset \mathbb{R}^{n+1}$ are spin, all 3-manifolds are spin and

*simply connected n-manifolds with trivial second homotopy groups are spin.*

The simplest non-spin manifolds are the complex projective spaces $\mathbb{C}P^n$ of even complex dimensions $n$ and connected sums of other manifolds with these $\mathbb{C}P^n$.

The *spinor bundle* of a Riemannian spin manifold $X$ of dimension $n$, denoted $\mathbb{S}(X)$, is a unitary vector bundle of vector bundle of rank $2^n$ with a unitary connection associated to the Levi-Civita connection in $T(X)$. If $n$ is even, the bundle $\mathbb{S}(X)$ splits, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$.

*The Dirac operator* is a canonically defined first order differential operator $D$ represented as a certain "natural" linear combination of covariant derivatives which act on $\mathbb{S}(X)$. (See [BHMM 2015] for definitions, basic results and geometric applications of the Dirac operator.)

When $n$ is even the Dirac operator splits:

$D = D^+ \oplus D^-$, where the operators $D^+$ and $D^-$ are mutually adjoint for $D^+ : C^\infty(\mathbb{S}^+) \to C^\infty(\mathbb{S}^-)$ and $D^- : C^\infty(\mathbb{S}^-) \to C^\infty(\mathbb{S}^+)$ and where $ind(D) = dim(ker D^+) - dim(ker D^-)$. The solutions of $D(s) = 0$ are called harmonic spinors on $X$.

*Twisted Dirac operator.* Given a complex vector bundle $L = (L, \nabla)$ with a

[14] See [Ros 2007], [BER 2014], [Gr2017] for more elaborated technics and examples.

[15] "Spin" makes sense also for non-orientable bundles and manifolds but we do not need them at this point.

linear connection, one naturally defines

$$D^{\pm}_{\otimes L} : C^{\infty}(\mathbb{S}^{\pm} \otimes L) \to C^{\infty}(\mathbb{S}^{\mp} \otimes L),$$

where the sections of $S^{\pm} \otimes L$ in the kernel of $D_{\otimes L} = D^{-}_{\otimes L} \oplus D^{+}_{\otimes L}$ are called *L-twisted harmonic spinors.*

**(2) Chern character and Todd Genus.** The Chern character of a complex vector bundle $L$ over $X$ is a certain polynomial in the Chern classes $c_i \in H^i(X;\mathbb{Z})$ of $L$ in the rational cohomology $H^*(X;\mathbb{Q})$ starting from $c_0 = rank(L)$,

$$ch(L) = c_0 + c_1 + \frac{1}{2}(c_1^2 + c_2) + \frac{1}{6}c_1^3 + c_1c_2 + 3c_3 + ... + \frac{1}{i!}(c_1^i + ... + k_ic_i) + ...$$

where, observe, all $k_i \neq 0$. The basic properties of $ch$ (which essentially define it) are additivity and multiplicativity:

$$ch(L_1 \oplus L_2) = ch(L_1) + ch(L_2) \text{ and } ch(L_1 \otimes L_2) = ch(L_1) \smile ch(L_2).$$

*The $\hat{A}$-genus* is another polynomial, now in the Pontryagin classes $p_i = p_i(T(X)) \in H^*(X;\mathbb{Q})$,

$$\hat{A}(X) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(-4p_2 + 7p_1^2) + \frac{1}{967680}(-16p_3 + 44p_2p_1 - 31p_1^3) + ...$$

where again the coefficients at $p_i \in H^{4i}(X;\mathbb{Z})$ are non-zero.

**(3) Topological Index $I$.** Let $\hat{X}$ be an oriented Riemannian spin $\Gamma$-manifold, which means $\hat{X}$ is acted upon by a group $\Gamma$ and let let $\hat{L}_1$ and $\hat{L}_2$ be vector bundles with linear connections such that $\Gamma$ also on $\hat{L}_1$ and on $\hat{L}_2$ by fiber-wise linear transformations compatible with the action of $\Gamma$ on $\hat{X}$, such that the following conditions are satisfied.

(i) The action of $\Gamma$ on $\hat{X}$ is *proper, isometric and orientation preserving*, where "proper" mains that there are at most finitely many $\gamma \in \Gamma$, such that there at for all compact subsets $K \subset \hat{X}$ the intersections $K \cap \gamma(K)$ are empty for all but finitely many $\gamma \in \Gamma$.

(ii) The action of $\Gamma$ on $\hat{L}_1$ and $\hat{L}_2$ preserves the connections in these bundles.

(iii) There exists an isometric connections preserving $\Gamma$-equivariant isomorphism between the bundles $\hat{L}_1$ and $\hat{L}_2$ restricted to the complement of a $\Gamma$-invariant subset $V \subset \hat{X}$ such that $V/\Gamma$ is compact.

Let

$$I = I(\hat{X}, \hat{L}_1 \ominus \hat{L}_2) = (\underline{\hat{A} \smile (ch(\hat{L}_1) - ch(\hat{L}_2))}[\hat{X}/\Gamma]$$

be defined by representing $\hat{A}$, $ch(\hat{L}_1)$ and $ch(\hat{L}_2)$ by the Chern-Weil differential forms on $\hat{X}$, call them $\alpha, \lambda_1, \lambda_2 \in \bigwedge^*(\hat{X})$ which, clearly, are $\Gamma$-invariant and where $\lambda_1 - \lambda_2$ vanishes outside $V$.

Since the form $\iota = \alpha \wedge (\lambda_1 - \lambda_2)$ vanishes outside $V$, since it is $\Gamma$-invariant and since the action of $\Gamma$ on $\hat{X}$ is proper, $\iota$ descends to a form $\underline{\iota}$ on the quotient space $\hat{X}/\Gamma$, which vanishes outside a compact subset[16] and defines the cohomology class $(\underline{\hat{A} \smile (ch(\hat{L}_1) - ch(\hat{L}_2))}$ of $\hat{X}/\Gamma$ with compact supports,

$$[\underline{\iota}] = (\underline{\hat{A} \smile (ch(\hat{L}_1) - ch(\hat{L}_2))} \in H^n_{comp}(\hat{X}/\Gamma;\mathbb{R}).$$

[16]We do not assume the action of $\Gamma$ on $\hat{X}$ to be free and the space $\hat{X}/\Gamma$ may be singular but our forms are defined on it anyway.

Then the index $I = \underline{\iota}[\hat{X}/\Gamma]$ can be defined as the integral

$$\int_{\hat{X}/\Gamma} \underline{\iota} = \int_{\Delta} \iota$$

for a fundamental domain $\Delta \subset \hat{X}$.

(4) Atiyah's $L_2$-Index Theorem. Let the following conditions be satisfied.

(a) The manifold $\hat{X}$ is *complete.*

(b) The connections in $\hat{L}_1$ and in $\hat{L}_2$ are *unitarizable.* This means these bundles admit unitary structures, i.e. fiberwise Hermitian scalar products $\langle ... \rangle$, preserved by the parallel transport in these connections.

(c) The above unitary structures, (which are unique up to scaling) are Γ-*invariant.*

(d) The operators $D^2_{\otimes \hat{L}_1}$ and $D^2_{\otimes \hat{L}_2}$ are *uniformly positive at infinity*/Γ, where a differential operator $\mathcal{D}$ on sections $s = s(\hat{x})$ of a unitary bundle on a manifold $\hat{X}$ with a Γ action is called uniformly positive at infinity/Γ, if

$$\int_{\hat{X}} \langle \mathcal{D}(s(\hat{x})), s(\hat{x}) \rangle_{\hat{x}} d\hat{x} \geq c \cdot \int_{\hat{X}} \|s(\hat{x})\|^2$$

for a constant $c > 0$ and all sections $s$ with compact supports outside a certain subset $V \subset \hat{X}$ such that $V/\Gamma$ is compact.

*If the topological index $I = I = I(\hat{X}, \hat{L}_1 \ominus \hat{L}_2)$ does not vanish, then there exists either an $\hat{L}_1$- or $\hat{L}_2$-twisted harmonic square integrable spinor on $\hat{X}$.*

In fact,

*the von-Neumann dimensions of the kernels $\hat{K}^{\pm}_{1,2}$ of the operators $D^{\pm}_{\otimes \hat{L}_{1,2}}$ satisfies*

$$dim_\Gamma(\hat{K}^+_1) - dim_\Gamma(\hat{K}^-_1) - dim_\Gamma(\hat{K}^+_2) + dim_\Gamma(\hat{K}^-_2) = I(\hat{X}, \hat{L}_1 \ominus \hat{L}_2).$$

*About the Proof.* The equality

$$dim_\Gamma(ker D^+_{\otimes \hat{L}}) - dim_\Gamma(ker D^-_{\otimes \hat{L}_2}) = I(\hat{X}, \hat{L})$$

in the case of *compact* $\hat{X}/\Gamma$ is proven in [At 1976].

(If $\hat{X}/\Gamma$ is compact only a single bundle $\hat{L} = \hat{L}_1$ is needed, since one may take the trivial bundle of rank zero for $\hat{L}_2$; then the conditions (a) and (d) are irrelevant.)

The case of *non-compact manifolds* with no Γ-actions is treated in [GL 1983].

The compatibility of the two arguments was pointed out in [Gr 1986], where one finds further references.

*Suggestion.* It would be interesting to remove or to relaxe some of the conditions in the formulation of the index theorem.

$\widetilde{spin}$-Example. Let $\hat{X}$ be the universal covering $\tilde{X}$ of a manifold $X$. If $X$ is spin then the spin bundle $\mathbb{S}(X)$ and the Dirac operator in it are defined and lift Γ-equivariantly to $\hat{X} = \tilde{X}$ for the Galois action of $\Gamma = \pi_1(X)$ on $\tilde{X}$.

But if $X$ is non-spin, yet $\tilde{X}$ is spin, then the group which acts on $\mathbb{S}(\tilde{X})$ is the semidirect product $\mathbb{Z}_2 \rtimes \Gamma$ where $\mathbb{Z}_2$ acts by the ±1-involution on spinors which corresponds to the Galois involutive transformation on the double covering of the principal bundle associated to the tangent bundle $T(\tilde{X})$.

Thus,

*Atiyah's $L_2$-index theorem applies to the Galois coverings $\hat{X}$ of non-spin manifolds $X$ whenever these $\hat{X}$ are spin.*

**(5) Twisted Schroedinger-Lichnerowicz-Weitzenboeck Formula.** This formula relates the squares of $L$-twisted Dirac operators with the *rough Laplacians* $\nabla^*\nabla$ in the bundles $L = (L, \nabla)$ on $X$ with unitary connections, where, recall, the operators $\nabla^*\nabla$ acts on sections of $L$; they are (non-strictly) positive

$$\int \nabla^*\nabla = \int ||\nabla||^2;$$

thus their kernels consist of $\nabla$-parallel sections of $L$ and $rank(ker(\nabla^*\nabla) \leq rank(L)$.

Here is the formula.

$$D^2_{\otimes l} = \nabla^*\nabla + \frac{1}{4}Sc(x)\cdot Id + \mathcal{R},$$

where $\mathcal{R}$ is a linear self adjoint endomorphism (zero order operator) of $\mathbb{S}\times L$ defined by the *operator valued curvature form $R$ of $L$* coupled by the *Clifford multiplication* in $\mathbb{S}$ as follows.

$$\mathcal{R}(s\otimes l) = \frac{1}{2}\sum_{1\leq i<j\leq n} e_ie_js\otimes R(e_i\wedge e_j)(l),$$

where $Id : \mathbb{S}\times L$ is the identity operator, where $e_i \in T_x(X) \subset T(X)$ is an orthonormal frame at the point $x\in X$, where the above formula applies and where $s \in \mathbb{S}_x$ and $l\in L_x$.

Since the Clifford multiplication operators $e_i : s\mapsto e_is$ are unitary,

$$||\mathcal{R}(s\otimes l)l| \leq \frac{n(n-1)}{4}||R||\cdot||s||\cdot||l||$$

where $||R||$ is the supremum of the norms of the curvature operator over all unit bivectors in the tangent spaces $T_x(X)$.

It follows then the norm of the operator $\mathcal{R}$ is bounded by

$$||\mathcal{R}|| \leq const_n||R||$$

for

$$const_n = \frac{n(n-1)}{4}\sqrt{rank(\mathbb{S})} = n(n-1)2^{n-2}.$$

**(6)** Let $\hat{X}$ be a $\Gamma$-manifold with $\Gamma$-invariant bundles $\hat{L}_{1,2}$, such that the assumptions (a), (b) and (c) in the above Atiyah's $L_2$-index theorem are satisfied.

Let, moreover, the norms of the curvature operators $R_1$ and $R_2$ of the (unitary) connections in $\hat{L}_1$ and $\hat{L}_2$ be bounded by

$$Sc(\hat{X})(\hat{x}) \geq \varepsilon + 4const_n\cdot max(||R_1||_{\hat{x}}, ||R_2||_{\hat{x}})$$

for the above $const_n = n(n-1)2^{n-2}$, some $\varepsilon > 0$ and all $\hat{x}\in\hat{X}\smallsetminus V$ for a subset $V\subset\hat{X}$ with compact quotient $V/\Gamma$.

Then the above **(4)** and **(5)** yield the following.

**Teorem.** *If the topological index*

$$I = I(\hat{X}, \hat{L}_1 \ominus \hat{L}_2) = \underline{\hat{A} \smile (ch(\hat{L}_1) - ch(\hat{L}_2))}[X/\Gamma]$$

*doesn't vanish, then there exists a point $\hat{x} \in \hat{X}$, where*

$$Sc(\hat{X})(\hat{x}) \leq 4const_n \cdot max(||R_1||_{\hat{x}}, ||R_2||_{\hat{x}}).$$

**(7)** Area Enlargeable Manifolds Recall that an $n$-dimensional Riemannin manifold $X$ is called *area enlargeable* if it admits a sequence of orientable coverings $\tilde{X}_i \to X$ and of smooth maps $f_i \tilde{X}_i \to S^n$ which are

$\bullet_1$ constant at infinity,

$\bullet_2$ have non-zero degree,

$\bullet_3$ contract the areas of the surfaces $\Sigma \subset \tilde{X}_i$ by

$$area(f_i(\Sigma)) \leq \alpha_i area(\Sigma) \text{ for } \alpha_i \underset{i\to\infty}{\to} 0$$

for

Observe that area enlargeablility is a weaker condition than enlargeability, where instead of $\bullet_3$ one requires $Lip(f_i) \to 0$ (see section 4), and that area enlargeability, similarly to enlargeability, is a *homotopy invariant* of *compact* manifolds $X$.

Let us show of that area enlargeability is incompatible with $Sc > 0$.

[□] *Complete area enlargeable manifolds $X$ the universal coverings of which are spin can't have $Sc(X) \geq \varepsilon > 0$.*

*Proof.* Let's first assume that $n = 2m$ and let $L$ be a complex vector bundle of some rank $N$ over $S^n$ with *non-zero* Chern class $c_m \in H^n(S^n)$.

Let $X$ be the universal covering $\tilde{X}$ acted upon by $\Gamma = \pi_1(X)$, let $L_1$ be the trivial bundle $X \times \mathbb{C}^N$ and let $L_i$ be induced from $L$ by the composed map

$$X = \tilde{X} \to \tilde{X}_i \underset{f_i}{\to} S^n.$$

It is easy to see that non-vanishing of $c_m$ implies non-vanishing of the topological index $I$ and that the curvature of $L_i$ tends to zero for $i \to \infty$

Therefore, the above **(6)** applies to $(X, L_1, L_i)$ for a sufficiently large $i$ and yields the proof for even $n$, while the case of $n = 2m - 1$ reduces to $n = 2m$ by taking $X \times S^1$. [17]

**(8)** Llarull's Rigidity Theorem. The above, as it is shown in [Ll 1998] can be rendered sharp by taking the positive (or negative) spin bundle $\mathbb{S}^+(S^n)$ for $\underline{L}$.

The Chern character of $\mathbb{S}^+(S^n)$ for $n = 2m$ is equal to the fundamental cohomology class $[S^n] \in H^n(S^n)$ and the norm of the Levi-Civita connection in this bundle equals $\frac{1}{2}$ – all this is more or less obvious.

What is less obvious (see Ll 1998], [Min-Oo 2002] is that the lowest eigenvalue of the operator $\mathcal{R}$ on $\mathbb{S} \otimes \mathbb{S}^+$ on $S^n$ is equal $-\frac{n(n-1)}{4}$, which, by**(6)** (and a trifle of linear algebra) implies the following

[17] "Area enlargeable" appears as "$\Lambda^2$-enlargeable" in [GL 1983], where the coverings $\tilde{V}_i$ are assumed spin.

○ Let $X$ be a Riemannin manifold, such that
• $X$ is complete,
• $Sc(X)(x) \geq \varepsilon > 0$ for all $\underline{x}$ outside a compact subset in $X$.
• the universal covering $\tilde{X}$ of $X$ is spin.
Let a continuous map $f : X \to S^n$ satisfy the following conditions.

$(*_{/\infty})$ $f$ is constant at infinity (i.e. constant outside a compact subset in $X$);

$(*_{deg})$ $f$ has non-zero degree;

$(*_{C^1})$ $f$ is $C^1$-smooth;

$(*_{ar})$ The map $f$ (non-strictly) decreases integrals of the scalar curvature of $X$ over all smooth surfaces $\Sigma \subset X$. (Since $S^n$ has constant scalar curvature $n(n-1)$ this amounts to the inequality

$$\int_S Sc(X)(\sigma)d\sigma \geq n(n-1)area(f(\Sigma).)$$

Then

*The map $f$ is a homothety: there exists a constant $\lambda > 0$, such that*

$$dist_{S^n}(f(x_1), f(x_1)) = \lambda \cdot dist_X(x_1, x_1) \textit{ for all } x_1, x_2 \in X.$$

*About the Proof.* Here, the Dirac operator on $X$ is twisted with the bundles $L = L_1$, which is induced by $f : X \to S^n$ from $\mathbb{S}^+(S^n)$, and where one takes the trivial bundle of the same rank as $L$ for $L_2$.

In this case, the formula for $\mathcal{R} : \mathbb{S} \otimes L \to \mathbb{S} \otimes L$ from the above **(5)**, that is

$$\mathcal{R}(s \otimes l) = \frac{1}{2} \sum_{1 \leq i < j \leq n} e_i e_j s \otimes R(e_i \wedge e_j)(l),$$

written in the frames of vectors $e_i \in T_{\underline{x}}$, which simultaneously diagonalize the Riemannian metric of $X$ and the metric induced by $f$ from $S^n$ effectively describes the action of $\mathcal{R}$ on the corresponding (Clifford) basis in $\mathbb{S}(X) \otimes f^!(\mathbb{S}^+(S^n)$, which is $\{\underline{e}_{i_1}\underline{e}_{i_2}...\underline{e}_{i_n} \otimes \underline{e}_{j_1}e_{j_2}...\underline{e}_{j_n}\}$. Then a straightforward computation in [Ll 1998] (and/or a more conceptual argument in [Min-Oo 2002]) shows that the spectrum of $\mathcal{R}$ is bounded from below by $-\frac{n(n-1)}{4}$ and the above **(6)** applies.

The above settles the case of even $n$.

If $n$ is odd one uses area contracting maps $X \times S^1(R) \to S^{n+1}$ for large $R$ where the corresponding $\mathcal{R}$ is still bounded by by $\frac{1}{4}(n(n+1) = \frac{1}{4}Sc(S^n)$ because the natural splitting of metric in $X \times S^1 \to S^{n+1}$ (see [Ll 1998]).

Alternatively, one can construct (non-split) metrics $g_\varepsilon$ for on $X \times S^1 \to S^{n+1}$, for all $\varepsilon > 0$, with $Sc(g_\varepsilon) \geq (n+1)(n+2) - \varepsilon = Sc(S^{n+1}) - \varepsilon$, such that area non-increasing maps $X \to S^n$ suspend to area non-increasing maps $(X \times S^1, g_\varepsilon) \to S^{n+1}$.

*Generalisation.* It is shown in [GS 2002] that the above remain valid for $S^n$ if the standard metric $g$ on $S^n$ is replaced by $g'$ with *positive curvature operator*. This, shows, in particular, that Llarull's theorem is *stable under small perturbations* of the spherical metric $g_0$.

**(9) Discussion.** There are two drawbacks of the above results compared to what can be done with minimal hypersurfaces.

I. Spin. In the original paper [Ll 1998] the manifold $X$ was assumed spin, which we have relaxed to requiring the universal covering of $X$ to be spin. Yet, we still can't proof, ◯ or even □ for all complete manifolds.

II. Completeness. Neither ◯ or □ hold true as they stand for non-complete manifolds and it is unclear what their correct reformulations should be.

And even if the area decreasing condition for maps $f : X \to S^n$ is strengthened to to $Lip(f) \leq 1$, one can't get any bound on $Sc(X)$ with Dirac operator methods for non-complete $X$, while minimal hypersurface do allow such bounds (see section 3).

On the other hand, the Dirac operator results also have two advantages over those achieved with minimal hypersurfaces.

[i] Area Versus Length. Application of minimal hypersurfaces depends on distance rather than area estimates of metrics involved.

[ii] Non-Abelian Symmetries. Dirac operator effectively accommodates symmetries of underlying (model) manifolds.

For instance, one can not prove with minimal hypersurfaces that *no metric* $g \geq g_0$ on $S^n$, where $g_0$ is the standard metric with the sectional curvature 1, *can have* $Sc(g) \geq n(n-1) = Sc(g_0)$. [18]

Specific Problem. Let $Z \subset S^n$ be a closed subset of codimension $k \geq 2$, let $X$ be an orientable $n$-dimensional Riemannian manifold and let

$$f : X \to S^n \setminus Z$$

be a smooth proper map of non-zero degree which is distance decreasing or, more generally, area decreasing.

When and how can one bound the scalar curvature of $X$?

*Example.* If $Z$ is a piecewise smooth one-dimensional subset (graph) with trivial Levi-Civita holonomies along all it cycles, e.g. a disjoint union of trees, and if $X$ complete, then – compare with remark (a) in section 3,

$$\inf_{x \in X} Sc(X)(x) < n(n-1) = Sc(S^n).$$

*Proof.* Let $\epsilon : S^n \to S^n$ be an arbitrarily small perturbation of the identity map which sends a small neighbourhood of $Z$ to $Z$. Then the bundle $L$ on $X$ which is induced from $\mathbb{S}^+(S^n)$ by the composed map $\epsilon \circ f : \underline{X} \to S^n$ is trivial at infinity and the above proof of ◯ applies.

More generally, the same argument applies to closed subsets $Z \subset S^n$ admit sequences of maps

$$\epsilon_i : S^n \to S^n$$

such that

• the maps $\epsilon_i$ send small neighbourhoods of $Z$ in $S^n$ to subsets $Z_i \subset S^n$ as above, namely i.e. piecewise smooth with trivial holonomies over all cycles in $Z_i$;

• the maps $\epsilon_i$ converge, for $i \to \infty$, to the identity map in the $C^1$-topology.

*Questions.* (a) Can one more effectively describe these $Z$ e.g. those of the topological dimension zero?

(b) Does the above inequality $\inf_x Sc(X)(x) < n(n-1)$ holds true for smooth closed curves $Z \subset S^n$, $n \geq 3$, with *non-trivial* holonomy?.

[18] Such a proof may be possible for $n = 3$.

(c) Does $S^n$ minus a point admits a *non-complete* metric $g \geq g_0$ with $Sc(g) > n(n-1) = Sc(g_0)$ (where $g_0$ is the spherical metric)?

Let us generalise the class of overtorical manifolds $X$, where non-zero multiples of the fundamental cohomology classes, denoted $[X]^\circ \in H^n(X;\mathbb{Z})$, decompose into products of one dimensional classes,

$$k[X]^\circ = h_1 \smile \ldots \smile h_n, \;\; h_i \in H^1(X;\mathbb{Z}),$$

as follows.

**(10) Oversymplectic Manifolds.** A compact orientable $n$-dimensional manifold $X$ is oversymplectic if a multiple of the fundamental cohomology class of $X$, decomposes into product of one and two dimensional classes,

$$k \cdot [X]^\circ = h_1 \smile \ldots \smile h_m,$$

and such an $X$ is called $[\tilde{\uparrow}0]$-oversymplectic, if

*the classes $h_i$ vanish in the cohomology of the universal covering $\tilde{X}$ under the natural homomorphism $H^*(X) \to H^*(\tilde{X})$.*

Notice that $[\tilde{\uparrow}0]$ is automatic for 1-dimensional classes.

Also note that if $n = 2m$, then, by grouping 1-dimensional $h_i$ into pairs, one can make all $h_i \in H^2(X;\mathbb{Z})$, $i = 1, \ldots, m$, and if $n = 2m+1$ all but one among $h_i$ can be brought to $H^2(X;\mathbb{Z})$.

Moreover, the a priori different 2-dimensional classes $h_i$, can be replaced by a single one, namely by a generic linear combination $h$ of $h_i$, since $\smile_i h_i = k' \cdot h^{\smile m}$.

It follows that $X$ of dimension $n = 2m$ is oversymplectic if and only if it admits a map of *non-zero degree* to the complex projectiv

e space $\mathbb{C}P^m$, where the condition $[\tilde{\uparrow}0]$ says in effect that *the pull back of the symplectic (Kähler) 2-form on $\mathbb{C}P^m$ to the universal covering $\tilde{X}$ of $X$ is exact.*

And if $n = 2m+1$ is odd, there is such a map $X \times S^1 \to \mathbb{C}P^{m+1}$.

Observe that $[\tilde{\uparrow}0]$-oversymplecticity, similarly to overtoricity and to iso-enlrageability of manifolds $X$ is inherited by $X'$ which admit maps $X' \to X$ of non-zero degrees and also by the products $X' = X \times \mathbb{T}^k$.

Still, $[\tilde{\uparrow}0]$-oversymplicity seems significantly different from isoenlargeability, and, probably, there are many examples of $[\tilde{\uparrow}0]$- $[\tilde{\uparrow}0]$-oversymplectic manifolds, even among projective algebraic ones, which are not (iso)enlargeable.

The reason we brought forth this oversymplecticity is the following proposition.

$(\star\tilde{\uparrow}0)$ *If $X$ is $[\tilde{\uparrow}0]$-oversymplectic, then it admits no metric with $Sc > 0$, provided the universal covering $\tilde{X}$ is spin.* [19]

(This, as it was mentioned earlier, implies that the only possibility for $Sc(X) \geq 0$ is $X$ being flat.

Also recall that vanishing of the second homotopy group $\pi_2(X)$ implies that $\tilde{X}$ to be spin and observe that $\pi_2(X) = 0$ also implies $[\tilde{\uparrow}0]$.)

*Proof of* $(\star)$. Let $n = 2m$ and $\tilde{l}$ be the lift of the canonical line bundle of $\mathbb{C}P^m$ to $\tilde{X}$. Because of $[\tilde{\uparrow}0]$, this bundle is trivial there are the $p$-th order roots

[19] $\tilde{X}$ is spin if and only if the restrictions of the tangent bundle $T(X)$ to all 2-spheres in $X$ are trivial; if $n \geq 5$ this is the same as triviality of the normal bundles of embedded 2-spheres in $X$.

$\sqrt[p]{\tilde{l}}$ for all $p = 1, 2, ...$, which are represented by the $p$-sheeted coverings of the total space of the circle bundles associated to $\tilde{l}$.

And albeit the Galois' actions of the fundamental group $\Gamma = \pi_1(X)$ on $\tilde{X}$ and on $\tilde{l}$ does not extend to $\sqrt[p]{\tilde{l}}$, the semidirect product $\mathbb{Z}_p \rtimes \Gamma$ does act on $\sqrt[p]{\tilde{l}}$.

Since $\tilde{X}$ is spin, the twisted Dirac operator $D_{\otimes \sqrt[p]{\tilde{l}}}$, i.e. $D$ with coefficients in $\sqrt[p]{\tilde{l}}$, is defined and the corresponding space of harmonic $L_2$-spinors is acted upon by the group $\mathbb{Z}_p \times \mathbb{Z}_2 \rtimes \Gamma$.

Then an elementary computation shows that the topological index $D_{\otimes \sqrt[p]{l}}$ does not vanish for infinitely many $p$ and then, by the Atiyah $L_2$-index theorem, $D_{\otimes \sqrt[p]{\tilde{l}}}$-harmonic $L_2$-spinors exist for arbitrarily large $p$.

But since the curvatures of the bundles $\sqrt[p]{\tilde{l}}$ tend to 0 for $p \to \infty$, uniform positivity of the scalar curvature of $\tilde{X}$ would not allow such spinors for large $p$ according to the twisted Schroedinger- Lichnerowicz-Weitzenboeck vanishing theorem. QED.

**11 Continuation of Discussion.** On the surface of things, $(\star\tilde{\uparrow}0)$ generalizes Schoen-Yau theorem on non-existence of metrics with $Sc > 0$ on overtorical manifolds, but...

(1)Here again there is an annoying spin condition in the statement of $(\star\tilde{\uparrow}0)$, which, for all we know must be unnecessary.

(2) More seriously, we can say preciously little about non-complete manifolds.

For instance,

one can't bound with the present day technics the width of product bands $(Y \times [-1, 1], g)$ with metrics $g$ which have $Sc(g) \geq \sigma > 0$ for $[\tilde{\uparrow}0]$-oversymplectic manifolds $Y$.

(The same can be said about all other non-$\mathcal{SYSE}$-manifolds $Y$ which are known not to to admit metrics with $Sc > 0$).

Because of this,

one is unable to rule out complete metrics with $Sc > 0$ on $Y \times \mathbb{R}$ and complete metrics with $Sc \geq \sigma > 0$ on $X \times \mathbb{R}^2$ for $[\tilde{\uparrow}0]$-oversymplectic manifolds $Y$.

What is not hard to show, however, is the following

$(\star_{\times\mathbb{R}})$ *Products $X = Y \times \mathbb{R}$ carry no complete metrics $g$ with $Sc(g) \geq \sigma > 0$ for all $[\tilde{\uparrow}0]$-oversymplectic manifolds $Y$ the universal coverings of which are spin.*

*Sketch of the Proof.* Since $X = (X, g)$ is complete and two-ended, it admits a proper 1-Lipschitz function onto $\mathbb{R}$, which we call it $\phi : X \to \mathbb{R}$.

Let $X' = X \times \mathbb{R}$ and let

$$\Phi_\varepsilon = (\varepsilon \cdot \phi, \varepsilon \cdot \phi') : X' \to \mathbb{R}^2$$

where $\phi' : X' = X \times \mathbb{R} \to \mathbb{R}$ is the coordinate projection.

Let $l'_\varepsilon$ be the $\Phi_\varepsilon$-pullback of $l_0$ to $X'$ and let $l^\circ_\varepsilon$ be the formal difference between $l'_\varepsilon$ and the trivial complex line bundle with the trivial connection.

Let $l_0$ be a complex line bundle over $\mathbb{R}^2$ with a unitary connection, which is isomorphic to the trivial bundle outside a compact subset in $\mathbb{R}^2$ and such that the curvature $\omega_0$ of $l_0$ is $\omega_0 = p_0(t_1, t_2) dt_1 \wedge dt_2$ for a non vanishing function $p_0 \geq 0$.

Let $dim(Y) = 2m$, let $l$ be the line bundle over $X'$ induced by the composed map $X' \to Y \to \mathbb{C}P^m$ from the canonical line bundle and let

$$l^\circ_\varepsilon = l \otimes l^\circ_\varepsilon.$$

Pass to the universal covering $\tilde{X}$ and observe as earlier, that Atiyah's $L_2$-index theorem, applied to the Dirac operator twisted with $\sqrt[p]{\tilde{l}^\circ_\varepsilon}$ and combined with Schroedinger- Lichnerowicz-Weitzenboeck vanishing theorem for small $\varepsilon <<\sigma$ and for $p \to \infty$, rules out $Sc \geq \sigma > 0$ for complete metrics on $X$. QED

*Generalisation to Non-Compact $X$.* The above $(\star)$ generalises to complete $[\tilde{\uparrow}0]$-oversymplectic manifolds $X$, where the fundamental class of $X$ in the cohomology with *compact supports*, denoted $H^n(X, [\infty])$,[20] decomposes into 1- and 2-classes also with compact supports and where these classes must vanish in the cohomology $H^{1,2}(\tilde{X}, \widetilde{[\infty]})$.

For instance, if $X$ of dimension $2m$ admits a proper map of non-zero degree to a complement of a subset $Z \subset \mathbb{C}P^m$, this condition is satisfied if $H_1(Z) = 0$ and the symplectic form of $\mathbb{C}P^m$ vanishes on $Z$.

**12 Min-Oo - Goette - Semmelmann Rigidity Theorem.** A (very) special case of this theorem (2.10 in [GS 2001]) reads as follows.

Let $X$ b a compact orientable Riemannian manifold of dimension $2m$ and let $f : X \to \mathbb{C}P^m$ be a $C^1$-smooth area non-increasing *spin map* of non-zero degree where $f$ is called *spin* if the restriction of the tangent bundle $T(X)$ to $\Sigma \subset X$ is trivial if an only the restriction $T(\mathbb{C}P^n)_{|f(\Sigma)}$ is trivial for all surfaces $\Sigma \subset X$.

For instance, if $m$ is odd and $X$ is spin then all maps $X \to \mathbb{C}P^n$ are spin.

⊛ *If*

$$Sc(X)(x) \geq Sc(\mathbb{C}P^m)(f(x)) \textit{ for all } x \in X,$$

*then $f$ is an isometry.*

*About the Proof.* The $\mathcal{R}$-term in the Schroedinger-Lichnerowicz-Weitzenboeck formula **(5)**) for $D$ twisted with *line bundles $l$* shows (see [Hit 1974]) that that if the curvature form $\omega$ of an $l$ (where the cohomology class of $2\pi\omega$ equals the Chern class $c_1(l)$) on a Riemannin manifold of dimension $2m$ diagonalises as

$$\sum_{i=1,...,m} \lambda_i e_{2i-1} \wedge e_{2i}$$

for an orthonormal frame $e_1, e_2, ..., e_{2m}$, then

$$||\mathcal{R}|| \leq 4 \sum_i \lambda_i.$$

If $l$ equals the (anti)canonical bundle $l_0 = \bigwedge^m T(\mathbb{C}P^m)$ then, its curvature form for the Levi- Civita connection of the Fubini Studi metric $g_0$ has

$$\lambda_1 = \lambda_2 = ... = \lambda_m = m + 1$$

and $g_0 = m(m+1)$.

On the other hand, an easy homological computation shows that the topological index $I(D, l_0)$ does not vanish on $\mathbb{C}P^m$ and since $deg(f) \neq 0$ it doesn't

[20]This $[\infty]$ stands to the complement to a (large) non-specified compact subset in $X$.

vanish on $X$ either. This shows that $f$ can't be *strictly* area decreasing, while the equality case needs an additional argument (see [GS 2001]).

The above applies, strictly speaking, to odd $m$, where $\mathbb{C}P^m$ is spin, and if $m$ is even, one twists $D$ with the *virtual square root of* $l_0$ (see [Hit 1974], [Min-Oo 1995], [GS 2001]).

**(13) (Interpolating between $(\star\tilde{\uparrow}0)$ with $(\circledast)$** Unlike the (obvious) implication $\bigcirc \Rightarrow \square$ the (sharp) theorem $(\circledast)$ by no means implies (rough) $(\star\tilde{\uparrow}0)$.

But an obvious combination of the proofs of these theorems brings the two together as follows.

Let $X$ be a complete oriented Riemannin $2m$-manifold and $f : X \to \mathbb{C}P^m \setminus Z$ be a proper $C^1$-smooth *area non-increasing* map of *non-zero* degree, where $Z \subset \mathbb{C}P^m$ a smooth submanifold on which the *symplectic form of* $\mathbb{C}P^m$ *vanishes* and which has $H_1(Z) = 0$.

Let the composed map $\tilde{X} \to X \to \mathbb{C}P^m \setminus Z \subset \mathbb{C}P^m$ from the universal covering of $X$ to $\mathbb{C}P^m$, call it $\tilde{f} : \tilde{X} \to \mathbb{C}P^m$, be spin.

$\left(\circledast\tilde{\uparrow}\frac{1}{p}\right)$ *If the $\tilde{f}$-pullback of the generator $c \in H^2(\mathbb{C}P^m);\mathbb{Z})$, that is $\tilde{f}^*(c) \in H^2(\tilde{X};\mathbb{Z})$, is divisible by a positive integer $p$, then*

$$\inf_x Sc(X)(x) < \frac{1}{p} Sc(\mathbb{C}P^m),$$

*unless $X$ is compact, $Z$ is empty, $p = 1$ and $f$ is an isometry.*

One can only wonder if there is anything of this kind that may come from minimal hypersurfaces.

# 11 References.


[Al 1985] Sebastiao Almeida, Minimal Hypersurfaces of a Positive Scalar Curvature. Math. Z. 190, 73-82 (1985).

[AndMinGal 2007] Lars Andersson, Mingliang Cai, and Gregory J. Galloway, Rigidity and positivity of mass for asymptotically hyperbolic manifolds, Ann. Henri Poincaré 9 (2008), no. 1, 1-33.

[At 1976] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. Astérisque 32-3 (1976), 43-72.

[Bam 2016] R. Bamler . A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature. Mathematical Research Letters Volume 23 (2016). Number 2, Pages 325 - 337.

[BER 2017] Boris Botvinnik, Johannes Ebert, Oscar Randal-Williams, Infinite loop spaces and positive scalar curvature, Inventiones mathematicae Volume 209, Issue 3, 2017, pp 749-835.

[BH 2009] M. Brunnbauer, B. Hanke, Large and small group homology , J. Topology 3 (2010) 463-486.

[BHMM 2015] Jean-Pierre Bourguignon, Oussama Hijazi, Jean-Louis Milhorat, Andrei Moroianu and Sergiu Moroianu A Spinorial Approach to Riemannian and Conformal Geometry,EMS Monographs in Mathematics 2015.

[DFW 2003] A. N. Dranishnikov, Steven C. Ferry, and Shmuel Weinberger, Large Riemannian manifolds which are flexible, Annals of Mathematics 157 (2003), 919-938.

[Dr 2000] A. Dranishnikov, Asymptotic topology , Russian Math. Surveys 55:6 2000), 71-116.

[Dr 2006 ] A. N. Dranishnikov, On hypereuclidean manifolds Geom. Dedicata 117 (2006), 215-231.

[FCS 1980] D. Fischer-Colbrie, R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Comm. Pure Appl. Math. 33 (1980) 199-211.

[GL 1980] M. Gromov, B. Lawson, Spin and Scalar Curvature in the Presence of a Fundamental Group I Annals of Mathematics, 111 (1980), 209-230.

[GL 1983] M. Gromov and H. B. Lawson, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Inst. Hautes Etudes Sci. Publ. Math.58 (1983), 83-196.

[Gr 1986] M. Gromov, Partial differential relations, Springer 1986

[Gr 1996] M. Gromov. Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993) ,volume 132 of Progr. Math., pages 1-213, Birkhäuser, 1996.

[Gr 2014] M. Gromov, Dirac and Plateau billiards in domains with corners, Central European Journal of Mathematics, Volume 12, Issue 8, 2014, pp 1109-1156

[Gr 2017] 100 Problems around Scalar Curvature, In preparation.

[GS 2001] S. Goette, U. Semmelmann, Spin$^c$ Structures and Scalar Curvature Estimates, Annals of Global Analysis and Geometry Volume 20, Issue 4, pp 301-324, 2001

[GS 2002] S. Goette and U. Semmelmann, Scalar curvature estimates for compact symmetric spaces. Differential Geom. Appl. 16(1):65-78, 2002.

[Han 2011] Positive scalar curvature, K-area and essentialness, Global Differential Geometry pp 275-302, 2011

[HaSch 2006] B. Hanke, T. Schick, Enlargeability and index theory,J. Differential Geom. 74 (2) (2006), 293-320.

[HaPS 2015] Hanke, Bernhard; Pape, Daniel; Schick, Thomas Codimension two index obstructions to positive scalar curvature, Annales de l'institut Fourier, Volume 65 (2015) no. 6 , p. 2681-2710.

[Hit 1074] N. Hitchin, Harmonic spinors, Advances in Math. 14 (1974), 1-55.

[Ll 1998] Llarull Sharp estimates and the Dirac operator, Mathematische AnnalenJanuary 1998, Volume 310, Issue 1, pp 55-71.

[Loh 1999] J. Lohkamp, Scalar curvature and hammocks, Math. Ann. 313, 385-407, 1999.

[Loh 2006] J. Lohkamp, The Higher Dimensional Positive Mass Theorem I, arXiv math.DG/0608795.

[Loh 2008] J. Lohkamp, Inductive Analysis on Singular Minimal Hypersurfaces, arXiv:0808.2035.

[lok 2016] J. Lohkamp, The Higher Dimensional Positive Mass Theorem II arXiv:1612.07505

[McFSzk 2012] On the positive mass theorem for manifolds with corners Donovan McFeron, Gábor Székelyhidi Communications in Mathematical PhysicsVolume 313, Issue 2, 2012 pp 425-443

[Mia 2002] P. Miao. Positive mass theorem on manifolds admitting corners along a hypersurface. Adv. Theor. Math. Phys., 6(6):1163-1182, 2002.

[Min-Oo 1988] M. Min-Oo. Scalar curvature rigidity of certain symmetric spaces. In Geometry, topology, and dynamics (Montreal, PQ, 1995), volume 15 of CRM Proc. Lecture Notes , pages 127- 136

[Min-Oo 1989] M. Min-Oo, Scalar curvature rigidity of asymptotically hyperbolic spin manifolds. Math. Ann. 285, 527- 539 (1989)

[Min-Oo 2002] M. Min-Oo, K-Area, mass and asymptotic geometry.
http://ms.mcmaster.ca/minoo/mypapers/crm −es.pdf.

[NS1993)] N. Smale, Generic regularity of homologically area minimizing hyper surfaces in eight-dimensional mani- folds, Comm. Anal. Geom. 1, no. 2 (1993), 217-228.

[Ros 2007] J. Rosenberg, Manifolds of positive scalar curvature: a progress report , in: Surveys on Differerential Ge- ometry, vol. XI: Metric and Comparison Geometry, Internati onal Press 2007.

[Sch1998] Thomas Schick, A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture, Topology 37 (1998), no. 6

[SY - Inc 1979] R. Schoen and S. T. Yau, Existence of incompressible minimal surfaces and the topology of three dimensional manifolds of non-negative scalar curvature, Ann. of Math. 110 (1979), 127-142.

[SY - Str 1979-Man] R. Schoen and S. T. Yau, On the structure of manifolds with positive scalar curvature, Manuscripta Math. 28 (1979), 159-183.

[SY 2017] Positive Scalar Curvature and Minimal Hypersurface Singularities Richard Schoen, Shing-Tung Yau, arXiv:1704.05490 [math.DG]