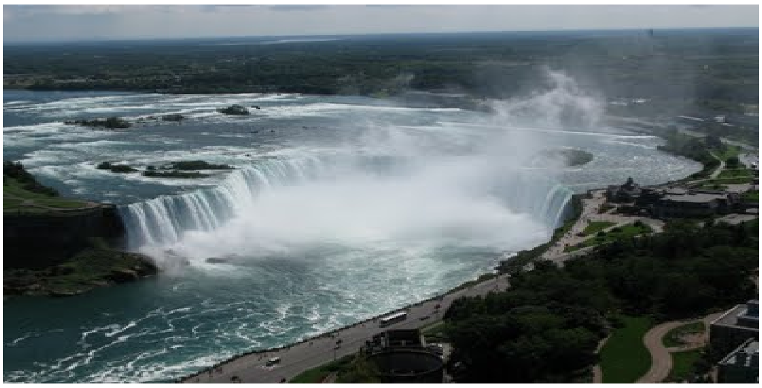

Super Stable Kählerian Horseshoe?

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Contents

Abstract

Following the prints of Smale's horseshoe, we trace the problems originated from the interface between hyperbolic stability and the Abel-Jacobi-Albanese construction.

1 Abelianization, Super-stability and Universality.

Let V be a compact manifold (or, more generally, a compact locally contractible space), denote by $\tilde{A}$ the real 1-homology $H_1(V;\mathbb{R})$; thus, $\tilde{A}$ is an $\mathbb{R}$-vector space of finite dimension, say of $N = rank(H_1(V;\mathbb{R}))$, and let $A = A(V)$ be the flat 1-homology (Abel-Jacobi-Albanese) torus $A = H_1(V;\mathbb{R})/H_1(V;\mathbb{Z})$.

Strictly speaking, we factorize not by $H_1(V;\mathbb{Z})$ but by $H_1(V;\mathbb{Z})/torsion$, and we denote by h^{Ab} be the canonical (Abel's) homomorphism from the homology of V to that of A, i.e. $h^{Ab} : H_1(V;\mathbb{Z}) \to H_1(V;\mathbb{Z}) = \mathbb{Z}^N$, where h^{Ab} is an *isomorphism* from $H_1(V;\mathbb{Z})/torsion$ onto $H_1(A;\mathbb{Z})$.

Since the universal covering $\tilde{A} = H_1(V;\mathbb{R})$ of A is *contractible*, and since the fundamental group $\pi_1(A) = \mathbb{Z}^N = H_1(V;\mathbb{Z})/torsion$ is *Abelian*, there exists a *unique (Abel's) homotopy class* $[f]^{Ab}$ of continuous maps $f : V \to A$, such that the induced homology homomorphism $[f]^{Ab}_{*1} : H_1(V;\mathbb{Z}) \to H_1(A;\mathbb{Z})$ equals h^{Ab}.

There are two remarkable instances of classes Σ of "geometric structures" with the following property.

For every structure $\sigma \in \Sigma$ on an arbitrary V, there exists a unique Σ-structure on $A = A(V)$, say $\sigma_A = \sigma_A(\sigma)$, and an *essentially unique σ/σ_A-compatible map* $f_\sigma : V \to A$ in the class $[f]^{Ab}$, (where, moreover, σ_A "commutes" with the group translations in A in the examples **A** and **B** below).

A. Holomorphic Abelianization Theorem. Let σ be a complex structure on V, such that (V,σ) is *Kähler*. e.g. complex *algebraic*. Then A admits a *unique translation invariant complex structure* σ_A, such that the homotopy class $[f]^{Ab}$ contains a *holomorphic* map $f_\sigma : (V,\sigma) \to (A,\sigma_A)$, where this f_σ is *unique up-to A-translations.*

B. Dynamical Superstability/Universality Theorem. Let $\sigma : V \to V$ be a self-homeomorphism, such that the induced homology automorphism

$$\sigma_{*1} : H_1(V;\mathbb{R}) = \mathbb{R}^N \to \mathbb{R}^N = H_1(V;\mathbb{R})$$

is *hyperbolic*, i.e. all (real and complex) eigenvalues λ of σ_{*1} satisfy $|\lambda| \neq 1$. Then the torus $A = A(V)$ admits a *unique (continuous group) automorphism* $\sigma_A : A \to A$, such that the class $[f]^{Ab}$ contains a *continuous* (typically, *non-smooth* even for real analytic σ) *map* $f_\sigma : (V,\sigma) \to (A,\sigma_A)$ *commuting with the two* σ which means the commutativity of the diagram

$$\overset{\sigma}{\circlearrowleft} V \overset{f_\sigma}{\to} A \overset{\sigma_A}{\circlearrowright}, \text{ that is } f_\sigma \circ \sigma = \sigma_A \circ f_\sigma,$$

where this f_σ is *unique, up-to translations by the (finite) subgroup* $fix(\sigma_A) \subset A$ *of the fixed points of* σ_A.

The complex torus (A, σ_A) in **A** is called the *Albanese variety*; it generalizes the classical Jacobian of an algebraic curve.

If $dim_{\mathbb{R}}(V) = 4$ and $N = rank(H_1(V))$ is *even*, then, Kodaira proved **A** *without* the assumption of V being Kähler. (Later on, these V were shown to be Kähler anyway as was was pointed to me by Domingo Toledo.) On the other hand, this fails to be true in the non-Kähler case for $dim_{\mathbb{R}}(V) \geq 6$.

Theorem **B** is due to John Franks [20]; it was preceded by a similar result by Michael Shub [70] for *expanding* (rather than hyperbolic) endomorphisms. The idea goes back to *Smale's horse-shoe* [74] – the first example of a *structurally stable* diffeomorphism with uncountable closure of the set of periodic points. Later on, Smale announced the stability of hyperbolic automorphisms of the 2-torus but his proof remained unpublished. The first accepted proof of the stability of *locally split hyperbolic* diffeomorphisms is due to Anosov [2].

C. Universality Problem. Let Σ be a class of "geometric structures" and Γ be a group, possibly with additional data expressible in the group theoretic terms. Construct a space $\tilde{A} = \tilde{A}(\Gamma, \Sigma)$ with a geometric structure $\sigma_{\tilde{A}} = \sigma_{\tilde{A}}(\Gamma) \in \Sigma$, or a "canonical family" of such $(\tilde{A}, \sigma_{\tilde{A}})$, with the following properties.

• Γ-*Invariance.* The space A is acted upon by Γ where this action is compatible with $\sigma_{\tilde{A}}$.

•• (Γ, Σ)-*Universality.* Let $\tilde{V}$ be a space with a structure $\tilde{\sigma} \in \Sigma$ and with a Γ-action on $\tilde{V}$ which is compatible with $\tilde{\sigma}$ and let $\tilde{f} : \tilde{V} \to \tilde{A}$ be a Γ-equivariant map. Then $\tilde{f}$ is Γ equivariantly homotopic to an essentially unique Γ-equivariant map $\tilde{f}_\sigma : \tilde{V} \to \tilde{A}$ which is compatible with the two Σ-structures: $\tilde{\sigma}$ on $\tilde{V}$ and $\sigma_{\tilde{A}}$ on $\tilde{A}$.

In examples **A** and **B**, the group Γ is isomorphic to $\mathbb{Z}^N = H_1(V)/torsion$ and $\tilde{A} = H_1(V; \mathbb{R}) = \mathbb{Z}^N \otimes \mathbb{R} = \mathbb{R}^N$, while $\tilde{V}$ is the maximal covering of V with a free Abelian Galois group, where this Galois group equals our Γ, (isomorphic to $\mathbb{Z}^N$ in the present case) such that $\tilde{V}/\Gamma = V$ and $\tilde{f}_\sigma : \tilde{V} \to \tilde{A}$ equals a lift of $f_\sigma : V \to A$ to $\tilde{V}$.

In the Albanese case, Σ is a subclass of complex analytic structures. These structures on $\tilde{A}$ are translation invariant. They make a family parametrized by $GL_N(\mathbb{R})/GL_{N/2}(\mathbb{C})$, where N is necessarily even.

In the Franks case, the structures $\sigma_{\tilde{A}}$ on $\tilde{A}$ are hyperbolic automorphisms of $\tilde{A}$, i.e. hyperbolic linear self-maps of $\mathbb{R}^N = \tilde{A}$, while $\tilde{\sigma}$ are lifts of self-homeomorphisms of $V = \tilde{V}/\mathbb{Z}^N$ to $\tilde{V}$.

What we want to is a characterization of groups Γ and of classes Σ, where our problem would be solvable and where the apparent similarity between **A** and **B** would be embodied into a general functorial framework. We expose below the basic ideas underlying **A** and **B** in the hope they would direct one toward such a general theory.

Acknowledgement. I thank Domingo Toledo who acquainted me with his unpublished $\mathbb{C}$-*convexity* result and also suggested several corrections and improvement in the present manuscript.

I am also thankful to John Franks and Jarek Kwapisz who instructed me

on the (relatively) recent development in the theory of hyperbolic, in particular pseudo-Anosov, systems associated with **B**.

2 Symbols of Shadows.

The construction of Franks' map

$$f_\sigma : V \to A = H_1(V;\mathbb{R})/H_1(V;\mathbb{Z})$$

satisfying the equation $f_\sigma \circ \sigma = \sigma_A \circ f_\sigma$ ultimately depends on *Markov (symbolic) shadowing of quasi-orbits of group actions* (see 2.4, 2.3), but we start with a slightly different argument due to Shub and Franks with no explicit mentioning of symbols and shadows.

2.1 Contraction, Expansion, Split Hyperbolicity and Fixed Points.

Recall that $\sigma_A : A \to A$ corresponds to the action of the self-homeomorphism σ of V on $H_1(V;\mathbb{R})$.

Since $\sigma_A : A \to A$ is invertible, the equation $f \circ \sigma = \sigma_A \circ f$ can be rewritten as the fixed point condition in the space F of maps $f : V \to A$,

$$\sigma^\bullet(f) = f \text{ for } \sigma^\bullet(f) =_{def} \sigma_A^{-1} \circ f \circ \sigma.$$

This equation is solved by using the standard (and obvious)

U*nique* ***f****ixed* ***p****oint property* for ***u****niformly* ***e****ventulally contracting* self-maps of complete metric spaces, where:

a self map φ of metric space X is called ***ue*** *contracting* if there exists a locally bounded function $i(d) = i_\varphi(d)$, $0 < d < \infty$, such that $diam(\varphi^i(U)) \leq \frac{1}{2} diam(\varphi(U))$ for all subset $U \subset X$ and all $i \geq i_\varphi(diam(U))$.

Usually, this property is formulated for contracting maps; the advantage of "virtually" is a low sensitivity to the metric involved.

Namely, define the *expansion* (control) function, $e(d) = e_f(d)$, $d \geq 0$ of a map f between metric spaces, say $f : X \to Y$ by

$$e(d) = \sup_{dist_X(x_1,x_2) \leq d} dist_Y(f(x_1), f(x_2)),$$

and say that an f has *controlled expansion* or, just, that f *is controlled* if

f is *controlled at infinity*, that means $e_f(d)$ is a *locally bounded* function, and

f is *uniformely continous*, that is $\lim_{d\to 0} e(d) = 0$.

Clearly, the **ue**-contraction condition is invariant under controlled homeomorphisms between metric spaces.

Local ***UFP****.* The unique fixed point property remains valid for *partially defined* (uniformly eventually) contracting maps $\varphi : U \to X$, where $U \subset X$ is a ρ-ball, $\rho > 0$, around some point $x_0 \in X$. Namely,

There exists an $\varepsilon_0 = \varepsilon_0(i(d), \rho) > 0$, such that if x_0 is ε-fixed by φ, i.e. $dist_X(x, \varphi(x_0)) \leq \varepsilon \leq \varepsilon_0$ then x_0 is accompanied by a unique fixed point $x_\bullet \in U$, where $dist_X(x_0, x_\bullet) \leq \delta = \delta(\varepsilon) \to 0$ for $\varepsilon \to 0$.

In fact, if $\varepsilon > 0$ is small enough, then the *forward orbit*

$$x_0,\ \varphi(x_0),\ \varphi^2(x_0) = \varphi\circ\varphi(x_0),\ \varphi^3(x_0) = \varphi\circ\varphi^2(x_0),...,\varphi^i(x_0),...$$

is defined for all $i = 1,2,...$, and the limit $x_\bullet = \lim_{i\to\infty}\varphi^i(x_0)\in U$ is fixed by φ.

Given a set X and a metric space Y, define $DIST = DIST_F = \sup_X dist_Y$ in the space F of maps $f : X\to Y$ by

$$DIST(f_1,f_2) = \sup_{x\in X} dist_Y(f_1(x), f_2(x)).$$

This $DIST$ is *not* a true metric in F since it may be infinite for infinite X and unbounded Y, but it *is* a true metric on every *$DIST$-finiteness component* F, that is a maximal subset where $DIST < \infty$. Accordingly, the usual "metric language" applies to these components.

For example, we say that F is *complete* if every its finiteness component is complete and observe that if Y is complete, then the space F is complete. Furthermore, given a metric in X, the subspaces of continuous, of uniformly as well as of controlled maps are also complete.

Call an invertible self-map $\varphi : X\to X$ **ue** *expanding* if the reciprocal map $\varphi^{-1} : X\to X$ is **ue** contracting.

Observe that **ue** *expanding* maps φ have (locally as well as globally) the **ufp** property which follows from that for φ^{-1}.

Call a self-map $\varphi : X\to X$ *split hyperbolic* if (X,φ) topologically decomposes into a Cartesian product,

$$(X,\varphi) = (X^+,\varphi^+)\times(X^-,\varphi^-),$$

where the spaces $X^\pm$ admit metrics, say $dist^+$ and $dist^-$, such that the map $\varphi^+ : X^+\to X^+$ is **ue** expanding while $\varphi^- : X^-\to X^-$ is **ue** contracting.

The **ufp**-property of the **ue** *contracting* self-maps $(\varphi^+)^{-1}$ and φ^- implies that

*split hyperbolic maps, of complete metric spaces enjoy the **ufp**-property: every such map has a unique fixed point, provided the metric spaces $(X^\pm, dist^\pm)$ are complete.*

Notice that this fixed point is of somewhat different nature than that for contracting map: if $\varphi : X\to X$ is contracting, then the forward orbit of every point $x\in X$ converges to the fixed point, but in the (nontrivially) split hypebolic case, the forward and backward orbits of almost all points go to infinity.

Another relevant and (equally obvious) property of split hyperbolicity is that it is inherited by spaces $F = X^Y$ of maps of an arbitrary Y to X.

Indeed, if $\varphi : X\to X$ is a **ue** contracting self-map, then the corresponding self-map, say $\varphi_F : F\to F$, on the space F of maps $f : Y\to X$ with the "metric" DIST is also **ue** contracting and the same is true for **ue** expanding maps.

Therefore,

*if X is complete, and $\varphi : X\to X$ is either **ue** contracting or **ue** expanding, then every φ_F-invariant DIST-finiteness component $F_0\subset F$ has a unique fixed point $f_0\in F_0$ of the map $\varphi_F : F\to F$*

If φ is split hyperbolic for $X = (X^+, dist^+)\times(X^-, dist^-)$ then, obviously, φ_F is split hyperbolic for $F = (F^+, DIST^+)\times(F^-, DIST^-)$ where $F^\pm = (X^\pm)^Y$ with

$DIST^{\pm}$ in $F^{\pm}$ corresponding to $dist^{\pm}$ in $X^{\pm}$. Moreover, this remains true for subspaces of continuous, uniformly continuous and controlled maps in F.

Warning. Usually, the space X comes with its own metric $dist_X$, but we can not claim that every φ_F-invariant DIST-finiteness component of F for $DIST$ associated to $dist_X$ has a fixed point, since our hyperbolic splitting $X = X^+ \times X^-$ is *not* necessarily a *metric* splitting.

However,

if the two projections $(X, dist_X) \to (X^{\pm}, dist^{\pm})$, are controlled at infinity, then they preserve the finiteness components of $DIST$; therefore, every such φ_F-invariant component $F_0 \subset F$ has a ***u****nique* ***f****ixed* ***p****oint of the map $\varphi_F : F \to F$, provided the metric spaces $(X^+, dist^+)$ and $(X^-, dist^-)$ are complete.*

Remark. In what follows, our self-map $F \to F$ for $F = X^Y$ depends on a given self-map $Y \to Y$ as well as on $X \to X$, but F inherits the hyperbolic splitting from X anyway and the above **ufp** property for the self maps of the $DIST$-finiteness components of F remained valid.

Proof of Franks' Super-stability for Self-homomorphisms $\sigma : V \to V$. Let $\tilde{V} \to V$ denote the Abelian covering of V that is induced by Abel's map $V \to A$ from the covering map $H_1(V;\mathbb{R}) = \tilde{A} \to A = H_1(V;\mathbb{R})/H_1(V;\mathbb{Z})$, i.e. the Galois group Γ of this covering equals $H_1(V;\mathbb{Z})/torsion = \mathbb{Z}^N$.

Denote by $\tilde{F}$ the space of continuous maps $\tilde{V} \to \tilde{A}$, let

$$\tilde{\sigma}^{\bullet}(\tilde{f}) = \sigma_{\tilde{A}}^{-1} \circ \tilde{f} \circ \tilde{\sigma},$$

where $\sigma_{\tilde{A}} : \tilde{A} \to \tilde{A}$ is the homology automorphism $\sigma_{*1} : H_1(V;\mathbb{R}) = \tilde{A} \to \tilde{A} = H_1(V;\mathbb{R})$ and $\tilde{\sigma}$ is a lift of $\sigma : V \to V$ to the covering $\tilde{V}$ of V.

We look for an $\tilde{f} \in \tilde{F}$ which satisfies the equation

$$\tilde{\sigma}^{\bullet}(\tilde{f}) = \tilde{f},$$

where, moreover, this $\tilde{f} : \tilde{V} \to \tilde{A}$ must be *equivariant* for the (Galois) actions of $\Gamma = H_1(X)/torsion$ on $\tilde{V}$ and $\tilde{A}$, i.e. $\tilde{f}$ must be a lift of some $f : V \to V$ from our homotopy class $[f]^{Ab}$ of maps $V \to A$.

Start by observing the following obvious.

1. *Quasi-morphism Property.* Since V is compact, the lift $\tilde{f} : \tilde{V} \to \tilde{A}$ of *every continuous* map $f : V \to A$ in the class $[f]^{Ab}$ *almost commutes* with the two lifted σ in the sense that the diagram

$$\overset{\tilde{\sigma}}{\circlearrowleft}\ \tilde{V} \overset{\tilde{f}}{\to} \tilde{A}\ \overset{\sigma_{\tilde{A}}}{\circlearrowright}$$

commutes *up-to a bounded error.* This means

$$DIST(\sigma_{\tilde{A}} \circ \tilde{f}, \tilde{f} \circ \tilde{\sigma}) =_{def} \sup_{\tilde{v} \in \tilde{V}} dist_{\tilde{A}}(\sigma_{\tilde{A}} \circ \tilde{f}(\tilde{v}), \tilde{f} \circ \tilde{\sigma}(\tilde{x})) < \infty,$$

or, equivalently (for $\sigma_{\tilde{A}}$ being invertible),

$$DIST(\tilde{\sigma}^{\bullet}(\tilde{f}), \tilde{f}) = \sup_{\tilde{v} \in \tilde{V}} dist_{\tilde{A}}(\sigma_{\tilde{A}}^{-1} \circ \tilde{f} \circ \tilde{\sigma}(\tilde{v}), \tilde{f}(\tilde{v})) < \infty.$$

In other words,

$\tilde\sigma^\bullet$ *sends the DIST-finiteness component of the lift* $\tilde f$ *of every* $f \in [f]^{Ab}$ *into itself.*

Such $\tilde f : \tilde V \to \tilde A$ are regarded as *quasi-morphisms* in the category of metric spaces with self-mappings where morphisms are maps where the corresponding diagram is truly commutative.

2. *Split Hyperbolicity.* The hyperbolicity of the action $\sigma_{\tilde A} = \sigma_{*1}$ of $\sigma : V \to V$ on the homology $\tilde A = H_1(V;\mathbb{R})$ amounts to the existence of a unique splitting of $\circlearrowright_{\sigma_{\tilde A}} \tilde A$ into a Cartesian sum of an *expanding* and a *contracting* self-mappings,

$$(\tilde A, \sigma_{\tilde A}) = (\tilde A^+, \sigma^+_{\tilde A}) \times (\tilde A^-, \sigma^-_{\tilde A}),$$

where $\tilde A^+ (= \mathbb{R}^{N_+})$ corresponds to the sum of the eigen-spaces of $\sigma_{\tilde A} = \sigma_{*1}$ regarded as a linear operator on $\mathbb{R}^N = H_1(X;\mathbb{R}) = \tilde A$ with the eigen-values λ where $|\lambda| > 1$ and $\tilde A^- (= \mathbb{R}^{N_-})$ represents $|\lambda| < 1$.

Since the hyperbolic splitting $\tilde A^+ \times \tilde A^- = \mathbb{R}^{N_+} \times \mathbb{R}^{N_-}$ of $\tilde A = \mathbb{R}^N$ is a *metric* one, the corresponding hyperbolic splitting for $\tilde\sigma^\bullet : \tilde F \to \tilde F$, where $\tilde F$ is the space $\tilde F, \tilde\sigma^\bullet)$ of continuous maps $\tilde f : \tilde X \to \tilde A$, is a "metric" splitting for DIST in $\tilde F$,

$$(\tilde F, \tilde\sigma^\bullet) = (\tilde F^+, \tilde\sigma^{\bullet +}) \times (\tilde F^-, \tilde\sigma^{\bullet -})$$

where

the self-map $\tilde\sigma^{+\bullet} : \tilde F^+ \to \tilde F^+$ is *contracting* with respect to $DIST_{\tilde F^+}$, that is $\sup_{\tilde x \in \tilde X} dist_{\tilde A^+}$, due to the contraction by $\sigma^{-1}_{\tilde A^+}$ in the decomposition $\sigma^{-1}_{\tilde A^+} \circ \tilde f \circ \tilde\sigma = \tilde\sigma^{+\bullet}(\tilde f)$, while $\tilde\sigma^{\bullet -} : \tilde F^- \to \tilde F^-$ is *expanding* for $DIST_{\tilde F^-}$, since $\tilde\sigma$ *invertible.*

In other words

the self-map $\tilde\sigma^\bullet : \tilde F \to: \tilde F$ *is split hyperbolic.*

Therefore,

the DIST-finiteness component $F_0 \subset F$ *of every map* $\tilde f_0 : \tilde V \to \tilde A$ *lifted from a map* $V \to A$ *contains a unique fixed point, say* $\tilde f_\bullet \in F_0$ *that is a map* $\tilde f_\bullet : \tilde V \to \tilde A$, *such that* $DIST(\tilde f_\bullet, \tilde f_0) < \infty$.

To conclude the proof of Franks' theorem let us show that $\tilde f_\bullet$ equals a lift of some map $V \to A$, where, observe, the "lifted" maps $\tilde f : \tilde V \to \tilde A$ are exactly the Γ-*equivariant* ones for the group $\Gamma = H_1(V)/torsion = \mathbb{Z}^N$ which acts on $\tilde V$ and on $\tilde A$.

Define the action of Γ on $\tilde F$ by $\tilde f \overset{\gamma}{\to} \gamma_{\tilde F}(\tilde f) = \gamma^{-1}_{\tilde A} \circ \tilde f \circ \gamma_{\tilde V}$, where $\gamma_{\tilde A}$ and $\gamma_{\tilde X}$ denote the γ-transformation of $\tilde V$ and $\tilde A$ for all $\gamma \in \Gamma$. Thus,

the fixed point set fix_Γ *of this action equals the subset* $\tilde E_\Gamma \subset \tilde F$, *of equivariant maps, where, moreover, this* $\tilde E_\Gamma$ *is contained in a single DIST-finiteness component of* $\tilde F$.

The actions of Γ on $\tilde F$ and $\tilde\sigma^\bullet : \tilde F \to \tilde F$ define an action of the normal extension $\Gamma' \supset \Gamma$ generated by translation of Γ on itself together with the automorhism $\sigma_\Gamma : \Gamma \to \Gamma$ induced by $\sigma : X \to X$ on $\Gamma = H_1(X)/torsion$.

Since $\Gamma \subset \Gamma'$ is normalized by $\tilde\sigma^\bullet$, the fixed point set $fix_\Gamma \subset \tilde F$ is $\tilde\sigma^\bullet$-invariant; hence, the *unique* $\tilde\sigma$-fixed point $\tilde f_\bullet \in \tilde F$ is contained in $\tilde E_\Gamma = fix_\Gamma$. QED.

This is not especially surprising since *globally* split hyperbolic self-maps are rather primitive dynamical creatures and the above proof might appear a pure tautology.

On the other hand, Franks' theorem easily implies (see below) that hyperbolic automorphisms σ_A of tori $A = \mathbb{T}^N$ are *structurally C^1-stable.*

This may appear paradoxical, since these σ_A are topologically and measure theoretically *ergodic*: naively intuitively, ergodicity and stability seem incompatible. (The idea that such σ could be structurally stable goes back to Thom and Smale, but a realization of this idea has undergone a few unsuccessful attempts at the proof by several great mathematicians.)

To derive structural stability from super-stability all one needs to show is that if a $\sigma : A \to A$ is *C^1-close* to σ_A then Franks' morphism $f_\sigma : (A, \sigma) \to (A, \sigma_A)$ is *one-to-one.*

But σ_A, and, therefore, *every σ which is sufficiently C^1-close to σ_A*, satisfy the following obvious

Infinitesimal Expansiveness Property. There as an integer $k = k(\sigma_A) \geq 0$ such the norms of the differentials of the iterated maps σ^i, $i = -k, ..., -1, 0, 1, ..., k$, satisfy

$$\sup_{|i| \leq k} \|D_{\sigma^i}(\tau)\| \geq (1 + \varepsilon)\|\tau\|$$

for some $\varepsilon = \varepsilon(\sigma_A) > 0$ and all tangent vectors τ of A.

It follows by the implicit function theorem that σ is *locally expansive*: the $sup_{\mathbb{Z}}$-distance in A between every two distinct σ-orbits $\mathbb{Z} \to A$ is $\geq \varepsilon' > 0$; hence, the map f_σ, being $\mathbb{Z}$-equivariant and C^0-close to the identity map $A \to A$, is, necessarily, locally one-to-one.

Since A is a *closed* manifold and f_σ is homotopic to a *homeomorphism*, that is to σ_A, it *is* globally one-to-one. QED

Similar stability theorems for general group actions often go under the heading of "rigidity" borrowing from the fame of the Mostow-Margulis-Zimmer (super)rigidity theory for semisimple groups, while what we call *"super*-stable" is called *"semi*-stable" by people in dynamics, who, apparently, are disgruntled with non-injectivity, rather than being excited by uniqueness and universality.

2.2 Shadowing Lemmas.

The idea underlying the above argument, due to Smale, Anosov, Tate, Shub and Franks, can be vaguely formulated as follows:

•

if a group (sometimes a semigroup) Σ of self-maps σ of a metric space X "strongly contracts/expands in many directions" then every Σ-quasi-orbit in X is shadowed by an orbit. Moreover, the space of this shadows is "small", i.e. it consists of a single orbit.

•

Recall that *orbits* of an action of Σ on X are Σ-equivariant maps $o : \Sigma \to X$, say, for the left action of Σ on itself, i.e. $o(\sigma \cdot \sigma') = \sigma_X(o(\sigma'))$, where $\sigma_X : X \to X$ denotes the action of $\sigma \in \Sigma$ on X.

These orbits are the same as the *fixed points* of the $\bullet$-*action* of Σ on the space X^Σ of all maps $q : \Sigma \to X$, defined by

$$\sigma^\bullet(q) = \sigma_X^{-1} \circ q \circ \sigma.$$

The deviation of a general $q \in X^\Sigma$ from being an orbit can be measured, for example, with a given (usually, finite generating) subset $\Theta \subset \Sigma$, by

$$DI_\Theta(q) =_{def} \sup_{\sigma\in\Theta} DIST(\sigma(q), q),$$

where, recall, $DIST$ in X^Σ is defined as $\sup_\Sigma dist_X$. (One can use various "weighted versions" of $DIST$ which lead to somewhat different "quasi"s and/or "shadows".)

Call $q : \Sigma \to X$ a *quasi-orbit* if $DI_\Theta(q) < \infty$; more generally, *ε-orbit* signifies $DI_\Theta(q) < \varepsilon$ (where this $\varepsilon \geq 0$ may be large in the present context).

In other words quasi-orbits of an action of Σ on X are the same as *quasi-fixed points* for the corresponding $\bullet$-action of the group Σ on X^Σ.

An orbit o is said to *δ-shadow* a q if $DIST(o, q) < \delta$, where the plain "shadow" refers to $\delta = \infty$.

The essence of Franks' argument is the following

Shadowing for Split Hyperbolic Actions of $\Gamma = \mathbb{Z}$: Every quasi-orbit of such an action on a complete metric space is shadowed by a unique orbit.

Indeed split hyperbolicity *passes from X to the corresponding $\bullet$-action on* $X^\mathbb{Z}$ and every almost fixed point of the resulting (split hyperbolic!) action is accompanied by a unique fixed point in $X^\mathbb{Z}$ that is an orbit of the original action in X.

Moreover, split hyperbolicity implies

Uniform Shadowing. Every ε-orbit q is δ-shadowed by an orbit where δ depends on ε but not on q.

The key property of the hyperbolic splitting $\tilde{A} = \tilde{A}^+ \times \tilde{A}^-$ in Franks' argument (where $\tilde{A}$ is a Euclidean space with a linear hyperbolic self-map $\sigma_{\tilde{A}}$) is that this splitting is *metrically controlled*, i.e. the projections $p^\pm : \tilde{A} \to \tilde{A}^\pm$ are *conrtolled at infinity*, for the (Euclidean) metrics in $\tilde{A}$ and $\tilde{A}_\pm$,

$$dist_{\tilde{A}}(\tilde{a}_1, \tilde{a}_2) \leq d < \infty \Rightarrow dist_{\tilde{A}^\pm}(p^\pm(\tilde{a}_1), p^\pm(\tilde{a}_2)) \leq e(d) < \infty;$$

hence, *quasi-orbits project to quasi-orbits.*

And on the bottom of all this lies **ufp** – the *unique fixed point* property of (uniformly eventually) *contracting* self-maps of *complete* metric spaces that are the spaces of $\sigma^\pm_{\tilde{A}}$-quaisi-orbits in $(\tilde{A}^\pm)^\mathbb{Z}$ in Franks' case.

Question. What are the most general *algebraic* properties of a group Σ and *geometric* properties of a metric space X and the action of Σ on X that would guaranty the (unique) fixed point and shadowing property of the action?

For example, which (semi)groups of linear operators in a Banach (e.g. Hilbert) space have the shadowing property?

Ultimately, one looks for criteria that would imply, for example, *Kazhdan's T-property* of Σ:

by definition of T, every *isometric* action of a T-group Σ on a *Hilbert* space has a fixed point, where the non-trivial point is proving T for particular groups Σ, such as $SL_N(\mathbb{Z})$, for $N \geq 3$.

Also Margulis' super rigidity theorem can be viewed as a fixed point theorem for semisimple groups acting on some spaces of "quasi-representations".

In what follows, we only look at "hyperbolic-like" actions with "strong contraction" in certain directions.

Definition of Stable Partitions. A family of self-maps $\{\sigma_i : X \to X\}_{i\in I}$ *eventually contracts* a subset $S \subset X$ if all subsets $S' \subset S$ with $diam_X(S') < \infty$ satisfy

$$diam_X(\sigma_i(S)) \to 0 \text{ for } i \to \infty;$$

this means that, for each $D > 0$, there are at most finitely many $i \in I$, such that $diam_X(\sigma_i(S')) > D$.

Thus, X is *partitioned* into maximal $\{\sigma_i\}$*-stable* subsets, called $\{\sigma_i\}$*-stable slices (leaves)* that are eventually contracted by $\{\sigma_i\}$, where the corresponding quotient space $X/partition$ is denoted $X^+ = X^+_{\{\sigma_i\}}$.

Observe that every uniformly continuous map $\sigma : X \to X$ which *commutes* with all σ_i sends stable subsets to stable ones; thus, σ induces a self-map of X^+, say $\sigma^+ : X^+ \to X^+$. Moreover, this remains true for σ which *eventually commute* with σ_i, i.e.

$$DIST(\sigma \circ \sigma_i, \sigma_i \circ \sigma) \to 0 \text{ for } i \to \infty.$$

An eventual $\{\sigma_i\}$-contraction is called *uniform* if for each $d > 0$ there is a *cofinite* (i.e. with finite complement) subset $I(d) \subset I$, such that every $\{\sigma_i\}$-stable subset $U \subset X$ of diameter$\leq d$ satisfies

$$diam(\sigma_i(U)) \leq \frac{1}{2} diam(U) \text{ for all } i \in I(d).$$

Call $x \in X$ an *eventual fixed point* of σ if

$$dist(\sigma_i(x), x) \to 0 \text{ for } i \to \infty.$$

The set of eventually fixed points is (obviously) *invariant* under every uniformly continuous map $\sigma : X \to X$ which eventually commutes with σ_i.

In fact, this is also true if the set $\{\sigma_i\}$ is *eventually normalized* by σ, i.e. there exists a *proper* map $j : I \to I$, (i.e. the pull-backs of finite set are finite) such that

$$DIST(\sigma \circ \sigma_i, \sigma_{j(i)} \circ \sigma) \to 0 \text{ for } i \to \infty.$$

Let us spice these definitions,with the two (obvious) observations.

Relative FP Property. Let X be a complete metric space and Σ be a semigroup of uniformly continous maps $\sigma : X \to X$ such that Σ admits a family of maps $\sigma_i : X \to X$ which *eventually commute with all* $\sigma \in \Sigma$ and such that

$\star$ *the eventual contraction (if any) of the family* $\{\sigma_i\}$ *is uniform;*

$\star\star$ *the induced action of* Σ *on* X^+*, for* $\sigma \mapsto \sigma^+ : X^+ \to X^+$ *has a fixed point* $x^+_\bullet \in X^+$

Then Σ *has a fixed point* $x_\bullet \in X$.

Observe that uniqueness of $x^+_\bullet$ *does not*, in general, imply uniqueness of $x_\bullet$ and that the ordinary **ufp** for contracting maps σ reduces to the above with $\Sigma = \{\sigma_i\} = \{\sigma^i\}_{i>0}$ and X^+ being a single point, where the uniqueness of the fixed point of a *contracting* σ is obvious.

Relative Unique Shadowing Property. Let X, X^- be metric spaces, $p_- : X \to X_-$ be a continuous map controlled at infinity and let a semigroup Σ act on X, where this action preserves the partition of X into the fibers (i.e. pullbacks of points) of p_- and, thus, Σ acts on X_- as well.

Let $\{\sigma_i\}_{i\in I} \subset \Sigma$ be a subset of invertible maps $\sigma_i : X \to X$, such that

• the maps $\sigma_i^{-1} : X \to X$ eventually contract the fibers of p_- and this contraction is uniform;

•• the family $\{\sigma_i^{-1}\}$ eventually commutes with every $\sigma \in \Sigma$.

Let $\Theta \subset \Sigma$ be a generating subset.

If every quasi-orbit q_- with respect to Θ of the action of Σ on X_- is shadowed by an orbit o_-, then also every quasi-orbit q in X is followed by an orbit o; moreover, if $o_- = o_-(q_-)$ is unique for every q_-, then $o = o(q)$ is also unique.

2.3 Applications of Shadows.

Nilpotent Example. Let $\tilde{G}$ be a simply connected nilpotent Lie group and $\sigma_0 : \tilde{G} \to \tilde{G}$ be a *hyperbolic* automorphism which means that the corresponding automorphism $d\sigma_0$ of the Lie algebra $\tilde{g}$ of $\tilde{G}$ is a hyperbolic linear self-map. Then

*every group $\Sigma \ni \sigma_0$ of automorphisms of $\tilde{G}$ commuting with σ_0 has the unique shadowing property. Furthermore, this remains valid (and becomes more obvious) for **semi**groups Σ, provided σ_0 is an expanding map.*

Indeed, since the differential $d\sigma_0$ (of σ_0 at $id \in \tilde{G}$) is a *linear* self-map of the Lie algebra $\tilde{g}$ of $\tilde{G}$ and since the center $\tilde{c} \subset \tilde{g}$ is $d\sigma_0$-invariant, the action of $d\sigma_0$ on $\tilde{c}$ is also hyperbolic.

Let $\tilde{C}^+ \subset \tilde{G}$ be the central subgroup corresponding to the subspace in $\tilde{c}$ with eigenvalues having $|\lambda| > 1$. If all eigenvalues λ of the action of σ on $\tilde{c}$ have $|\lambda| < 1$ and if σ_0 is invertible in Σ (e.g. if Σ is a group), we just replace σ_0 by σ_0^{-1}.

Since the quotient map $\tilde{G} \to \tilde{G}^- = \tilde{G}/\tilde{C}^+$, being a Lie group homomorphism, is controlled at infinity, the relative shadowing property above yields the unique shadowing property in $\tilde{G}$ by induction on $dim(\tilde{G})$,

(One may equally use the induction on the *nilpotency degree*, rather than on dimension; thus, the above applies to *infinite dimensional* nilpotent Lie groups and also to *projective limits* of such groups.)

Remark. The hyperbolic splittings $G = G^+ \times G^-$ are *not*, in general, metrically controlled for "natural" (e.g. left-invariant Riemannian) metrics in G and $G^\pm$, since the subgroups $G^\pm \subset G$ are not necessarily normal; possibly, the control can be regained with some "unnatural" metrics.

Infra-nilpotent Shub-Franks Super-stability Theorem. Let B be a (possibly non-compact) infra-nil-manifold, that is a quotient of a simply connected nilpotent Lie group $\tilde{G}$ by a group Γ freely and discretely acting on $\tilde{G}$, such that Γ equals an extension of a discrete subgroup $\Gamma_0 \subset \tilde{G}$ by a finite group of automorphisms of $\tilde{G}$ preserving Γ_0.

Let $\sigma_{\tilde{G}} : \tilde{G} \to \tilde{G}$ be an automorphism which descends to a self-map of B with a fixed point $b_\bullet$, say $\sigma_B : B \to B$. Denote by σ_Γ the induced endomorphism of $\Gamma = \pi_1(B, b_\bullet)$.

Let V be a compact locally contractible space with a continuous self-map $\sigma : V \to V$ with a fixed point $v_\bullet$ and let $\sigma_* : \pi_1(V, v_\bullet) \to \pi_1(V, v_\bullet)$ denote the induced endomorphism of the fundamental group.

Let $h : \pi_1(V, v_\bullet) \to \Gamma$ be a homomorphism compatible with the two endomorphisms, i.e. the following diagram commutes

$$\overset{\sigma_*}{\circlearrowright} \pi_1(V, v_\bullet) \overset{h}{\to} \Gamma \overset{\sigma_\Gamma}{\circlearrowleft}, \text{ that is } h \circ \sigma_* = \sigma_\Gamma \circ h.$$

In the following two cases there exists a unique continuous σ/σ_B-morphism $f:(V,\sigma)\to(B,\sigma_B)$, such that $f(v_\bullet)=b_\bullet$ and the induced homomorphism $f_*:\pi_1(V,v_\bullet)\to\pi_1(B,b_\bullet)=\Gamma$ equals h.

Shub Case. *The self-map $\sigma_{\tilde G}:\tilde G\to\tilde G$ is expanding.*

Franks Case. *The differential $d\sigma_{\tilde G}$ of $\sigma_{\tilde G}:\tilde G\to\tilde G$ at $id\in\tilde G$ is hyperbolic and $\sigma:X\to X$ is a homeomorphism.*

Proof. Since $\tilde B=\tilde G$ is contractible, there exists a continuous map $(V,v_\bullet)\to(B,b_\bullet)$ which implements h.

The lift of this map to the respective Γ-coverings $\tilde V$ of V and $\tilde B=\tilde G$ sends $\tilde\sigma$-orbits to $\sigma_{\tilde G}$-quasi-orbits.

The shadowing orbits of these, which are provided by the nilpotent example, transform this lift to a $\tilde\sigma/\sigma_{\tilde G}$-morphism.

Since Γ is normalized by $\sigma_{\tilde G}$, this morphism is Γ-equivariant. QED.

Remarks and Questions.

(a) The fixed point $x_\bullet\in X$ of σ is not truly needed as one can always extend σ to a self-map σ' of a path connected $X'\supset X$ *with* a fixed point $x'_\bullet\in X'$.

(b) The contractibility of $\tilde B$ and the freedom of the action of Γ are used only for an implementation of $h:\pi_1(X)\to\Gamma$ by an equivariant map $\tilde f_0:\tilde X\to\tilde B$. Granted such an $\tilde f_0$, one may drop the freedom of the Γ-action on B. Thus, for example, one obtains intersting self-homeomorphisms of simply connected spaces, such as the sphere $S^2=\mathbb{T}^2/\pm1$.

(c) Possibly, infra-nilpotent (B,σ_B) are the only "superstable" compact *topological manifolds.* But, probably, there are interesting "superstable" B with a complicated, e.g. non-locally compact and/or fractal, local topology.

(d) The images of **F**ranks- **A**bel morphisms $(X,\sigma)\to(B,\sigma_B)$ are particular *closed connected σ_B-invariant* subsets in (B,σ_B).

What are, for example, these images for the FA maps of closed surfaces X^2 with, say, *pseudo-Anosov* homeomorphisms $X^2\to X^2$?

There are lots of other such closed connected σ_B-invariant subsets, e.g. the "walls" of *Markov partitions.* Can one explicitly describe the "topologically simplest" of them, e.g. locally contractible and/or having equal topological and Hausdorff dimensions, say for hyperbolic automorphisms of the tori $A=\mathbb{T}^N$?

Jarek Kwapisz pointed out to me that the description problem for closed invariant subsets for hyperbolic automorphisms of tori was raised by M.Hirsh [43] and that, according to a conjecture attributed to Smale in [3], hyperbolic automorphisms of tori admit no invariant *compact topological submanifolds*, except for unions of subtori.

Can one systematically describe n-dimensional spaces, $n<N$, with self-homeomorphisms, say (X,σ), with a given Franks-Abel Jacobian (A,σ_A), i.e. with $rank(H_1(X))=N$ and where σ induces a given automorphism of $H_1(X)$?

(e) Some pseudo-Anosov homeomorphisms (X,σ) are obtained from hyperbolic automorphisms σ_0 of 2-tori $\mathbb{T}^2$ by taking ramified coverings X^2 of $\mathbb{T}^2$ with the ramification locus contained in the fixed-point set of σ_0, where the resulting ramified covering $X\to T^2$ equals the corresponding Franks map.

This map is non-injective, but the full Franks-Abel map $X=X^2\to A=\mathbb{T}^N$, $N=rank(H_1(X^2))$ may seem injective for pseudo-Anosov $\sigma:X^2\to X^2$ with hyperbolic $\sigma_*:H^1(X^2)\to H^1(X^2)$. In fact, it is shown in [4] that these maps are almost everywhere one-to-one.

However, this I learned from Jarek Kwapisz, certain FA maps may contain horseshoes of double points [3] and, it was recently proven by Jarek Kwapisz with Andy Bouwman, these maps are never injective for genus two surfaces X (where the torus has dimension $4 = 2dim(X)$).

This strikes contrast with the complex analytic Abel-Jacobi maps which *are injective* by Torelli theorem. Yet, one wonders if Franks-Abel maps and invariant subsets of hyperbolic toral automorphisms come as limit sets of "nice complex analytic somethings", similarly to how quasi-circles appear as limit sets of complex analytic actions of discrete groups on the Riemann sphere. (The formula 2.7 in [4] is indicative of such an "analytic connection" as was pointed out to me by Jarek Kwapisz.)

(f) What are "ramification constructions" for automorphisms of $\mathbb{T}^N$ for $N > 2$, and of infra-nilmanifolds in general?

(Besides unions of codimension two subtori one, probably, can ramify $\mathbb{T}^N$ along "wild" codimension two subsets, such as the images of FA-maps of genus two surfaces in $\mathbb{T}^4$.)

What is a "good natural" class of self-homeomorphisms encompassing Anosov along with $2D$-pseudo-Anosov maps and closed under Cartesian products and ramified coverings?

"Connected Hyperbolicity". Let $\{\Sigma_j\}_{j\in J}$ be a collections of transformation groups of an X and denote by $(E = E_{fix}, J)$ the graph on the vertex set J where the edges $e \in E$ correspond to the pairs of subgroups (Σ_j, Σ_k), $j,k \in J$, such that

the intersection $\Sigma_{jk} = \Sigma_j \cap \Sigma_k$ is *normal* in Σ_j as well as in Σ_k,

the action of Σ_{jk} on X has the **ufp** property.

Similarly, define the graph (E_{shad}, J) with the unique shadowing property in lieu of **ufp**.

It is obvious that

*If the set E_{fix} is non-empty and the graph (E_{fix}, J) is connected, then the action of the group Σ generated by all Σ_j satisfies the **ufp** property.*

Consequently, if the graph (E_{shad}, J) is connected, then Σ has the unique shadowing property.

This applies to certain "sufficiently hyperbolic" homeomorphisms groups, e.g. linear groups Σ with "many" hyperbolic $\sigma \in \Sigma$ but the "connected hyperbolicity", as it stands, has limited applications, since the connectedness often fails for (E_{shad}, J); however, it may be satisfied by a larger graph of "virtual subgroups" in Σ, e.g. corresponding to subgroups in a group $S \supset \Sigma$.

For example, such "virtual connectedness" holds for lattices in semisimple groups of $\mathbb{R}$-rank≥ 2.

Questions. (A) Let a group Σ act by linear transformations σ of a Banach (e.g. Hilbert) space X, where all (or, at least, "many") $\sigma \neq id$ are split hyperbolic.

Is there a general "virtual connectedness" criterion for the unique shadowing property of such an action in the spirit of Kazhdan's T and/or of the Mostow-Margulis (super)rigidity?

(Mostow's proof of the rigidity of cocompact lattices for $rank_{\mathbb{R}} \geq 2$ depends on connectedness of *Tits' geometries* of the ideal boundaries of the orresponding symmetric spaces, and similar ideas underly "softer" arguments by Kazhdan, Margulis and Zimmer.)

Let Σ be a group of linear transformations of $\mathbb{R}^n$. Then every Σ-quasi-orbit q is shadowed by σ-orbits for hyperbolic $\sigma \in \Sigma$ and Σ acts on the space $H(q)$ of these orbits; yet, it may fail to have a fixed point in there. For example, this typically happens for free non-Abelian groups Σ.

(B) What is the dynamics of the action of Σ on the closure of $H(q)$? When does it admit a *compact* invariant subset?

Suppose Σ is a *word hyperbolic* group and let $\partial_\infty = \partial_\infty(\Sigma)$ denote its *ideal boundary.*

(C) Is there a natural map from $\partial_\infty \times \partial_\infty \setminus$ *diagonal* to the clousure of $H(q)$?

(D) Are there meaningful examples of the unique shadowing for actions of *non-elementary hyperbolic* groups Σ? Is Kazhdan's T relevant for this?

Let Σ be a group of hyperbolic conformal transformations of the sphere S^n and let q be a quasi-orbit of the corresponding action of Σ on the tangent bundle $T = T(S^n)$ or on some associated bundle, e.g. on $\bigwedge^n T$. Let Q denote the space of Σ-quasiorbits in T.

(E) What are "interesting" invariant subsets (minimal? compact? finite?) of the $\bullet$-action of Σ on Q, e.g. those contained in the closure of the Σ-orbit of a single $q \in Q$? (Recall that the bullet action of a $\sigma \in \Sigma$ on $q : \Sigma \to T$ is given by $\sigma_\bullet(q)(\sigma') = \sigma^{-1} \circ q(\sigma \cdot \sigma') : \sigma' \mapsto T$).

What happens if we enlarge Σ by a group of automorphisms of the bundle $T \to S^n$?

2.4 Combinatorial Reconstruction of Shub-Franks Group Actions.

Franks, (Shub) superstability theorem, applied to a hyperbolic automorphism (expanding endomorphism) of compact infra-nil-manifold, $\sigma : B \to B$, shows that the space B and the transformation σ are uniquely determined by purely algebraic data encoded in the automorphism $\overset{\sigma_*}{\circlearrowright} \Gamma = \pi_1(B)$.

Let us describe a *functorial construction* , called *combinatorial reconstruction*

$$\circlearrowright \Gamma \rightsquigarrow \circlearrowright B$$

which is built into the Shub-Franks argument.

Let Γ be a group with a left invariant metric, e.g. coming with a finite generating subset in Γ and let $\Sigma = \Sigma_\Gamma$ be a semigroup of endomorphisms (e.g. group of automophisms) $\sigma = \sigma_\Gamma : \Gamma \to \Gamma$.

Denote by $Q_\varepsilon = Q_\varepsilon(\Gamma, \Sigma, \Theta)$ the space of ε-*orbits* that are maps $q : \Sigma \to \Gamma$, such that

$$DI(q) = DI_\Theta(q) = \sup_{\sigma \in \Theta} \sup_{\sigma' \in \Sigma} dist_\Gamma(\sigma(q(\sigma')), q(\sigma \cdot \sigma')) \le \varepsilon$$

where Θ is a given generating subset in Σ, which we assume being *finite* in most of what follows.

Endow Q_ε with the topology of point-wise convergence (or, rather, point-wise *stabilization* for Γ discrete) in the space of maps $q : \Sigma \to \Gamma$

Observe that Γ embeds into Q_ε via the orbit map $\gamma \mapsto o = o_\gamma$ for $o_\gamma(\sigma) = \sigma(\gamma)$; thus, Γ acts on Q_ε by $\gamma(q)(\sigma) = o_\gamma(\sigma) \cdot q(\sigma)$ and this action commutes with the action of Σ on Q_ε, that is $\sigma(q(\sigma')) = q(\sigma\sigma')$.

Let

$$\tilde{B}_\varepsilon = \tilde{B}_\varepsilon(\Sigma_\Gamma) = Q_\varepsilon/[DIST_\Gamma < \infty]$$

that is the space of $DIST$-finiteness component of Q_ε (for our $DIST(q_1, q_2) = \sup_\Sigma dist_\Gamma$) with the quotient space topology, and let

$$\tilde{B} = \tilde{B}(\Sigma_\Gamma) = \bigcup_{\varepsilon > 0} \tilde{B}_\varepsilon, \text{ and } B = B(\Sigma_\Gamma) = \tilde{B}/\Gamma,$$

where Σ naturally acts on B since the action of Σ on $\tilde{B}$ commutes with the action of Γ.

Say that

Σ_Γ *is rigid* if there exists a (possibly large, but finite) $\varepsilon = \varepsilon(\Gamma, dist_\Gamma, \Sigma_\Gamma, \Theta) > 0$, such that, for every quasi-orbit q, there is an ε-orbit, say $q_\varepsilon = q_\varepsilon(q)$, such that $DIST_\Gamma(q, q_\varepsilon) < \infty$,

and

Σ_Γ *is Divergent* if there is a constant $d = d(\Gamma, dist_\Gamma, \Sigma_\Gamma, \Theta, \varepsilon) > 0$, such that the inequality $DIST_\Gamma(q_1, q_2) \leq 2d$ for two ε-orbits, implies that $DIST_\Gamma(q_1, q_2) \leq d$.

This implies that the inequality $DIST_\Gamma(q_1, q_2) \leq d$ is a *transitive* (i.e. equivalence) relation between ε-orbits.

Divergence can be also regarded as a *convexity-type* inequality or a *maximum principle* satisfied by the function

$$di(\sigma) = dist_\Gamma(\sigma(q_1(\sigma)), q_2(\sigma)) :$$

if two ε-orbits, $q_1, q_2 : \Sigma \to \Gamma$, are $2d$-close on a (large) ball in Σ then they are d-close at the center of the ball.

Shadowing $\Rightarrow$ Rigidity. Let $\Gamma = \pi_1(B, b_\bullet)$ for a locally contractible space B and let Σ be induced by a semi-group of self-maps of $(B, b_\bullet)$. If B is *compact*, then, obviously, the (not necessarily unique)

shadowing property of the lifted action of Σ to the universal covering $\tilde{B}$ of B implies rigidity of Σ_Γ.

Furthermore,

if the shadowing is unique and uniform, the action is divergent as well as rigid.

In particular "connectedly hypebolic" (e.g. cyclic hyperbolic) automorphisms groups Σ of compact infra-nil-manifolds $B = \tilde{G}/\Gamma$ are rigid and divergent. (We did not pay much attention to uniformity of shadowing in these examples, but the proofs of unique shadowing automatically deliver the uniformity as well since hyperbolic actions are *globally expansive*: the sup_Σ distance (i.e. $DIST$) between every two distinct orbits is infinite.

Remarks. (a) The above construction is just a "discrete time" counterpart to the Efremovich-Tikhomirova-Mostow-Margulis description of the ideal boundary of a hyperbolic group via quasi-geodesic rays.

(b) If we drop the rigidity and/or the divergence condition then the resulting space $B = B(\Sigma_\Gamma)$ and $B(\sigma_\Gamma)$ may become non-compact and/or *non-Hausdorff*; yet, such a B may may have non-trivial Hausdorff Σ-equivariant quotient spaces; besides, the geometry of "non-Hausdorffness" of a B may be interesting in its own right.

(c) There are by far (?) more rigid divergent actions then those on infra-nil-manifolds, especially if we allow infinitely generated groups, e.g. $\Gamma = \mathbb{Z}^\infty$, where the construction must incorporate a choice of a suitable metric on such a Γ.

Questions. (a) Find/classify rigid divergent (semi)groups of automorphisms (endomorphisms) of *finitely generated* (finitely presented?) groups. For example:

Which automorphism groups Σ of $\mathbb{Z}^N$ are rigid?

Are, for instance, "virtually connectedly hyperbolic" Σ rigid? (Some of these are known to be locally rigid [51].)

Which automorphisms groups of free groups and of surface groups are rigid?

Are automorphisms of fundamental groups of surfaces induced by pseudo-Anosov maps rigid? (This, probably, follows from [18] and [21].)

Let A be a ramified covering of $\mathbb{T}^N$ with ramification locus being the union of flat codimension two subtori in general position. The fundamental group of such an A often admits hyperbolic-like automorphisms.

Are these ever rigid?

Notice that there are lots of homeomorphisms in the world and typical groups generated by several homeomorphisms are *free*; "intersting" Σ-actions of groups Σ with "many" relations can not be constructed at will – such actions are usually associated with specific geometric structures on corresponding spaces.

Also, there is no systematic way to produce *finitely presented* groups Γ with "large" automorphisms groups Σ. Notice that every Γ acted by Σ is obtained from the free group $F(\Sigma)$ freely generated by $\sigma \in \Sigma$ which is factorized by a Σ-invariant set $\mathcal{R}$ of relations. If $\mathcal{R}$ is "small", the resulting Γ remains infinitely generated; if $\mathcal{R}$ is too large, Γ becomes trivial. It seems difficult to strike the right balance.

Yet, there is a construction by Rips [67] of "generic" finitely generated (but not finitely presented) groups Γ with a given Σ in their outer automorphism groups.

On the other hand, the quotient groups $F(\Sigma)/\mathcal{R}$ acted upon by Σ look interesting even for infinitely generated Γ.

(b) When does $\tilde{B} \supset \Gamma$ admit a (natural?) group structure extending that of Γ?

(c) The group Γ enters the definition of "rigid" and/or "divergent" only via its (word) metric, but eventually, Γ acts on $\tilde{B}$. Can one incorporate the action of Γ into "rigid/divergent" to start with?

(d) The notion of a quasi-orbit make sense for *partially defined* actions on Γ, such as the obvious partial action of $GL_N(\mathbb{Q})$ on $\mathbb{Z}^N$, e.g. the (contracting) inverse σ^{-1} of an expanding endomorphism $\sigma : \mathbb{Z}^N \to \mathbb{Z}^N$.

Namely, an ε-orbit is a map $q : \Sigma \to \Gamma$ such that for each $\sigma \in \Sigma$ there exists $\gamma = \gamma(q, \sigma) \in \Gamma$ with $dist_\Gamma(\gamma, q(\sigma)) \leq \varepsilon$ such that γ belongs to the domain of definition of all $\theta \in \Theta$, for a given generating subset $\Theta \subset \Sigma$, and such that $dist_\Gamma(\theta(\gamma), q(\theta \cdot \sigma) \leq \varepsilon$ for all $\theta \in \Theta$.

However, there are no full-fledged orbits and the corresponding action of Γ on Q, is only partially defined.

What kind of spaces B with Σ-dynamics can be constructed for partial actions? (The above contracting σ^{-1} serves as an encouraging example.)

If Σ acts by *injective* homomorphisms defined on subgroups of *finite index* in

Γ one defines the (generalized) *semidirect product* $\Gamma \rtimes \Sigma$ that is the set of partial transformations of Γ generated by, say, left translations $\gamma : \Gamma \to \Gamma$ and all partial $\sigma : \Gamma \to \Gamma$.

Probably, the rigidity of the original action, can be adequately addresses in terms of the group $\Gamma^\star = \Gamma \rtimes \Sigma$, as suggested by the construction from the next section. and /or in the spirit of *permutational bimoduli* of Nekrashevych [63], [45] where one works, instead of partial quasi-orbits, with the, say right, action of Σ on the product $\Gamma^\star \times \Delta$ and the obvious left action of Γ, for some finite set Δ, where the two actions commute.

(If Δ is the one element set, these correspond, modulo $DIST < \infty$ condition, to the action of Σ on $\Gamma^\star$ by conjugations.)

(e) There is a remarkable class of groups Γ discovered by Grigorchuk, where such a Γ admits subgroups of finite index, say $\Gamma_1, \Gamma_2 \subset \Gamma$, and a *contracting* (in a suitable sence) isomorphism of a Cartesian power of Γ_1 to Γ_2, say $\sigma : \Gamma_1^k \to \Gamma_2$, that is a partially defined contracting "isomorphism" $\sigma : \Gamma^k \to \Gamma$.

(See [63], [45] for a combinatorial (re)construction of a dynamical system associated to this kind of σ.)

Observe that every *k-bracketing* of a finite ordered set I, e.g.

$$\big([(\bullet\bullet)\bullet](\bullet\bullet)\big)\big(([(\bullet\bullet)(\bullet\bullet)]\bullet)\bullet\big) \text{ where } k = 2 \text{ and } card(I) = 11,$$

provides a partial map $\Gamma^I \to \Gamma$. The totality $\Sigma^\star$ of such maps make what is called an *operad*.

Can one make the full use of the this operad $\Sigma^\star$ generated by σ in Nekrashevych' construction of B with $\Sigma^\star$ "acting on B"?

Is there, in general, a meaningful dynamical theory of operads acting on (compact?) spaces?

(A system of N bracketing on a set I of cardinality N gives one a partial endomorphism $\Gamma^N \to \Gamma^N$ while the full operad comprises a "functorially coherent" family of all such maps for all possible bracketing.)

2.5 Symbolic Dynamics, Markov Coding and Markovian Presentations.

Recall the obvious *Bernoulli shift action* of a discrete (semi)group Σ on the space Δ^Σ of maps of Σ to a set Δ, let $\Theta \subset \Sigma$ be a finite subset and take some $M \subset \Delta^\Theta$.

Denote by $C = C_M \subset \Delta^\Sigma$ the pullback of M under the obvious (restriction) map $\Delta^\Sigma \to \Delta^\Delta$ and define the corresponding *Markov (sub)shift* (of finite type) $Q = Q_M \subset \Delta^\Sigma$ as

$$Q = \bigcap_{\sigma \in \Sigma} \sigma(C).$$

Next, observe that $Q \times Q$ is a Markov (sub)shift in $(\Delta \times \Delta)^\Sigma$ and let $R \subset Q \times Q$ be a Markov subshift in this $Q \times Q$.

If this R, regarded as a binary relation on Q, is symmetric and transitive, let $X = Q/R$ be the quotient space of Q by this *equivalence relation* R and observe that the (semi)group Σ naturally acts on this X.

Such an (X, Σ) is called *Markov hyperbolic* (dynamical system), usually for *finite* sets Δ, and the corresponding surjective map $Q \to X = Q/R$ is called (finitery) *Markovian presentation.*

Let us explain why

"rigid Markovian" for a finitely generated (semi)group Σ *of automorphisms (endomorphisms) of a finitely generated group* Γ *implies "rigid hyperbolic" for* $(B(\Sigma_\Gamma), \Sigma)$.

Given subsets $\Delta \subset \Gamma$ and $\Theta \subset \Sigma$ take a subset $M \subset \Delta^\Theta$ and define *M-orbits* $q : \Sigma \to \Gamma$ by the condition

$$m(\theta) = (q(\theta \cdot \sigma))^{-1} \cdot \theta(q(\sigma)) \in \Delta \text{ for all } \sigma \in \Sigma \text{ and } \theta \in \Theta$$

and this function $m : \theta \mapsto \delta \in \Delta$ belongs to M for all $\sigma \in \Sigma$. The orbits $o : \Sigma \to \Gamma$ correspond to $\Delta = id \in \Gamma$, i.e. $(o(\theta \cdot \sigma))^{-1} \cdot \theta(o(\sigma)) = id$ and the space $\tilde{Q}_M$ of M-orbits is invariant under the multiplication by orbits, since

$$(o(\sigma \cdot \sigma') \cdot q(\sigma \cdot \sigma'))^{-1} \cdot \sigma(o(\sigma')) \cdot \sigma(q(\sigma')) = (q(\sigma \cdot \sigma'))^{-1} \cdot \sigma(q(\sigma')).$$

(We switched from the notation Q_ε used earlier to $\tilde{Q}_M$ but our ε-orbits are special cases of M-orbits)

Thus every $M \subset \Delta^\Theta$, for given finite subsets $\Theta \subset \Sigma$ and $\Delta \subset \Gamma$, define a Markov Σ-shift $Q_M = \tilde{Q}_M/\Gamma$.

Given an arbitrary finite generating subset $\Theta \subset \Sigma$, a *sufficiently large* finite $\Delta \subset \Gamma$ and $M = \Delta^\Theta$, then the quotient $\tilde{B}_M$ of $\tilde{Q}_M$ by the equivalence relation $\tilde{R} \subset \tilde{B}_M \times \tilde{B}_M$ given by $[DIST_\Gamma < \infty]$ does not depend on Δ: this is our old $\tilde{B}$.

The group Γ (embedded into the space of maps $\Sigma \to \Gamma$ via Σ-orbits as earlier, i.e. by $\gamma \mapsto \sigma(\gamma)$) acts on this R and the quotient $R/\Gamma \subset B_M \times B_M$ is Markov by the Marcovian property of Σ_Γ which, recall, corresponds to the uniformity of shadowing. QED.

Remarks. (a) "Markov symbolic coding" can be traced to the work by Hadamard (1898) and Morse (1921) on geodesics in hyperbolic surfaces. The above argument is essentially the same as the derivation of Markov property for locally split hyperbolic (Bowen-Anosov) actions of $\mathbb{Z}$ (defined below) on compact spaces from Anosov's *local shadowing lemma* and local expansiveness. This was exploited/refined by Sinai and Bowen in their *Markov partition theory* [5] [72] and then extended to general Markov hyperbolic transformations in [22] (where these were called "finitely presented dynamical systems".)

(b) The major advantage of "Markov hyperbolicity" (originally called just "hyperbolicity" [25]) over "split hyperbolicity" is the applicability of "Markov" to arbitrary (semi)group Σ, not only to $\Sigma = \mathbb{Z}$ and/or $\mathbb{Z}_+$.

However, non-trivial examples of Markov hyperbolic actions of non-cyclic groups Σ in [25] were limited to *word hyperbolic groups* Σ acting on their *ideal boundaries.* (These groups were called "coarse hyperbolic" in [25].)

We shall see in section 3 below further examples that became available due to the recent progress in the *geometric rigidity theory.*

Anosov-Bowen Systems. An action of $\mathbb{Z}$ by uniformly continuous homeomorphisms of a metric space X is called *locally split hyperbolic*, if there exists a $\rho > 0$ such that every ρ-ball in X is contained in a split neighbourhood $U = U^+ \times U^- \subset X$ such that

the fibers of the projection $U \to U^+$ are $\{\sigma^i\}_{i \to +\infty}$-stable, i.e. uniformly eventually contracted by the positive powers of σ, that is the transformation corresponding to $1 \in \mathbb{Z}$, the fibers of $U \to U^-$ are $\{\sigma^i\}_{i\to-\infty}$-stable,

the projections $U \to U^\pm$ are uniformly continuous with the moduli of continuity independent of U.

Anosov Shadowing Lemma, says that if all metric spaces $U^\pm$ are complete then

there exists an $\varepsilon_0 > 0$, such that every ε-orbit $q : \mathbb{Z} \to X$ with $\varepsilon \le \varepsilon_0$ is $\delta = \delta(\varepsilon)$-shadowed by a unique orbit $o : \mathbb{Z} \to X$ where $\delta = \delta(\varepsilon) \to 0$ for $\varepsilon \to 0$.

Proof. Let $X^{\mathbb{Z}}$ be the space of maps with the (possibly infinite) metric $DIST = \sup_{\mathbb{Z}} dist_X$ and observe that the local split hyperbolicity of an action on X implies that the corresponding $\bullet$-action of $\mathbb{Z}$ on $X^{\mathbb{Z}}$ is also split hyperbolic.

Thus we need to show that every ε-fixed point of σ is accompanied by a nearby fixed point.

This is done exactly as in the globally split case with a little caveat that the two contracting actions are only *partially* defined and one needs to use the *local* **ufp** property of contracting maps from section 2.

The Markov partition theorem of Sinai-Bowen says, in effect, that

every locally split hyperbolic action of $\mathbb{Z}$ on a compact space X admits a cofinite Markov presentation $Q \to X$, i.e. where the cardinalities of the pullbacks of all $x \in X$ are bounded by a constant $< \infty$.

This, as it was shown by [22], remains true for *all Markov hyperbolic $\mathbb{Z}$-actions.*

Question. Which Markov hyperbolic Σ-actions admit cofinite Markov presentations?

This is, apparently, so for the *boundary actions* of the word hyperbolic groups Σ and for many (all?) Markov hyperbolic actions of free groups.

There is nothing of "holomorphic" in all this so far. Of course, endomorphisms $\sigma : \mathbb{T}^N \to \mathbb{T}^N$ analytically extend to holomorphic endomorphisms $\sigma_{\mathbb{C}} : (\mathbb{C}^\times)^N \to (\mathbb{C}^\times)^N$ where $(\mathbb{C}^\times)^N$ are affine *toric* (hence, rational) varieties, where, small rational deformations of $\sigma_{\mathbb{C}}$ may be of some interest.

Questions. (a) Does $\mathbb{C}P^N$ admit a birational action by the group $SL_{N+2}(\mathbb{Z})$ which does not factor through a finite group?

(b) Does it admits such an action by $SL_{N+1}(\mathbb{Z})$ which is not conjugate to a projective action?

(It is "No" for (b), hence, for (a), see [14].)

Similarly, compact real nil-manifolds G/Γ complexify to $G_{\mathbb{C}}/\Gamma$ where these quotients of complex nilpotent groups $G_{\mathbb{C}}$ are, apparently, algebraic (affine? rational?) as well.

But this is not quite a kind of "holomorphic connection" we are looking for – the first whiff of "Kähler" can be felt, however, in the examples we shall see presently.

3 Inner Rigidity and Markov Coding.

A finitely generated subgroup Σ in a finitely generated group Γ is called *inner rigid* if the action of Σ on Γ by inner automorphisms is rigid. In particular, Γ is called inner rigid, if its action on itself by inner automorphisms is rigid.

Let us express this in geometric language and show that actions of discrete rigid groups on homogeneous spaces are Markov.

3.1 Quasi-isometries of Lie Groups and Combinatorial Reconstruction of Homogeneous Spaces.

A (possibly discontinuous) map $f : X \to Y$ is called *d-eventually λ-Lipschitz* if $dist_Y(f((x_1), f(x_2)) \leq \lambda \cdot dist_X(x_1, x_2)$ for all those $x_1, x_2 \in X$ where $dist_X(x_1, x_2) \geq d$. for some constant $d = d(f, \lambda) < \infty$.

An eventually Lipschitz map is called a *quasi-isometry* if it is invertible modulo *translations*, where a map $\tau : X \to X$ is called a δ-translation if $supdist(\tau, id) \leq \delta < \infty$.

More specifically, an $f : X \to Y$ is a (d, λ)*-quasi-isometry* if there is a $g : Y \to X$ such that both f and g are d-eventually λ-Lipschitz and the composed self-maps $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are translations. Just "quasi-isometry" means λ-quasi-isometry for some $\lambda < \infty$. (See [66] and references therein for another concept of approximate isometry.)

Quasi-isometric Rigidity and Completeness. A metric space X is called *quasi-isometrically rigid* if there is some $\lambda < \infty$ such that every quasy-isometry $q : X \to X$ lies within bounded distance from a λ-quasi-isometry, say q_λ such that $supdist(q, q_\lambda) < \infty$.

X is called *quasi-isometrically complete* if the group of quasi-isometries modulo translation of X, denoted $qis(X)$, equals the isometry group $iso(X)$.

Clearly, this *completeness implies rigidity* and

co-compact discrete isometry groups Σ of rigid spaces X are inner rigid.

What is non-trivial is that (see [64] [52]

Let X be a Cartesian product of irreducible symmetric spaces of non-compact type and irreducible Euclidean buildings. If X does contains among its factors trees, as well as real hyperbolic and complex hyperbolic spaces then X is quasi-isometrically complete.

Accidentally(?) this the same assumption which ensures the T-property of the group $G = iso(X)$.

Markov Corollary. *Let Σ, Γ be discrete cocompact subgroups in the above group $G = iso(X)$, e.g. $\Sigma = \Gamma$. Then the right action of Σ on the left quotient space G/Γ is Markov hyperbolic.*

Indeed, what one needs besides rigidity is the expansiveness property which is obvious in the present case since G has trivial center and $\sup_{x \in X} dist(g(x), x)) = \infty$ for all $g \neq id \in G$.

Remarks and Questions (a) Probably, "generic" finitely presented groups Γ are q.i. complete, i.e. $qis(\Gamma) = \Gamma$, but $qis(\Gamma)$ may be quite large for certain (which?) groups Γ [17], [79].

(b) Let Σ be a rigid (and divergent) group of automorphisms of Γ or, more generally, a rigid semigroup of partially defined endomorphisms.

Probably, the semidirect product $\Gamma \rtimes \Sigma$ is *inner rigid* except for a specific list of examples (possibly?) including certain (all) expanding endomorphisms of $\mathbb{Z}$ and of nilpotent groups of nilpotency degree 2 with cyclic center.

(b) The quasi-isometric completeness allows a canonical reconstruction of G in terms of a given co-compact subgroup $\Gamma \subset G = iso(X)$ which is, probably, functorial for *quasi-isometric embeddings* (injective homomorphisms?) $\Gamma_1 \to \Gamma_2$ in many cases.

Thus, for example, some compact locally symmetric Kähler manifolds Y can be reconstructed from their fundamental groups Γ in an essentially combinatorial Markovian fashion.

(c) What are non-cocompact $\Sigma \subset G$ for which the action on G/Γ is Markov hyperbolic? Is it true whenever $vol(G/\Sigma) < \infty$?

(d) What is the story for the real and, most interestingly, for the complex hyperbolic spaces X?

One still can reconstruct $iso(X)$ in terms of a $\Gamma \subset iso(X)$ as follows.

Let $\partial_\infty(X)$ be the ideal (hyperbolic) boundary of X. Assume for the simplicity sake, that the action of Γ on X is free and orientation preserving and let $\mathcal{C}_n$, $n = dim(X)$ denote the space of Γ-invariant Borel measures C on $(\partial_\infty(X))^{n+1}$ of finite mass, i.e. the (simplicial) L_1-norm $||C|| < \infty$, where $C \in \mathcal{C}$ represent the fundamental homology class $[X/\Gamma]$ of X/Γ, that is a generator of the (infinite cyclic!) group $H_n(\Gamma, \mathbb{Z})$.

Let $\mathcal{G}$ be the group of self-homeomorphisms h of $\partial_\infty X$ the action of which on $\mathcal{C}_n$ satisfies $||h(C)|| = ||C||$.

If $X = H^n_{\mathbb{R}}$ is real hyperbolic and $n \geq 3$, then the group $\mathcal{G}$ equals G since the support of C of minimal mass equals the set of *regular* ideal n-simplices in X by Milnor -Haagerup-Munkholm theorem.

Probably, $\mathcal{G} = G$ also in the complex hyperbolic case (and, properly stated, in *all* symmetric spaces with no flat and $H^2_{\mathbb{R}}$-factors).

But regardless of whether this is true or not for the complex hyperbolic spaces $X = H^m_{\mathbb{C}}$ of complex dimension $m \geq 2$, one can replace $[X/\Gamma] \in H_n(\Gamma; \mathbb{Z})$ by the *Kähler (Toledo) class* $C \in H_2(\Gamma; \mathbb{Z})$. that is the Poincare dual of ω^{m-1} for the Kähler class $\omega \in H^2(X/\Gamma; \mathbb{Z}) = H^2(\Gamma; \mathbb{Z})$. Equivalently, C is the class in $H_2(X/\Gamma)$ which maximizes the ratio $\omega(C)/||C||_{l_1}$ for the simplicial l_1-norm on homology (in the sense of [26]).

In either case, we need a distinguished class $\omega \in H^2(X/\Gamma; \mathbb{Z})$, and then, it is easy to see that the group $\mathcal{G} = \mathcal{G}(\omega)$ of self-homeomorphisms of $(\partial_\infty(X))^3$ preserving the norms of 2-cycles equals $G = iso(H^m_{\mathbb{C}})$.

(If we choose a "wrong" class $\omega \in H_2(\Gamma)$, then, most likely, $\mathcal{G}(\omega)$ will be equal to Γ itself or a finite extension of Γ.)

(e) What are a measure-theoretic (instead of quasi-isometric) counterparts to rigidity, quasi-orbits, etc. that would be applicable to *non-cocompact* lattices $\Gamma \subset G$ in the context of Margulis-Zimmer super-rigidity theory that would give, in particular, a canonical reconstruction of G from Γ?

Notice that the l_1-homology makes sense with the Poisson-Furstenberg boundary in lieu of the hyperbolic boundary and it is compatible with *measurable equivalences* between groups (that are "ergodic bimoduli" where Nekrashevitch type constructions may be possible) but all this was not studied systematically.

(f) Besides the simplicial l_1-norm on homology, there is another norm, which is defined via the *assembly map* α from the group of *reduced rational bordisms* $\mathcal{B}_*(\Gamma)$ to the *rational Wall surgery groups* $\mathcal{L}_*(\Gamma)$.

Recall that each $L \in \mathcal{L}_i$ is represented by a free module M over the rational group ring of Γ with some extra structure (e.g. a non-singular quadratic form for $i = 4j$) on M. Let $rank(L)$ denote the minimum of $rank(M)$ over all M representing L and

$$||L|| =_{def} \lim_{n\to\infty} \frac{1}{n} rank(n \cdot L).$$

Non-vanishing of this norm on $\alpha(B)$, $B \in \mathcal{B}_*(\Gamma)$, strengthens *Novikov higher signatures conjecture* that claims just non-vanishing of $\alpha(B)$ for $B \neq 0$. It is known, that that this norm does not vanish on the fundamental classes of compact locally symmetric Hermitian spaces of non-compact type with non-zero Euler class, [33], [57] but its role in rigidity remains unclear.

3.2 Stable Factorization of Rigid Flows.

Actions of such (softish) groups as $\mathbb{R}$ can not be stable in the above sense, since one can always reparametrize an action along the orbits. Accordingly, (super)stability is defined as preservation of the *partitions (foliations) into orbits* rather than of the actions themselves.

(This notion of stability is poorly adjusted to many "real life systems", e.g. to the amazing stability of the 24-hour *circadian rhythm* under variation of temperature, which, at the same time, can be greatly perturbed by a physically insignificant factor, e.g. by a bad news said in a low voice. Finding an adequate functorial concept of "selective stability" is a challenge for mathematicians.)

Examples of such (super)stable actions of $\mathbb{R}$ are *suspensions* of (super)stable actions of $\mathbb{Z} \subset \mathbb{R}$ and geodesic flow on manifolds of negative curvature.

(The Σ^+-suspension of an action of Σ on X for $\Sigma^+ \supset \Sigma$ is the natural action of Σ^+ on $(X \times \Sigma^+)/\Sigma_{diag}$.)

The "combinatorial reconstruction" in this context, say for $\mathbb{R}$-actions, defines the space of *unparameterized* orbits of such an action.

For example, if X is a δ-hyperbolic space X, then the corresponding (space of ideal) unparameterized geodesics correspond to (the space of) pairs of disjoint points in the ideal boundary of X. But this does not automatically deliver the "ideal geodesic flow space" Ω with an actual action of $\mathbb{R}$ with these orbits.

What one does have, however, at least in the cases at hand, is a larger space, say Ξ such that Ω, if it exists, comes as a quotient space of Ξ.

For example, if X is a geodesic hyperbolic space, one may take the space of isometric maps $\mathbb{R} \to X$ for Ξ. The images of these maps are distance minimizing geodesics in X and the natural action of $\mathbb{R}$ corresponds to the *geodesic flow.*

The quotient map $\Xi \to \Omega$ must bring two geodesics together if they have the same ends at the ideal boundary of X but this does not tell you which points on geodesics must be actually identified.

But this ambiguity is "homotopically trivial" and can be easily resolved by a (soft) partition of unity argument as follows.

Let Ξ be a metric space with a free continous action of a locally compact and compactly generated group Σ (that is either $\mathbb{R}^n$ or $\mathbb{Z}^n \subset \mathbb{R}^n$ in what follows) and let $\lambda \geq 0$ be constant such that the following three conditions are satisfied.

(1) the orbits maps $\Sigma \to \Xi$ are λ- bi-Lipschitz with respect to, say, maximal left-invariant metric, on Σ which equals a given metric on a compact subset generating Σ.

(2) If two orbits are *parallel* i.e. the Hausdorff distance between them is finite, then this distance is $\leq \lambda$.

(3) Every $2R$-ball in Ξ can be covered by at most $N = N(R)$-balls of radius R.

Let $h : \Sigma \to H = \mathbb{R}^n$ be a continuous homomorphism with compact kernel.

Then there exists a metric space Ω with a Lipschitz H-action on it and a continuous quasi-isometric map $P : \Xi \to \Omega$, which sends Σ-orbits to H-orbits and such that two orbits in Ξ go to the same orbit in Ω if and only if they are parallel.

Moreover, if Γ is a discrete isometry group of Ξ which commutes with the action of Σ, then Ω also admits a discrete isometric action of Γ which commutes with H and such that the map P is Γ-equivariant.

Proof (Compare with [27], [59]). An orbit preserving map $P : \Xi \to \Omega$ which identify parallel orbits defines a continuous *closed H-valued 1-cocycle* ρ on the set $\Pi \subset \Omega \times \Omega$ of the pairs of points that lie on *mutually parallel* Σ-orbits in Ξ, namely,

$$\rho(\chi_1, \chi_2) = P(\chi_1) - P(\chi_2)$$

which make sense for $(\chi_1, \chi_2) \in \Pi$, since $P(\chi_1)$ and $P(\chi_2)$ lie in the same H-orbit.

Conversely every continuous *closed H-valued 1-cocycle* ρ on $\Pi \subset \Omega \times \Omega$, that is an anti-symmetric function in two variables, $\rho(\chi_1, \chi_2)$, $(\chi_1, \chi_2) \in \Pi$, such that $\rho(\chi_1, \chi_3) = \rho(\chi_1, \chi_2) + \rho(\chi_2, \chi_3)$, defines a space Ω with an H action and a map $P : \Xi \to \Omega$.

For example if $h : \Sigma \to H = \mathbb{R}^n$ is an isomorphism, then Ω is defined as the quotient space of Ξ by the equivalence relation $R \subset \Xi$ that equals the zero set of ρ. In what follows, we stick to this case, since h *is* an isomorphism in most example and since the general case needs only extra notation.

Let us look at cocycles that are represented by locally finite sums

$$\rho = \sum_i \rho_i, \text{ where } \rho_i(\chi_1, \chi_2) = \phi_i(\chi_1) - \phi_i(\chi_2)$$

for some Lipschitz maps $\phi_i : \Xi \to H = \mathbb{R}^n$ with bounded (compact in the present examples) supports.

Such maps are easy to come by. Indeed every Σ-orbit, say $S \subset \Xi$, which is bi-Lipschitz to $\mathbb{R}^n$, admits a Lipschitz retraction $\Phi_S : \Xi \to S = H$ (i.e. $\Phi_S|S = id$) with the Lipschitz constant $\leq n \cdot \lambda$. This Φ_S can be cut off (i.e. made zero) to a $\phi_\xi = \phi_\xi(\chi)$, $\xi \in S$, $\chi \in \Xi$, by multiplying it with the function $f_\xi(\chi)$ which equals 1 on the (large) r-ball $B_\xi(R) \subset \Xi$ around ξ, which vanishes outside the concentric $2r$-ball and which equals $r^{-1}(2r - dist(\xi, \chi))$ in the annulus $B(2r) \setminus B(r)$.

What remains is to find a Γ-*invariant* collection of balls $B_i(r) = B_{\xi_i}(r)$ with a large r_i, such that the concentric balls $B_i(r/2)$ cover Ξ, while the intersection multiplicity of the balls $B_i(2r)$ is bounded by a constant $N < \infty$.

For example, if the action of Γ is cocompact, one may take the Γ orbit of a single (large) ball, and the general case needs a minor additional effort.

Remarks and Questions. (i) The main applications of Ω for hyperbolic groups Γ is showing that (certain) cohomology classes in Γ are bounded. Is there something similar for other *globally split hyperbolic $\mathbb{R}$-actions*, e.g. for the suspensions of hyperbolic $\mathbb{Z}$-actions?

The fundamental groups Γ of suspensions of known (all?) hyperbolic $\mathbb{Z}$-actions are amenable and their bounded cohomologies vanish. But possibly there is a meaningful notion of cohomology with *partially* defined cocycles.

(ii) Can one combinatorially reconstruct the $\mathbb{R}^k$-action on Weyl chambers in locally symmetric spaces X of $\mathbb{R}$-rank k in term of $\pi_1(X)$?

(iii) Does the above factorization property extend to *non-Abelian groups H*?

The above argument, possibly, can be extended to simply connected *nilpotent* Lie groups H. On the other hand, the conclusion may be even stronger for "rigid". i.e. (some) semisimple H.

3.3 Shadows of Leaves.

When is a "quasi-leaf" in a foliated manifold shadowed by a leaf?

We know this is so for the geodesic foliations of (finite and infinite dimensional) manifolds (and singular, i.e. $CAT(-\kappa)$, spaces) of negative curvature and, at least *locally*, for the orbit 1-foliations of general Anosov flows, such as suspensions of hyperbolic automorphisms (and expanding endomorphisms) of infranilmanifolds (where "locally" is, in fact, "semi-global").

Let us look at examples of k-foliations with $k > 1$.

Submanifolds Foliations. Given a smooth manifold X, let $\mathcal{F}_k(X)$ denote the space of pairs (x,S), where $x \in X$ and $S \ni x$ is a germ of an k-submainifold in X. Notice that $\mathcal{F}_k(X)$ is tautologically foliated, where the leaves correspond to *immersed* (with possible self-intersections) submanifolds in X.

We are mostly concerned with "small" $\mathcal{F}$*-saturated* submanifolds $G \subset \mathcal{F}_k(X)$ which are unions of leaves in $\mathcal{F}_k(X)$.

Examples. Let X is a Riemannian manifold. Then the the space marked geodesics in X, denoted $Geo_1 \subset \mathcal{F}_1(X)$, is an instance of our G. The space Geo_1 is foliated into geodesics in X – the orbits of the geodesic flow in (the unite tangent bundle of) X.

Similarly one defines the space $Geo_k \subset \mathcal{F}_k(X)$ the space of marked *totally geodesic* submanifolds in X.

Unlike Geo_1, the space $Geo_{k\geq 2}$ is empty for *generic* Riemannian manifolds X. This suggests another extension of Geo_1 to $k > 1$, namely $G = Plat_k \subset \mathcal{F}_k(X)$ – the space of marked *minimal* k-subvarieties in X that are solutions to the Plateau problem.

This G is infinite dimensional, but it may contain "nice" finite dimensional sub-foliations [28]

If X is a complex manifold and $k = 2l$ then one defines $Hol_l \subset \mathcal{F}_k(X)$ – the space of all complex analytic submanifolds $Y \subset X$ with $dim_{\mathbb{C}}(Y) = l$ which, for *Kähler* manifolds X, makes a subfoliation in $Plat_k$.

Also notice that Hol_1 is somewhat similar to Geo_1, since holomorphic curves can be parametrized either by $\mathbb{C}$ or by $H^2_{\mathbb{R}}$; accordingly, these leaves comes from the orbits of $\mathbb{C}$- and/or $SL_2(\mathbb{R})$-actions on the spaces of holomorphic maps of $\mathbb{C}$ and/or of $H^2_{\mathbb{R}}$ to X. (See [37], [58] for a study of the dynamics of such group actions.)

If $k > 1$, then "interesting" finite dimensional saturated $G \subset \mathcal{F}_1(X)$ are rather scarce – the main source of examples of foliations in finite dimensions comes from Lie groups. Namely, we assume that X is transitively acted by a Lie group L and let $Y \subset X$ be an orbit under a connected subgroup in L. This Y, can be regarded as a leaf, say $S_Y \subset \mathcal{F}_k(X)$ for $k = dim(Y)$, and we take the L-orbit of S_Y in $\mathcal{F}_k(X)$ for G.

Why "quasi-leaves" but not "quasi-orbits"? Apparently, smooth actions of Lie groups larger than $\mathbb{R}$, especially of semisimple ones, are very rare [80].

Such an action may be stable just because miracles do not happens twice. On the other hand, one entertains certain freedom in perturbing individual leaves, where stability seems more meaningful.

But what is a "quasi-leaf"?

If a foliated manifold G is endowed with a Riemannian metric, then an immersed k-submanifold $S_0 \to G$ with a *complete* induced Riemannian metric is called an ε-*leaf* if S_0 meets the leaves S of our foliation at the angles $\le \varepsilon$.

This definition is OK for small $\varepsilon > 0$ when we deal with the *local* shadowing/stability problem, but making sense of it for $\varepsilon \to \infty$, apparently, needs a specific definition case by case.

Next, we say that an $S_0 \to G$ is δ-*shadowed* by a leaf, if there is a δ-small C^0-perturbation of an original immersion $S_0 \to G$ which sends S_0 onto a true leaf $S \subset G$. We express this in writing by $DIST(S, S_0) \le \delta$. (This is almost, but not quite, the same as the δ-bound on the *Hausdorff distance* $dist_{Hau}(S, S_0)$.) Plain "shadowing" refers, as earlier, to $DIST(S, S_0) < \infty$.

"ε-Stable" ("prestable" in the terminology of [31]) means that every ε-leaf is δ-shadowed by a unique leaf where $\delta = \delta(\varepsilon) \to 0$ for $\varepsilon \to 0$, where we may forfeit the uniqueness of the shadow and where the the corresponding definition of "super-stability" i.e. where $\varepsilon \to \infty$ needs a special consideration.

A particular class of examples where the shadowing/stability problem can be, probably, fully resolved, at least for small $\varepsilon > 0$, is that of the foliations associated to *totally geodesic* submanifolds Y in (possibly infinite dimensional) *Riemannian symmetric* spaces X. This can be formulated without direct reference to $G \subset \mathcal{F}_k(X)$ as follows.

Let $Y \subset X$ be a totally geodesic k-submanifold. Say that a k-submanifold $Y_\varepsilon \subset X$ is an ε-*quasi-translate of* Y if for each point $x \in Y_\varepsilon$, there exists an isometry g of X by which Y is moved ε-close to Y_ε, where there several (essentially equivalent) possibilities for this "close". For example, this may mean the existence of an ε-diffeomorphism ϕ of the unite ball $B_x(1) \subset X$ which maps the intersection $g(Y) \cap B_x(1)$ onto $Y_\varepsilon \cap B_x(1)$, where the ε-property of ϕ signifies that $dist_{C^i}(\phi, id) \le \varepsilon$, for $dist_{C^i}$ being a certain "standard" metric in the space of C^i-smooth maps $B_x(1) \to X$ for a given i.

Shadowing/Stability Problem for Symmetric Spaces. Given the above X, $Y \subset X$ and $\varepsilon > 0$. What are the cases, where Y is ε-*geostable*, i.e, every ε-*quasi-translate of* Y lies within finite Hausdorff distance from the true g-translate of Y for some isometry $g : X \to X$?

Examples. (a) Let X be a symmetric space (possibly of infinite dimension) with non-positive curvature, $K(X) \le 0$, and $Y \subset X$ be the union of *all* geodesics parallel to a given one. Then

Y is ε-geostable for some $\varepsilon = \varepsilon(X) > 0$. In particular, the maximal flats in X, i.e. maximal subspaces isometric to a Euclidean space, are ε-geostable.

This follows by the standard "transversal (co)hyperbolicity" argument (suitably articulated in in [31], albeit with a few unnecessary extra assumptions on X), which equally applies to some non-symmetric spaces, e.g. to *the Euclidean buildings* and also to the Cartesian products $X = \mathbb{R}^m \times X_1 \times X_2 \times ... \times X_k$, where all X_i have $K(X_i \le -\kappa) < 0$.

The idea is to regard the leaves S corresponding to the translates of Y in the corresponding foliated space G as fixed points of the group Σ of diffeomorphisms

σ (or the semigroup of selfmappings) of G which send every leaf of our foliation, say $\mathcal{S}$, into itself.

"Transversal (co)hyperbolicity" signifies that there are "many" $\sigma \in \Sigma$ which "strongly contract" X in "many" directions transversal to the leaves. Such a σ defines a foliation $\mathcal{S}_+$, every leaf S_+ of which is consist of entire S-leaves and σ acting on S_+ brings these leaves closer together. It follows that it also brings closer together quasi-leaves modulo the non-contracting directions, which allows a local shadowing argument a la Anosov.

The simplest case of this is where $X = \mathbb{R} \times X_0$, where $K(X_0) \leq -\kappa < 0$ and dim(Y)=2.

In fact, let $Y \subset X$ be a "quasi-flat" surface, i.e. such that

Y is *quasi-vertical*: its angles with the vertical lines $\mathbb{R} \times x$, $x \in X_0$, are separated away from $\pi/2$;

the intersection of Y with each horizontal slice $r \times X_0 \subset X$, $r \in \mathbb{R}$, is *quasi-geodesic* in $r \times X_0 = X_0$.

Since the (Hausdorff) distance between every two intersections $Y \cap (r_1 \times X_0)$ and $Y \cap (r_2 \times X_0)$ is (obviously) finite, the projection of Y to X_0 lies within a finite distance from a geodesic, say $Y_0 \subset X_0$; hence, Y lies within a finite distance from $\mathbb{R} \times Y_0 \subset X$. QED.

(b) Let X be a Cartesian square, $X = Z \times Z$ and $Y = Z_{diag} \subset Z \times Z = X$, where Z is a simply connected symmetric space with *no Euclidean, no real hyperbolic and no complex hyperbolic factors.*

Then Y is geostable: there exists an $\varepsilon = \varepsilon(Z) >$), such that every ε-quasi-translate $Y_\varepsilon \subset X$ of Y is shadowed by a translate of Y.

Indeed, as we know, these Z are quasi-isometrically rigid, i.e. their the isometry groups are inner rigid, where the general case easily reduces to that where $K(Z) \leq 0$.

Then, if $K(Z) \leq 0$, all one needs of "ε-quasi" is that the angles at which Y_ε meets the fibers of the two projections $X \to Z$ are confined to an interval $[\alpha_1, \alpha_2]$ for $0 < \alpha_1 \leq \alpha_2 < \pi/2$, because such a Y_ε serves as the graph of bi-Lipschitz map $Z \to Z$; hence, it is close to an isometry $g_Z : Z \to Z$. Then the graph of g_Z in X serves as the required translate of Y which lies within finite Hausdorff distance from Y_ε.

Counterexamples. If X is a non-rigid symmetric space then, apparently, the shadowing/stability property fails to be true for most totally geodesic $Y \subset X$ with a notable exception for those from the above (a) and some similar (split) examples.

Questions. (a) Are there other sources of the failure of geostability in symmetric spaces?

Namely, is every *non-geostable* $Y \subset X$ *contained in a non-rigid* $Y' \supset Y$ in X?

Is this, at least, true for graphs $Y \subset X = Y_0 \times Z$ of isometric embeddings $Y_0 \to Z$?

(Here, $Y_\varepsilon \subset X$ with arbitrarily *large* $\varepsilon < \infty$ essentially correspond to quasi-isometric embeddings $Y_0 \to Z$.)

Are totally geodesic quaternionic geodesic subspaces in the quaternionic hyperbolic space $H^{4n}_{\mathbb{H}}$ geostable?

Is the complex hyperbolic $H^{2n}_{\mathbb{C}} \subset H^{4n}_{\mathbb{H}}$ (of complex dimension n) geostable?

(b) Is there a version (or versions) of geostability in dimensions $k > 2$ similar to that for $k = 1$, which would be stable under *perturbations of the Riemannian metrics* in X which destroy all totally geodesic submanifolds of dimension > 1?

A tangible possibility is offered by the Plateau foliated space $Plat_k$, which, moreover, may be of use in the geostability problem for symmetric spaces X, since many quasi-geodesic (and sometimes even quasi-minimal) k-subvarieties are shadowed by k-volume minimizing subvarieties $Y_{min} \subset X$ (see [28]).

One can show in some cases that such a Y_{min} is *unique*, and if Y is geostable, then Y_{min} is, *a posteriori,* totally geodesic.

Is there an *a priori* criterion for a $Y_{min} \subset X$ to be totally geodesic?

Do (the norms of) the second fundamental forms of some minimal subvarieties Y_{min} in symmetric spaces X of non-compact type ever satisfy some maximum principle or enjoy a Bochner-Simons type formula?

(c) Suppose that an ε-quasi-translate $Y_\varepsilon \subset X$ of (a possibly non-geostable) $Y \subset X$ is *periodic*, i.e. invariant under an isometry group Γ_0 of X, such that the quotient Y_ε/Γ_0 is compact. Is Y_ε shadowed by an isometric translate of our (totally geodesic) $Y \subset X$?

This is not true for the real hyperbolic space $X = H^n_{\mathbb{R}}$, even if Γ_0 includes into a discrete isometry group Γ with compact quotient X/Γ; one can "bend" closed totally geodesic submanifold $Y/\Gamma \subset X/\Gamma$ along totally geodesic codimension one submanifolds $Z \subset Y$. But we shall see in the next section 4.1 that a *theorem of Grauert* delivers a *holomorphic periodic stability* in certain cases, which makes the *periodic geo-stability* rather likely, e.g. for some $Y \subset H^m_{\mathbb{C}}$.

Sometimes "periodicity" can be relaxed to "quasiperiodicity" where the action of Γ_0 only "slightly" move Y_ε. E.g. one may allow invariant Y_ε with $Y - \varepsilon\Gamma_0$ having finite volume rather than being compact.

This can be studied in a foliated measure-theoretic set-up (see [80], [29] [19]) where one has a (suitably understood) measurable family of submanifolds, (i.e. a foliation with transversal measure mapped to X) rather than an individual Y_ε, and where the local and global closeness are understood "on the average", but this picture has not been fully clarified yet.

4 Kähler Stability and Kähler Universality.

Let look at the stability/rigidity problem from the angle of the Cauchy-Riemann equations, where we try to obtain "holomorphic objects" e.g. subvarieties, maps or sections of bundles, from (generously understood) *approximately holomorphic* ones.

Ultimately, starting from a "suitable" group Γ, we want to identify/construct some "universal" (generalized) complex analytic space (e.g an algebraic or a Kähler manifold) $B = B(\Gamma)$, or a holomorphic family of such B, such that all (many) other complex analytic spaces (e.g. Kähler manifolds) V with given homomorphisms $\pi_1(V) \to \Gamma$ would admit canonical holomorphic maps $V \to B$.

Besides Abel-Jacobi-Albanese construction there are two fundamental "superstability" results in the complex geometry: criteria for the existence of certain complex subvarieties (Grauert) and of holomorphic maps (Siu).

4.1 $\mathbb{C}$-Convexity and the Existence of Complex Subvarieties.

Recall that a complex manifold X with a boundary is called $\mathbb{C}$-*convex (at the boundary)* or having a *pseudoconvex boundary* if no (local) holomorphic curve (i.e. Riemann surface) $S \subset X$ can touch the boundary ∂X at a non-boundary point $s \in S$.

A complex manifold X without boundary is called $\mathbb{C}$-*convex at infinity* if it can be exhausted by compact $\mathbb{C}$-convex domains $X_i \subset X$.

X is called *strictly* $\mathbb{C}$-*convex at infinity* if there exists an exhaustion of it by X_i with C^2-smooth boundaries, such that these X_i are C^2*-stably* $\mathbb{C}$-*convex*, i.e. the convexity property persists under all sufficiently small C^2-perturbations of their boundaries

Let $k = dim_{hmt}(X)$ denote the *homotopy dimension* of X, that is the minimum of dimensions of locally contractible topological spaces which are homotopy equivalent to X.

Grauert Exceptional Cycle Theorem. *Let X be a complex manifold which is strictly $\mathbb{C}$-convex at infinity. If its homotopy dimension satisfies $k = dim_{hmt}(X) > m = dim_{\mathbb{C}}(X)$, then k is even and X contains a unique maximal compact complex subvariety of complex dimension $k/2$, say $Y_0 \subset X$, such that the homology inclusion homomorphism $H_k(Y_0) \to H_k(X)$ is an isomorphism.*

Idea of the Proof. The strict $\mathbb{C}$-convexity of the boundary of a relatively compact domain $X_0 \subset X$ implies that *the coherent sheaves* over X_0 have *finite dimensional* cohomology. This implies that there are "many" holomorphic functions in the interior X_0 which extend to a small neigbourhood $X_1 \supset X_0$ with a pole at a complex hypersurface $Z \subset X_1$ which is tangent to the boundary ∂X_0 at a single point.

These functions provide a proper holomorphic map from X_0 to $\mathbb{C}^N$ (with large N) which is an embedding away from a subset $Y_0 \subset X_0$ where this map is locally constant.

The homomorphisms is $H_k(Y_0) \to H_k(X)$ is an isomorphism by the Lefschetz theorem for (possibly) singular complex *Stein spaces*, i.e. properly embedded complex subvarieties in some $\mathbb{C}^N$.

Periodic Hol-Stability Corollary. Let X be a Hermitian (thus, Kählerian) symmetric space with non-positive sectional curvatures and let $Y \subset X$ be a totally geodesic subspace, such that

no $\mathbb{C}$-line tangent Y admits a parallel translation in X normal to Y,

where, observe, such a line at a point $y \in Y$, say $L_{\mathbb{C}} \subset T_y(Y) \subset T_y(X)$ admits a parallel translation in the direction of a unit normal vector $\nu \in N_y(Y) = T_y(X) \ominus T_y(Y) \subset T_y(X)$, if and only if the sectional curvatures of X vanish on the bivectors (ν, l) for all $l \in L_{\mathbb{C}}$. (These curvatures *do not vanish*, for example, if $rank_{\mathbb{R}}(Y) = rank_{\mathbb{R}}(X)$.)

If $dim(Y) > \frac{1}{2}dim_X$ then Y is periodically holomorphically stable: every periodic ε-quasi-translate Y_ε is δ-close to a complex analytic submanifold $Y' \subset X$. Moreover, such a complex analytic Y' close to Y is unique; hence, periodic.

Proof. The "no-parallel translate condition" is equivalent to the *strict* $\mathbb{C}$-convexity of all ρ-neighbourhoods of Y in X and if "ε-quasi" is understood in some C^2-norm, then this property passes to the ρ'-neighbourhoods of Y_ε, for all $\rho' \geq \rho'(\varepsilon) \to 0$ for $\varepsilon \to 0$. QED.

Remarks. (a) We did not assume Y being complex analytic itself but this follows by sending $\varepsilon \to 0$.

(b) The actual "C^2-quasi" condition can be expressed by a bound on the second fundamental form of Y_ε. This can be relaxed to C^1 (and, probably, to C^0) by smoothing Y_ε.

(c) Probably, "periodic" can be replaced by a suitable "quasi-periodic" with an extension of the L_2-techniques from [35] to the foliated framework but it is unclear if "periodicity" it can be fully removed.

Singular Generalization. The exceptional cycle theorem remains valid for singular complex spaces X, where a hypersurface $\partial \subset X$ is called (strictly) $\mathbb{C}$-convex if the intersection of ∂ with a small neighbourhood $U_x \subset X$ of each point $x \in \partial$ equals the pullback of a (strictly) $\mathbb{C}$-convex hypersurface in $\mathbb{C}^N$ under a holomorphic embedding from U_x into some complex *manifold* U'_x (where one may assume $dim_{\mathbb{C}}(U'_x) = 2dim_{\mathbb{C}}(X)$).

Basic Question. We want to eventually address the followin global rather than ε-local holomorphic stability problem.

Let V be a closed (compact with no boundary) complex analytic space (e.g. manifold) and $\Gamma_0 \subset \Gamma = \pi_1(V)$ be a subgroup.

Let $X_{\Gamma_0} \to V$ denote the Γ_0-*covering* of V, i.e. $\pi_1(X_{\Gamma_0})$ is isomorphically send onto Γ_0 by this covering map and let $k_0 = dim_{\mathbb{Q}hmt}(\Gamma_0)$ be

the minimal number such that Γ_0 *admits a discrete action on a contractible locally contractible metric space* X (which may have fixed points under finite subgroups in Γ) *such that* $dim_{hmt}(X/\Gamma_0) = k_0$, where "discrete" means that, for every bounded subset $B \subset X$ there are at most *finitely many* $\gamma \in \Gamma$, such that $\gamma(B)$ intersects B.

If Γ_0 has no torsion, then

$$dim_{\mathbb{Q}hmt}(\Gamma_0) = dim_{hmt}(\Gamma_0) =_{def} dim_{hmt}(K(\Gamma_0;1))$$

for the Eilenberg-MacLane classifying space $K(\Gamma_0;1)$ of Γ_0.

Also, there is a counterpart of this dimension for locally compact groups. For example, if G is a Lie group then $dim_{\mathbb{Q}hmt}(G)$ can be defined as the dimension of the quotient space of G by the maximal compact subgroup in it. More generally, one may look at *proper* actions of G on contractible locally contractible metric spaces X and take the minimal dimension of such an X for $dim_{\mathbb{Q}hmt}(G)$, where "proper" means that for every bounded subset $B \subset X$ the set of those $g \in G$, such that $\gamma(B)$ intersects B is *precompact.*

When does the homology group $H_{k_0}(\Gamma_0;\mathbb{Q})$ *admit a basis which comes (via the Eilenberg-MacLane classifying map* $X_{\Gamma_0} \to K(\Gamma_0,1)$*) from compact complex subspaces of dimensions* $k_0/2$ *in* X_{Γ_0} ?

Subquestions. What properties of V and/or of Γ_0 could ensure that the homotopy dimension $k_0 = dim_{\mathbb{Q}hmt}(\Gamma_0)$ is *even* and what are conditions for *non-vanishing* of $H_{k_0}(\Gamma_0;\mathbb{Q})$?

Below is an instance of a partial answer for Kähler manifolds V which globalizes the above hol-stability in X, at least in the case of $K(X) < 0$.

($\star$) Let X be a complete simply connected complete Kähler manifold and let Γ_0 be an *undistorted* finitely generated isometry group of X, where "undistorted" means that some (hence every) orbit map $\Gamma_0 \to X$ for $\gamma \mapsto \gamma(x)$ is a *quasi-isometry* on its image for the word metric in Γ_0.

If X has pinched negative curvature, $-\infty < -\kappa_- \le K(X) \le -\kappa_+ < 0$, if Γ_0 and if

$$k_0 = dim_{\mathbb{Q}hmt}(\Gamma_0) = dim_{hmt}(X/\Gamma_0) > m = dim_{\mathbb{C}}(X),$$

then

- *k_0 is even,*
- *the homology group $H_{k_0}(\Gamma_0;\mathbb{R}) = H_{k_0}(X/\Gamma_0;\mathbb{R})$ does not vanish. Moreover, the $k_0/2$-power of the Kähler class of X/Γ does not vanish on $H_{k_0}(\Gamma_0)$,*
- *the homology $H_{k_0}(\Gamma_0;\mathbb{R}) = H_{k_0}(X/\Gamma_0;\mathbb{R})$ is generated by the fundamental classes of irreducible components of a compact complex analytic subspace in $V_{hol}^{k_0/2} \subset X/\Gamma_0$.*

Proof. Since Γ_0 is undistorted and X has *pinched negative* curvature, every orbit $\Gamma_(x_0) \subset X$ admits a Γ_0-invariant neighbourhood $U \subset X$ which lies within finite Hausdorff distance from this orbit and the boundary of which is smooth strictly convex by *Anderson's lemma* (see [1] and a sketch of the proof in [⌢] below).

Since X is Kähler, (*strict*) *convexity* ⇒ (*strict*) $\mathbb{C}$-*convexity*, and Grauert theorem applies to the quotient space X/X_0 (which may be singular as we do not assume the freedom of the action action of Γ_0 on X). QED

$(\star\star)$. This $(\star)$ is most interesting where X admits a cocompact isometry group, e.g.

(1) *X equals the universal covering $\tilde{V}$ of a compact manifold V without boundary.*

In this case the lower bound on $K(X)$ is automatic, all one needs is

(2) *the strict negativity of sectional curvatures, $K(V) \le -\kappa < 0$.*

What is especially pleasant for $X = \tilde{V}$ is that the "undistorted" condition becomes a purely algebraic one:

(3) the orbit maps $\Gamma_0 \to X$ are undistorted if and only if

the imbedding $\Gamma_0 \subset \Gamma = \pi_1(V)$ is a quasi-isometry for some (hence all) word metrics in Γ and Γ_0.

Thus, our basic question gets a satisfactory answer under the (1+2+3)-condition, i.e. for

undistorted subgroups $\Gamma_0 \subset \Gamma = \pi_1(V)$ with $k_0 = dim_{\mathbb{Q}hmt} > m = dim_{\mathbb{C}}(V)$, where V is compact Kähler manifold with strictly negative sectional curvatures.

Discussion. What are these $(\star)$ and $(\star\star)$ good for?

As for X, the primely example is the complex hyperbolic space $H^m_{\mathbb{C}}$, $m = dim_{\mathbb{C}}(X)$. But are there good candidates for Γ_0?

It seems hard to come up with examples where the assumptions of $(\star)$ are satisfied, but the existence of the complex subvariety $V_{hol}^{k_0/2} \subset X/\Gamma_0$ was not apparent beforehand.

A pessimistic conjecture (which may thrill champions of "rigidity") is that every top-dimensional irreducible component of $V_{hol}^{k_0/2}$ equals the image of a totally geodesic $H_{\mathbb{C}}^{k_0/2} \subset H^m_{\mathbb{C}}$ in the present case.

On the other hand, $(\star)$ implies that certain groups Γ_0, e.g. with *odd* $k_0 = dim_{\mathbb{Q}hmt}(\Gamma_0) > m$, admit *no undistorted actions* on $H^m_{\mathbb{C}}$.

Notice that the bound $k_0 > m$ is sharp:

cocompact groups Γ_0 of isometries on $H^m_{\mathbb{R}}$ naturally act on $H^m_{\mathbb{C}} \supset H^m_{\mathbb{R}}$, where this action is, clearly, undistorted. On the other hand, according to $(\star)$, *these Γ_0 admit no undistorted actions on $H^{m-1}_{\mathbb{C}}$ for odd k_0.*

We shall see in the next section that this remains true for *even* $k_0 \geq 4$ as well, but it remains unclear if one can drop the "undistorted" condition in this case.

Questions and Conjectures. (a) Possibly, on can relax $K(X) < 0$ to $K(X) \leq 0$ complemented by some extra condition, on Γ_0, e.g. by requiring that the action is

fully undistorted, that is some, (hence, every) orbit $O \subset X$ of Γ_0 admits an *eventually contracting* retraction $R : X \to O$, i.e. such that

$$dist(R(x_1), R(x_2)) \leq \varepsilon \cdot dist(x_1, x_2),$$

where ε depends on $d = dist(x_1, x_2)$ and $D = \min(dist(x_1, O), dist(x_2, O))$, such that $\varepsilon \to 0$ for $d, D \to \infty$.

This is akin to convexity but it has an advantage of being a quasi-isometry invariant. This is equivalent to "undistorted" if $K(X) \leq -\kappa < 0$, but may be strictly stronger, e.g. it is so for non-cocompact lattices acting on irreducible symmetric spaces of $\mathbb{R}$-ranks ≥ 2 according to (by now confirmed) Kazhdan's conjecture.

It is, in general, strictly stronger than "undistorted" but is equivalent to it for $K(X) \leq -\kappa < 0$.

The degenerate case of X/Γ_0 being quasi-isometric to the real line $\mathbb{R}$ allowed by this condition needs to be excluded, e.g. by insisting that X/Γ_0 is *δ-hyperbolic* with the ideal boundary of strictly positive topological dimension.

(b) It seems harder to replace "$K < 0$" by some *purely topological* conditions.

The strongest topological/algebraic substitute(s) for "$K < 0$" is the assumption that X is δ-hyperbolic with the ideal boundary homeomorphic to the sphere S^{2m-1}, $m = dim_{\mathbb{C}}(X)$.

This becomes a "true topology" if we also assume that X admits a cocompact isometry group, say a discrete group Γ freely acting on X.

Also, we need to assume that X is contractible or, at least that the fundamental homology class $[V]_{2m} \in H_{2m}(V)$ of $V = X/\Gamma$ does not vanish in the homology $H_{2m}(\Gamma : \mathbb{R})$.

Would all these allow the conclusions of $(\star)$ and $(\star\star)$ to remain valid?

Can, at least, one show that X is Stein?

(This seems likely even for **semi**hyperbolic X.)

(c) Can one do anything without "Kähler"?

For example, suppose that all assumptions from (b) are satisfied.

Does then the fundamental group $\Gamma = \pi_1(V)$ admit a *formal Kähler class* $\kappa \in H^2(\Gamma) = H^2(V)$, i.e. such that $\kappa^m = [V] \neq 0$?

If not, is this condition of any use for us?

Probably one handle "non- Kähler" for $m = 2$ in view of *astheno-Kählerian* approach to *Kodaira theory of complex surfaces* [47]

(d) Probably, $(\star)$ and/or $(\star\star)$ (as well as their conjectural generalizations) remains valid for *singular* Kähler spaces X, where, by definition, a singular Kähler metric is locally induced from an ordinary Kähler structure, (i.e. each point $x \in X$ admits a small neighbourhood $U_x \subset X$ and a holomorphic embedding of U_x to a complex manifold U'_x, such that our singular metric on U_x comes from a smooth Kähler metric on U'_x) and where the condition $K(V) < -\kappa$ must be understood as $CAT(-\kappa)$ in the singular case.

The bottleneck here is Anderson's lemma which, in fact, holds (this is easy) for all $CAT(-\kappa)$ spaces which have the following $[\frown]$-*property* that is obviously satisfied by Riemannian manifolds with *both side bounded curvatures*, $-\kappa \le K(X) \le \kappa$, and with the *injectivity radius bounded from below by some* $R > 0$.

$[\frown]$ There exists, for all $\varepsilon_1, \varepsilon_2 > 0$ and each point $x_0 \in X$, a function $\varphi_\frown : X \to \mathbb{R}_+$ with the support in the ε_1-ball $B_{x_0}(\varepsilon_1) \subset X$, such that

the second derivative of $\varphi_\frown$ *on every geodesic in* X *is bounded in the absolute value by* $\varepsilon_2 > 0$ *and*

$$\varphi_\frown(x_0) \ge \delta = \delta(X, \varepsilon_1, \varepsilon_2) > 0.$$

If a $CAT(-\kappa)$-space X satisfies $[\frown]$, then, by an easy argument,

the geodesic convex hull an ε*-quasiconvex subset in* X *is contained in the* δ*-neighbourhood of* U *with* $\delta \to 0$ *for* $\varepsilon \to 0$.

This, applied to the ρ-neigbourhood of U with a large ρ, implies Anderson's lemma:

the convex hull of a quasi-convex U *lies within finite Hausdorff distance from* U.

(It is unknown if Anderson's lemma holds for *all* $CAT(-\kappa)$ spaces X, where the test questions are the following. Let $U = B_{x_1}(R) \cup B_{x_2}(R) \subset X$.

Is the convex hull of U contained in the ε-neigbourhood of U where $\varepsilon \to 0$ for $R \to \infty$?

Does, at least, ε admit a bound independent of $dist(x_1, x_2)$ for $R \to \infty$?

The above $[\frown]$ has an obvious $\mathbb{C}$-counterpart, where the bound on the full Hessian (second derivatives) of $\varphi_\frown$ is replaced by such a bound on the complex Hessian.

The relevant version of Anderson's lemma holds (by $[\frown]$ or by an additional argument) for some (all?) singular Kählerian $CAT(-\kappa)$-spaces X with cocompact isometry groups, e.g. for X with isolated singularities.

Singular Kähler Examples. Compact non-singular quotients of the $2m$-ball, $V = H^m_\mathbb{C}/\Gamma$ often have many complex *totally geodesic* submanifolds $V_0^{m_0} = H^{m_0}_\mathbb{C}/\Gamma_0 \subset V$.

By the Grauert blow-down theorem [24] the space $\underline{V} = V/V_0$ obtained from V by shrinking V_0 to a point has a complex analytic structure for which the obvious map $V \to \underline{V}$ is complex analytic outside V_0.

These $\underline{V}$ carry natural structures of *Artin algebraic spaces* moreover, some of these $\underline{V}$ admit singular Kähler metrics with $K < 0$ and often (always?) these $\underline{V}$ are *projective algebraic*.

On the other hand, the universal covers $\tilde{\underline{V}}$ of these are generically hypebolic by the (generalized) small cancellation theory. However, only relatively few among these are known to carry singular metrics of negative curvature, where the sufficient condition for this a lower bound on the size of the maximal ρ*-collar* $U_\rho \supset V_0$ of V_0 that is the maximal ρ-neighbourhood of V_0 in V which admits a homotopy retraction to V_0. Namely, if $\rho \ge \rho_0$ for some universal ρ_0, (something about $\pi/2$), then $\underline{V} = V/V_0$ carries a singular metric of negative curvature which, apparently (I did not carefully check this) can be chosen Kählerian.

More interestingly, this also applies to singular subvarieties $V_0 \subset V$ that are *immersed* (i.e. with non-trivial selfintersections) totally geodesic in $V = H^m_\mathbb{C}/\Gamma$, provided their self-intersection loci are "sufficiently sparse".

Namely, suppose that the self-intersection angles are bounded from zero by some $\alpha > 0$ and let V_0 admits a ρ-collar with $\rho \geq \rho_0(\alpha)$

Then such a collar can be approximated by a locally convex $U \supset V_0$ and by Grauert theorem V/V_0 is complex analytic. Also, such a V has a singular Riemannian (probably, Kählerian) metric of negative curvature. But it is unclear if there are other complex analytic subsets in these V with large approximately locally convex collars.

The picture is somewhat opposite for $V_0 \subset X$ with *many* self-intersections, where such V_0 tend to be "mobile" (ample) rather than exceptional. For example, let $V = H^m_{\mathbb{C}}/\Gamma$ and let V_0 be a immersed totally geodesic (reducible or irreducible) subvariety of complex dimension $m-1$ with $vol_{2m-2}(V_0) \geq const \cdot vol_{2m}(V)$ for a large *cost*. Then most (probably, all) of such V_0 are *connected* and

(1) the homomorphism $\pi_1(V_0) \to \pi_1(V)$ is *onto*;

(2) V_0 can be included into a *family* of divisors $V_q \subset V$ which are generically *non-singular.*

It is usually easy to see how (2)⇒(1) but the the opposite implication, probably, needs (?) extra conditions on V_0.

It is also not fully clear what happens if $dim_{\mathbb{C}}(V_0) \leq dim_{\mathbb{C}}(V) - 2$, where the cases $dim_{\mathbb{C}}(V_0) \geq dim_{\mathbb{C}}(V)/2$ and $dim_{\mathbb{C}}(V_0) < dim_{\mathbb{C}}(V)/2$ need separate treatments. (Possibly, every $V_0 \subset V$ with $dim_{\mathbb{C}}(V_0) < dim_{\mathbb{C}}(V)/2$ and very many self-intersections is contained in an immersed totally geodesic $V_1 \subset V$ with $dim(V_0) < dim(V_1) < dim(V)$.)

Also the arithmetics of the self-intersection loci of totally geodesic $V_0 \subset V = H^m_{\mathbb{C}}$ (e.g. their *definition fields* and/or arithmetic Galois groups) seems poorly understood.

Besides shrinking subvarieties in compact quotients $H^m_{\mathbb{C}}/\Gamma$ one obtains a attractive class of analytic spaces with "interesting" fundamental group by compactifying $V = H^m_{\mathbb{C}}/\Gamma$ for *non-cocompact* lattices Γ, where the fundamental group $\Gamma_\bullet$ of such a compactification $V_\bullet$ is hyperbolic if V has sufficiently large cusps. In fact, these $V_\bullet$ carry singular (Kählerian?) metrics with negative curvatures.

There are two somewhat opposite constructions of a different kind which deliver spaces of negative (or close to that) curvatures both, in (singular) Riemannian and the Kählerian categories. (Specific instances of this will come up in the next section).

(1) *Ramified Coverings* $V_1 \to V$ tend to be more negatively curved than V.

This is literally true if the branching locus $\Sigma \subset V$ (that is a possibly singular subvariety of codimension two) is totally geodesic; immersed totally geodesics Σ (with self-intersections) also serves well in many cases.

The above $V_0 \subset H^m_{\mathbb{C}}/\Gamma$ as well as unions of translates of coordinate complex $(m-1)_{\mathbb{C}}$-subtori in $\mathbb{C}^m/\mathbb{Z}^{2m}$ provide examples of such Σ.

(2) *Quotient Spaces of non-Free Group Actions.* If the action of Γ, say on $X = H^m_{\mathbb{C}}$, has fixed points, the quotient space $V = X/\Gamma$ may be (not necessarily) singular. However, if the fixed point locus is "sparse" this V may still carry a metric of negative (or close to that) curvature.

Probably, there are lots of a smooth projective algebraic varieties (defined over number fields) which are biholomorphic to quotients $H^m_{\mathbb{C}}/\Gamma$. (Possibly, there are Kählerian counterexamples in view of [77].)

More modestly, does every irreducible (smooth?) algebraic $\mathbb{C}$-variety V admit a dominating regular (only rational?) map from some $H^m_{\mathbb{C}}/\Gamma$ to some deformation V' of V?

We shall look closer on specific instances of (1) and (2) in the next section.

Non-holomorphic Problems. It seems that groups Γ_0, e.g. subgroups in a given discrete or a Lie group Γ, with large $dim_{\mathbb{Q}hmt}(\Gamma_0)$ are rather exceptional. (Not nearly as exceptional as arithmetic groups, but something in the same spirit.)

In fact, such subgroups $\Gamma_0 \subset G$ in semisimple (real and p-adic) Lie groups G seems to be associated with (possibly reducible) Γ_0-invariant "subvarieties" in the space G/K (in the Euclidean building for p-adic G) for the maximal compact subgroup $K \subset G$ (and in the Euclidean building in the p-adic case).

On the other hand if a group Γ has large $dim_{\mathbb{Q}hmt}$, then most (sometimes all) infinite quotient groups $\underline{\Gamma}$ of Γ have $dim_{\mathbb{Q}hmt}(\underline{\Gamma}) \geq dim_{\mathbb{Q}hmt}(\Gamma)$. Moreover, these Γ are unlikely to have isometric actions on low dimensional spaces, such as trees, for instance.

Besides large $dim_{\mathbb{Q}hmt}$, what makes a group Γ_0 "rigid" and its actions on X "special" is its connectivity at infinity.

The standard condition of this kind for hyperbolic groups Γ_0 is that its ideal boundary $\partial_\infty(\Gamma_0)$ is connected, locally connected and has no local *cut-points*: every connected subset $U \subset \partial_\infty(\Gamma_0)$ remains connected upon removing a finite subset $S \subset U$.

Alternatively one may bound from below the *cut dimension of* Γ_0, i.e. the minimal topological dimension of an $S \subset \partial_\infty(\Gamma_0)$ such that removal of S from U *does change* the connectedness of U.

(There several candidates for dim_{cut} for non-hyperbolic Γ_0, e.g. in terms of asymptotic dimensions of subsets in Γ_0) which disrupt the connectedness of Γ_0 at infinity, but I am not certain what the working definition should be.)

Questions. Does the "no cut point" condition, or, at least, a stronger bound $dim_{cut} \geq d_0$ for some $d_0 > 0$, imply that every embedding of Γ_0 into a hyperbolic group Γ is *undistorted*?

What characterizes symmetric spaces (and Euclidean buildings) X, such that every discrete isometric action of a Γ_0 on X with $dim_{\mathbb{Q}hmt} \geq d_0$ and $dim_{cut}(\Gamma_0) \geq d_1$ is undistorted? Fully undistorted?

The "undistorted" and "fully undistorted" conditions are accompanied by similar ones, such as absence of parabolic elements in $\Gamma_0 \subset G$ (e.g. satisfied by all Γ_0 in cocompact discrete $\Gamma \subset G$) and/or by *stability* of the action (see the next section and [15], [54], [9], [39]).

But the full scope of relations between such properties remains unclear, where such relations, probably, become more pronounced for "large" $dim_{\mathbb{Q}hmt}$ and/or dim_{cut}.

Call a subgroup Γ_0 in a Lie group (or a p-adic Lie group) a *quasi-lattice* if "an essential part" of Γ_0 is a *lattice* in a Lie subgroup $G' \subset G$, i.e. $G'/(G' \cap \Gamma_0)$ *has finite volume*, and where the "essential part" condition is expressed by

$$dim_{\mathbb{Q}hmt}(G' \cap \Gamma_0) = dim_{\mathbb{Q}hmt}(\Gamma_0).$$

What is the maximal dimension $dim_{\mathbb{Q}hmt}$ of discrete *non-quasi-lattices* $\Gamma_0 \subset G$?

For example, what is the maximal dimension $d_{max} = dim_{\mathbb{Q}hmt}$ of non-quasi-lattices Γ_0 in the isometry group of the hyperbolic quaternion space $H_{\mathbb{H}}^{4m}$?

Does every discrete non-quasi-lattice $\Gamma_0 \subset iso(H_{\mathbb{H}}^{4m})$ with no parabolic elements has $dim_{\mathbb{Q}hmt} \leq 2m$?

How many (undistorted) subgroups Γ *from a given class a locally compact (e.g. discrete) group* G *may contain?*

Let us formulate this precisely for hyperbolic groups G and a given set $\mathcal{G}$ of subgroups Γ as follows.

Take the closure $\mathcal{C} = \mathcal{C}(G, \mathcal{G})$ of the set of the ideal boundaries $\partial_\infty(\Gamma) \subset \partial_\infty(G)$, $\Gamma \in \mathcal{G}$, in the Hausdorff (distance) topology and ask ourselves:

What condition(s) on $\Gamma \in \mathcal{G}$ would imply a bound on the dimension of $\mathcal{C}$?

Notice in this regard that if G equals the isometry group of a symmetric space X of negative curvature and $\Gamma \subset G$ is an undistorted subgroup with $dim_{\mathbb{Q}hmt} < dim X$, then there are lots of undistorted subgroups in G isomorphic to the free product of as many copies of Γ as you wish by the (obvious) *Schottky combination theorem.* This makes $dim(\mathcal{C}) = \infty$ in this case.

In order to rule this out, one needs, besides a lower bound on $dim_{\mathbb{Q}hmt}$ for all $\Gamma \in \mathcal{G}$ also such a bound on dim_{cut}, or something like that.

For instance let $\mathcal{G}$ consist of all hyperbolic groups Γ with $dim_{\mathbb{Q}hmt}(\Gamma) = d$, for a given d, and such that the ideal boundaries of these Γ are homeomorphic to the sphere S^{d-1}.

Can one describe the symmetric spaces of a given dimension n (not very large compared to d) and/or hyperbolic groups G with $dim_{\mathbb{Q}hmt}(G) = n$ such that $dim(\mathcal{C}(G, \mathcal{G})) \geq D$ for a given (large) D?

Now a few words about the quotient problem. The only (known) systematic construction of quotients $\underline{\Gamma}$ of hyperbolic groups Γ depends on "collapsing" undistorted subgroups in Γ, (or suitable subgroups in something like a free product $\Gamma * \Gamma'$) where, the generalized small cancellation theory (see 1.5 in [39]) shows that the dimension $dim_{\mathbb{Q}hmt}$ may only *increase* in the process (e.g. if $dim_{\mathbb{Q}hmt}(\Gamma') > dim_{\mathbb{Q}hmt}(\Gamma)$ and Γ' *injects* into $\underline{\Gamma}$.)

What are condition on Γ which would rule out other kind of infinite quotient groups?

For example, does every infinite quotient group $\underline{\Gamma}$ of a Kazhdan's T hyperbolic Γ has $dim_{\mathbb{Q}hmt}(\underline{\Gamma}) \geq dim_{\mathbb{Q}hmt}(\Gamma)$?

In particular, let Γ be a cocompact lattice in $iso(H_{\mathbb{H}}^{4m})$ for $m \geq 2$ and $\underline{\Gamma}$ be an infinite quotient group of Γ.

Can the induced homology homomorphism $H_{4m}(\Gamma) \to H_{4m}(\underline{\Gamma})$ vanish?

Finally, does the above Γ admit an isometric action with *unbounded* orbits on a 2-dimensional simply connected space X with $K(X) \leq 0$ (i.e. $CAT(0)$-space which is not assumed locally compact)?

About $dim = \infty$. Is there an infinite dimensional version of Grauert's exceptional cycle theorem?

To formulate this one needs a notion of "middle dimension" in an infinite dimensional complex variety X.

One option, e.g. for the space X of smooth maps of the circle S^1 to a Kähler manifold V, is suggested by "quasi-splitting" of this X into two "halves" similarly how the space of functions $S^1 \to \mathbb{C}$ "splits" into two *Hardy spaces* of *holomorphic functions* on the Northern and to the Southern hemispheres into which an equatorial $S^1 \subset S^2$ divides the sphere S^2.

Then "$dim_{hmt}(X) > dim_{\mathbb{C}}(X)$" may signify that the gradient flows of C-convex functions on X from a suitable class, can not bring all of X to (a Fredholm perturbation) of such a "half".

Another possibility is suggested by the concept of "mean dimension" for infinite dimensional spaces acted upon by transformation groups [37], [58]

"Philosophical" Questions. (a) Are there examples where something like the above ($\star\star$) makes sense but which are not ultimately depend on locally symmetric spaces?

Is there something of the kind in the Riemann moduli space of curves, or rather in its universal orbi-covering space X acted upon by the mapping class group Γ?

(b) The $\mathbb{C}$-*convexity* condition is vaguely similar to the *contraction* property in dynamics and Grauert's blow down theorem has a unique fixed point flavour to it. Accordingly, one may draw a similarity between $\mathbb{C}$-*concavity* (associated with *ampleness* or "mobility" of subvarieties) and *expanding* maps (or vice versa?).

What is the holomorphic counterpart of split hyperbolicity? Is there a holomorphic [$\mathbb{C}$-concave]$\times$[$\mathbb{C}$-convex] version of Frank's super-stability theorem?

(*Non-super* stability seems easy: "[exceptional]$\times$[ample]" is, apparently, a stable property of complex subvarieties.)

(c) Looking from a different angle, the existence/nonexistence of a subvariety (algebraic cycle) Y_0 in an algebraic variety V is an essentially *Diophantine* question which can be often resolved by reducing it to linear algebra where the rationality of a solution is automatic.

Grauert's theorem gives a convexity inequality criterion for the existence of certain Y_0 where the proof goes via *infinite* dimensions, similarly to the proof of the Shub-Franks superstability theorem which depends on a contraction/expansion inequality in an *infinite* dimensional space.

Is there a general picture where both arguments would be simultaneously visible?

(d) Is there an algebraic-geometric version of ($\star\star$), at least for locally symmetric V, where everything should be expressed in terms of *finite* coverings $\tilde{V} \to V$ (or of *finite dimensional* representations of Γ) and where "homotopy" must refer to the Grothendieck étale (and/or Nisnevich) topology?

(Probably, it is not hard to identify totally geodesic submanifolds Y_0 and their fundamental (sub)groups $\Gamma_0 \subset \Gamma$ in the purely algebraic/arithmetic language in the spirit of *Kazhdan's theorem on arithmetic varieties.*)

(e) Is there anything in common between (finitary!) Markov presentations of hyperbolic systems and the approximation of the first order theory of algebraic varieties over $\mathbb{C}$ by that over finite fields?

4.2 Existence and non-Existence of Holomorphic Maps.

Define the *homotopy rank* of (the homotopy class of) a continuous map $f : X \to Y$, denoted $rank_{hmt}[f]$, as the minimal number r such that f is homotopic to a composed map $X \to P^r \to Y$, where P^r is an r-dimensional cellular space. For example, if Y itself is a cellular (e.g. triangulated) space, then $rank_{hmt}[f] \leq r$ if and only if f is homotopic to a map into the r-skeleton of Y. (At the end of the next section 4.3, we shall refine the concept of $rank_{hmt}[f]$ for maps into non-compact, possibly, infinite dimensional, spaces Y.)

Call a homotopy class of maps $f : V \to W$ between complex analytic spaces *±-holomorphic.* and write $[f] \in \pm\mathcal{HOL}$ if f is homotopic to a *holomorphic or to an anti-holomorphic* map.

Recall that the Galois group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ of $\mathbb{R}\backslash\mathbb{C}$ acts on the category of complex spaces V by conjugation, $V \leftrightarrow \overline{V}$, where the arrow "$\leftrightarrow$" establishes a homeomorphism between V and $\overline{V}$.

For example, if $V \subset \mathbb{C}P^N$ is a complex projective subvariety, then $\overline{V}$ equals the image of V under the conjugation involution $\mathbb{C}P^N \to \mathbb{C}P^N$ given by $z_i \mapsto \overline{z}_i$, $i = 0, 1, ..., N$.

A map $V \to W$ is antiholomorphic, if the corresponding map $V \to \overline{W}$ (for $\overline{W}$ topologically identified with W) is holomorphic.

Complex Structures on Symmetric Spaces. The main targets of holomorphic maps in relevant examples are *Hermitian symmetric spaces* Y with non-positive sectional curvatures, $K(Y) \le 0$, and their quotients $W = Y/\Gamma$ by groups of holomorphic isometries $\Gamma \subset iso_{hol}(Y)$. These Y are Kähler and the groups $iso_{hol}(Y)$ are transitive on Y.

The simplest such Y is the *complex hyperbolic space* $H^m_{\mathbb{C}}$ which is biholomorphic (but by no means isometric) to the unit ball in $\mathbb{C}^m$.

If a Hermitian symmetric Y is *irreducible as a Riemannian space*, i.e. does not (non-trivially) isometrically split as $Y = Y_1 \times Y_2$ then it carries *exactly two* complex analytic (Hermitian Kähler) structures invariant under $iso_{hol}(Y)$ – the original one and its conjugate, since the action of the isotropy subgroup $iso_{hol}(Y, y_0)$ on the tangent space $T_y(Y)$ is *irreducible.*

If $Y = Y_1 \times Y_2 \times ... \times Y_k$, where Y_i are irreducible Hermitian, then, obviously, Y admits 2^k invariant complex structures. Sometimes, maps into Y and the quotient spaces $W = Y/\Gamma$, which are holomorphic with respect to some of these structures, are called ±holomorphic or just holomorphic maps.

(The complex Euclidean space $\mathbb{C}^n$ is also Hermitian symmetric, but this space, as well as $Y = Y_0 \times \mathbb{C}^n$, must be addressed slightly differently.)

These definitions are justified by a theorem of Y. T. Siu who, in 1980, found a Hodge-Bochner type formula for *harmonic maps* from Kählerian to Riemannian manifolds which has led to a variety of results by Siu and his followers. (see [73], [75] and references therein).

Example: the ***S****ui-****S****ampson-****C****arlson-****T****oledo* $[H^m_{\mathbb{C}}/\Gamma]$*-Theorem.* Let W be covered by the complex hyperbolic space, i.e. $W = H^m_{\mathbb{C}}/\Gamma$ for a free discrete isometry group $\Gamma \subset iso_{hol}(H^m_{\mathbb{C}})$, let V be a compact Kähler manifold and $f_0 : V \to W$ be a continuous map. Denote by $\Gamma_0 \subset \Gamma$ the image of the fundamental group $\pi_1(V)$ in $\Gamma = \pi_1(W)$ under the induced homomorphism $\pi_1(V) \to \Gamma = \pi_1(W)$ (for some choice of the base points in V and W).

Let Γ_0 contains no nilpotent subgroup $\Gamma'_0 \subset \Gamma_0$ of finite index (i.e. with $card(\Gamma_0/\Gamma'_0) < \infty$) and with the nilpotency degree of Γ'_0 at most 2 (e.g. $H^m_{\mathbb{C}}/\Gamma$ is compact and Γ_0 contains no cyclic subgroup of finite index).

If $rank_{hmt}(f_0) > 2$, then $[f_0] \in \pm\mathcal{HOL}$, i.e. f_0 is homotopic to a holomorphic or anti-holomorphic map. Moreover this ±-holomorphic map is unique.

Furthermore, if $rank_{hmt}[f_0] = 1, 2$, then

f_0 is homotopic to an "almost holomorphic" map $f : V \to W$, in the sense

that f decomposes as $V \to S \to W$, where S is a Riemann surface and the map $V \to S$ is holomorphic.

Moreover, if $H^m_{\mathbb{C}}/\Gamma$ is compact, "no nilpotent subgroup of finite index" can be relaxed to "no cyclic subgroup of finite index".

(In other words, the covering $\tilde{V} \to V$ with the Galois group Γ_0 contains a one parameter family of compact complex subvarieties $V'_s \subset \tilde{V}$ with $dim_C(V'_s) = dim_{\mathbb{C}}(V) - 1$.)

Corollaries. (1: $[V]_{sing}$) This theorem, as stated, remains valid for *singular locally irreducible* Kählerian, e.g. algebraic, varieties V.

Indeed, by Hironaka's theorem, V admits a surjective holomorphic map $V' \to V$ with *connected* fibers, where V' is non-singular.

The composition of a continuous map $f_0 : V \to W$ with $V' \to V$ contains a holomorphic representative $f' : V' \to W$ in its homotopy class, where f' sends every fiber $S_v \subset V'$, $v \in V$, of the map $V' \to V$ to a single point in W, since every contractible holomorphic map of a *connected* S to W with $K(W) \leq 0$ is constant. This gives us a holomorphic map $f : V \to W$ which is, clearly, homotopic to f_0.

(2: $[W_0 \subset W]$) Let $W = H^m_{\mathbb{C}}/\Gamma$, for a discrete torsion free isometry group Γ of $H^m_{\mathbb{C}}$, contain a *unique* maximal compact *connected* complex analytic subspace $W_0 \subset W$ of positive dimension and let the inclusion $W_0 \subset W$ be a homotopy equivalence.

Let f_0 be a continous map from a compact Kähler manifold V to W_0 and let $\Gamma_0 \subset \Gamma = \pi_1(W_0)$ denote the image of $\pi_1(V)$ under f_0. If $rank_{hmt}[f_0] > 2$ and if Γ_0 contains no nilpotent subgroup of finite index (this seems redundant), then f_0 is homotopic to a holomorphic map $V \to W_0$.

Possibly, there are lots of such W and $W_0 \subset W$ but the only examples I see offhand are immersed totally geodesic $W_0 \subset W$.

If the selfintersection of such a W_0 in W is sufficiently sparse (as in "Singular Kähler Examples" of 4.1) then the inclusion homomorphism $\pi_1(W_0) \to \pi_1(W)$ is injective and, by passing to the $\pi_1(W_0)$-covering of W if necessary, we achieve the above "homotopy equivalence" property.

Also one can show in the "sparse case" that W admits a proper positive $\mathbb{C}$-convex function which vanishes on W_0 and is strictly $\mathbb{C}$-convex away from W_0, so that W_0 is *the only* compact analytic subspace in W.

These W_0, in the interesting cases where they do have self-intersections, are singular with locally reducible singularities; hence, every holomorphic map from a *non-singular* V to W_0 lifts to the normalization W'_0 of W_0 that, if connected, equals $H^{m_0}_{\mathbb{C}}/\Gamma'_0$ for $m_0 = dim_{\mathbb{C}}(W_0)$. (If W'_0 is disconnected, then its each connected component equals $H^{m_i}_{\mathbb{C}}/\Gamma'_i$.)

(3:$[V \times H^m_{\mathbb{C}}]_{cycle}$) Let $X \to V \times W$ be the "diagonal" covering map, i.e. $\pi_1(X)$ *isomorphically* projects onto $\pi_1(V)$ and *surjectively* onto $\Gamma_0 \subset \pi_1(W)$. Observe, that the projection $X \to V$ is a homotopy equivalence – a fibration with the fibers $H^m_{\mathbb{C}}$.

If the generator $[V]_X \in H_{2k}(X) = H_{2k}(V) = \mathbb{Z}$, $k = dim_{\mathbb{C}}(V)$, goes to a *non-zero* class in $H_{2k}(W)$ under the projection $X \to W$ and $k > 1$, then $[V]_X$ can be realized a *compact k-dimensional complex subvariety $V' \subset X$*, namely by the *graph of the holomorphic map $V \to W$ (lifted to X)* which is guaranteed by the SSCT theorem.

On the other hand,

if V is *non-singular*, then, *every* complex subvariety $V' \subset V \times W$ which *projects to* V *with degree one*, equals the *graph of a holomorphic map* $V \to W$.

Indeed, the fibers of the projection $V' \to V$, that is a regular rational map, are *rationally connected*, while W contains *no rational curves.*

But singular V (e.g. those indicated in 4.1) may admit non-regular rational maps into W.

(4:[$H^m_{\mathbb{R}}$]) Let $W = H^n_{\mathbb{R}}/\Gamma$, let V be a compact (possibly singular) locally irreducible Kähler space, let $f_0 : V \to W$ be a continuous map and $\Gamma_0 \subset \Gamma$ be the image of $\pi_1(V)$ under f_0.

If Γ_0 *contains no Abelian subgroup of finite index, then* $rank_{hmt}(f_0) \le 2$*; moreover, if* $rank_{hmt}(f_0) = 0, 1, 2$*, then* f_0 *is homotopic to a map* f *which factors through a holomorphic map* $V \to S$ *for a Riemann surface* S*.*

To see this observe that $\Gamma \subset iso(H^m_{\mathbb{R}}) \subset iso(H^m_{\mathbb{C}})$ for $H^m_{\mathbb{R}} \subset H^m_{\mathbb{C}}$ and that "no Abelian" implies "no nilpotent" in $iso(H^m_{\mathbb{R}})$.

Since the squared distance function from $H^m_{\mathbb{R}}/\Gamma \subset H^m_{\mathbb{C}}/\Gamma$ is *strictly* $\mathbb{C}$*-convex* on $H^m_{\mathbb{C}}/\Gamma$, every complex submanifold $V \subset H^m_{\mathbb{C}}/\Gamma$ of positive dimension must be contained in $H^m_{\mathbb{R}}/\Gamma$. But $H^m_{\mathbb{R}}/\Gamma$ is totally real in $H^m_{\mathbb{C}}/\Gamma$; hence, $H^m_{\mathbb{C}}/\Gamma$ receives no non-constant holomorphic map from a compact connected analytic space.

We conclude by a corollary that combines SSCT with the Grauert's exceptional cycle theorem. (See 4.1. It is unclear if the linear analysis, which underlies Grauert's argument, is truly necessary here.)

(5) Let X be a complete simply connected Kähler manifold with strictly negative sectional curvature, $K(X) \le -\kappa < 0$ (e.g. $X = H^m_{\mathbb{C}}$) and Γ_X be a discrete undistorted torsion free isometry group of X. Assume that $dim_{hmt}(\Gamma_X) = dim_{hmt}(X/\Gamma_X) > m = dim_{\mathbb{C}}(X)$, i.e. $V = X/\Gamma_X$ is not contractible to the middle dimensional skeleton of some (hence, any) triangulation of V.

Let Γ_Y be a a discrete torsion free isometry group of $Y = H^N_{\mathbb{C}}$, such that the quotient space Y/Γ_Y contains no compact complex analytic subvariety of positive dimension, e.g. $\Gamma_Y \subset iso(H^N_{\mathbb{R}}) \subset iso(H^N_{\mathbb{C}})$.

Let $h : \Gamma_X \to \Gamma_Y$ *be a homomorphism such that the image* $\Gamma_0 = h(\Gamma_X) \subset \Gamma_Y$ *contains no nilpotent subgroup of finite index, (where "no Abelian" suffices if* $\Gamma_Y \subset iso(H^N_{\mathbb{R}})$*). Then* $dim_{hmt}(\Gamma_0) \le 2$*, i.e. the quotient space* $H^N_{\mathbb{C}}/\Gamma_0$ *is contractible to the* 2*-skeleton of some triangulation of this space.*

In particular, no cocompact lattice in $iso(H^n_{\mathbb{R}})$*,* $n \ge 3$*, admits an undistorted action on* $H^{n-1}_{\mathbb{C}}$*.*

Proof. Apply the SSCT $[H^N_{\mathbb{C}}/\Gamma]$-theorem to the normalization of the subvariety $V \subset W$ delivered by Grauert's exceptional cycle theorem (see 4.1).

Disclaimer. The $[H^m_{\mathbb{C}}/\Gamma]$-theorem looks even prettier than the **A**bel-**J**acobi-**A**lbanese maps into tori – you do not have to bother with choosing a complex structure in the target. It may look as a possible tool comparable to AJA for constructing "new" holomorphic objects, e.g. for a realization of homology classes in a complex manifold W by *complex analytic* subvarieties – the images of *holomorphic* maps $V \to W$.

However, this possibility seems as remote as in the Grauert case from the previuos section, since producing *Kähler* manifolds satisfying given requirements on their homotopy types seems more difficult than finding complex subvarieties in a given W.

How, on earth, for instance, can you construct a compact Kähler manifold V with fundamental groups Γ admitting a discrete co-compact actions on $H^m_{\mathbb{C}}$, rather than by *first* constructing $H^m_{\mathbb{C}}/\Gamma$ and *then* taking a *subvariety* $V \subset H^m_{\mathbb{C}}/\Gamma$?

Apparently, the only realistic message one can extract from the $[H^m_{\mathbb{C}}/\Gamma]$-theorem in the available examples is "just" *holomorphic rigidity*:

"the only holomorphic representatives in a certain class of continuous objects are the obvious ones if at all."

In fact, "holomorphic realization" of continuous maps $V \to W$ imposes strong restriction on the topologies of both manifolds. For example, the image $[V]_* \in H_{2n}(W)$ of the fundamental class $[V] \in H_{2n}(V)$, $n = \dim_{\mathbb{C}}(V)$ must be (n,n) (as a current) in the Hodge decomposition in W. In particular, if a class $h \in H^{2n}(W)$ is representable by a *holomorphic* $2n$-form on W, then $h[V]_* = 0$. (Compact manifolds $W = H^m_{\mathbb{C}}/\Gamma$ usually carry lots of such n-forms for $m = 2n$.)

Yet, there is some reason for optimism as we shall see by looking at *infinite dimensional* examples.

4.3 Dirichlet Flow into Harmonicity for $K \leq 0$.

Recall that the *Dirichlet energy* of a C^1-smooth map between Riemannian manifolds, $f : V \to W$ is

$$E(f) = \int_V ||Df||^2 dv.$$

Equivalently, let $UT(V) = Geo_1(V)$ be the unit tangent bundle regarded as the space of marked geodesics $g : \mathbb{R} \to V$. Then

the energy of f equals the energy of the curves $f \circ g : \mathbb{R} \to V$ integrated against the Liouville measure in $UT(V) = Geo_1(V)$,

where, observe, so defined energy makes sense for an *arbitrary* metric space W and a geodesic metric space V with a distinguished (Liouville-like) measure on the space of geodesics in V invariant under the geodesic flow. (Loosely speaking, $E(f) = E(f \circ g)$ for a random geodesic g in V.)

If $dimV = 2$, i.e. V is a surface, then the energy of an f is a *conformal invariant* of the metric in V: it does not change if the Riemannian length (metric) is multiplied by a positive function $\phi(v)$, since the squared norm of the differential $||Df||^2$ is divided by ϕ^2, while dv, being a 2-form on surfaces, is multiplied by ϕ^2 which keeps the integrant $||Df||^2 dv$ unchanged.

It is also clear that $E(f) \geq area(f)$, where the equality holds if and only the map f is conformal.

If W is *Kählerian* with the fundamental 2-form denoted ω_W, then every surface $f : V \to W$ satisfies

$$E(f) \geq area(f) \geq \Big| \int_V f^*(\omega_W) \Big|,$$

where $E(f) = \int_V f^*(\omega_W)$ if and only if the map f is ±-holomorphic, and where, if V is a *closed oriented* surface, the integral $\int_V f^*(\omega_W)$ is a homotopy invariant of f which, in fact, depends only the homology class $f_*[V] \in H_2(W; \mathbb{R})$.

Thus, holomorphic maps $V \to W$ for $dim_{\mathbb{R}}(V) = 2$ are energy minimizing in their homotopy classes.

An elementary computation shows that this remains true for all *Kähler* manifolds V of an arbitrary dimension.

If V and W are Kählerian and V is compact, then every holomorphic map $f : V \to W$ is energy minimizing in its homotopy class, where the minimal energy $E_{min}[f]$ depends only on the homomorphism $f_* : H_2(V) \to H_2(W)$, namely

$$E_{min}[f] = \Big| \int_V \omega_V^{n-1} \wedge f^*(\omega_W) \Big| \text{ for } n = dim_{\mathbb{C}}(V).$$

If $V = V^n \subset \mathbb{C}P^N$ is a projective algebraic variety, and $C \subset V$ is an algebraic curve which equals the intersection of V with a generic $\mathbb{C}P^{N-n+1} \subset \mathbb{C}P^N$, then $E_{min}[f]$ equals the energy of $f : C \to W$. In particular, this energy of $f|C$ does not depend on C for holomorphic maps $V \to W$.

This suggests the definition of the energy for arbitrary (non-holomorphic) f associated to a probability measure μ on the space $\mathcal{C}$ of these curves C, (that equals the Grassmannian $Gr_{N-n+1}(\mathbb{C}P^N)$), as "the energy of f on a random holomorphic curve in V", namely,

$$E_\mu(f) = \int_{\mathcal{C}} E(f|C) d\mu.$$

If f is holomorphic all these energies are equal $E_{min}[f]$; thus,

if a holomorphic f in the homotopy class $[f_0]$ of a continous map $f_0 V \to W$ exists, it necessarily equals the energy minimizing map in this class (where one needs the measure μ to have full $\mathcal{C}$ for its support).

The energy minimizing maps are especially appealing if W has *non-positive* sectional curvature, where the energy $f \mapsto E(f)$ is, obviously, *a geodesically (almost strictly) convex function* on the space of maps $f : V \to W$ in a given homotopy class, and as we shall explain below,

if V is compact, then every homotopy class $[f_0]$ of maps contains a smooth energy minimizing representative f_{min} under a (necessary) mild restriction on this class; moreover, this f_{min} is unique up to properly understood "translations by constants".

But even if one a priori knows that a homotopy class $[f_0]$ does contain a holomorphic map, which necessarily equals f_{min}, one can not (?) directly show that f_{min} is holomorphic, unless one imposes rather stringent assumptions on the local geometry of W. Yet, these assumptions are satisfied for a variety of meaningful examples.

The Euler-Lagrange equations for the Dirichlet energy on C^2-smooth maps f between Riemannian manifolds make a second order non-linear elliptic written as $\Delta f = 0$, where the solutions of this are called *harmonic* maps.

It is easy to see that a C^2-smooth map $f : V \to W$ is harmonic if and only if its second differential at each point $v \in V$, that is the quadratic map between the tangent spaces, $\phi = D_f^2 : T_v(V) \to T_{f(v)}(W)$ satisfies $\Delta f(v) =_{def} \Delta\phi(0) = 0$ for $0 \in \mathbb{R}^n = T_v(V)$, where the second differential is defined with the local geodesic coordinates in (V, v) and $(W, f(v))$, and where the Laplace operator $\Delta = \Delta_{\mathbb{R}^n}$ on C^2-maps from a Euclidean space $\mathbb{R}^n$ to a linear space T (that is the tangent space $T_{f(v)}(W)$ in the present case) is defined in the usual way.

Since $\Delta_{\mathbb{R}^n}$ does not depend on the metric in the target space T, but only on the affine structure in T, the equation $\Delta f(v) = 0$ does not fully uses the metric in W but rather the corresponding affine connection. On the other hand, the equation $\Delta f = 0$ is Euler-Lagrange for maps between Riemannian (and pseudo-Riemannian) manifolds, i.e it represents the stationary points of

Dirichlet's energy $E(f) = \int_V ||Df||^2 dv$ and global minima of $E(f)$ is the major (but not the only) source of harmonic maps in geometry.

Similarly to the stationary Euler-Lagrange equation $\Delta f = 0$, one sees that the vector field for the *downstream Dirichlet energy gradient flow* f_t on the space $\mathcal{F} \ni f$ of maps $f : V \to W$ is $f \mapsto \Delta f$ where the vectors $\Delta f(v) \in T_{f(v)}(W)$ are defined as above via the local geodesic coordinates at the points $v \in V$ and with the geodesic coordinates in W at $w = f(v)$.

If W is a Riemannian manifold with *non-positive curvature,* $K(W) \leq 0$ then, by a *theorem of Eells-Sampson*, this flow behaves very much the same as the usual heat flow on functions $V \to \mathbb{R}$.

In fact, $K(W) \leq 0$ makes this flow "more contracting" then for $W = \mathbb{R}^n$; in particular, it is strictly contracting away from a finite dimensional space of directions.

For example, if $K(W < 0)$, then the flow is strictly contracting on the subspace of maps f with $rank(Df) \geq 2$, which implies that the fixed point set of the flow – the set of harmonic maps in the homotopy class of $f_0 : V \to W$, consists of a *single* point (or it is empty if the flow slides to infinity in W which makes "strictness not fully strict" after all).

Furthermore the flow f_t, whenever it exists (i.e. if the image $f_t(V) \subset W$, does not "slide to infinity" in W and/or does not "hit the boundary" of W), satisfies the usual *parabolic estimates*, where the most important one is the bound on the norm of the differential of f at each point $v_0 \in V$ in terms of the full energy

$$||Df_{t+1}(v_0)||^2 \leq const_V \int_V ||Df_t(v)||^2 dv.$$

Notice, that the dimension of W does not enter these estimates, as it is especially clear for maps $V \to \mathbb{R}^N$; therefore, this flow is defined and satisfies all Eels-Sampson estimates for *infinite dimensional* Riemannian-Hilbertian manifolds W with $K(W) \leq 0$.

If V and W are compact manifolds without boundaries, then, according to Eells-Sampson,

the energy gradient flow is defined for all $t \geq 0$ *and* f_t *converges, for* $t \to \infty$ *to a harmonic map* $f_\infty : V \to W$.

If W is non-compact, then $f_t(V) \subset W$ may "slide to infinity" as it happens, for example, in $W = S^1 \times \mathbb{R}$ with the metric $\phi^2(t)ds^2 + dt^2$ for the obvious maps $f_t = S^1 \to S^1 \times t \subset W$ if the function $\phi(t) > 0$ is decreasing.

If $\lim_{t\to\infty} \phi(t) = l_{inf} > 0$, then one still obtains a harmonic map f_∞ from the circle S^1 (that is a closed geodesic in the present case), however, not into W but into *the marked Hausdorff limit space* $W_\infty = lim_{t\to\infty}(W, f_t(s_0))$ that equals $S^1(l_{inf}) \times \mathbb{R}$ for the circle $S^1(l_{\text{inf}})$ of length l_{inf}.

But if $l_{inf} = 0$ the family f_t, $t \to \infty$, "collapses" to a constant map into $\mathbb{R}$.

In general, for more complicated V and W, e.g. for $V = S^1 \times S^1$ and $W = V \times R$ with the metric $\phi_1^2(t)ds_1^2 + \phi_2^2(t)ds_2^2 + dt^2$, one may have a partial collapse of the limit map.

If $dim(W) = \infty$ then $f_t(V)$ may slide to infinity "dimension-wise" rather than "distance-wise". For example, let $\phi : \mathbb{R}^\infty \to \mathbb{R}_+$ be a convex function with bounded sublevels $\{\phi(\overline{x}) \leq l\} \subset \mathbb{R}^\infty$ which *does not* achieve its minimum $l_{inf} \geq 0$ on our Hilbert space $\mathbb{R}^\infty$.

Accordingly, the harmonic flow on the circles in $W = S^1 \times \mathbb{R}^\infty$ with the metric $\phi^2(\overline{x})ds^2 + d\overline{x}^2$ (for $d\overline{x}^2$ denoting the Hilbert metric) will not converge in W, even if it remains in a bounded regions. Yet, the situation here is no worse than that in the $S^1 \times \mathbb{R}$-example: the limit of maps $f_t : S^1 \to W$ goes to the limit space, that is to $S^1(l_{inf}) \times \mathbb{R}^\infty$, except that "limits of pointed metric spaces" must be understood as *ultralimits* as in [60], [41], [53], and in sections 6.A.III, $6D_3$, 7.A.IV in [32] (See more on this "stability" at the end of this section.)

If W is complete, finite or infinite dimensional, manifold of strictly negative curvature, $K(W) \leq -\kappa < 0$, or even, a possibly singular, complete $CAT(-\kappa)$-space for this matter, then a family of *uniformly Lipschitz maps* $f_t : V \to W$, e.g. finite energy f_t of an Eells-Sampson Dirichlet flow,

can not slide to infinity, except for the case where the action of the group Γ_0 – *the image of* $\pi_1(V)$ *in* $\Gamma = \pi_1(W)$ – *on the universal covering* $X = \tilde{W}$ *is parabolic*,

where, recall, "parabolic" means, that this action fixes a point, say b, in the ideal boundary $\partial_\infty(X)$ and preserves the *horospheres* centered at b, that are limits of spheres $S_x(R) \subset X$, where $R = dist(x, x_0)$ for a fixed point $x_0 \in X$ and where $x \to b$.

If W is a complete simply connected *finite dimensional* manifold with δ-*pinched* negative curvature, i.e. $-\infty < -\kappa \leq K(X) \leq -\delta\kappa < 0$, then every *discrete parabolic* group $\Gamma \subset iso(X)$ is *virtually nilpotent*, i.e. it contains a nilpotent subgroup of finite index, according to *Margulis' lemma.* Moreover the nilpotency degree of such a Γ is bounded by $\delta/2$.

For example, if $\delta < 4$, e.g. X has constant curvature, then Γ is *virtually Abelian* (which explains the presence of these conditions in the SSCT theorems $[H^m_{\mathbb{C}}]$ and $[H^m_{\mathbb{R}}]$ of the previous section).

Equivariant Maps. It is appropriate to regard maps $V \to W$ as equivariant maps between their respective universal coverings.

In fact, all of the above automatically translates and generalizes to equivariant maps $f : X \to Y$, where X and Y are acted by (not necessarily discrete) isometry groups Γ_X and Γ_Y and "equivariant" refers to a given homomorphism $h : \Gamma_X \to \Gamma_Y$.

One needs (?) the action of Γ_X to be *proper*, e.g. discrete in order to define the energy $E(f)$ as the integral of the (Γ_X-invariant!) energy density $||Df||^2 dv$ over X/Γ_X and one needs no assumptions on the action of Γ_Y what-so-ever, except of being isometric.

Despite the apparent *technical* triviality of such a generalization, it significantly broadens the range of *applications* of the Eells-Sampson theorem.

Foliated Harmonicity. The equivariant setting for "periodic" metrics and maps admits the following "almost periodic" generalization.

Let X and Y foliated spaces with Riemannian leaves where the leaves in Y have $K \leq 0$. If the foliation in X comes with a *transversal measure*, on may speak of the energy of such maps and, under suitable (and not fully understood) stability conditions the leaf-wise Dirichlet gradient flow in the space of maps $X \to Y$ which send leaves to leaves converges to a leaf-wise harmonic map $f : X \to Y$ (see [29])

This is relevant for our present purpose if the leaves in both spaces are Kählerian, where we are after leaf-wise holomorphic maps (see next section).

Harmonic Maps with Infinite Energy. The radial projection f_{mdl} from $\mathbb{C} \setminus 0$ to the unit circle $S^1 \subset \mathbb{C}$, and/or the projection $S^1 \times \mathbb{R} \to S^1$ serve as models for general maps with infinite energy, where one exercises a sufficient control on the energy-density.

The existence of similar harmonic maps f from *quasi-Kählerian*, e.g. *quasi-projective*, varieties $V = \overline{V} \setminus \Sigma$ (that are complements to complex subvarieties Σ in compact Kählerian $\overline{V}$, e.g. projective algebraic varieties $\overline{V}$) to spaces with $K \leq 0$ is established in [56], [78] [48], [49].

These maps, in the directions transversal to V_0 near V_0, behave like f_{mdl} and satisfy the natural bounds on the energy density.

Since, eventually, one wants to prove that these f are *pluriharmonic* (see below) it is immaterial which Kähler metric on V is used for harmonicity. Yet, it is technically convenient to work with a *complete* Kähler metric on V, where such metrics are readily available. (See [48], [49].. where all this is implemented for maps into finite dimensional spaces with the techniques which, probably, apply to $dim = \infty$ as well.)

About Stability. Let X and Y be acted by isometry groups Γ_X and Γ_Y, let $h : \Gamma_X \to \Gamma_Y$ be a homomorphism, where one can assume without loss of generality that Γ_Y equals the full isometry group $iso(Y)$ the most (but not only) relevant part of it is the image $h(\Gamma_X) \subset iso(Y)$.

Let us formulate several stability properties of h-equivariant maps $f : X \to Y$ which allow a harmonic limit f_∞ of the Eells-Sampson flow $f_t : X \to Y$, where, in the best case, f_∞ sends $X \to Y$, or, if this flow "slides to infinity" in Y, we want to guarantee a very similar harmonic map $X \to Y_\infty$, where the limit space Y_∞ should not be much different from Y.

We assume that Y is a complete simply connected (the latter can be dropped) manifold with $K(Y) \leq 0$, where we do allow $dim(Y) = \infty$. Besides it is convenient (and possible) to admit singular $CAT(0)$-spaces into the picture, keeping in mind that even if Y itself is non-singular the space Y_∞ may be singular.

Also, we can afford Y with *convex* boundary; moreover, if we expect the limit map f_∞ to be holomorphic, then just $\mathbb{C}$-*convexity* of the boundary will do.

As for X, we concentrate on the *co-compact* case, i.e. where the action of Γ_X on X is *proper* (i.e. $\gamma \to \infty \Rightarrow \gamma(x) \to \infty$) and X/Γ_X is *compact.* In fact, everything equally applies to the case where X is complete, the action of Γ_X is proper and the starting map $f_0 : X \to Y$ has Γ_X-*finite* energy – the integral of the energy density over X/Γ_X is finite.

The strongest stability condition, which, due to the point-wise bound on $||Df_t||$, i.e a uniform Lipschitz bound on f_t, prevents "sliding to infinity" in Y and, thus, insures the limit map $f_\infty : X \to Y$, reads as follows (see [15], [54],[9], [39]).

$[Stab_{strong}]$ There are a geodesically convex subset $Y_0 \subset Y$ (which may have $dim(Y_0) < dim(Y)$) invariant under some isometry group Γ_0 of Y such that $h(\Gamma_X) \subset \Gamma_0 \subset \Gamma_Y$ and a finite (compact if Γ_X is locally compact rather than just discrete) subset $\Delta \subset \Gamma_X$, such that, for every $C > 0$, the subset $U(\Delta, C) \subset Y_0$ defined by

$$U(\Delta, C) \ni y_0 \Leftrightarrow dist_Y(h(\gamma)(y_0), y_0) \leq C \text{ for all } \gamma \in \Delta$$

is Γ_Y-*precompact* in Y_0, i.e. it is covered by a Γ_0-orbit of a compact subset in Y_0.

For example, this condition is (obviously) satisfied by the actions of the above non-purely parabolic $h(\Gamma_X) \subset \Gamma_Y$ on spaces Y with $K(X) \le -\kappa < 0$.

Notice that $Stab_{max}$ makes sense for *infinite dimensional* Y but this is not particularly interesting, since the normal projection $Y \to Y_0$, which is distance *decreasing*, brings the flow f_t to $Y_0 \subset Y$ and the full geometry of Y (most of which is situated away from Y_0) remains out of the picture.

One can do slightly better with *non-locally compact* convex $Y_0 \subset Y$, $i = 1, 2, ...,$ which isometrically split, $Y_0 = \times Z_i$, $i = 1, 2, ...,$ with locally compact factors Z_i, but this is not very exciting either.

What serves better and covers a wider class of examples, is (as in [75], [7] , [8]) a lower bound on the *k-volume* $vol_k[f_0]$ *of the equivariant homotopy class* of $f_0 : X \to Y$ which is similar to but more accurate than the homotopy rank $rank_{hmt}[f_0]$ defined earlier (at the beginning of 4.2).

Namely, if $n = dim(X/\Gamma_X)$ and $f : X \to Y$ is a smooth equivariant map, we define $vol_n(f)$ as the volume of the integral over X/Γ_X of the absolute value of the Jacobian of f. (Typically, Γ_X is discrete and $dim(X/\Gamma_X) = dim(X)$, but we do not, a priori, exclude $dim(\Gamma_X)$, where maps f must be regarded locally as maps from X/Γ_X rather than from X in order to have correct Jacobians.)

The volume of the equivariant homotopy class of f, denoted $infvol_n[f]$, is defined as the infimum of all smooth maps in the homotopy class $[f]$.

Furthermore, this is especially relevant where X/Γ_X is non-compact, we denote by $[f]_L$ the set of maps which can be joined with f by a homotopy of equivariant *L-Lipschitz* maps (if f is *not* L-Lipschitz itself this class is empty) and let $infvol_n[f]_L$ denote this infimum of the volumes of the maps in this class.

Define $infvol_k[f]_L$ for $k \le dim(X/\Gamma_X)$ by restricting maps f to Γ_X-invariant piece-wise smooth $X' \subset X$ with $dim(X'/\Gamma_X) = k$ and setting

$$infvol_k[f]_L = \sup_{X'} \inf_{f \in [f]_L} vol_k(f|X')$$

Finally, let $rank_{hmt}[f]_{Lip}$ be the maximal k such that $infvol_k[f]_L > 0$ for all $L > 0$.

Notice that $rank_{hmt}[f]_{Lip} \ge rank_{hmt}[f]_{Lip}$, where the inequality is *strict*, for example, for the identity maps on complete *non-compact* manifolds with *finite* volumes.

The role of this Lipschitz homotopy rank, is to bound from below the rank of the (differential of the) limit map $f_\infty : X \to Y_\infty$, since, clearly,

$$rank(f_\infty) =_{def} \max_{x \in X} rank(D_x f_\infty) \ge rank[f_0]_{Lip}$$

for the Eells-Sampson equivariant Dirichlet flow $f_t : X \to Y$, which starts with a Lipschitz f_0 and converges to harmonic $f_\infty : X \to Y_\infty$.

Notice that

if Y *is a (possibly infinite dimensional) symmetric space then the (ultra)limit space* Y_∞ *is isometric to* Y,

but the homomorphism $h_\infty : \Gamma_X \to iso(Y)$ does nor have to be equal to the original h.

Also observe that

if Y *has* $-\kappa_1 \le K(Y) \le -\kappa_2$, *then the limit space* Y_∞ *also has its sectional curvatures pinched between* $-\kappa_1$ *and* $-\kappa_2$.

4.4 From Harmonic to Pluriharmonic for $K_{\mathbb{C}} \leq 0$.

A map f from a complex V manifold to Riemannian one is called *pluriharmonic* if its restriction to *every holomorphic curve (Riemann surface)* in V is harmonic.

Equivalently, pluriharmonicity can be expressed by $Hess_{\mathbb{C}} f = 0$ for $Hess_{\mathbb{C}} f(v)$ being "one half" of the second differential $D_f^2 : T_v(V) \to T_{f(v)}(W)$ made of the values of Laplacians of D_f^2 at $0 \in T_v(V)$ on all holomorphic lines in $T_v(V)$.

Unlike harmonicity, pluruharmonicity is a very stringent condition on maps $f : V \to W$ at the points $v \in V$ where the ranks of the differentials $D_v f : T_v(V) \to T_{f(v)}(W)$ are > 2 – generic W do not receive such maps at all. In fact, maps $V_1 \times V_2 \to W$ which are harmonic on all coordinate "slices" $v_1 \times V_2$ and $V_1 \times v_2$ satisfy *two* determined PDE systems, which make such maps exceptionally rare.

On the other hand, there is the following

List of Standard Pluriharmonic Maps.

$\bullet_{\pm holo}$ ±-*Holomorphic* maps from complex manifolds to Kählerian ones are pluriharmonic.

$\bullet_{plu \circ hol}$. Composed maps $V \overset{holo}{\to} S \overset{harmo}{\to} W$, where if S is a Riemann surface are pluriharmonic.

In fact, composed maps $V \overset{holo}{\to} S \overset{pluri}{\to} W$ are pluriharmonic. for all complex manifolds V and S.

$\bullet_{subm}$ *Riemannian submersions* $f : V \to W$, where V is Kähler and the fibers $f^{-1}(w) \in V$ are complex analytic, are pluriharmonic. ("Riemannian submersion" means that the differential Df *isometrically* sends every the normal space to each fiber say $N_v \subset T_v(V)$, $v \in f^{-1}(w)$, *onto* $T_w(W)$.)

$\bullet_{geod \circ plu}$ *Composed* $geod \circ pluri$ *maps* $V \overset{pluri}{\to} W_0 \overset{geod}{\to} W$, are pluriharmonic, where "geod" refers to *locally isometric geodesic maps* which sends geodesics to geodesics.

$\bullet_{\times}$ *Cartesian products* of pluriharmonic maps $pluri_i : V_i \to W_i$ are pluriharmonic $\times_i pluri_i : \times_i V_i \to \times_i W_i$.

$\bullet_{dia}$ *Diagonal* Cartesian products of pluriharmonic maps $pluri_i : V \to W_i$, that are composed maps $diag : V \to \times_i (V_i = V) \to \times_i W_i$, are pluriharmonic $V \to \times_i W_i$.

[∗] *Basic Class of Examples.* If W contains a totally geodesic split submanifold, $W \supset W_1 \times W_2$, where W_1 is Kählerian and where W_2 receives a harmonic maps from a Riemann surface, $harmo : S \to W_2$, then, for every holomorphic map $holo : V \to W_1 \times S$, the resulting composed map $V \overset{holo}{\to} W_1 \times S \overset{id \times harmo}{\to} W_1 \times W_2$ is pluriharmonic $V \to W$ with the image in $W_1 \times W_2 \subset W$.

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The ***H**odge-**B**ochner-**S**ui-**S**ampson formula* for *harmonic* maps f of a *Kählarian* V to a (possibly infinite dimensional) Riemannian W can be schematically written as

$$(\star) \qquad \|Hess_{\mathbb{C}}\|^2 dv = d\langle Df \cdot Hess_{\mathbb{C}} f\rangle_{k-1} + [K_{\mathbb{C}}(W)(Df(\tau_1), ..., Df(\tau_4))]dv,$$

where $\langle ... \cdot ... \rangle_{k-1}$ is a certain bilinear form which takes values in $(k-1)$-forms on V, $k = dim_{\mathbb{R}}(V)$, and $K_{\mathbb{C}}(W)(t_1, t_2, t_3, t_4)$, $t_i \in T(W)$, is "a certain part" of the curvature tensor of W, where [...] means that this $K_{\mathbb{C}}(W)$ applies to the images of the tangent vectors $\tau \in T(V)$ under the differential $Df : T(V) \to T(W)$ and then averages in a certain way at each point $v \in V$.

Since

$$\int_V d\langle Df \cdot Hess_{\mathbb{C}} f\rangle_{k-1} = \int_{\partial V} \langle Df \cdot Hess_{\mathbb{C}} f\rangle_{k-1}$$

by Stoke's formula, the inequality $K_{\mathbb{C}}(W) \le 0$, i.e.

$$K_{\mathbb{C}}(W)(t_1, t_2, t_3, t_4) \le 0 \text{ for all } t_1, t_2, t_3, t_4 \in T(W),$$

implies

★HBSS: *Every harmonic map f from a Kähler manifold V into W is pluriharmonic,*

provided V is compact without boundary, or it can be exhausted by compact domains with "small" boundaries so that the boundary term in the Stoke's formula goes to zero.

Miraculously, a seemingly impossible problem of the existence of a *pluriharmonic* map $f : V \to W$ is reduced to a realistic one of finding a harmonic map, where, if also $K(X) \le 0$, one knows (Eells-Sampson) that every map $f_0 : V \to W$ is homotopic to a harmonic one under suitable stability assumptions, e.g. if W is compact.

(Explicit writing down $K_{\mathbb{C}}(W)$, which we have no intention of doing, see [75] for this, shows that $K_{\mathbb{C}}(W) \le 0 \Rightarrow K(W) \le 0$; thus we do not have to make any additional assumption on W if $K_{\mathbb{C}}(W) \le 0$.)

But how could it happen that the Dirichlet flows that, in general, *do not commute* in the space of maps $V \to W$ for *different* Kählerian metric in V, have the *same* fixed point set?

Apparently, these flows "semi-commute" in a certain way, which is partly reflected in $[Tol]$-*convexity* (see 4.6) and which, if one could expressed this *fully and precisely*, would be useful in other contexts, e.g. for harmonic maps from Riemannian manifolds with special holonomy groups (which for locally symmetric spaces amounts to Margulis superrigidity, [61], [46], [19]) and for actions of groups with "connected families of virtually split subgroups" (similar to those in 2.3.) on $CAT(0)$ spaces of finite and infinite dimensions.

Also, it would be nice to find more sophisticated (semi)group actions defined by PDE in function spaces that would display (super)stability similar to that in sections 2 and 3.

Since pluriharmonicity is a very restrictive condition, one concludes that the map f must be a very special one, which, in turn, imposes strong constrains on the *homotopy class* of f. (see below).

Flat Targets. If W is a flat manifold, e.g. $W = \mathbb{R}^n/\mathbb{Z}^n$, and the $K_{\mathbb{C}}$-term in $(\star)$ vanishes, then the implication *harmonic* $\Rightarrow$ *pluriharmonic* for maps $f : V \to W$ follows from the *Hodge decomposition* on 1-forms, applied to the differential $\phi = df$,

every harmonic $\mathbb{R}^n$-valued (including $n = \infty$ [60]) 1-form ϕ on a compact Kähler manifold V, that is locally equals the differential of a harmonic map $f : V \to \mathbb{R}^n$, *decomposes into the sum $\phi = \phi_+ + \phi_-$, where the forms $\phi_\pm$ are $\pm$-holomorphic:*

locally, $\phi_\pm = df_\pm$*, where the (local) map* $f_+ : V \to \mathbb{C}^n$ *is holomorphic while* $f_- : V \to \mathbb{C}^n$ *is antiholomorphic.*

In particular, the homotopy class $[f]_{Ab}$ of continuous Abel's maps f from V to the torus $W = A(V) = H_1(V;\mathbb{R})/H_1(V;\mathbb{R})$ admits a *pluriharmonic* representative $f_{plu} : V \to A(V)$ for every compact Kähler manifold and hence, for every variety with normal singularities, where this map, as we know, does not depend on any metric in W, but only on the flat affine structure in it.

Moreover, the Hodge decomposition provides the complex structure in this W for which f_{plu} is holomorphic, thus, furnishing the proof of the Abel-Jacobi-Albanese theorem.

This can be also seen by translating the classical

"pluriharmonic function equals the real part of a holomorphic function"

to the Hodge theoretic language:

the complex valued 1*-form* $df_{plu} + \sqrt{-1} d_J f_{plu}$ *for* $d_J f_{plu}(\tau) = df_{plu}(\tau/\sqrt{-1})$, $\tau \in T(V)$*, is a closed holomorphic* 1*-form.*

Therefore, $f_{plu} : V \to A(V)$ is holomorphic for the complex structure on $A(V)$ corresponding to the linear anti-involution $J : \phi \to \phi_J$ on the linear space $H_1(V;\mathbb{R})$ which is realized by pluriharmonic 1-forms ϕ on V.

Since the harmonic map theory is invariant under isometries, one automatically obtains, as was explained at the end of the previous section, the following

Equivariant Hodge-Albanese Theorem. Let Γ_Y be an isometry group of the Euclidean space $Y = \mathbb{R}^n$, let X be a complete normal Kähler space with a proper cocompact isometric action of a group Γ_X, let $h : \Gamma_X \to \Gamma_Y$ be a homomorphism and $f_0 : X \to Y = \mathbb{R}^n$ be a continuous h-equivariant map. ("Proper" means $[\gamma \to \infty] \Rightarrow [\gamma(x) \to \infty]$) Then

there exists an isometric action of Γ_Y *on some* $\mathbb{C}^N$*, an* $\mathbb{R}$*-affine surjective equivariant map* $\mathbb{C}^N \to \mathbb{R}^n$*, and a holomorphic equivariant map* $X \to \mathbb{C}^N$ *such that the composed map* $X \to \mathbb{C}^N \to \mathbb{R}^n$ *is equivariantly homotopic to* f_0*.*

This theorem is limited by scarcity of "interesting" isometric group actions on $\mathbb{R}^n$ for finite n. But there are by far more such actions on $\mathbb{R}^\infty$ where the above applies under suitable *stability conditions*, see [60] [13] for instances of this.

Another kind of generalization of Hodge-Albanese concerns mappings of *quasi-projective* algebraic (and *quasi-Kähler*) varieties $\overline{V} \setminus V_0$ to commutative algebraic groups build of Abelian varieties and the multiplicative group $\mathbb{C}^\times$ ([71],[44]); probably, there is a full equivariant $\mathbb{R}^\infty$-valued version of this.

Discussion on $K_\mathbb{C} \neq 0$*.* The general HBSS formula $(\star)$ is not that surprising in view of the corresponding Hodge formula where $K_\mathbb{C} = 0$, since

every invariant Euclidean formula involving at most second derivatives has its Riemannian counterpart with an extra curvature term.

Actually, $K_\mathbb{C}$ could be *defined* as such an "extra term" in the Hodge formula.

What is remarkable, however, that this $K_\mathbb{C}$ is *non-positive* in a variety of *significant* cases where one can use the Eels-Sampson theorem.

For example, one shows by a local/infinitesimal (sometimes quite involved, [7] [8]) computation that the following spaces do have $K_\mathbb{C} \leq 0$.

List of Spaces with $K_{\mathbb{C}} \leq 0$.

$\bullet_2$ 2-*Dimensional* Y with $K(Y) \leq 0$, i.e. *surfaces* with non-positive curvatures.

$\bullet_{-1/4}$ Riemannian manifolds Y with $-1/4$-*pinched* curvature, i.e. where $-1 \leq K(Y) \leq -1/4$ have $K_{\mathbb{C}}(Y) \leq 0$. [42]

Furthermore, if the pinching is *strict*, i.e. $-1 < K(Y) < -1/4$, then also $K_{\mathbb{C}} < 0$, where this inequality, is, by definition (whatever it is), stable under C^2-perturbation of metrics.

Moreover, the above remains true for *the local* $1/4$-pinching condition. i.e. where $-\kappa(y) \leq K_y(Y) \leq -(1/4)\kappa(y)$ for some positive *function* $\kappa(y)$, $y \in Y$.

$\bullet_{sym}$ *Symmetric* spaces Y with $K(Y) \leq 0$ have $K_C(Y) \leq 0$, [73] [68] [7], [75].

Notice that the real hyperbolic spaces $H^n_{\mathbb{R}}$, $n = 2, 3, ..., \infty$, of *constant* negative curvature have strictly negative $K_{\mathbb{C}}$, while the inequality $K_{\mathbb{C}}(Y) \leq 0$ is non-strict for all other symmetric spaces Y, not even for $Y = H^n_{\mathbb{C}}$ (which have $-1 \leq K(Y) \leq -1/4$) – an arbitrary small perturbation of the metric may bring $K_{\mathbb{C}} > 0$.

$\bullet_{WP}$ *Weil-Peterson* metric on the moduli space of curves. [69] (This metric is non-complete but it is *convex at infinity* which is enough for Eells-Sampson, where, however, the stability condition needs a special attention [11] and where the measurable dynamics provides an alternative to harmonic maps [50].)

$\bullet_{\times}$ Cartesian products of manifolds with $K_{\mathbb{C}} \leq 0$ also have $K_{\mathbb{C}} \leq 0$.

$\bullet_{sing}$ There are some singular spaces, e.g. Euclidean and hyperbolic buildings where $K_{\mathbb{C}} \leq 0$ and the HBSS formula $(\star)$ applies, [41] [10].

$\bullet_{\Lambda}$ Let Y, topologically a ball, be a smooth Riemannian manifold and let $\Sigma \subset Y$ be a union of k mutually orthogonal totally geodesic submanifolds of real codimension 2.

Let $\widetilde{Y \setminus \Sigma}$ be the obvious $\mathbb{Z}^k$-covering and Y_{Λ} be the completion of $\widetilde{Y \setminus \Sigma}/\Lambda$ for some discrete isometry group Λ of $\widetilde{Y \setminus \Sigma}$ isomorphic to $\mathbb{Z}^k$.

The simplest example of this is $\Lambda = \lambda\mathbb{Z}^k$, where λ is an integer and Y_{Λ} equals a *ramified* covering of Y.

A pleasant instance of $Y_{\lambda\mathbb{Z}^k}$ being defined for *all* $\lambda > 0$, is where Y equals the Euclidean space as well a real or complex hyperbolic space.

If $K_{\mathbb{C}}(Y) \leq 0$, then the natural singular metric on such Y_{λ} for $\lambda \geq 1$, probably, also has $K_{\mathbb{C}} \leq 0$ which is easy to see for $Y = \mathbb{R}^n$, $H^n_{\mathbb{R}}$ and $H^n_{\mathbb{C}}$.

One can not have $K_{\mathbb{C}}(Y_{\lambda\mathbb{Z}^k}) \leq 0$ for $\lambda < 1$, e.g. where $\lambda = 1/m$ and $Y_{\lambda\mathbb{Z}^k} = Y/\mathbb{Z}^k_m$. But one can, in some cases (where Y harbor much negative curvature away from Σ) make $K_{\mathbb{C}} \leq 0$ by suitably smoothing the metric, where this is sometimes possible even in the Kähler category, e.g. for the *Mostow-Siu* examples (see [73] and references therein).

All being said, the range of possibilities and applications of spaces *locally isometric* to these Y_{Λ}, which are abundant, especially (but not only) for (finite and infinite dimensional) symmetric spaces Y, remains mainly unexplored.

$\bullet_{dim=\infty}$ Since the condition $K_{\mathbb{C}} \leq 0$ is expressible with quadruples of vectors it makes sense (like sectional curvature and unlike, say, Ricci curvature) for all infinite dimensional Riemannian-Hilbertian manifolds Y, and all of the above applies to these Y.

Our essential examples are infinite dimensional symmetric spaces Y which are completions of the unions of $Y_1 \subset Y_2 \subset ...$ for increasing chains of finite dimensional symmetric spaces Y_i and geodesic isometric embeddings $Y_i \subset Y_{i+1}$ (see [55], [65] and references therein).

The simplest infinite dimensional irreducible symmetric space is the real hypebolic $H^\infty_\mathbb{R}$, which, as we know, has $K_\mathbb{C} < 0$. It admits, for instance, a natural isometric action of every countable subgroup Γ in the *Cremona group* $Bir_2 = Bir(\mathbb{C}P^2)$ of birational automorphisms of the projective plane $\mathbb{C}P^2$. (The group Bir_2 naturally acts on the cohomology $\overline{H}^2$ of the projective limit of all rational surfaces over $\mathbb{C}P^2$ and regular rational maps between them, where this action preserves the intersection form of the type $(+-------...)$ on $\overline{H}^2$.)

This is used in [14] to show, among other things, that, for instance, "most" such Γ are *not* Kazhdan T and that they *can not* serve as fundamental groups of Kähler manifolds.

4.5 From Pluriharmonic to Holomorphic.

Despite the fact that pluriharmonic maps are very special, it is not apparent what they actually are in specific examples.

There are several cases, however, where one knows that such maps either have small ranks or they are holomorphic ([73], [75],[7], [8]), where these properties can be expressed with the following *local* invariants of Y.

Define $rank_{plu}(Y)$ of a Riemannian manifold Y (possibly infinite dimensional and/or singular) as the maximal number r such that Y receives a pluriharmonic map of rank r from a complex manifold.

Define $rank_{plu/hol}(Y)$ of a locally irreducible Hermitian manifold Y as the maximal number r such that Y receives a *non-±holomorphic* pluri-harmonic map of rank r from a complex manifold.

If Y (locally) reducible, i.e. if the universal covering $\tilde{Y}$ isometrically splits into a Cartesian product of $\tilde{Y} = \times \tilde{Y}_i$, where Y_i are Hermitian manifolds, then "±holomorphicity" of a map $f : X \to Y$ means ±holomorphicity of the local projections of f to each Y_i.

Siu Rank. These two ranks equal 2 for *generic* Y; they are most significant for *symmetric* spaces Y.

Their main role is to provide a lower bound on $rank_{Siu}(W) = rank_{hmt/hol}(W)$ of a complex manifold W, where $rank_{hmt/hol}$ is the minimal number k such that, for every compact Kähler manifold V, the space of continuous maps $f : V \to W$ with $rank_{hmt} > k$ contracts to the the subspace of ±holomorphic maps i.e. the inclusion $\pm\mathcal{HOL} \subset \mathcal{CONT}$ is a homotopy equivalence.

Here, as earlier, we need to make a provision for locally split W, i.e. where the universal covering $Y = \tilde{W}$ admits a complex analytic splitting $Y = \times_{i=1,...,j} Y_i$ such that the j foliations of W into the Y_i-slices, $i = 1, ..., j$, are invariant under the Galois group of the covering $Y \to W$ (and so W comes with j mutually transversal holomorphic foliations which locally split W).

Here ±holomorphicity, is understood for such a W, as ±-holomorphicity of the j local projections of our maps on the Y_i-factors.

Remark about the h-Principle. Non-trivial lower bounds on $rank_{hmt/hol}$ are formally similar to the *Oka-Grauert h-principle*, but the underlying mechanisms of the proofs are opposite in nature: the h-principle is a manifestation of "softness" of holomorphic maps, while $rank_{hmt/hol}$ is about "rigidity".

Examples. ($A_{1/4}$) If a Riemannian manifold Y has $-1 < K(X) < -1/4$, then $rank_{plu}(Y) = 2$; moreover every pluriharmonic map $X \to Y$ locally factors as $X \overset{holo}{\to} S \overset{plu}{\to} Y$ for $dim(S) = 2$. Moreover, this remains true under the corresponding *local* strict 1/4 negative pinching assumption (as on the $K_{\mathbb{C}}$-list in the previous section.)

This implies, with the ★HBSS from 4.4 and the stability discussion in 4.3 the following

Kählerian 1/4*-non-Pinching Theorem.* (see [7]) Let X be a normal Kähler space with a proper isometric cocompact action of a group $\Gamma_X \subset iso_{hol}(X)$, let Y be a complete, possibly infinite dimensional, manifold which is *strictly negatively* 1/4 *pinched* i.e. $-1 < -\kappa_1 \leq K(Y) \leq -\kappa_2 < -1/4$, and let $h : \Gamma_X \to iso(Y)$ be a homomorphism. Then

every h*-equivariant map* $f_0 : X \to Y$ *has* $rank[f_0]_{Lip} \leq 2$.

(Probably, this remains true for $K_{\mathbb{C}} \leq -\kappa < 0$, e.g. for strictly *locally* negatively 1/4-pinched manifolds, i.e. for $-(1+\varepsilon)\kappa(y) \leq K_y(Y) \leq -\kappa(y)/4$, where a technical difficulty arises if $\inf_{y \in Y} K_y(Y) = -\infty$ and the limit space Y_∞ is singular.)

($B_{H^n_{\mathbb{C}}}$) The complex hyperbolic spaces $Y = H^n_{\mathbb{C}}$, $n = 1, 2, ..., \infty$, satisfy

$$rank_{plu/hol}(H^n_{\mathbb{C}}) = 2;$$

moreover, every *non-±holomorphic* local pluriharmonic map into $H^n_{\mathbb{C}}$ factors as $X \overset{holo}{\to} S \overset{plu}{\to} Y$ for $dim_{\mathbb{R}}(S) = 2$. (see [75] [7])

We have indicated essential corollaries of this in section 4.2; the present terminology allows the following reformulation.

Given a Kählerian X as in the above (A), an $h : \Gamma_X \to iso_{hol}(H^n_{\mathbb{C}})$ and an h-equivariant map $f_0 : X \to H^n_{\mathbb{C}}$.

If $rank[f_0]_{Lip} \geq 3$*, then there exists a homomorphism* $h_\infty : \Gamma_X \to iso_{hol}(H^n_{\mathbb{C}})$ *and an* h_∞*-equivariant ±holomorphic map* $f_\infty : X \to H^n_{\mathbb{C}}$*, such that*

$$rank_{\mathbb{R}}(f_\infty) \geq rank[f_0]_{Lip}.$$

(A×B) Let X/Γ_X do not fiber over Riemann surface in the sense that X admits no holomorphic equivariant map for any homomorphism h (of Γ_X to the isometry group of the target space) neither to the complex line $\mathbb{C}$ nor to the hyperbolic plane $H^2_{\mathbb{R}}$.

Let $f_0 : X \to Y = Y_1 \times Y_2$ be an equivariant map (for some homomorphism $h : \Gamma_X \to iso(Y)$), let Y_1 be strictly negatively 1/4-pinched, e.g. $Y_1 = H^\infty_{\mathbb{R}}$, let $Y_2 = H^\infty_{\mathbb{C}}$ and let $rank[f_0]_{Lip} > 0$.

Then the corresponding limit map f_∞ *±-holomorphically send* X *to a single "holomorphic slice"* $y_1 \times H^\infty_{\mathbb{C}}$.

Question. There are, apparently, lots of "interesting" isometry groups Γ acting on $H^\infty_{\mathbb{R}} \times H^\infty_{\mathbb{C}}$ but are there any candidates among them for our kind of groups Γ_X that also act on a Kählerian X, e.g. being the fundamental groups of a complex projective manifolds V?

(C) If Y is a Hermitian symmetric space with $K(X) \leq 0$ and with no flat (i.e. Euclidean) factor, then $rank_{plu/hol}(Y)$ equals the maximum of the real dimensions of totally geodesic subspaces $Y' = Y_0 \times H^2_{\mathbb{R}} \subset Y$, with a Hermitian Y_0

In fact, every non-±-holomorphic pluriharmonic map $X \to Y$ of maximal rank = $rank_{plu/hol}(Y)$ lands in some $Y_0 \times Y_1 \subset Y$ and equals $[X \overset{holo_0}{\to} Y_0] \times [X \overset{holo}{\to} S \overset{harm}{\to} Y_1]$ ([75], [7], [8]).

(C′) A similar evaluation of $rank_{plu/hol}(Y)$ in terms of split totally geodesic $Y' \subset Y$ is available (albeit the proofs are more involved, see [8]) for most (non-Hermitian) symmetric spaces Y.

Question. Let $f_0 : V \to W$ be a pluriharmonic map, where V is Kähler and W is Hermitian locally symmetric. Suppose that f_0 respects the Hodge filtrations on the cohomologies of the two manifolds. Does it help, this map to be holomorphic? Would it be useful to look at the corresponding the cohomologies with coefficient in flat finite and infinite dimensional unitary bundles with more cohomology available?

Discussion on Infinite Dimensional X and Y. Irreducible symmetric spaces Y with $rank_{\mathbb{R}} \geq 2$ admit more "interesting", e.g. discrete and proper, isometric group actions than $H^\infty_{\mathbb{R}} \times H^\infty_{\mathbb{C}}$, e.g. proper actions of Kazhdan's T-groups.

On the other hand the ranks $rank_{plu/hol}(Y)$ for these spaces Y go to infinity for $dim(Y) \to \infty$ (see [7] [8]) and it is unclear under which (global) conditions pluriharmonic maps from *finite dimensional* X into such Y are holomorphic.

There are several possibilities for *infinite dimensional* X, e.g. one can make such X by taking unions of increasing families $X_1 \subset X_2 \subset ...$ (and inductive limits in general); however, it is unclear what are *meaningful* examples.

(A potentially more promising path to infinite dimensions via "symbolic" varieties is indicated 4.10.)

4.6 [Tol]-Convexity and Deformation Completeness.

The essential property of a W and/or of a continuous map $f_0 : V \to W$ we are concerned with is $[f_0] \in \pm\mathcal{HOL}$, that is, we look for a ±holomorphic (preferably unique) representative in the homotopy class of f_0.

This has nothing to do with any metric on W and with its curvature, and we naturally wish to have a condition expressible in global, purely complex analytic, even better topological, terms.

Are there global/asymptotic conditions on the universal covering Y of W adequately reflecting $K_{\mathbb{C}} \leq 0$ and $K_{\mathbb{C}} < 0$, similarly to how hyperbolicity captures $K < 0$?

Is there some "generalized CR-structure" on the ideal boundary $\partial_\infty(Y)$?

What does $K_{\mathbb{C}}$ tell you about *Pansu's conformal dimension* of $\partial_\infty(Y)$?

Suppose W is a topological manifold such that the universal covering Y is bi-Lipschitz equivalent to a complete Riemannian manifold Y' with $K_{\mathbb{C}}(Y') \leq -\delta < 0$.

Does W admit maps f_0 from compact Kähler manifolds V with $rank_{hmt}[f_0] \geq 3$?

Would it help to require $K(Y') \geq -const > -\infty$?

Can one relax "bi-Lipschitz" to "quasiisometric" in the case where Y is contractible?

A step toward this direction is suggested by the following

$\mathbb{C}$-*Convexity Lemma* [76]. Let W be a complete Riemannian manifold let C be a smooth connected projective algebraic curve (Riemann surface) and let $[f_0]$ be a homotopy class of maps $C \to W$. The minimum of the Dirichlet energies of maps $f \in [f_0]$ (which, as we know, depends only on the conformal structure on C) defines a function, call it E_{min}, on the moduli space $\mathcal{M}$ of conformal structures on C. If E_{min} is *assumed* by some (harmonic) map $f \in [f_0]$ with $rank = 2$ ("assumed" is probably, unnecessary in most cases).

The Toledo Lemma says that *if* $K_{\mathbb{C}}(W) \le 0$, *then*

[Tol]$_{conv}$ *the function* E_{min} *is* $\mathbb{C}$*-convex on* $\mathcal{M}$.

($\mathbb{C}$-Convex = pluri-sub-harmonic in the traditional terminology).

Furthermore, let C_b be a family of compact connected Riemann surfaces parametrized by a holomorphic curve $B \subset \mathcal{M}$ and $f_b : C_b \to W$ be *non-constant harmonic* maps with *non-equal images* in W,

$$f_{b_1}(C_{b_1}) \ne f_{b_2}(C_{b_2}) \subset W \text{ for } b_1 \ne b_2.$$

If $K_{\mathbb{C}} < 0$, *then*

the energy $E(f_b)$ *is strictly* $\mathbb{C}$*-convex in* b *at almost all* $b \in B$,

where "$\mathbb{C}$-convex" is the same as subharmonic and "almost all" means "from an *open dense* subset in B".

This Toledo's "almost strict" $\mathbb{C}$-convexity is, essentially, as good as $K_{\mathbb{C}} < 0$ for ruling out Kähler subgroups in $\pi_1(W)$, since, obviously,

every continous map f_0 *from a projective algebraic manifold* V *to such a [Tol]*$_{alm.strict}$ *space* W *has* $rank[f_0]_{hmt} \le 2$.

Remarks. (a) If the condition $f_{b_1}(C_{b_1}) \ne f_{b_2}(C_{b_2})$ is not satisfied, then, this was pointed out to me by Toledo, the energy $E(f_b)$ may be *constant* in b, where the basic examples are families of ramified holomorphic covering maps between Riemann surfaces, $C_b \to W$, $dim_{\mathbb{C}}(W) = 1$. Probably, if $dim_{\mathbb{C}}(W) > 1$, then the only way the strict convexity fails is where the maps f_b factor through such ramified coverings over a minimal surface in W.

(b) The condition [Tol]$_{conv}$ (unlike $K \le 0$) makes sense for an *arbitrary* metric space W and this also applies to (properly reformulated) property [Tol]$_{alm.strict}$.

Also both conditions have a global flavour and they are (slightly) more robust then the corresponding $K_{\mathbb{C}}$-curvature inequalities.

However, they limited to harmonic maps of *closed* surfaces into W, and it would be nice to have something similar for open surfaces – non-compact and/or with boundaries. This would, in particular, allow a local version of [Tol]-convexity and might imply stability of this property, and consequently of $K_{\mathbb{C}} \le 0$, under weak (Hausdorff?) limits of metric spaces.

From now on, [Tol]$_{conv}$*-manifolds* W are those satisfying this $\mathbb{C}$-convexity property.

"Holomorphic" Corollary. Let V be a projective algebraic manifold, let $F : V \to W$ be a *continuous* map and let $C_q \subset V$ be an algebraic family of algebraic curves in V, where q runs over a smooth connected projective algebraic curve Q, such that C_q is connected non-singular for generic $q \in Q$.

If W *is [Tol]-convex, then the minimal energy* $E_{min}(q)$ *of the (homotopy class of the) map* F *restricted to non-singular curves is constant in* q.

Proof. If C_q is a singular curve in our family and $f_q : C_q \to W$ is a continuous map which is smooth away from the singular points of C_q, then the energy of this map is defined by integrating $||Df_q||^2$ over the non-singular locus of C_q. Thus, the function $E_{min}(q)$ is defined for all $q \in Q$.

Let us show that the function $E_{min}(q)$ is continuous on Q for all (not necessarily [Tol]-convex) Riemannian manifolds W.

This is obvious at those q where the curves C_q are non-singular. It is also clear that E_{min} is semicontinuous at all q: the energy may only jump up for $q \to q_0$.

To see that, in fact, there is no "jump", i.e. $E_{min}(q_0) \leq \lim E_{min}(q)$, assume (the general case trivially reduces to this) that the curve $C_0 = C_{q_0}$ has a double point singularity, say at $c_0 \in C_0$. Let $\tilde{C}_0 \to C_0$ be the non-singular parametrization (by normalization) of C_0 and observe that $E_{min}(\tilde{C}_0) = E_{min}(C_0)$. (The extremal harmonic map $\tilde{C}_0 \to W$ does not, typically, factors through the parametrizing map $\tilde{C}_0 \to C_0$, which makes the extremal map $C_0 \to C_0$ discontinuous).

The curves C_q, $q \to q_0$, are obtained near c_0 by attaching "arbitrarily narrow" 1-handles H_ε to $\tilde{C}_0 = C_0$ by gluing the two branches of C_0 along the two infinitesimally small circles in C_0 around c_0.

These handles can be implemented by maps $H_q \to W$ with the energies $E(H_q) \to 0$, that makes

$$(\circ) \qquad \limsup_{q \to q_0} E_{min}(C_q) \leq E_{min}(\tilde{C}_0) = E_{min}(C_0).$$

All one needs to construct such $H_q \to W$ is a family ϕ_δ, $\delta > 0$, of smooth functions $\phi_\delta : D \to \mathbb{R}$, where $D \subset \mathbb{C}$ is the unit disc, where $E(\phi_\delta) \leq \delta$ for all $\delta > 0$ and where

- the functions ϕ_δ vanish on the boundary of D.
- the functions ϕ_δ equal 1 at the center $0 \in D$.

(Such ϕ_δ are made out of $\delta \log |z|$ in an obvious way.)

Thus, by ($\circ$), the function $E_{min}(q)$ is continuous at all, including singular, curves C_q in our family, and if W is [Tol]-convex $E_{min}(q)$ is constant on Q, since *every continuous* function on D which is smooth $\mathbb{C}$-convex on Q minus a finite subset is constant on Q. (I am not certain, but this is, undoubtedly, known, if every bounded $\mathbb{C}$-convex function on $D \smallsetminus 0$ is continuous at 0.)

***C**urve **D**eformation **C**ompleteness.* A complex manifold W is called cdc, if the following holds for all above (V, C_q, f_0):

if the continuous map $f_{q_0} = f_0 | C_{q_0} : C_{q_0} \to W$ is homotopic to a holomorphic one for some $q_0 \in Q$, then all $f_q : C_q \to W$ are homotopic to holomorphic maps.

Observe that the cdc property, unlike [Tol]-convexity and/or $K \leq 0$, does not depend on any metric, but only on the complex structure in W.

On the other hand, the above Corollary implies that

Kählerian [Tol]$_{conv}$-manifolds are cdc – curve deformation complete.

Indeed, if a smooth map $f_q : C_q \to W$ is homotopic to a holomorphic $h : C_q \to W$, then $E(f_q) \geq E(h)$ by *Wirtinger's inequality*, where, a priori, the equality holds if and only if f_q is ±-holomorphic. Since the ±-involution amounts to the change of the orientation in C_q, a (+)-holomorphic map of positive rank into a *Kähler* manifold W can not be homotopic to a (−)-holomorphic one; hence all f_q are holomorphic. ("Kähler" is essential: Calabi-Eckmann-Hopf manifolds $H_{mn} = [(\mathbb{C}^m \smallsetminus 0) \times (\mathbb{C}^n \smallsetminus 0)]/\{e^z \times e^{\sqrt{-1}z}\}_{z \in \mathbb{C}}$ contain *contractible* holomorphic curves, $H_{11} \subset H_{mn}$.)

Let us indicate some simple properties of *curve deformation complete compact Kähler* manifolds W.

Holomorphic Extensions from Curves. Start with the case where a homotopy class $[e_0]$ of maps of a compact complex curve (Riemann surface) C to W admits *at most one holomorphic* representative $C \to W$ for every complex structure on C. This is so, for example, for all *non-contractible* maps $e_0 : C \to W$ if W admits a Kähler metric with strictly negative sectional curvature.

Remarks. (a) If the space E_0 of *continuous* maps $e : C \to W$ homotopic to e_0 has $H^2(E_0; \mathbb{R}) = 0$, then the space of *holomorphic* maps $C \to W$ homotopic to e_0 is *finite*.

Indeed, the space $H_0 \subset E_0$ of holomorphic maps $h : C \to W$ is a complex space, such that the c-evaluation map $\varepsilon_c : E_0 \to W$ for $\varepsilon_c : e \mapsto e(c) \in W$ is *holomorphic* on $H_0 \subset E_0$ for ecah $c \in C$. Since W is Kähler as well as compact the space H_0 is compact.

If $dim_{\mathbb{C}}(H_0) = d > 0$, then the pullback $\varepsilon_c^*(\Omega_W)$ of the Kähler class Ω_W of W to E_0 by ε does not vanish on H_0 for generic $c \in C$, since

$$\varepsilon_c^*(\Omega_W)^d[H_0] > 0;$$

for the fundamental class $[H_0]$ of (a d-dimensional irreducible component of) H_0; hence, $H^2(E_0) \neq 0$ for $d > 0$.

(b) If W is *aspherical*, i.e. the universal covering of W is contractible, then E_0 is homotopy equivalent to the Eilenberg-MacLane classifying space $K(Z_0; 1)$ of the centralizer Z_0 of the $[e_0]$-image of the fundamental group of C in $\pi_1(W)$. For instance,

(c) If W admits a Riemanninan metric with strictly negative curvature, then $H^2(E_0; \mathbb{R}) = 0$ for all non-contractible maps $e_0 : C \to W$.

(d) If W admits a Kähler metric with non-positive sectional curvature, then the space of holomorphic maps $h : C \to W$ in every homotopy class $[e_0]$ is connected; such an h is unique, if and only if the the e_0-image of $\pi_1(C)$ in $\pi_1(W)$, has trivial centralizer Z_0.

Probably, there are many algebraic manifolds W where the space of holomorphic maps $C \to W$ in a given homotopy class is disconnected, even for aspherical W. One could arrange this, for example, with an algebraic curve B in the moduli space $\mathcal{M}_g$ of curves C of a genus g, where B an irreducible component of the lift $\tilde{B}$ of B to the universal orbicovering $\tilde{\mathcal{M}}_g$ has a double point.

Assume the above algebraic manifold V is a non-singular compact complex surface, i.e. $dim_{\mathbb{C}}(V) = 2$, let the family C_q be *non-constant* so that the union of all curves $C_q \subset V$ equals V and let us show that all *compact Kähler cdc* manifolds W *without rational curves* satisfy the following *extension property.*

$\star$ *Every continuous map $f_0 : V \to W$ which is holomorphic on some non-singular curve $C_{q_0} \subset V$ is homotopic to a unique holomorphic map $V \to W$, provided the homotopy class of $e_0 = f_0|C_{q_0}$ contains at most one holomorphic representative $C_q \to W$ for all q in a small neighbourhood $U_0 \subset Q$ of q_0.*

Proof. Let $Z \subset V \times W$ be the union of (possibly singular) holomorphic curves $\tilde{C} \subset V \times W$ such that

• the projection $P_V : V \times W \to V$ biholomorphically sends each $\tilde{C}$ onto a curve $C_q \subset V$ for some $q \in Q$ and the projection $P_W : V \times W \to W$ is holomorphic on each $\tilde{C}$;

•• all curves $\tilde{C} \subset V \times W$ have "degrees" equal that of the graph $\Gamma(C_{q_0}) \subset V \times W$ of the map $f_0|C_{q_0} : C_{q_0} \to W$, i.e. the value of the Kähler class $\Omega = \Omega_{V\times W} = \Omega_V \oplus \Omega_W$ on each $\tilde{C}_q$, that is $\Omega[\tilde{C}_q]$, equals $\Omega[\Gamma(C_{q_0})]$.

Since • is comprised of sentences in the "first order holomorphic language" and •• is "linguistically Kähler", this Z is a *compact complex analytic* subset in $V \times W$. (One usually applies such "linguistic" argument to algebraic manifolds but it remains valid in the compact Kähler case as well.)

Let $Z_0 \subset Z$ be the irreducible component of Z which contains the graph of the (holomorphic!) map $f_0|C_{q_0} : C_{q_0} \to W$.

Since the projection $P_V : Z_0 \to V$ is one-to-one over U_0 it is *generically* one-to-one; hence, Z_0 serves as the graph of a rational map $V \to W$ that, in fact, is holomorphic since W contains no rational curves.

Remarks. The above argument does not work if "at most *one* holomorphic representative $C_q \to W$" ($q \in U_0$) is relaxed to "at most *finitely many* holomorphic representatives $C_q \to W$" or to "at most one holomorphic representative $C_{q_0} \to W$ (for a single q_0), but it seems hard to find an actual cdc manifold where the above extension property fails to be true under the so relaxed conditions.

It is also unclear whether "without rational curves" is relevant for cdc manifolds W.

Let us adjust the above to the case where there may be several mutually homotopic holomorphic maps of a curve C into W, keeping in mind split submanifolds $W' \times W''$ in a manifold W of non-positive curvature. (The proper condition was missing from the first version of this paper as was pointed out to me by Domingo Toledo).

Let W, assumed as earlier compact Kähler cdc, have contractible universal covering. Assume moreover, that W has the following

Split Deformation Property. Given a homotopy class $[e_0]$ of maps of a closed surface C_0 into W there exists a unique (possibly empty) compact split irreducible analytic space $W' \times W''$ and a holomorphic map $I : W' \times W'' \to W$ with the following property.

Let C_q be a complex curve (Riemann surface) homeomorphic to C_0 and $h_q : C_q \to W$ be a holomorphic map such that the composition of h_q with a homeomorphism $C_0 \leftrightarrow C_q$ belongs to the homotopy class $[e_0]$.

Then there exist unique *holomorphic* map $h'_q : C_q \to W'$ and a *constant* map $g''_q : C_q \to W''$, such that the map h_q factorizes as $C \overset{h'_q \times g''_q}{\to} W' \times W'' \overset{I}{\to} W$, i.e. $h_q = I \circ (h'_q \times g''_q)$.

Basic Example. The split deformation property is enjoyed by compact Kähler manifolds W with *non-positive* sectional curvatures.

Indeed, let $\Gamma_0 \subset \pi_1(W)$ be a subgroup let $\Gamma''_0 \subset \pi_1(W)$ be its centralizer and $\Gamma'_0 \supset \Gamma_0$ be the centralizer of Γ''_0.

Since $K(W) \leq 0$ – "Kähler" is irrelevant here – there exists, by the Gromoll-Wolf-Lawson-Yau splitting theorem, a split Riemannian manifold $W'_0 \times W''_0$ and a map $I_0 : W'_0 \times W''_0 \to W$ which is geodesic isometric on all coordinate slices $w'_0 \times W''_0$ and $W'_0 \times w''_0$ and such that the I_0-images of the fundamental groups of W'_0 and W''_0 in $\pi_1(W)$ are conjugate to Γ'_0 and Γ''_0 respectively.

Now, recall that W is Kähler and assume that W'_0 contains an irreducible complex analytic subset A_0 such that the image of the fundamental group of A_0 in $\Gamma'_0 = \pi_1(W'_0)$ contains Γ_0.

Then W''_0 identifies with the space of holomorphic maps $A \to W$ which send $\pi_1(A)$ onto Γ_0 while W'_0 appears as the space of holomorphic maps $W''_0 \to W$ homotopic to $I_0|w'_0 \times W''_0$. Thus, the manifolds W'_0 and W''_0 acquire complex analytic structures for which the map I is complex analytic.

Questions. Given a compact Kähler (e.g algebraic) manifold W, call (the conjugacy class of) a subgroup $\Delta \subset \Gamma = \pi_1(W)$ *holomorphic*, if it equals the image of the fundamnetal group of a compact connected Kähler (possibly singular) space A under a *holomorphic* map $A \to W$.

What is the structure $\mathcal{H} = \mathcal{H}(\Gamma) = \mathcal{H}(\Gamma, W)$ of (the set of) holomorphic subgroups Δ in Γ?

For instance, for which W is the centralizer of a holomorphic subgroup is holomorphic (as for the above W with $K(W) \le 0$)?

When can one reconstruct the complex structure in W in terms of $\mathcal{H}(\Gamma)$?

Which manifolds W have many and which few holomorphic subgroup?

For example, what are aspherical W (e.g. with $K(W) \le 0$), where every holomorphic Δ either has finite index in Γ or equals $\{id\} \subset \Gamma$?

What is the corresponding structure for the *algebraic* fundamental group that is the profinite completion of Γ?

If W is an algebraic manifold, what is a similar structure on the full geometric Galois group Γ_{geo} of W that encodes all ramified coverings of W?

Does $\Gamma_{geo}(W)$ allow a reconstruction of W?

***S**plit **D**eformation **P**roperty $\Rightarrow$ Extension Property.* Return to the above non-singular compact complex surface V and a non-constant family of curves $C_q \subset V$, parametrized by an irreducible algebraic curve $Q \ni q$.

$\star\star$ *Let W be a compact Kähler cdc (curve deformation complete) manifold which also enjoy SDP and let $f_0 : V \to W$ be a continuous map which is holomorphic on some curve $C_{q_0} \subset V$. Then f_0 is homotopic to a holomorphic map $V \to W$ in the following two cases:*

(1) every continuous map $Q \to W$ is contractible, e.g. Q is a rational curve and W is aspherical;

(2) the inclusion homomorphism $\pi_1(C_{q_0}) \to \pi_1(V)$ is onto.

Proof. Because of SDP everything reduces to maps into $W' \times W''$ and the proof of the above $\star$ applies to maps $V \to W'$.

Questions (a) Let W be $[\mathrm{Tol}]_{conv}$ and let $f_q : C_q \to W$ be a family of harmonic (rather than holomorphic) maps for generic points $q \in Q$ for which the curves $C_q \subset V$ are non-singular).

When does such a family *continuously* extend to all of V with *no assumption* of having a holomorphic member among them?

If so, this would imply that every $f_0 : V \to W$ is homotopic to a pluriharmonic map, as in the case $K_{\mathbb{C}} \le 0$.

(b) Can the curve deformation completeness property (or some slight modification of it) be expressed *algebraically* for algebraic manifolds W, i.e. is it invariant under the action of the Galois group?

Notice that the main (but not only) source of (known) $\mathcal{C}$-deformation complete manifolds are arithmetic varieties, the class of which *is* Galois invariant by a theorem of Kazhdan.

So we ask whether Kazhdan's theorem extends to the class of curve deformation complete manifolds.

Also this his question may be asked about $rank_{hmt/hol}(W)$: is it an algebraic invariant for algebraic manifolds W?

(c) Another question motivated by Kazhdan's theorem is as follows. Let W be an arithmetic variety, (possibly one has to assume it admits positive solution to the *congruence problem*) let V be a smooth projective manifold (defined over a number field?) and let h be a homomorphism of the profinite completion $\overline{\pi}_1(V)$ of the fundamental group of V to $\overline{\pi}_1(W)$.

Does there exist, under a suitable lower bound on some "topological rank" of h, a regular map f from V to a Galois transform of W (kind of ±-holomorphic, but now discontinuous, map), such that the homomorphism $\overline{\pi}_1(V) \to \overline{\pi}_1(W)$ induced by f equals h up to a Galois automorphism of $\overline{\pi}_1(W)$.

(d) There is a purely algebraic counterpart to $rank_{hmt/hol}(W)$ which is defined as follows.

Let $p : U \to B$ be a surjective holomorphic map between non-singular algebraic varieties and $V_b = p^{-1}(b) \subset U$ be (necessarily non-singular) fiber $V_b = p^{-1}(b) \subset U$ over a non-critical point $b \in B$.

Define $rank^{dfm}_{hmt/hol}(W)$ as the minimal number k, such that every holomorphic map $V_b \to W$ of $\mathbb{R}$-rank $> k$ extends to a holomorphic map $U \to W$ for *all* U, B, p and V_b as above (where we do not assume beforehand the existence of any *continuous* map $V \to W$ that extends our holomorphic $V_b \to W$).

It is easy to show that

$$rank^{dfm}_{hmt/hol}(W) \le rank_{hmt/hol}(W).$$

Are there algebraic manifolds W with $rank^{dfm}_{hmt/hol}(W) < rank_{hmt/hol}(W)$?

(e) Let Y be a symmetric, say Hermitian, space with $K(Y) \le 0$ and $\Gamma_C = \pi_1(C)$ be a surface group. Let $\mathcal{M}$ be the space of conformal structures on C and $\mathcal{R}$ the space of conjugacy classes of homomorphisms $\Gamma \to iso_{hol}(Y)$.

The minimal energy $E_{min} = E_{min}(\mu, \rho)$ of Γ_C-equivariant maps $\tilde{C} \to Y$, for $\tilde{C} = H^2_{\mathbb{R}}$, is a real analytic function on $\mathcal{M} \times \mathcal{R}$.

What is the algebraic/analytic nature of this function?

What are "natural" PDE satisfied by it?

What is the set of the critical points of $E_{min}(\mu, \rho)$?

Does the set of these functions for variable Y and/or Γ_S carry some meaningful structure?

Can one *effectively* describe pluriharmonic/holomorphic maps from Kähler manifolds into Y/Γ in terms of $E_{min}(\mu, \rho)$?

Example Let V be a non-singular projective variety, $V \subset \mathbb{C}P^M$, and let $C \subset V$ be a "generic mobile" curve, e.g. the intersection of V with an M'-plane $\mathbb{C}P^{M'} \subset \mathbb{C}P^M$, $M' = M - dim(V) - 1$, in general position.

If W is a cdc (curve deformation complete) manifold (e.g. W is $[\mathrm{Tol}]_{conv}$), which also has SDP (split deformation property), and if W contains no rational curve (e.g. $K(W) \le 0$), then *every continuous map $f_0 : V \to W$ which restricts to a holomorphic map on C is homotopic to a holomorphic map $V \to W$.*

Furthermore let cdc be relaxed to $rank^{dfm}_{hmt/hol}(W) \geq 2k$ and let $V_0 \subset V$ be the intersection of $V \subset \mathbb{C}P^M$ with a generic M_0-plane $\mathbb{C}P^{M_0} \subset \mathbb{C}P^M$ for $M_0 > M - dim(V) + k$.

Then every holomorphic map $V_0 \to W$ of $\mathbb{C}$-rank $> k$ extends to a holomorphic map $V \to W$.

Questions. (1) We assumed from time to time that certain manifolds (varieties) were algebraic. Was it truly needed or would "Kähler" suffice?

(2) Let W be a closed Riemannian $[\mathrm{Tol}]_{conv}$ manifold and suppose it receives a continuous map f_0 from an algebraic (Kählerian?) manifold V, such that $rank_{hmt}[f_0] = dim(W)$.

Is (the Riemannian metric on) W "essentially" Kählerian?

Can one non-trivially deform the Riemannian metric on a locally symmetric Kählerian W with $K_{\mathbb{C}} \leq 0$ keeping it $[\mathrm{Tol}]_{conv}$?

For example, does the n-torus admit a non-flat $[\mathrm{Tol}]_{conv}$-metric?

(3) Which "slowly growing" harmonic maps $X \to Y$ are pluriharmonic?

For instance, are *Lipschitz* harmonic functions $Y \to \mathbb{R}$ on Abelian coverings Y of compact Kähler manifolds pluriharmonic?

(4) Do non-trivial lower bounds on $rank_{hmt/hol}$ or $rank^{dfm}_{hmt/hol}$, e.g. the strongest $rank_{hmt/hol}(W) = 2$ or the weakest $rank^{dfm}_{hmt/hol}(W) < dim_{\mathbb{R}}(W)$, say for projective algebraic manifolds W, imply that the universal covering $\tilde{W}$ of W is *contractible*?

(5) Conversely, let W be a compact Kähler manifold W with *contractible* $\tilde{W}$. Does it satisfy $rank_{hmt/hol}(W) < dim_{\mathbb{R}}(W)$ or at least $rank^{dfm}_{hmt/hol}(W) < dim_{\mathbb{R}}(W)$?

Here one has to keep in mind certain exceptions/modifications. e.g. for $\pi_1(W) = \Gamma_0 \ltimes \mathbb{Z}^k$ as in 4.9.

On the other hand, there are, apparently, no examples of Kähler manifolds W with *contractible* $\tilde{W}$ and *word hyperbolic* fundamental group $\pi_1(W)$ where one would be able to show that $rank_{hmt/hol}(W) > 2$.

(6) Can you tell in terms of Γ when a compact Kähler space W with $\pi_1(W) = \Gamma$ and $rank_{hmt/hol} < dim_{\mathbb{R}}(W)$ is non-singular?

When does a complete Kähler manifold W with $rank_{hmt/hol} < dim_{\mathbb{R}}(W)$ contain a compact (or just finite dimensional for $dim(W) = \infty$) "core" , i.e. an analytic subspace $W_0 \subset W$ (similar the above $\mathcal{B}$ in the moduli space of curves) such that the inclusion $W_0 \subset W$ is a homotopy equivalence, or, at least, such that the inclusion homomorphism $\pi_1(W_0) \to \pi_1(W)$ is an isomorphism?

For instance, let Γ admit a proper discrete (free?) action on a given (some?) infinite dimensional symmetric space Y.

Granted such an action when does Y/Γ admit a finite dimensional (compact?) complex analytic "core" $W_0 \subset W$?

There are few examples of "interesting" isometric group actions on infinite dimensional symmetric spaces Y with $K(Y) \leq 0$. The most studied are *Haagerup's a-T-menable* groups that *properly* act on $Y = \mathbb{C}^\infty$; these, according to A. Valette, include all *amenable*, e.g. solvable groups.

These actions are used to derive the following constrains on the fundamental groups $\Gamma = \pi_1(W)$ of compact Kähler manifold W.

if Γ *is* $1/6$*-small cancelation group, then it contains a surface group of finite index* [13],

and

if Γ *is solvable then it contains a nilpotent subgroup of finite index* [12] (where it remains unknown which nilpotent groups can serve as $\Gamma = \pi_1(W)$).

Also, there are interesting isometric actions of $PSL_2(\mathbb{R})$ and of the Cremona group $Bir(\mathbb{C}P^2)$ on $H^\infty_\mathbb{R}$ [14].

4.7 Algebra-Geometric Abel-Jacobi-Albanese Construction.

The following classical construction of the Jacobian $W = A(V)$ of a projective algebraic variety V is vaguely similar to the combinatorial reconstruction of Shub-Franks group actions (see 2.4) where orbits of an action come as equivalence classes of quasi-orbits modulo $DIST < \infty$ equivalence relation.

Let $Z_0(V)$ be the group of 0*-cycles in* V that are formal integer combinations $\sum_{v\in V} m_v$, for functions $m : V \to \mathbb{Z}$ with *finite* supports. and Observe two tautological maps $\pm : V \to Z_0(V)$ for $v \to \pm v = \pm 1 \cdot v$.

Every holomorphic map $\alpha : V \to A$ to a commutative algebraic group A extends, via the summation in A, to $\alpha^+ : Z_0(V) \to A$.

Assume that the image $\alpha(V) \subset A$ generates A as a group. Then α^+ is *onto* and in order to reconstruct A from V it remains to identify the equivalence relation on $Z_0(V)$ which reduces $Z_0(V)$ to A.

This can be equivalently seen in terms of the symmetric powers $V^{\frac{N}{sym}} = V^N/\Pi(V)$ of V (for the permutation group $\Pi(N)$ acting on the Cartesian power V^N) as follows.

Extend the above $\pm$maps $V \to Z_0(V)$ by summation in $Z_0(V)$ to the corresponding maps $\pm_N : V^{\frac{N}{sym}} \to Z_0(V)$ and compose these maps with α^+.

Thus, for every pair of integers (N_+, N_-), we obtain a map $\alpha^{N_\pm} : V^{\frac{N_+}{sym}} \times V^{\frac{N_-}{sym}} \to A$, that are

$$[(v_1, ..., v_{N_+}), (v'_1, ..., v'_{N_-})] \to v_1 + ... + v_{N_+} - v'_1 - ... - v'_{N_-}.$$

which are onto for large $N_\pm$ and try to identify the fibers of these maps.

A rational curve in $Z_{v_0}(V) = Z_0(V)/\mathbb{Z}v_0$, is the image of a *non-constant holomorphic* map ϕ from the projective line $P^1 = \mathbb{C}P^1$, where "holomorphic" means that $\phi : P^1 \to Z_0(V)$ descends from a pair of (ordinary) holomorphic maps $\phi_\pm : P^1 \to V^{\frac{N_\pm}{sym}}$ for some (large) integers $N_\pm$ via the summation map on 0-cycles, $\pm_{N_\pm} : V^{\frac{N_+}{sym}} \times V^{\frac{N_-}{sym}} \to Z_0(V)$.

Two zero cycles z_1 and z_2 in $Z_0(V)$ are called *rationally equivalent* if they can be joined by chain of *rational curves* between them and the corresponding quotient space is denoted by $Z_0(V)/\sim_{rat}$

Metrically speaking, let $DIST(z_1, z_2) = DIST_{rat}(z_1, z_2)$ be the length of the shortest chain of rational curves between z_1 and z_2 (e.g. $DIST(x_1, z_2) \leq 1$ if and only if z_1 and z_2 lie on a rational curve in $Z_0(V)$) and rewrite $Z_0(V)/\sim_{rat} = Z_0(V)/[DIST_{rat} < \infty]$.

Since A contains no rational curve, every map $Z_0(V) \to A$ factors via a map $Z_0(V)/\sim_{rat} \to A$.

If V is a *curve*, i.e. $dim_{\mathbb{C}}(V) = 1$, then, classically, $Z_0(V)/rat$ equals the Jacobian $A(V) = H_1(V;\mathbb{R})/H_1(V;\mathbb{Z})$.

In fact, the map $\alpha^{N_+} : V^{\frac{N_+}{sym}} \to A(V)$ associated to an Abel-Jacobi map $\alpha : V \to A(V)$ (which is defined up-to translations in $A(V)$) is *onto* for $N_+ \geq \frac{1}{2}rank(H_1(V))$ and if $N_+ >> rank(H_1(V))$, in fact, $N_+ > rank(H_1(V))$ suffices, then the fibers of α^{N_+} are complex projective spaces.

(This agrees with algebraic topology: the homotopy types of the symmetric powers $V^{\frac{N}{sym}}$, for any V, converge as $N \to \infty$ to the product of the *Eilenberg-MacLane spaces*, $K(H_1(V),1) \times K(H_2(V),2) \times ... \times K(H_{2m}(V),2m)$ for $m = dim_{\mathbb{C}}(V)$ by the *Dold-Thom theorem.*)

However if $dim_{\mathbb{C}}(V) \geq 2$, then the quotient space $Z_0(V)/\sim_{rat}$ may be much greater than the Albanese variety $A(V)$, in fact $Z_0(V)/\sim_{rat}$ is *infinite dimensional* in most cases (see [62]). I wonder if the rate of growth of the dimension of $V^{\frac{N}{sym}}/\sim_{rat}$ for $N \to \infty$ has ever been looked at. Also, the geometry of $(Z_0(V), DIST_{rat})$ may be (?) interesting.)

Possibly, one can remedy this by limiting $Z_0(V)$ to a certain subspace $Q_0(V) \subset Z_0(V)$ (similar to quasi-orbits in the Shub-Francs case) and/or by strengthening the $\sim_{rat}$ relation. For example, instead of chains of rational curves, one may try chains of *surfaces* $S \subset Z_0(V)$ *such that* $H_1(S;\mathbb{R}) = 0$, but this does not look pretty.

Traditionally, if $V \subset \mathbb{C}P^M$ is projective algebraic, one makes $A(V)$ from $A(C)$ for generic curves $C \subset V$, that are intersections of V with M'-planes $\mathbb{C}P^{M'} \subset \mathbb{C}P^M$, $M' = M - dim(V) - 1$, in general position, since

the Albanese variety $A(V)$ *equals the maximal common quotient space (Abelian variety) of the Jacobians* $A(C)$ *of generic curves* $C \subset V$.

(The suitably augmented category of the Abelian varieties that are Jacobians of *all* non-singular curves in V seems a nice comprehensive invariant of V; it, probably, has been studied by algebraic geometers, but I am not an expert.)

To see this, let $f_C : C \to A$ be a holomorphic map from a generic $C \subset V$ to a flat Kählerian torus A. Such a map extends to a continuous map $f : V \to A$ if and only if the homology homomorphism $(f_C)_* : H_1(C) \to H_1(A)$ factors as $H_1(C) \overset{emb_*}{\to} H_1(V) \overset{h}{\to} H_1(A)$ for some h, where emb_* is the inclusion homomorphism for $C \subset V$.

Granted such an f, we restrict it to all curves $C_g = V \cap \mathbb{C}P_g^{M'}$ in V, for $V \subset \mathbb{C}P^M \supset \mathbb{C}P_g^{M'}$ which are obtained by varying M'-planes in $\mathbb{C}P^M \supset V$ (parametrized by the corresponding Grassmannian $G \ni g$) passing through a given point $v_0 \in C \subset V \subset \mathbb{C}P^M$, where we want this v_0 to go to $0 \in A(V)$.

Deform the resulting maps $C_g \to A$ to *harmonic* ones, say $f_g : C_g \to A$, all sending $v_0 \to 0 \in A$. These maps, as we know, must be all holomorphic because of $[\mathrm{Tol}]_{conv}$ (see the end of previous section) and, consequently, holomorphically depending on q.

It follows, that if two such curves, say C_{g_1} and C_{g_2} intersect at a point $v \in V$, then $f_{g_1}(v) = f_{g_2}(v)$, since C_{g_1} and C_{g_2} can be joined by a *rational* curve, say P, in the space of all C passing through v_0 and v and, since *every holomorphic* map, such as $p \mapsto f_p(v)$, from a rational P to A is constant, these f_g define a holomorphic map $V \to A$.

However, it is not apparent (at least to the author) how to see in a simple geometric way (without using Hodge theory or Picard varieties) that the maximal common factor (or, rather, the maximal quotient space) of the Jacobians $A(C)$ has the $\mathbb{C}$-dimension equal one half of the rank of $H_1(V)$.

4.8 Deformation Completeness and Moduli Spaces.

Let us look from a similar perspective at other curve deformation complete, e.g. $[\mathrm{Tol}]_{conv}$, spaces W, such as $W = H^m_{\mathbb{C}}/\Gamma$ for a free discrete undistorted (e.g. cocompact) group $\Gamma \subset iso_{hol}(H^m_{\mathbb{C}})$.

Let C be a non-singular projective algebraic curve (Riemann surface) of positive (preferably large) genus and $\alpha_C : C \to W$ be a non-constant holomorphic map.

Such a map may be non-unique in its homotopy class, as in the above case of flat Kählerian tori and then we need to normalize this map as we did it with SDP in 4.6. This, however, is a minor issue and we assume α_C *is unique in its homotopy class* as for $W = H^m_{\mathbb{C}}/\Gamma$.

Denote by $\mathcal{M} = \mathcal{M}(C)$ the *Riemann moduli space* of deformation of this curve C and let $\mathcal{C} \to \mathcal{M}$ be the "universal curve " over $\mathcal{M}$ that is the space of C with marking points $c \in C$.

Let $b_C \in \mathcal{M}$ be the point corresponding to C and denote by $\mathcal{B} = \mathcal{B}(C, W)$ the union of all algebraic curves $B \subset \mathcal{M}$, such that $B \ni b_C$ and such that the map $\alpha_C : C \to W$ extends to a *continous* map α_S of the complex surface $S = \mathcal{C}|B$ over B, that is the restriction of the universal curve to B, to W.

Recall that these *continuous* maps $\alpha_S : S \to W$ are homotopic to *holomorphic* ones $\alpha_S^{hol} : S \to W$ in the curve deformation complete case and we additionally assume these are unique.

Denote by $\mathcal{D} = \mathcal{D}(C, W) \subset \mathcal{C}$ the universal curve restricted to $\mathcal{B} \subset \mathcal{M}$ and denote by $\mathcal{A} : \mathcal{D} \to W$ the map compiled by α_S^{hol} for all above $B \subset \mathcal{M}$ and S over B.

(Since $\mathcal{M}$ is an *orbifold*, rather than a manifold, where the orbi-singular points in $\mathcal{M}$ correspond to curves with non-trivial symmetries, one should, to be faithful to the truth, pass to the universal orbi-coverings $\tilde{\mathcal{M}}$ take the corresponding $\tilde{\mathcal{C}} \to \tilde{\mathcal{M}}$ and to formulate everything in terms of equivariant maps from $\tilde{\mathcal{C}}$ to the universal covering of W. And if one wants to stay in the algebraic category, one may use finite coverings of sufficiently high level.)

Notice that for $[\mathrm{Tol}]_{conv}$ manifolds W the subspace $\mathcal{B}$ equals the minimum set of the ($\mathbb{C}$-convex) minimal energy function $b \mapsto E_{min}(C_b)$, $b \in \mathcal{M}$, for maps from the fibers C_b of the universal curve into W.

Also observe that if W is finite dimensional, then $\mathcal{B} \subset \mathcal{M}$ is a $\mathbb{C}$-*algebraic* subvariety in the moduli space of curves and that $\mathcal{A} : \mathcal{D} \to W$ is a holomorphic map.

Let $W \subset \mathbb{C}P^M$ be projective algebraic, apply the above to a generic curve $C = W \cap \mathbb{C}P^{M'}$, $M' = M - m + 1$, $m = dim_{\mathbb{C}}(W)$, and observe that the closure of $\mathcal{B}$ in a suitably compactified $\mathcal{M}$ equals, in this case, the Grassmann manifold $G_k(\mathbb{C}P^M)$ of k-planes $\mathbb{C}P^k \subset \mathbb{C}P^M$ and that the fibers of the map $\mathcal{A} : \mathcal{B} \to W$ are *rational* varieties.

The essential specific feature of our W, besides the (assumed) injectivity of the tautological classifying map $G_k(\mathbb{C}P^M) \to \mathcal{M}$ for $g \mapsto C_g = \mathbb{C}P^k_g \cap W$ (defined on the Zariski open subset in $G_k(\mathbb{C}P^M)$ corresponding to non-singular curves) is that the image $\mathcal{B}$ of this map has a "semitopological" description, being the *maximal* algebraic subset in $\mathcal{M}$, such that $\mathcal{D} \subset \mathcal{C}$ over it admits a *continuous* map to W extending this from some curve $C \in \mathcal{D}$.

Questions. Can one *intrinsically,* in terms of the moduli spaces $\mathcal{M}$ of curves of *all* genera, relate these $\mathcal{B}$ and the maps $\mathcal{A} : \mathcal{D} \to W$ associated to different projective embeddings of W?

Is the *canonical embedding* $W \to \mathbb{C}P^{M_h - 1}$ associated to the space $\mathcal{H}_m$ (of dimension M_h) of holomorphic m-forms on W is of a particular interest in this picture?

Can all this (which is just a reformulation of Sui-Sampson-Carlson-Toledo theorem) be actually used for *(re)construction* of spaces like $H^m_{\mathbb{C}}/\Gamma$?

"Internalization" of Deformation of Surfaces. One can do a little of such "reconstruction" with moduli spaces of surfaces rather than of curves (where, of course, these spaces are not so readily available for sightseeing.)

Namely, let $\mathcal{S} \to \mathcal{M} = \mathcal{M}(S)$ be a "universal surface" where $\mathcal{S}$ and $\mathcal{M}$ are projective algebraic varieties and $\mathcal{P} : \mathcal{S} \to \mathcal{M} = \mathcal{M}(S)$ a surjective holomorphic *parametrization* map, where $\mathcal{S}$ is non-singular over a generic point in $\mathcal{M}$, where generic fiber $S \subset \mathcal{S}$ is a smooth surface and such that *all* (isomorphism classes of) smooth deformations of S are among the fibers of $\mathcal{P} : \mathcal{S} \to \mathcal{M}$.

Let W have $rank^{dfm}_{hmt/hol}(W) = 2$, e.g. $W = H^m_{\mathbb{C}}/\Gamma$. Then, by the very definition of $rank^{dfm}_{hmt/hol}$,

every holomorphic map $S \to W$ extends to a rational map $\mathcal{S} \to W$; moreover if $\mathcal{S}$ is non-singular and W contains no rational curve (which may follow from [Tol]–conv) then this rational map is holomorphic.

Notice that universality of $\mathcal{S}$ is non-essential for this statement but is relevant for identification/realization of spaces like $H^m_{\mathbb{C}}/\Gamma$.

For example, (this is significant starting from $dim(W) = 3$) assume $W \subset \mathbb{C}P^M$ is projective and take the intersections of W with all $(k+1)$-planes $\mathbb{C}P^{M'+1}_g \subset \mathbb{C}P^M$, $g \in G_{M'+1}(\mathbb{C}P^M)$, $M' = M - dim(W) + 1$.

Then these intersection *surfaces* make the *full moduli* spaces $\mathcal{M}$ of these surfaces (i.e. every "abstract deformation" of $S = W \cap \mathbb{P}^{k+1}$ is "internal" – it comes from moving the k-plane $\mathbb{C}P^{k+1} \subset \mathbb{C}P^M$) and W comes as a quotient of the universal surface $\mathcal{S}$ over this $\mathcal{M}$ where the quotient map $\mathcal{S} \to W$ has rational fibers.

As we mentioned earlier, this "internalization" is a purely algebraic property which makes sense for varieties over an arbitrary field, but it is unclear, for example, in what form it holds (if at all) for mod p reductions of $W = H^m_{\mathbb{C}}/\Gamma$.

Questions. Let the inclusion homomorphism $\pi_1(S) \to \pi_1(W)$, for a non-singular surface $S \subset W = H^m_{\mathbb{C}}/\Gamma$, is an isomorphism.

Is such an S "mobile" in the sense that its deformations cover all of W?

Can *mobile generic* surfaces in other locally symmetric Hermitian spaces W without flat factors have "external" deformations?

(The probable answer is "No", which may (?) lead to examples where $rank^{dfm}_{hmt/hol} < rank_{hmt/hol}$)

A related group of questions is motivated by the concept of *Kähler hyperbolicity* (see [6], [30].)

Let Γ be a finitely presented group with even dimension $dim_{\mathbb{Q}hmt}(\Gamma) = 2m$ and such that $H^{2m}(\Gamma; \mathbb{Q}) = \mathbb{Q}$. Let $\kappa \in H^2(\Gamma; \mathbb{R})$ be a *hyperbolic* "Kähler" class, i.e. $\kappa^m \neq 0$ and where "hyperbolic" means that it is representable as the differential (coboundary) of a *bounded* (non-invariant) 1-cochain. For example, if Γ is *word hyperbolic*, then all 2-cocycles are hyperbolic.

Basic Questions. When does there exist a Kähler space Y with a discrete isometric action of Γ, such that the Kähler class of Y corresponds (in an obvious way) to κ?

If such spaces Y do exist, can one evaluate $rank_{hmt/hol}(Y/\Gamma)$ and identify those where this rank is minimal?

One knows (see [30]) that if such a Y is finite dimensional, topologically contractible and non-singular and if the quotient space $W = Y/\Gamma$ is compact, (probably these conditions can be significantly relaxed), then Y supports a huge Hilbert space $\Omega^m_{L_2} = \mathbb{C}^\infty$ of square integrable holomorphic m-forms, $m = dim_{\mathbb{C}}(Y)$, where the Γ-equivariant Bergman evaluation map $B : Y \to \mathbb{C}P^\infty$ (for this $\mathbb{C}P^\infty$ being the space of hyperplanes in $\mathbb{C}^\infty = \Omega^m_{L_2}$) is, generically, one-to-one.

When does the Bergman metric on Y induced by B (from the Fubini-Studi metric on $\mathbb{C}P^\infty$) has $K_{\mathbb{C}} \leq 0$?

Is there an *effective* criterion (criteria) expressible *entirely in terms of the group* Γ for the Γ-*invariant* $2m$-dimensional homology class in this $\mathbb{C}P^\infty$, corresponding to the B_*-image of the Γ-*invariant* fundamental class $[Y]_\Gamma$ of Y, (that is, essentially, the same as $[W] \in H^{2m}(W)$) to be representable by an m-dimensional Γ-invariant complex analytic subvariety $Y' \subset \mathbb{C}P^\infty$ with compact quotient $W' = Y/\Gamma$?

The first difficulty to overcome is to identify the Hodge component $\Omega^m_{L_2}$ of holomorphic L_2-forms in the full L_2-cohomology $H^m_{L_2}(Y;\mathbb{C})$ in terms of Γ *alone.*

Then you, probably, need to pinpoint the correct Γ-invariant Hermitian-Hilbert metric on the linear space $\mathbb{C}^\infty = \Omega^m_{L_2}$ in order to have a useful metric on our $\mathbb{C}P^\infty$.

Granted this, you can formulate the necessary and sufficient condition in term of the *minimal* $2m$*-volume* of the homology class corresponding $[Y]_\Gamma$. but it remains unclear how to compute this volume in specific examples.

Alternatively, one can consider *all* (suitably homologically normalized) Hermitian-Hilbert metrics HH on $H^m_{L_2}(\Gamma;\mathbb{C})$ and *minimize* this minimal $2m$-volume in the corresponding projective space (acted upon by Γ) over all HH.

But it remains highly problematic how to compute these minimal volumes in specific examples.

A more general and promising class of questions is as follows:

Which complex invariants of a Kähler manifold W "essentially" depend on the fundamental group $\Gamma = \pi_1(W)$?

For example, is there a Kählerian counterpart to *Novikov's higher signature conjecture*?

Namely, let $W \to K(\Gamma, 1)$ be the Eilenberg-MacLane classifying map and $h_{2i} \in H^{2i}(W;\mathbb{Q})$ be a cohomology class coming from $H^{2i}(K(\Gamma,1);\mathbb{Q})$ via this map. Let $c_{n-i} \in H^{2n-2i}(W;\mathbb{Q})$, $n = dim_{\mathbb{C}}(W)$, be a linear combinations of product of certain Chern classes of W.

Does the value $(h_{2i} \smile c_{n-i})[W]$, for the fundamental class $[W] \in H_{2n}(W) = \mathbb{Z}$, actually depend on the complex structure in W or only on the homotopy type of W?

If not, what should one add to make it true?

For example, do the Chern numbers of a W with *contractible* universal covering depend *only* on $\pi_1(W)$?

4.9 On Kählerian and Hyperbolic Moduli Spaces.

The Abel-Jacobi-Albanese construction needs *a choice* of a complex structure in the target torus covered by $\mathbb{C}^n$. This can be compensated by considering the moduli space $\mathcal{B}_n$ of all such structures, where, however, some caution is needed, since some complex tori, e.g. $\mathbb{C}^n/\mathbb{Z}^{2n}$ for $n \geq 2$, admit *infinite* groups of complex automorphisms.

Accordingly, the moduli space of the complex $2n$-tori A, that is an orbispace which is locally at a point $b_0 = b_{A_0}$ corresponding to A_0 equals the quotient of a complex analytic space of deformations of the complex structure in A_0 by the automorphism group of A, has a pretty bad singularity at this b_0.

To remedy this, one fixes a *polarization* i.e. a translation invariant non-singular 2-form ω_0 on a torus and considers the moduli space of the isomorphism classes of the invariant complex structures where this ω_0 serves as the imaginary part of an invariant Hermitian metric.

The resulting moduli space $\mathcal{B}_n$ of Kählerian tori is a non-compact locally symmetric Hermitian orbi-space of finite volume, that is a quotient of a Hermitian symmetric space Y by a discrete isometry group $\Gamma = \Gamma_n = \pi_1^{orbi}(\mathcal{B}_n)$, also denoted Γ_Y.

These tori A themselves, parametrized by $\mathcal{B}_n$, make the *universal family*, say $\mathcal{A}_n \to \mathcal{B}_n$ where the fibers represent the isomorphism classes of all these A.

The Abel-Jacobi-Veronese theorem says in this language that

every continuous map map from a compact Kähler manifold V to a fiber $A_{b_0} \subset \mathcal{A}_n$, $b_0 \in \mathcal{B}$, which induces an isomorphism $H_1(V)/torsion \to H_1(A_{b_0})$, is homotopic to a holomorphic map $V \to \mathcal{A}$ with the image in a single fiber A_b, $b = b(V) \in \mathcal{B}_n$.

The universal orbi-covering space $\tilde{\mathcal{A}}_n$ of $\mathcal{A}_n$ has a natural structure of a holomorphic vector bundle over Y, where this bundle carries a Γ invariant flat $\mathbb{R}$-linear (but not $\mathbb{C}$-linear) connection.

On the other hand, since Y is *topologically contractible* as well as *Stein*, this bundle is holomorphically (non-canonically) isomorphic to the trivial bundle $Y \times \mathbb{C}^n$.

The Galois group $\Gamma_{\mathcal{A}}$ of the covering map $\tilde{\mathcal{A}}_n \to \mathcal{A}_n$ is the semidirect product $\Gamma \ltimes \mathbb{Z}^{2n}$, for the *monodromy action* of Γ on $\mathbb{Z}^{2n}$ and where the action of $\Gamma_{\mathcal{A}}$ on $\tilde{\mathcal{A}}_n = Y \times \mathbb{C}^n$ is $\mathbb{C}$-*affine* on the fibers $y \times \mathbb{C}^n \subset \tilde{\mathcal{A}}_n$

This monodromy action Γ on $\mathbb{Z}^{2n}$ is of the kind we met in section 2.3 (where we discussed/conjectured the super-stability of such actions) and the Siu theorem for equivariant map $X \to Y$ can be reformulated in "dynamics" terms as well.

Namely, let X be a complex analytic space with a discrete action of $\Gamma_X \subset iso_{hol}(X)$ and $f : X \to Y$ be an h-equivariant continuous map for a homomorphism $h : \Gamma_X \to \Gamma = \Gamma_Y$. Regard the trivial bundle $X \times \mathbb{C}^n \to X$ as that induced from $\tilde{\mathcal{A}}_n \to Y$ and, thus, lift the action of Γ_X on X to a continous fiber-wise $\mathbb{C}$-linear action of Γ_X on $X \times \mathbb{C}^n \to X$.

If the map f is holomorphic, then so is this lifted action, and, whenever Siu theorem applies, the continuous action of Γ_X on $X \times \mathbb{C}^n \to X$ is equivariantly homotopic to a holomorphic action.

(The moduli space $\mathcal{B}_n$ and its finite orbi-covers contain lots of compact Hermitian totally geodesic subspaces, e.g. some W locally isometric the complex hyperbolic spaces $H^m_{\mathbb{C}}$. The Siu theorem applies, for example, if the image of h is contained in the fundamental (sub)group of such a W and $rank_{hmt}(f) > 2$.)

Besides the action of Γ_X, the map f induces an action of $\mathbb{Z}^{2n}$ on $X \times \mathbb{C}^n$ by parallel translations in each fiber $x \times \mathbb{C}^n$, where the two actions together define an action of the semidirect product $\Gamma'_X = \Gamma_X \ltimes_h \mathbb{Z}^{2n}$ where Γ_X acts on $\mathbb{Z}^{2n}$ as $h(\Gamma_X) \subset \Gamma_Y$.

If the map f is holomorphic, so is the action of Γ'_X on $X \times \mathbb{C}^n$.

Conversely,

if such an action is holomorphic to start with, it defines, by the universality of $\mathcal{A}_n$, an equivariant holomorphic map $f : X \to Y$.

This reformulation does not seem to help in proving Siu-type properties, but it raises the following

Questions (a) Are there other instances of holomorphic Γ_X-spaces X, such that a continuous lift of a discrete Γ_X-action from X to a fiber-wise linear and/or fiber-wise affine action on a holomorphic vector bundle over X can be *predictably* deformed to a holomorphic lift?

(b) Let $W' \to W$ be a holomorphic fibration where the fibers are Kählerian tori. Let $f_0 : V \to W'$ be a continuous map, such that its projection to W is (known to be) homotopic to a holomorphic map $V \to W$. (We may additionally insists that this fibration admits a holomorphic section.)

Under what geometric conditions on W' (in particular, concerning the classifying map from W to the moduli space $\mathcal{B}$ of Kählerian tori) and homotopy conditions on f_0 is the map f_0 itself homotopic to a holomorphic map?

(We look for an answer that would embrace the Albanese theorem corresponding to W being a single point along with the Siu theorem where one has to require, in particular, that the homotopy rank $rank_{hmt}(f_0)$ equals that of the projection of f_0 to W.)

(c) The latter question has a counterpart in hyperbolic dynamics that we formulate below in the simplest case.

Let W be a manifold (or orbifold) of negative curvature and let $W' \to UT(W)$ be an n-torus fibration over the unite tangent bundle of W.

Let the geodesic flow on $UT(W)$ lift to an $\mathbb{R}$-action on W' which maps $\mathbb{T}^n$-fibers to fibers and such that the return map $\mathbb{T}^n_u \to \mathbb{T}^n_u$, $u \in G \subset UT(W)$, over every periodic orbit (closed geodesic) G in $UT(W)$, is a linear hyperbolic automorphism of $\mathbb{T}^n_u$.

(An inspiring example of such a W' is the lift of the universal elliptic curve $\mathcal{A}_1$ over the modular curve $W = \mathcal{B}_1$ to $UT(W)$, where the $\mathbb{R}$-action is Anosov hyperbolic and where the return maps $\mathbb{T}^2 \to \mathbb{T}^2$ simultaneously and coherently represent *all* hyperbolic automorphisms of $\mathbb{T}^2$ by monodromy transformations.)

What is the (super)stability range of this $\mathbb{R}$-action on W'?

Namely, let V' be a topological, say compact, space with an $\mathbb{R}$-action and $f_0 : V' \to W'$ be a continuous map. (If W is an *orbi*fold, one has to formulate this in terms of equivariant maps between universal orbicoverings of our orbispaces.)

Under what circumstances is f_0 homotopic to a continuos map $V' \to W'$ which sends $\mathbb{R}$-orbits in V' to $\mathbb{R}$-orbits in W'?

4.10 Symbolic and other Infinite Dimensional Spaces.

Interesting (semi)group actions on compact complex spaces X appear only sporadically, where some of these, e.g. holomorphic actions of $\mathbb{Z}_+$ on the 2-sphere (rational curve), have been extensively studied under the heading of *complex (holomorphic) dynamics*.

On the other hands there are lots of infinite dimensional spaces holomorphically acted upon by pretty large groups. Below is a particular construction of such spaces.

$\underline{\Gamma}$-*Power Categories.* Let $\mathcal{K}$ be a "geometric" category, e.g. the category of Kähler or complex algebraic spaces, or a category of dynamical systems and let $\underline{\Gamma}$ be a countable group. Define the category $\underline{\Gamma}^{\mathcal{K}}$ by Markovian recipe as follows.

The objects $X \in \underline{\Gamma}^{\mathcal{K}}$ are projective limits of finite Cartesian powers $K^{\underline{\Delta}}$ for $K \in \mathcal{K}$ and finite subsets $\underline{\Delta} \subset \underline{\Gamma}$. These X are naturally acted upon by $\underline{\Gamma}$ and the admissible *finitery morphisms* in our $\underline{\Gamma}$-category are $\underline{\Gamma}$-*equivariant* projective limits of morphisms in $\mathcal{K}$, where such a morphism $F : X = K_1^{\underline{\Gamma}} \to Y = K_2^{\underline{\Gamma}}$ is defined by a single morphism in $\mathcal{K}$, say by $f : K_1^{\underline{\Delta}} \to K_2$ where $\underline{\Delta} \subset \underline{\Gamma}$ is a *finite* (sub)set.

Namely, if we think of $x \in X$ and $y \in Y$ as K_1- and K_2-valued functions $x(\underline{\gamma})$ and $y(\underline{\gamma})$ on $\underline{\Gamma}$ then the value $y(\underline{\gamma}) = F(x)(\underline{\gamma}) \in K_2$ is evaluated as follows:

translate $\underline{\Delta} \subset \underline{\Gamma}$ to $\underline{\gamma\Delta} \subset \Gamma$ by γ, restrict $x(\gamma)$ to $\gamma\Delta$ and apply f to this restriction $x|\underline{\gamma\Delta} \in K_1^{\gamma\underline{\Delta}} = K_1^{\underline{\Delta}}$.

In particular, every morphism $f : K_1 \to K_2$ in $\mathcal{K}$ tautologically defines a morphism in $\mathcal{K}^{\underline{\Gamma}}$, denoted $f^{\underline{\Gamma}} : K_1^{\underline{\Gamma}} \to K_2^{\underline{\Gamma}}$, but $\mathcal{K}^{\underline{\Gamma}}$ has many other finitery morphisms in it.

(Apparently, the right setting for $\mathcal{K}^{\underline{\Gamma}}$ is where $\mathcal{K}$ is a *multi-category* and where, in particular, $\underline{\Gamma}$-equivariant transformations of spaces $\mathcal{K}^{\underline{\Gamma}}$ represent certain "branches" of operads in $\mathcal{K}$.)

Moreover, there may exist extra (non-finitery) $\underline{\Gamma}$-equivariant morphisms in $\mathcal{K}^{\underline{\Gamma}}$ for certain categories $\mathcal{K}$. For instance, if $\mathcal{K}$ is the category of topological spaces, then continuos $\underline{\Gamma}$-equivariant maps do not need be, in general, finitery. Also one can naturally define continuos holomorphic maps $K_1^{\underline{\Gamma}} \to K_2^{\underline{\Gamma}}$ for complex analytic K_1 and K_2.

Let us enrich $\mathcal{K}^{\underline{\Gamma}}$ by adding new objects to it defined by "equivariant systems of equations" in $X = K^{\underline{\Gamma}}$, e.g. by $F_1(x) = F_2(x)$ for two morphisms $F_1, F_2 : X \to Y$ for some $Y \in \mathcal{K}^{\underline{\Gamma}}$. Then introduce quotients of (old and new) objects X by equivalence relations $R \subset X \times X$, that are subobjects in our category.

Most of finite dimensional *questions* in algebraic geometry and complex analysis, as well as in dynamics, e.g. concerning $rank_{hmt/hol}$, superstability, etc, automatically extend to the corresponding $\underline{\Gamma}$-categories, where they acquire a dynamics component coming from the $\underline{\Gamma}$-actions. (See last section in [40] and references therein.)

Furthermore, some finite dimensional *results* pass to $\underline{\Gamma}$-categories by just taking projective limits, and, in rare cases, finite dimensional *proofs* also extend to $\mathcal{K}^{\underline{\Gamma}}$.

But the predominant number of *obviously formulated* problems in $\underline{\Gamma}$-categories still wait their solutions [38].

Sample Questions. (a) Let $\mathcal{K}$ be the category of complex projective algebraic manifolds, let $X \subset K_1^{\underline{\Gamma}}$ be a subobject and $Y = K_2^{\underline{\Gamma}}$, where $K_2 = H_{\mathbb{C}}^n/\Gamma$.

If the group $\underline{\Gamma}$ is amenable then one can, probably, define $meanrank_{hmt}(F)$ (similarly to *meandim* in [37],) for continuous $\underline{\Gamma}$-equivariant maps $F : X \to Y$, and show that every continuous $\underline{\Gamma}$-equivariant maps $F : X \to Y$ with $meanrank_{hmt}(F) > 2$ is $\underline{\Gamma}$-equivariantly homotopic to a holomorphic map.

(b) An alternative to the $\mathcal{K}^{\underline{\Gamma}}$ setting is that of compact infinite dimensional *concentrated mm-spaces* (see [38]) foliated by Hermitian-Hilbertian manifolds. Can one develop the full fledged non-linear Hodge theory for such spaces and apply this for proving Siu-type theorems in the "maximally extended/completed" Kähler $\underline{\Gamma}$-categories?

(c) Let $\mathcal{K}$ be the category of algebraic varieties defined over $\mathbb{Z}$. Then the $\mathbb{F}_q$-points of an object X in the (extended) category $\mathcal{K}^{\underline{\Gamma}}$ make an ordinary Markov hyperbolic $\underline{\Gamma}$-dynamical system, say $X(\mathbb{F}_q)$.

What are the patterns in the behaviour, say, of the *topological entropies* of $X(\mathbb{F}_q)$ for $q = p^i$ and $i \to \infty$, or, more interestingly, for $p \to \infty$?

(One may expect a satisfactorily general/concise answer for $\underline{\Gamma} = \mathbb{Z}$, but one, probably, needs strong assumptions on X for more general amenable and *sofic* groups $\underline{\Gamma}$.)

(d) The Markovian dynamical $\underline{\Gamma}$-systems $X(\mathbb{F}_{p^i})$ converge in the model theoretic sense to $X(\mathbb{C})$. Is it has anything in common with Markovian presentations in 2.5 and 3.1?

Is there anything special from the dynamics point of view about the $\underline{\Gamma}$-systems $X(\mathbb{F}_q)$?

Notice that the $\underline{\Gamma}$-spaces $X(\mathbb{F}_{p^i})$ converge in the model theoretic sense to $X(\mathbb{C})$ for $p, i \to \infty$, [36], but it is unclear if there is any link between this and Markovian presentations in 2.5.

The above $\underline{\Gamma}$-spaces also make sense for continuous, e.g. Lie, groups $\underline{\Gamma}$, where the relevant equations defining "subobjects" are partial differential ones.

A particular instance of such a space is that of holomorphic maps from $\underline{\Gamma} = \mathbb{C}$ to an algebraic manifold, or more generally, an *almost complex* manifold K (see dyn-inv, [58]).

The dynamics of the action of $\mathbb{C}$ as well as of the group of $\mathbb{C}$-affine transformations of $\mathbb{C}$, on spaces of holomorphic maps $\mathbb{C} \to K$, e.g. the structure of invariant measures on such spaces can be seen as a part of the *Nevanlinna value distribution theory* (See [58], [23] for the first steps along these lines.)

Concluding Remarks. There are other parallels between complex geometry and dynamics. For example, conjectural lower bounds on the topological entropy and on the full spectrum of *intermediate entropies*, (defined in 0.8.F in[34]) of continuous actions on an X in terms of the corresponding asymptotic invariants of the induced actions on $\pi_1(X)$ (a correct/ definition of such invariants is not so apparent) are vaguely similar to the *Hodge conjecture.*

But the main question remains open:

Is there something more to all these "parallels" than just the universality of the categorical/functorial language?.

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