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Infinite Groups as Geometric Objects

1.

Let a group Γ faithfully and isometrically act on a metric space X. We are interested in those algebraic properties of Γ which are reflected in the geometry of X and the action of Γ on X.

Examples. 1. A. The word-metric. If $\gamma_1, \ldots, \gamma_k$ is a finite system of generators of Γ, then there is a unique maximal left-invariant metric on Γ for which $\operatorname{dist}(e, \gamma_i^{\pm 1}) = 1$, $i = 1, \ldots, k$, where e is the identity element in Γ. This is called the *word-metric* and Γ acts on $X = (\Gamma, \operatorname{dist})$.

1.A′. If Γ is f.p. (finitely presented), then there exists a closed manifold V of any given dimension $n \geqslant 4$, such that $\Gamma \approx \pi_1(V)$. Indeed, take a 2-dimensional cell complex P for which $\pi_1(P, p_0) \approx \Gamma$, then embed $P \hookrightarrow \boldsymbol{R}^n$, $n \geqslant 5$, and take the boundary of a regular neighborhood of P in $\boldsymbol{R}^n$ for V. By fixing a Riemannian metric in V we obtain the induced metric on the universal cover $X' = \tilde{V}$ and Γ acts on X' by isometric deck transformations.

These examples are tied together with the following

1.B. Definitions. The *dilation* of a map $f\colon X \to X'$ between two metric spaces is

$$\operatorname{dil} f = \sup_{x,y} \operatorname{dist}\big(f(x), f(y)\big)/\operatorname{dist}(x, y).$$

The *distance* between two maps f_1 and f_2 is

$$\operatorname{dist}(f_1, f_2) = \sup_{x} \operatorname{dist}\big(f_1(x), f_2(x)\big).$$

A map $\tau\colon X \to X$ is called a *tranlation* if $\operatorname{dist}(\tau, \mathrm{id}) < \infty$. For instance, central elements in Γ are translations for the word-metric.

A map $f\colon X \to X'$ is called *v.L.* (*virtually Lipschitz*) if there exists a translation $\tau\colon X \to X$ such that $\mathrm{dil}(f \circ \tau) < \infty$. We call f *quasi-isometry* (*q.i.*) if there is a v.L. map $g\colon X' \to X$, for which $f \circ g$ and $g \circ f$ are translations.

Every isomorphism $f\colon \Gamma \to \Gamma'$ between finitely generated groups obviously is a quasi-isometry for arbitrary word-metrics in Γ and Γ'.

The orbit map $f = f_{\tilde v}\colon \Gamma \to \tilde V$ for $f\colon \gamma \to \gamma(\tilde v)$ for the above deck action of $\Gamma = \pi_1(V)$ on $\tilde V$ clearly is a quasi-isometry for all $\tilde v \in V$ and for all compact manifolds V.

Let Γ and Γ' be discrete finitely generated subgroups in an arbitrary locally compact group G. Suppose Γ and Γ' are *cocompact* in G which means the existence of a compact subset $D \subset G$ such that $\Gamma \cdot D = \Gamma' \cdot D = G$. Then there exists a map $f\colon \Gamma \to \Gamma'$, such that $f(\gamma) \in \gamma \cdot D$ for all $\gamma \in \Gamma$, and every map f with this property clearly is a quasi-isometry $\Gamma \to \Gamma'$.

2.

Many (algebraic) properties of f.g. groups are q.i. invariants of the word-metric as the following examples show.

2.A. *If a torsion-free group Γ is quasi-isometric to a (nontrivial) free product, then Γ itself is such a product.*

Indeed, a famous theorem of Stallings [14] provides a *quasi-isometry invariant* description of free products:

A torsion free group Γ is a (nontrivial) free product if and only if Γ has infinitely many ends.

If Γ is finitely presented and so $\Gamma \approx \pi_1(V)$ for a compact manifold V, then ends of Γ correspond to ends of the universal covering $\tilde V$ of V and Stallings' theorem takes the following geometric form suggested by Matthew Brin [3].

2.A′. *If $\tilde V$ is dis connected at infinity, then there exists a closed hypersurface $\tilde H \subset \tilde V$ with the properties:*

(a) *The complement $\tilde V \setminus \tilde H$ contains at least two infinite components.*

(b) *If $\gamma \tilde H \subset \tilde V$ meets $\tilde H$ for some $\gamma \in \Gamma$, then $\gamma \tilde H = \tilde H$.*

Proof of 2A′. The isometric cocompact action of Γ on $\tilde V$ insures the solvability of *Plateau's problem* in $\tilde V$. Thus one gets a hypersurface $\tilde H$ of *minimal volume* which satisfies (a).

It is well known (and easy to prove) that no two codimension one (connected normally oriented and absolutely minimizing) solutions to Plateau's problem intersect unless they coincide (to grasp the idea look, for example, at minimal non-contractible closed geodesics in an infinite cyclic covering of the 2-torus T^2 with an arbitrary metric on T^2) and so H satisfies (b) as well.

Remark. This proof also applies if $\tilde{V}$ admits a (possibly noncompact) fundamental domain $\tilde{U} \subset \tilde{V}$ for which $\text{Volume}_{n-1}(\partial \tilde{U}) < \infty$. Therefore the f.p. condition can be relaxed to f.g. Also notice that the above geometric proof can be translated to the algebraic language and then one comes back to the original argument of Stallings.

2.B. *If an f.g. group Γ is q.i. to a nilpotent (for example an abelian) group, then there is a nilpotent (correspondingly abelian) subgroup $\Gamma' \subset \Gamma$ for which Γ/Γ' is finite.*

Indeed, virtually nilpotent groups are characterized by the polynomial growth property (see [8]) which is q.i. invariant.

2.C. *If a torsion-free f.g. group Γ is q.i. to a cocompact lattice in $O(n, 1)$, $n \geqslant 2$, then Γ is isomorphic to such a lattice.*

The proof depends upon the conformal structure on the ideal boundary of Γ (see [12], [11], [9] and a forthcoming paper by Tukia) and it probably generalizes to arbitrary lattices in all semisimple Lie groups.

3. The Euler characteristic and cohomology with estimates

It is unknown whether the sign of the *virtual Euler characteristic* $\chi(\Gamma)$ (whenever defined, see [13]) is a q.i. invariant. However, $\chi(\Gamma)$ can be expressed by the L^2-cohomology of Γ (see [1], [5]) and vanishing of this cohomology is q.i. invariant. This leads to geometric criteria for vanishing (and sometimes nonvanishing) of $\chi(\Gamma)$ (see [4]).

Example. Let a *contractible* polyhedron X admit a cocompact isometric action of a group Γ and let $\tau_i\colon X \to X$, $i = 1, 2, \ldots$, be an infinite sequence of translations such that $\text{dil}\,\tau_i \leqslant \text{const} < \infty$ for all i and $\text{dist}(\tau_i, \text{id}) \to \infty$ for $i \to \infty$. Then one easily sees that the L^2-cohomology of X vanishes and so $\chi(\Gamma) = 0$ in case Γ is discrete. This generalizes a theorem of Gottlieb [7] (which was brought to my attention by S. Rossete) on the vanishing of $\chi(\Gamma)$ for groups with infinite centers.

In fact, it is usually more useful to have $\chi(\Gamma) \neq 0$, but no general geometric condition insures nonvanishing of the relevant L^2-cohomology. However, there is a closely related invariant, called the *simplicial volume*, which can be defined with *bounded cohomology* of Γ (see [10]) and which is known to be nonzero for certain *hyperbolic groups* (see below).

4. Hyperbolic spaces

A metric space X is called *hyperbolic* (*coarse hyperbolic* in the terminology of [9]) if there exist three positive constants C_1, C_2 and C_3 with the following properties:

(a) Take two arbitrary balls of radii R_1 and R_2 around some points x_1 and x_2 in X. Then the diameter of their intersection satisfies

$$\delta - C_1 \leqslant \operatorname{Diam}[B(x_1, R_1) \cap B(x_2, R_2)] \leqslant C_2(\delta + C_1).$$

where $\delta = R_1 + R_2 - \operatorname{dist}(x_1, x_2)$.

(b) Take three balls in X of radii R_i, $i = 1, 2, 3$, such that every two of them have nonempty intersection. Then the intersection of the three concentric balls of the respective radii $R_i + C_3$ is nonempty.

Here we are interested in hyperbolic spaces X with "large" isometry groups Γ operating on them. Since the hyperbolicity is a q.i. invariant (easy to show), the hyperbolicity of X amounts to that of Γ with the word-metric, in case of X/Γ compact.

Examples. 4.A. Let X be a *tree* with some "singular Riemannian" metric, such that the distance equals the length of the shortest path between two points. Such an X obviously is hyperbolic (for $C_1 = 0$, $C_2 = 1$ and $C_3 = 0$) and so every f.g. free group (with the word-metric) is hyperbolic.

4.B. *Noncompact symmetric spaces of rank* 1 *are hyperbolic.* In fact, the whole conception of hyperbolicity is an attempt to encompass the basic properties of symmetric spaces and their groups of isometries (in particular arithmetic and S-arithmetic groups) into a general geometric framework. Notice that the definition of hyperbolicity needs a nontrivial modification in order to include symmetric spaces of rank $\geqslant 2$ and their combinatorial counterparts which are called *Bruhat–Tits buildings.*

4.C. *Complete simply connected manifolds X of strictly negative curvature, $K \leqslant -\varepsilon < 0$, are hyperbolic.* One may even allow such an X to have a boundary, provided the boundary is convex. Furthermore, one may extend the notion of curvature to *spaces with singularities*, and then again $K \leqslant -\varepsilon < 0$ implies hyperbolicity (see [9]). Here are specific examples which explain the meaning of "singular negative curvature".

4.C′. Start with a complete noncompact manifold Y of constant curvature -1, such that $\operatorname{Vol} Y < \infty$. A submanifold C of full dimension in Y is called a *cusp* if the universal covering of C is (isometric to) a *horoball* B in the universal covering $\tilde{Y}$ of Y. (A *horoball* by definition is a union of an increasing family of balls of radii $R \to \infty$, $B = \bigcup_{R\to\infty} B(R)$, such that the boundary spheres $\partial B(R)$ have a common point for all R, i.e. $\bigcap_{R\to\infty} \partial B(R) \neq \emptyset$). It is well known that by chopping away finitely many disjoint cusps in Y one obtains a compact submanifold Y_0 of full dimension in Y whose boundary ∂Y_0 consists of finitely many smooth closed connected hypersurfaces $H_i \subset Y$, $i = 1, \ldots, k =$ the number of cusps in Y. (Notice, that every H_i is covered by a *horosphere* and so the induced metric in H_i is Riemannian flat for $i = 1, \ldots, k$.) Let us attach the unit cone to each H_i, $i = 1, \ldots, k$ (this can be done by taking first an isometric imbedding of each H_i into the unit sphere $S^{N-1} \subset \mathbf{R}^N$ for large N and then by taking ordinary cones from the center) and consider the resulting singular space X. One can show (with appropriate definitions) that this X has negative curvature under the following geometric condition

$$\text{every closed curve in } \partial Y_0 \text{ of length} \leqslant 2\pi \text{ is contractible.} \qquad (*)$$

Hence, the fundamental group $\pi_1(X)$ is infinite hyperbolic, provided $(*)$ is satisfied. (Since the fundamental group $\pi_1(Y_0) = \pi_1(Y)$ is *residually finite* condition $(*)$ is always satisfied for sufficiently large finite coverings of Y_0. Thus one obtains many manifolds of constant curvature -1 to which the above applies.)

4.C″. Fix integers $k \geqslant 6$ and $l \geqslant 2$. Then there exists a unique simply connected 2-dimensional polyhedron X whose all 2-cells are plane regular k-gons and such that

(a) The intersection of any two k-gons in X is (if nonempty) a common edge or a vertex.

(b) Every edge in X has l adjacent k-gons.

(c) Every vertex in X has $l+1$ adjacent edges (and hence $l(l+1)/2$ adjacent k-gons).

In other words, the *link* of every vertex in X is the complete graph with l vertices. This X has nonpositive curvature and $K < 0$ for $k \geqslant 7$. The isometry group is cocompact on X.

4.C‴. There is a unique simply connected 3-dimensional polyhedron X whose 3-cells are regular dodecahedra and the link of each vertex is the 2-skeleton of the l-dimensional octahedron (for a given fixed $l = 3, 4, \ldots$). This X has $K < 0$ and the isometry group is cocompact. (If $l = 3$, one gets the hyperbolic 3-space $\boldsymbol{H}^3$ paved by dodecahedra).

4.D. Ramified covers. Take a totally geodesic submanifold V_0 of codimension 2 in a compact manifold V of negative curvature. There is a unique simply connected space X and a map $X \to V$ which is a covering over $V \setminus V_0$ and which *ramifies* at V_0 with a prescribed *ramification number* $\varrho = 1, 2, \ldots, \infty$. The induced singular metric in X has $K < 0$ and the isometry group is cocompact on X. If $\varrho = 1$, this is the ordinary universal covering $\tilde{V}$. If $\varrho = \infty$, then X is the (not locally compact) completion of the universal covering $\widetilde{V \setminus V_0}$. Furthermore, this construction applies to an arbitrary *locally convex subset* V_0 in V (compare [9]).

4.E. Small cancellation spaces. There are combinatorial criteria for hyperbolicity in terms of suitable covering of X by "small" subsets. These are systematically studied in the combinatorial group theory for the simplest case $\dim X = 2$. For example, $\frac{1}{6}$-groups are combinatorially hyperbolic and, hence, hyperbolic (for the word-metric). Many high-dimensional examples come from *Coxeter groups* (see [2], [6]). Let, for instance, V_0 be a compact oriented manifold of dimension n with a non-empty boundary. Then one can obtain (using an appropriate "sufficiently fine" triangulation of ∂V_0, see [6]) the following objects:

(i) A compact oriented n-dimensional manifold V *without boundary* and an embedding $a\colon V_0 \to V$ such that the fundamental group $\pi_1(V_0)$ *injects* into $\pi_1(V)$.

(ii) An aspherical compact oriented n-dimensional pseudomanifold W without boundary and a degree one map $b\colon V \to W$ such that the fundamental group $\pi_1(W)$ (being a finite index subgroup in a Coxeter group) is hyperbolic and the composed homomorphism $(b \circ a)_*\colon \pi_1(V_0) \to \pi_1(W)$ is zero.

(iii) If, furthermore, V_0 is aspherical, then V also is aspherical. Moreover, if $\pi_1(V_0)$ is hyperbolic, then $\pi_1(V)$ is hyperbolic as well.

4.F. There is a certain amount of surgery one can perform over hyperbolic spaces (like shrinking cusps in 4.C′, see [9]) and then starting with 4.A–4.E one has a huge amount of hyperbolic spaces and groups. The main problem is to classify them up to quasi-isometry. An important invariant of a hyperbolic space X is the *ideal boundary* ∂X, on which all isometries (and even quasi-isometries) of X act by homeomorphisms (see [9]). Most algebraic properties of hyperbolic groups (in particular, of small cancellation groups) Γ are immediate (see [9]) from (hyperbolic) dynamic behavior of the action of Γ on $\partial\Gamma$. There are, however, more subtle questions (such as evaluating the simplicial norm on $H_*(\Gamma)$, see [10], or the determination of all quasi-isometries of Γ modulo translations) which need additional ideas. Important results were announced (a private communication) by I. Rips (Jerusalem). His results would imply, for instance, nonexistence of a map $V \to V$ of degree $d \geq 2$, where V is an aspherical pseudomanifold without boundary with $\pi_1(V)$ hyperbolic (compare [10]). This, in fact, can be verified for the above examples with the techniques of [10].

5

In the limited space we could only mention a few geometric aspects in infinite groups. More examples can be found in the following references.

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