ON SIMPLEXES INSCRIBED IN A HYPERSURFACE

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The sufficient conditions are obtained for the existence, on a hypersurface $M \subset R^n$, of k points whose convex hull forms a (k − 1)-dimensional simplex, homothetic to a given simplex $\Delta \subset R^n$. In particular, it is shown that if M is a smooth hypersurface, homeomorphic to a sphere, such points will exist for any simplex $\Delta \subset R^n$. The proofs are based on simple topological considerations. There are six references.

§ 1. Introduction

Let $\Delta \subset R^n$ be a rectilinear m-dimensional simplex with vertices $\delta_0, \delta_1, \ldots, \delta_m$, and q a real number. We shall say that a simplex Δ can be q-inscribed in a set $A \subset R^n$ if there exists a vector $y \in R^n$ such that $q\delta_i + y \in A$ for $i = 0, 1, \ldots, m$. We shall say that a simplex Δ can be q-immersed in a closed region $\Omega \subset R^n$ if for any nonnegative continuous function $f(\omega)$, assigned in the region Ω and vanishing on its boundary, there exists a vector $y \in R^n$ such that $q\delta_i + y \in \Omega$ for $i = 0, 1, \ldots, m$ and

$$f(q\delta_0 + y) = f(q\delta_1 + y) = \ldots = f(q\delta_m + y),$$

or, what amounts to the same, the simplex $\Delta \subset R^n \subset R^{n+1}$ can be q-inscribed in a graph $\Gamma \subset R^{n+1}$ of the function $f(\omega)$.

<u>THEOREM 1.</u> Let $\Omega \subset R^n$ be a closed bounded region such that its boundary M is a C^1-submanifold of the space R^n and its Euler characteristic $\chi(\Omega)$ is nonzero. Let $\Delta \subset R^n$ be a rectilinear simplex. Then:

a) there exists a positive q such that the simplex Δ can be q-inscribed in M;

b) there exists a positive p such that for any positive $q < p$ the simplex Δ can be q-immersed in the region Ω.

If $V \subset R^n$ is a convex body with boundary W and $v \in W$, we shall denote by $s(v; V)$ the (closed) subset of the unit sphere $S^{n-1} \subset R^n$ that corresponds to the point v under a spherical mapping (see [1]) of the hypersurface W.

The boundary M of a convex body Ω is said to be smooth with respect to an n-dimensional simplex $\Delta \subset R^n$ if for any point $\omega \in M$ there exists a vertex δ_i of Δ such that $s(\omega; \Omega) \cap s(\delta_i; \Delta) = \phi$ (let us note that if the boundary M is smooth with respect to every n-dimensional simplex $\Delta \subset R^n$, then M will be a C^1-submanifold of the space R^n).

<u>THEOREM 2.</u> Let $\Omega \subset R^n$ be a bounded convex region with boundary M that is smooth with respect to an n-dimensional simplex $\Delta \subset R^n$. Then:

a) there exists a positive q such that the simplex Δ can be q-inscribed in M;

b) there exists a positive p such that for any positive $q < p$ the simplex Δ can be q-immersed in Ω.

The proof of Theorem 1 is based on simple and well-known topological results, set forth in § 2. Theorem 2 will be derived from Theorem 1 in § 4. In the remarks at the end of the paper we shall discuss (without proof) the uniqueness of an inscribed simplex and the importance of the restrictions involved in the conditions of Theorems 1 and 2.

In the following we shall denote by $M + \bar{e}$, where $M \subset R^n$ and $\bar{e} \in R^n$, the subset obtained from M by a shift by the vector $\bar{e}$.

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§ 2. Topological Lemmas

LEMMA 1. Let an oriented C^1-submanifold $M \subset R^n$ be the boundary (with allowance for the orientation) of a bounded closed region $\Omega \subset R^n$ whose orientation is induced from the space R^n. Let $t(M)$ be the absolute value of the degree of the tangential mapping $\tau : M \to S^{n-1}$, where $S^{n-1} \subset R^n$ is the unit sphere. Then

$$t(M) = |\chi(\Omega)|. \tag{1}$$

The proof of formula (1) can be found in Hopf's paper [4] (see also [6]).

Remark 1. If n is odd, formula (1) will be equivalent to another well-known formula of Hopf:

$$t(M) = \frac{1}{2} |\chi(M)|$$

(see [3]).

Let $Z_0, Z_1, \ldots, Z_k$ be integral cycles of the space R^n such that the intersection of their supports is empty and

$$\dim Z_0 + \dim Z_1 + \ldots + \dim Z_k = nk - 1.$$

Let us consider webs Π_i, $i = 0, 1, \ldots, k$, spanned over the cycles Z_i in the half-space $R_+^{n+1} = R_+ \times R^n \supset 0 \times R^n = R^n$ (R_+ being a positive ray). In [5] we can find the definition of the index of intersection $i(\Pi_0, \Pi_1, \ldots, \Pi_k)$ of webs Π_i, this index being independent of the choice of the webs.

Definition 1. The index of intersection $i(\Pi_0, \Pi_1, \ldots, \Pi_k)$ is called the linking coefficient $k(Z_0, Z_1, \ldots, Z_k)$ of the cycles $Z_0, Z_1, \ldots, Z_k$.

The various definitions of the linking coefficient and the proof of their equivalence for the case of two cycles can be found in [2]. The analysis for a larger number of cycles is entirely similar. In Lemmas 2 and 3 we present without proof the properties of the linking coefficient to be used by us below. The proof of these lemmas is based on the standard use of the properties of the index of intersection [5].

LEMMA 2. By $Z_0^* \subset R^{kn}$ let us denote the image of the cycle $Z_0 \subset R^n$ for a diagonal immersion

$$R^n \to \underbrace{R^n \times R^n \times \ldots \times R^n}_{k \text{ times}},$$

and by $\zeta \subset R^{kn}$ we shall denote the cycle

$$Z_1 \times Z_2 \times \ldots \times Z_k \subset R^n \times R^n \times \ldots \times R^n.$$

Then

$$|k(Z_0^*, \zeta)| = |k(Z_0, Z_1, \ldots, Z_k)|,$$

where $k(Z_0^*, \zeta)$ is the linking coefficient of the cycles Z_0^* and ζ in the space R^{kn}.

LEMMA 3. Let M^m and N^n be closed oriented C^2-submanifolds of the space R^{m+n+1}. By U_N^ε we shall denote a tubular neighborhood of N^n. Let M^m be contained in the set $U_N^\varepsilon \setminus N^n$. We shall assume that $U_N^\varepsilon = N^n \times D_\varepsilon^{m+1}$ (D_ε^{m+1} is a sphere of radius ε), and denote by $p\colon U_N^\varepsilon \to D^{m+1}$ the projection onto the component of the direct product. Let us consider the restriction $p_M\colon M^m \to D_\varepsilon^{m+1}$ of the projection p. Under the mapping p_M the image M_p of the manifold M^m does not intersect with the center of the sphere D_ε^{m+1}, so that it is possible to project the image M_p onto the boundary S_ε^m of the sphere D_ε^{m+1} from its center. Let $\theta : M^m \to S_\varepsilon^m$ be a mapping constructed in this way. Then the degree of the mapping θ will be equal to the linking coefficient of the cycles, determined by the submanifolds M^m and $N^n \subset R^{m+n+1}$.

Let M be a closed oriented $(n-1)$-dimensional C^1-submanifold of the space R^n, and let $E = (\bar{e}_1, \ldots, \bar{e}_n)$ be a basis in R^n. By M_{bi} (b real) we shall denote the submanifolds $M + b\bar{e}_i$, where $i = 1, 2, \ldots, n$. By a(M; E) we shall denote the largest number b such that for any positive c < b we have $M \cap M_{c,1} \cap \dots \cap M_{c,n} = \phi$. If such a large number does not exist, we shall write a(M; E) = ∞.

It is clear that for a sufficiently small positive b the intersection $M \cap M_{b,1} \cap \dots \cap M_{b,n}$ will be empty, and therefore a(M; E) > 0.

Definition 2. The self-linking coefficient $\varkappa(M)$ of a submanifold $M \subset R^n$ is defined as the absolute value of the linking coefficient of cycles, determined (oriented!) by the submanifolds M, $M_{b,1}, \dots, M_{b,n}$, where 0 < b < a(M; E). It is clear that the quantity $\varkappa(M)$ depends neither on the choice of the basis E, nor on the choice of a positive b < a(M; E), so that the definition is correct.

Proposition 1. The self-linking coefficient $\varkappa(M)$ is equal to the absolute value t(M) of the degree of the tangential mapping $\tau : M \to S^{n-1}$.

Proof. Without loss of generality it can be assumed that the submainfold M has smoothness of class C^2, and that the basis E is formed by orthogonal unit vectors $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$.

Let us denote by $M^* \subset R^{n^2}$ the image of the manifold $M \subset R^n$ in the case of a diagonal immersion

$$R^n \xrightarrow{\Delta} \underbrace{R^n \times R^n \times \dots \times R^n}_{n \text{ times}},$$

and by $N \subset R^{n^2}$ the direct product

$$\underbrace{M \times M \times \dots \times M}_{n \text{ times}} \subset \underbrace{R^n \times R^n \times \dots \times R^n}_{n \text{ times}}.$$

By

$$\bar{e} \in \underbrace{R^n \times R^n \times \dots \times R^n}_{n \text{ times}}$$

we shall denote a vector with components $(\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n)$, where $\bar{e}_i \in R^n$.

Let us consider a tubular neighborhood U_N^ε of the submanifold $N \subset R^{n^2}$. By selecting the direction of of the normal of the submanifold $M \subset R^n$, we specify the decomposition $U_N^\varepsilon = N \times D_\varepsilon^n$ and the projection $p : U_N^\varepsilon \to D_\varepsilon^n$. Let us denote by $M_b^* \subset R^{n^2}$ the submanifold $M^* + b\bar{e}$ and assume the positive number b to be so small that the inclusion $M_b^* \subset U_N^\varepsilon$ holds. Since the intersection $M_b \cap N$ is empty, it is possible to construct the mapping $\theta : M_b^* \to S_\varepsilon^{n-1}$, considered in Lemma 3. It follows from Lemmas 2 and 3 that deg (θ) coincides in absolute value with the self-linking coefficient $\varkappa(M)$.

Let us denote by $\bar{\nu}(m) \in R^n$ the vector of the unit normal to the manifold M at the point $m \in M$ and consider the vectors $\bar{\nu}_1(m) = (\bar{\nu}(m), \bar{0}, \dots, \bar{0})$, $\bar{\nu}_2(m) = (\bar{0}, \bar{\nu}(m), \bar{0}, \dots, \bar{0})$, ..., $\bar{\nu}_n(m) = (\bar{0}, \dots, \bar{0}, \bar{\nu}(m))$ in the space $R^{n^2} = R^n \times R^n \times \dots \times R^n$. The vectors $\bar{\nu}_1(m), \bar{\nu}_2(m), \dots, \bar{\nu}_n(m)$ form a basis of the space of normals of the manifold N at the point (m, m, ..., m) where $m \in M \subset R^n$. Let us consider mapping $\theta' : M \to S_\varepsilon^{n+1}$, defined as follows: To each point $m \in M$ we assign a set of scalar products $(\bar{e}, \varepsilon\bar{\nu}_1(m))$, $(\bar{e}, \varepsilon\bar{\nu}_2(m))$, ..., $(\bar{e}, \varepsilon\nu_n(m))$. For small positive b the mapping $\theta: M_b^* \to S_\varepsilon^{n-1}$ is close to the mapping $\theta' : M \to S_\varepsilon^{n-1}$ (let us note that the manifolds M and M_b^* are canonically diffeomorphic). But since $(e, \varepsilon\nu_i(m)) = \varepsilon(e_i, \nu(m))$, it follows that the mapping θ' coincides in fact (when the spheres S_ε^{n-1} and S^{n-1} are set identical) with the tangential mapping $\tau : M \to S^{n-1}$

Hence

$$|\deg(\tau)| = |\deg(\theta')| = |\deg(\theta)| = \varkappa(M),$$

which proves the proposition.

COROLLARY. Let $M \subset R^n$ be a closed oriented (n − 1)-dimensional C^1-submanifold, and let $t(M) \neq 0$. Let E be a basis of the space R^n. Then:

a) $a(M; E) < \infty$;

b) if $0 < b < a(M; E)$, then the intersection $\Pi \cap \Pi_1 \cap \dots \cap \Pi_n$ of arbitrary webs, spanned in the space R_+^{n+1} over the manifolds $M, M_{b,1}, \dots, M_{b,n}$, will not be empty.

Proof. If the intersection of some webs $\Pi, \Pi_1, \dots, \Pi_n$, spanned in the space R_+^{n+1} over the manifolds $M, M_{b,1}, \dots, M_{b,n}$, is empty for $0 < b < a(M; E)$, then the self-linking coefficient $\varkappa(M)$ will be equal to zero; but in this case $t(M) = 0$, which proves Assertion (b).

For a sufficiently large number b (we must take $b > \max \frac{d(M)}{|e_i|}$, where $d(M)$ denotes the diameter of the set M and the e , $i = 1, 2, \dots, n$, are vectors forming the basis E) there exist nonintersecting webs (for example, conical), spanned in R_+^{n+1} over the manifolds $M, M_{b,1}, \dots, M_{b,n}$. But since the definition of the number $\varkappa(M)$ does not depend on the choice of a positive $b < a(M; E)$, we would obtain for $a(M; E) = \infty$ the relation $\varkappa(M) = 0$, and hence $t(M) = 0$, which proves Assertion (a).

§ 3. Proof of Theorem 1.

Without loss of generality it can be assumed that the simplex $\Delta \subset R^n$ is n-dimensional. In the space R^n let us consider a basis E, formed by the vectors $\bar{e}_1 = \delta_0 - \delta_1, \bar{e}_2 = \delta_0 - \delta_2, \dots, \bar{e}_n = \delta_0 - \delta_n$. Let us write $q = a(M; E)$. From the corollary of Proposition 1 and Lemma 1 it follows that $0 < q < \infty$. From the definition of the number $a(M; E)$ it follows that the intersection $M \cap M_{q,1} \cap \dots \cap M_{q,n}$ is not empty. Let $m \in M \cap M_{q,1} \cap \dots \cap M_{q,n}$. Then $m, m - q\bar{e}_1, \dots, m - q\bar{e}_n \in M$. Let us write $y = m - q\delta_0$. Then $q\delta_0 + y = m \in M$ and $q\delta_i + y = m - q\bar{e}_i \in M$ for $i = 1, 2, \dots, n$, which proves Assertion (a).

Let us denote by $\Gamma, \Gamma_{q,1}, \dots, \Gamma_{q,n} \subset R_+^{n+1}$ the graphs of the functions $f(x), f(x - q\bar{e}_1), \dots, f(x - q\bar{e}_n)$, defined in the regions $\Omega, \Omega + q\bar{e}_1, \dots, \Omega + q\bar{e}_n$. The graph $\Gamma_{q,i}$ can be regarded as a web, spanned over the manifold $M_{q,i}$. By virtue of the corollary of Proposition 1 and of Lemma 1, the intersection $\Gamma \cap \Gamma_{q,1} \cap \dots \cap \Gamma_{q,n}$ is not empty for $0 < q < a(M; E)$. By reasoning in the same way as above, we can see that for $0 < q < a(M; E)$ the simplex $\Delta \subset R^n \subset R_+^{n+1} \subset R^{n+1}$ can be q-inscribed in the graph $\Gamma \subset R_+^{n+1} \subset R^{n+1}$, which completes the proof of Theorem 1.

Remark 2. In Assertion (b) of Theorem 1 we can take as the number p the smallest positive number q such that the simplex Δ can be q-inscribed in the manifold M.

§ 4. Proof of Theorem 2.

At first we shall formulate a simple geometrical proposition:

LEMMA 4. Let $\Omega \subset R^n$ be a closed bounded convex region with a boundary M that is smooth with respect to a simplex $\Delta \subset R^n$ with vertices $\delta_0, \delta_1, \dots, \delta_n$. Let $\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_k \supset \dots$ be a sequence of regions with C^1-smooth boundaries M_k such that $\bigcap_{k=1}^{\infty} \Omega_k = \Omega$. Let $q_k > 0$, $k = 1, 2, \dots$, be a sequence of numbers such that the simplex Δ can be q_k-inscribed in the set M_k for $k = 1, 2, \dots$. Then $\lim_{k\to\infty} \inf q_k > 0$.

Proof. By Δ^k we shall denote a simplex with vertices of the form $\delta_0^k = q_k\delta_0 + y_k, \dots, \delta_n^k = q_k\delta_n + y_k$ such that $\delta_0^k \dots, \delta_n^k \in M_k$. It is clear that the spherical images of the boundaries of the simplexes $\Delta, \Delta^1, \dots, \Delta^k \dots$ coincide. If $\lim_{k\to\infty} \inf q_k = 0$, there would exist a subsequence $\Omega_{k_1}, \Omega_{k_2}, \dots, \Omega_{k_j}, \dots$ with the following properties:

1) there exist limits $l_0 = \lim_{j\to\infty} \delta_0^{k_j}, \dots, l_n = \lim_{j\to\infty} \delta_n^{k_j}$, such that $l_0 = l_1 = \dots = l_n \in M$;

2) there exist limits $\sigma_0 = \lim_{j\to\infty} s_0^{k_j}, \dots, \sigma_n = \lim_{j\to\infty} s_n^{k_j}$, where we denoted by s_i^k ($i = 0, 1, \dots, n$; $k = 1, 2, \dots$) the unique point of which the set $s(\delta_i^k, \Omega_k)$.

On the one hand, $\sigma_i \in s(\delta_i, \Delta)$ for $0 \le i \le n + 1$, and on the other hand $\sigma_0, \sigma_1, \dots, \sigma_n \in s(l_0, \Omega) = s(l_1, \Omega) = \dots = s(l_n, \Omega)$, so that at the point $l_0 = l_1 = \dots\ l_n \in M$ the smoothness of the boundary M of Ω with respect to the simplex Δ is disturbed, which completes the proof of Lemma 4.

Proof of Theorem 2. For the region Ω we shall construct a sequence of regions Ω_1, Ω_2, ... with smooth boundaries M_1, M_2, ..., such that $\bigcap_{k=1}^{\infty} \Omega_k = \Omega$. Since $\chi(\Omega_k) = 1$, it follows from Assertion (a) of Theorem 1 that there exists a sequence of numbers $q_k > 0$ and vectors $y_k \in R^n (1 \leq k < \infty)$ such that $q_k\delta_i + y_k \in M_k (0 \leq i \leq n)$. There evidently exists a subsequence Ω_{k_1}, Ω_{k_2}, ... such that there exist the limits $q = \lim_{j\to\infty} q_{k_j}$ and $y = \lim_{j\to\infty} y_{k_j}$. Since $q\delta_i + y \in M (0 \leq i \leq n)$ and q is positive by Lemma 4, we thus proved Assertion (a).

Let us continue the function $f(\omega)$ by a zero to the region $\Omega_k \supset \Omega$. By applying Assertion (b) of Theorem 1 to each region Ω_k and going over to the limit in the same way as in the proof of Assertion (a), we obtain by virtue of Remark 2 and Lemma 4 the Assertion (b) of Theorem 2.

§ 5. Remarks.

If n is odd, then for a region $\Omega \subset R^n$ with a smooth boundary M the condition $\chi(\Omega) \neq 0$ will be equivalent by virtue of Remark 1 to the condition $\chi(M) \neq 0$; hence, the fulfillment of this condition will depend only on the manifold M, and not on its manner of immersion in the space R^n. If Ω' is the complement of the region Ω in the sphere S^n, we have for even n the following evident formula: $\chi(\Omega) + \chi(\Omega') = 2$; this formula shows that in general the fulfillment of the condition $\chi(\Omega) = 0$ depends on the manner of immersion of the manifold M in the space R^n for even n. This question is examined in more detail in [6].

The condition $\chi(\Omega) \neq 0$ in Theorem 1 is essential. The corresponding examples for Assertion (a) can be constructed beginning with n = 3, and for Assertion (b) beginning with n = 2.

The "smoothness" conditions in Theorem 2 are essential; moreover, if for any n-dimensional simplex $\Delta \subset R^n$ there exists a positive q such that the simplex Δ can be q-inscribed (q-immersed) in M (in Ω), then the boundary M will be C^1-smooth.

If $M \subset R^n$ is a smooth or convex closed hypersurface and if for any n-dimensional simplex $\Delta \subset R^n$ and any positive number q there exists not more than one vector $y \in R^n$ such that $q\delta_i + y \in M$ for i = 0, 1, ..., n, then for n > 2 the hypersurface M will be an ellipsoid.

L. A. Slutsman has reported that if $M \subset R^2$ is a strictly convex closed smooth curve, then there exist for any triangle $\Delta \subset R^2$ a unique positive number q and unique vector $y \in R^2$ such that $q\delta_i + y \in M$ for i = 0, 1, 2. Using this fact, Slutsman proposed an elementary proof of Assertion (a) of Theorem 1 for the case of a convex region $\Omega \subset R^3$.

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