Geometry, Topology and Spectra of Non-Linear Spaces of Maps - Wolfgang Pauli Lectures

Misha Gromov

May 25, 2009

"How Many" Curves of Length

$\leq \lambda$ are there in a domain U in the Plane?

Plane algebraic curves of degree d

$\sum a_{ij}u_1^{d_i}u_2^{d_j} = 0$, $d_i + d_j \leq d$.

Their intersections with the disk of radius R have

lengths at most $\lambda \approx Rd \approx area^{1/2}d$ (Buffon's needle formula)

The number of a_{ij} is

$1+...+(d+1) = \frac{(d+1)(d+2)}{2} \approx d^2$.

"The number" of curves of length λ (secretly of degree $d = \lambda/area^{1/2}$)

equals (secretetly $\approx d^2 \approx (\lambda/area^{1/2})^2$)

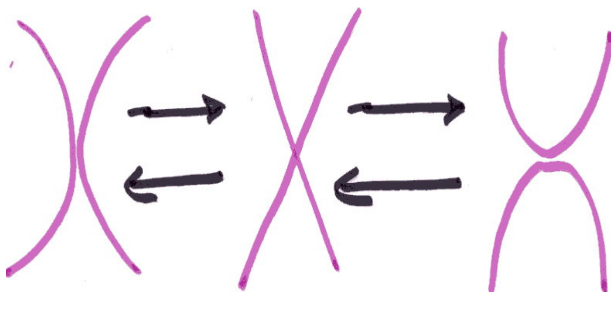
$const(\lambda)(area(U))^{-1}\lambda^2$

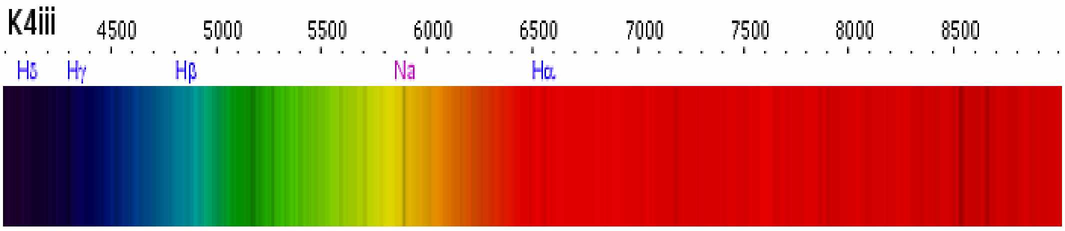
where $const(\lambda)$ converges to a non-zero limit, some universal constant

$c_1(2)$, for $\lambda \to \infty$.

Hermann Weyl Formula.

The number $N_{\leq\lambda}$ of eigenvalues





$\lambda_i \leq \lambda$ of the Laplace operator in a domain $U \subset \mathbb{R}^n$:

$N_{\leq\lambda} \sim c_n \lambda^{n/2}$

(If $n = 1$, then $\lambda_i \sim i^2$ and, for any n the eigenvalues are, roughly, the sums $i_1^2 + i_2^2 + ... + i_n^2$, where we pretend all sums are different.)

Dimension, non-Linear Spectra and Morse Theory.

A is a linear self-adjoint possibly unbounded (e.g., differential) operator on a Hilbert space H. The *normalized energy* (function) F on H:

$F(\bar{x}) = \langle A\bar{x}, \bar{x}\rangle / \langle \bar{x}, \bar{x}\rangle$.

This is defined for all non-zero $\bar{x} \in$

H in the domain of A.

Since this energy is homogeneous, $F(a\bar{x}) = F(\bar{x})$ for all $\bar{x} \in H$ it defines a function on the projective space $X = PH$ consisting of the lines in the domain of A. This function is still denoted $F(x) = \langle Ax, x\rangle / \langle x, x\rangle$ but now on $X = PH$.

Assume A is a positive operator with discrete spectrum and let us look at F from the *Morse theoretic* point of view.

Critical points of smooth functions $F : X \to \mathbb{R}$ *on arbitrary smooth manifolds* X:

a point $x \in X$ is called *singular* and the value (point) $F(x) \in \mathbb{R}$ is called *critical* for F if the differential (gradient) of F vanishes at x.

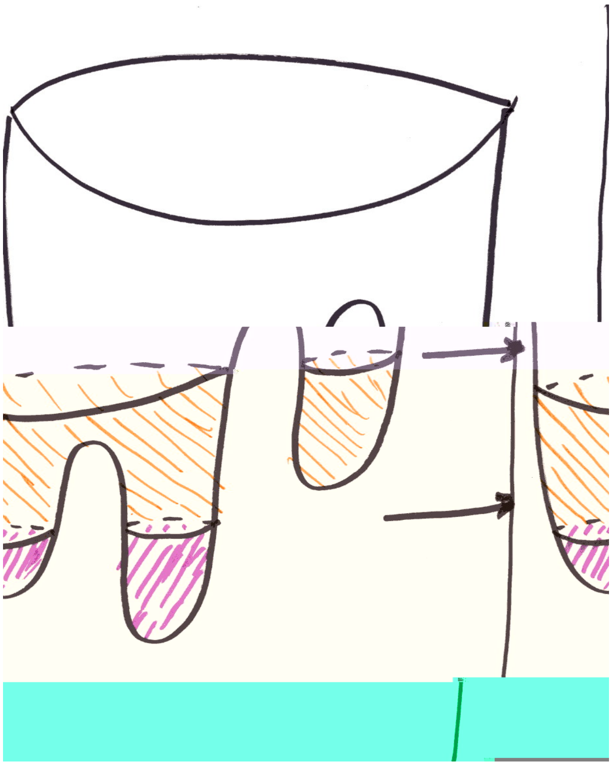
The cardinality $N_{crit} = N_{crit}(F)$ *of the critical value set* $\Sigma = \Sigma(F) \subset \mathbb{R}$ *of a generic (Morse) smooth function on a closed* n*-dimensional manifold,* $F : X \longrightarrow \mathbb{R}$*, is bounded from below by the sum of the Betti numbers of* X*,*

$N_{crit} \geq |H_*(X)|_{\mathbb{F}}$
$=_{def} \sum_{i=0,1,\ldots,n} rank(H_i(X;\mathbb{F}))$,
where the homology groups $H_i(X) = H^i(X;\mathbb{F})$ may be taken with any coefficient field $\mathbb{F}$.

The Morse-theoretic description of the spectrum.

$\tilde{\Sigma}(F) \subset X$ the *singular set* of F where the differential (or gradient) of F on X vanishes.

If $X = PH$ and



$F(x) = \langle Ax, x\rangle / \langle x, x\rangle$,

then $\tilde{\Sigma}(F)$ equals the union of the 1-dimensional eigenspaces of A.

In other words, $\bar{x}$ may serve as a non-zero vector in the line in H representing a point $x \in \tilde{\Sigma}(F) \subset X = PH$ if and only if $A(\bar{x}) = \lambda\bar{x}$ for some real λ and, in this case, $F(x) = \lambda$.

The critical point of F corresponding to a simple eigenvalue λ_i is non-degenerate and has Morse index i. More generally, the multiplicity of λ_i equals $dim\tilde{\Sigma}_i + 1$ for the component $dim\tilde{\Sigma}_i \subset \tilde{\Sigma}$ on which F equals λ_i, since $\tilde{\Sigma}_i$ consists of the lines in the eigenspace of λ_i.

Figure 1: default

The critical values and unstable under small perturbations (every point can be made critical by an arbitrary small C^0-perturbation of the energy function): look for another candidate for the non-linear spectrum.

The eigenvalue λ_i equals the minimal number, such that the sublevel

$X_{\leq\lambda} = F^{-1}[0,\lambda] \subset X = PH$
contains a projective subspace P' of dimension i.

Given $Y \subset X$ define $essdim(Y)$ as the the minimal d such that Y it can be contracted in X to a subspace Y' with $dim(Y') = d$.

The eigenvalue λ_i is the minimal number, such that the sublevel
$X_{\leq\lambda} = F^{-1}[0,\lambda] \subset X = PH$
has $essdim(X_{\leq\lambda}) = i$.

Lusternik–Schnirelmann–Borsuk–Ulam Theorem. *An i-dimensional projective subspace $P^i \subset X = PH$ can not be contracted to anything of dimension $< i$, i.e. $essdim(P^i) = i$; furthermore,*

$$essdim(A \cup B) \leq essdim(A) +$$

$essdim(B) + 1$

Examples of non-quadratic energy. $F(\bar{x}) = ||\bar{x}||_{L_p}/||d(\bar{x})||_{L_p}$ where $d(\bar{x})(= grad(\bar{x})$ is the differential (gradient) of a function $\bar{x} : V \to \mathbb{R}$ and

$||...||_{L_p} = \left(\int (...)^p\right)^{1/p}$

If $p = 2$ this gives the spectrum of the Laplacian Δ on V and if $p = 1$ the singular points correspond to minimal hypersurfaces in V.

Hermann Weil again. Let V be a compact n-dimensional Riemannian manifold and F(x) denotes the $(n-1)$-dimensional volume of the zero set of a function $\bar{x} : V \to \mathbb{R}$. Then

the spectrum of F satisfies $\lambda_i \sim i^{1/n}$ (where $a(i) \sim b(i)$ if the ratios

$a(i)/b(i)$ and $b(i)/a(i)$ stay bounded for $i \to \infty$).

Almost equivalently and more quantitatively:

$essdim(X_{\leq\lambda})/\lambda^n \to c_{n-1}(n)vol(V)^{-(n-1)}$ as $\lambda \to \infty$,

Guth' ε-Inequalities.

Let X be the space of k-dimensional submanifolds x in the n-ball V and $F(x) = vol_k(x)$.

Then $\lambda_i \leq const \cdot i^{\frac{1}{k+1}}$ *and* $\lambda_i \geq const(\varepsilon) \cdot i^{\frac{1}{k+1}-\varepsilon}$ *for all* $\varepsilon > 0$.

What is "the space X of submanifolds" in V ?

Points $x \in X$ are smooth maps of smooth k-manifolds $S = S_x \to V$.

An i-dimensional family x_y of k-

submanifolds, parametrized by a manifold $Y \ni y$ is declaired "continous" if

there is a smooth manifolds T of dimension $i + k$, a generic smooth map $p : T \to Y$ and a smooth map $q : T \to V$ such that $x_y = q(p^{-1}(y))$. (Some fibers $p^{-1}(y) \subset T$ may be singular.)

Problems (raised by Guth).

1. Can one take $\varepsilon = 0$?

2. The ε-inequality applies to *non-orieneted* submanifolds; the case of orieneted ones, even of curves in the 3-balls, remains unclear.

The first significant case of Guth' inequlities is that of curves $(k = 1)$ in the 3-ball and both problems are open in tis case.

Parametric Packing.

Given a domain $U \subset \mathbb{R}^n$, a number $N = 1, 2, ...,$ and a positive ρ.

Packing problem. Can one find N disjoint balls in U of radii ρ?

Parametric packing problem. Evaluate ($\Pi(N)$-equivariant) *essdim* of the space of N-tuples of disjoint ρ-balls in U.

($\Pi(N)$ is the permutation group on the N-point set.)

Here $X = U^N = U \times U \times ... \times U$ and the energy is

$F(x) = \min_{i \neq j} dist^{-1}(u_i, u_j)$

What happens to spectra and to Morse theory if we replace $\mathbb{R}$, the range of a map F from X, by a more general space Y, e.g. by $Y = \mathbb{R}^m$?

$$F : X \to Y$$

If we think of X as the "space of microstates" and Y is the coarse graining, then the entropy of a subset $Y_0 \subset Y$ is

$ent(Y_0) = \log$ "*number*" $(F^{-1}(Y_0))$

But what is "number" and why log?

If X and Y are smooth manifolds and F is smooth, then what we may see in Y is the *critical set* $\Sigma(F) \subset Y$

Definition of $\hat{\Sigma}$ *and* Σ. The *singularity* $\hat{\Sigma}(F) \subset X$ of a smooth map $F : X \longrightarrow Y$, for $dimY \leq dimX + 1$, is the set of the points in X where the rank of the differential of F is $< m = dimY$, while the image of $\hat{\Sigma}$, called the *critical set*, is denoted by

$$F(\hat{\Sigma}(F)) = \Sigma(F) \subset Y.$$

• • • • • • • • • • • • • • • • •

Klartag's Theorem. *Given a probability measure μ with a continuous density function on $\mathbb{R}^n$ and a number $k << n$, there exists a surjective affine map $P = P_\mu : \mathbb{R}^n \to \mathbb{R}^k$ such that the push-forward measure $P_*(\mu)$ is ε-round (in the natural sense) where $\varepsilon \to 0$ for $n \to \infty$.*