Entropy and Isoperimetry for Linear and non-Linear Group Actions.

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X may be equal to Γ with the left action, where Γ is a countable (or locally compact) group with a given finite (compact) subset $G \subset \Gamma$ that is usually (but not always) assumed to generate Γ.

$I_\circ(r)$ for real numbers r. It is convenient to have functions of positive integer arguments extended to all $r \in \mathbb{R}_+$; we tacitly assume such extension with the functions $I_\circ(r)$ (and similar functions we shall meet later on) being constant or linear on the integer intervals $(i, i+1)$.

Asymptotic Equivalence. If Γ is an infinite finitely generated group then the combinatorial isoperimetric profiles for different finite generating subsets $G, G' \subset \Gamma$ are *asymptotically equivalent*, i.e. their ratio is bounded away from 0 and ∞.

In fact if $G \subset G' \subset G^{\cdot m}$, where $G^{\cdot m} \subset \Gamma$ denotes the set of the words in the letters $g \in G$ of length $\leq m$ then the corresponding profiles satisfy $I \leq I' \leq m \cdot |G'| \cdot I$.

This equivalence class is then regarded as the combinatorial isoperimetric profile of Γ that does not depend on G.

Sublinearity of $I_\circ$. The function $I_\circ(r) = I_\circ(r; \Gamma, G)$ is bounded by $|G|r$ and it satisfies the inequality

$$(+)_\circ \qquad I_\circ(r_1 + r_2) \leq I_\circ(r_1) + I_\circ(r_2)$$

for all *infinite* groups Γ and finite subsets $G \subset \Gamma$.

Non-monotonicity of $I_\circ$ and $I_\circ^{incr}$. The profile $I_\circ(r)$ of a group Γ is not always monotone increasing in r. For example, $I_\circ(r \geq |\Gamma|) = 0$ for finite groups Γ. If we insist on the monotonicity, we define

$$I_\circ^{incr}(r) = inf_{s \geq r} I_\circ(s),$$

that is the maximal monotone increasing minorant of $I_\circ$.

Possibly, the ratio $I_\circ(r)/I_\circ^{incr}(r)$ is bounded for all infinite finitely generated groups.

Remark. The notion of the isoperimetric profile for groups was introduced by Anatoly Vershik [29] in the (essentially equivalent) language of the *Følner functions* defined below.

1.2 $I_\circ$ for Subgroups, for Factor Groups and for Measures on Groups.

The inequality $(+)_\circ$ implies that the G-isoperimetric profile of a free action of an infinite group (Γ, G) on an arbitrary space (set) X equals the G-profile of Γ, since the boundary of a subset in X equals the *disjoint* union of the boundaries of the orbits. Consequently,

the profile of every group extension $(\Gamma_1 \supset \Gamma, G_1 \supset G)$ is bounded from below by the profile of (Γ, G),

$$I_\circ(r; \Gamma_1, G_1) \geq I_\circ(r; \Gamma, G).$$

Also

$I_\circ$ decreases under epimorphisms of infinite groups .

Namely, let $h : \Gamma \to \underline{\Gamma}$ be a surjective group homomorphism, G a finite subset in Γ and let $\underline{G} = h(G) \subset \underline{\Gamma}$ be the image of G in $\underline{\Gamma}$. Then

$$I_\circ(r; \Gamma, G) \geq I_\circ(r; \underline{\Gamma}, \underline{G}).$$

Proof. Given $Y \subset \Gamma$, let $\underline{Y}_m \subset \underline{\Gamma}$, $m = 1, 2, ...$, be the subset of points with at least m preimages in Y. Clearly,

$$(\underline{+}) \qquad |\partial_G(Y)| \geq \sum_{m=1,2,...} |\partial_{\underline{G}}(\underline{Y}_m)|$$

and the subadditivity applies, since

$$\sum_{m=1,2,...} |\underline{Y}_m| = |Y|.$$

Subgraphs of Functions and their Boundaries. Given a positive function $f(x)$ on a space X, define the subgraph $Y^f \subset X \times \mathbb{R}_+$ as the set of pairs (x, r), such that $f(x) \leq r$. Then, for a set G of transformations of X, let

$$|\partial_G(f)| =_{def} |\partial_G(Y^f)|$$

for the obvious action of G on $X \times \mathbb{R}_+$.

If the function f is integer valued, then $|\partial_G(f)|$ equals the sum of the G-boundaries of its levels $Y_m =_{def} \{f(x) = m\} \subset X$, $m = 0, 1, 2, ...$,

$$|\partial_G(f)| = \sum_m |\partial_G(Y_m)|;$$

thus one obtains the isoperimetric lower bound on $|\partial_G(f)|$ by the l_1-*norm* $||f||_{l_1} =_{def} |Y^f|$ of f,

$$|\partial_G(f)| \geq I_\circ(||f||_{l_1}; G),$$

provided $I_\circ$ is subadditive. (A similar inequality for non-integer valued functions needs extra entropic data on f, see 5.12).

This applies, in particular to the function $\underline{f}(\underline{y}) = |h^{-1}(\underline{y})|$ for the above $h : \Gamma \to \underline{\Gamma}$, where the essential feature is the (obvious) inequality $|\partial_{\underline{G}}(\underline{f})| \leq |\partial_G(Y)|$.

Generalization to maps between Γ-Spaces. Let X and $\underline{X}$ be spaces acted upon by Γ and $X \to \underline{X}$ be a Γ-equivariant map, where a basic example is $X = \Gamma$ and $\underline{X} = \Gamma/\Gamma_0$. The function $I_\circ(r; \underline{X})$ is not always subadditive, e.g. for some quotients Γ/Γ_0 where Γ is the free group F_2. If this is the case we work with the maximal subadditive minorant $I_\circ^{sb.ad}(r)$ of $I_\circ(r)$ and observe with the above argument that

$$I_\circ^{sb.ad}(r; X) \geq I_\circ^{sb.ad}(r; \underline{X}).$$

1.3 Boundaries $|\partial|^{max}$, ∂_μ and the Associated Isoperimetric Profiles.

The boundary of a $Y \subset X$ can be regarded as a function of $g \in G$ for $g \mapsto \partial_g(Y)$ and, besides taking the union of these g-boundaries, one may take the supremum

$$(\partial_{G^{\cdot m}}) \qquad \frac{1}{m}|\partial|^{max}_{G^{\cdot m}}(Y)| \leq |\partial|^{max}_{G}(Y).$$

On the other hand, if the action of Γ on X is free, the above identity ΣG , for $H = G^{\cdot m}$ implies

$$|\partial|^{max}_{G^{\cdot m}}|(Y) \geq (1 - \frac{|Y|}{|G^{\cdot m}|})|Y|$$

and yields the following inequality due to Coulhon-Saloff-Coste [4] .

CSC Displacement Inequality

$$(DIS) \qquad |\partial|^{max}_{G}(Y) \geq \frac{1}{m}(1 - \frac{|Y|}{|G^{\cdot m}|})|Y|$$

for all $Y \subset \Gamma$ and all $m = 1, 2, ...$

Inverse $\partial_{G^{\cdot m}}$-Inequality. In most (if not all) known cases there is a converse inequality saying that the function $I(r, m) = I^{max}_{\circ}(r; G^{\cdot m})$ is, roughly, linear in m, i.e. $I(r, m) \sim m \cdot I(r, 1)$ for $m \leq r/2I(r, 1)$.

It is unclear if this is true in general and what are more general constrains on possible (Γ-invariant) metrics on the orbits $\Gamma(Y) \subset 2^{\Gamma}$ for various $Y \subset \Gamma$.

On the other hand, (DIS) admits the following (strengthened) generalization.

Intersection Inequality. A subset $H \subset \Gamma$, is called *(right) G-connected* if every two points h_0 and h in H can be joint by a chain of points $(h_0, h_1, ..., h_i, ..., h_k = h)$, such that $h_i = h_{i-1}g_i$ for some $g_i \in G$ and all $i = 1, 2, ..., k$. Observe that the subsets $H = G^{\cdot m} \subset \Gamma$ are G-connected for all $m = 1, 2, ...$; moreover, if $G^{-1} = G$ and $G \ni id$, then every two points in $G^{\cdot m}$ are contained in a G-connected subset of cardinality $\leq 2m$.

If a finite subset $H \subset \Gamma$ is G-connected for a given $G \subset \Gamma$, then all measurable subsets Y of an arbitrary Γ-space X (with the action written "on the left": $x \mapsto yx$) satisfy

$$|\bigcap_{h \in H} h(Y)| \geq |Y| - (|H| - 1)|\partial|^{max}_{G}(Y).$$

Proof. Observe that $|\cap_{h \in H} h(Y)| = |Y|$ for $|H| = 1$ and proceed by induction on $|H|$ for $|H| > 1$. To do this, decompose $H = H' \cup \{h_0 g\}$ for a *G-connected* subset $H' \subset H$ with $|H'| = |H| - 1$ and some $h_0 \in H'$ and $g \in G$ and notice that the intersection inequality is stable under the left translations of H.

Thus, we pass from H to $h_0^{-1}H$ and arrive at the case where $h_0 = id \in H'$ and $H = H' \cup \{g\}$, and where, obviously,

$$|(\bigcap_{h \in H} h(Y)) \bigcap g(Y)| \geq |\bigcap_{h \in H} h(Y)| - |(Y \setminus g(Y))| = |\bigcap_{h \in H} h(Y)| - |\partial_g(Y)|.$$

1.5 Linear Algebraic Isoperimetric Profiles I_* of Groups and Algebras.

Let L be a linear space over some field $\mathbb{F}$ and G a set of linear operators $L \to L$, where in our our examples these operators are invertible, injective as well

as surjective. We define the (joint exterior) *G-boundary* of a linear subspace $D \subset L$ as the quotient space

$$\partial_G(D) = \partial_G^{joint}(D) = (G * D)/D,$$

where $G * D \subset L$ denotes the linear span of the subspaces $g(D)$ for all $g \in G$ and where $A/B =_{def} A/A \cap B$. Set

$$|\partial_G(D)| = rank_{\mathbb{F}} \partial_G(D)$$

(that equals $rank_{\mathbb{F}}(G * D) - rank_{\mathbb{F}}(D)$ if $G \ni id$) and define *the isoperimetric profile of L with respect to G* as the maximal function I_* such that all finite dimensional subspaces $D \subset L$ satisfy the following "isoperimetric inequality",

$$(*) \qquad |\partial_G(D)| \geq I_*(|D|),$$

where $|D|$ stands for $rank(D) = rank_{\mathbb{F}}(D)$.

Main Examples: Group Algebras and their Extensions. Let $L = A$ be the group algebra of Γ over some field $\mathbb{F}$ regarded as the vector space of the $\mathbb{F}$-valued functions on Γ with finite supports. The group Γ linearly acts on A and, for a given subset $G \subset \Gamma$, one may speak of the G-boundary ∂_G in A as well as in Γ.

Supports of Linear Spaces and the Implication $(*) \Rightarrow (\circ)$. The support of a linear space D of functions on Γ is, by definition, the complement to the set where all functions from D vanish. If $|supp(D)| = |D|$ then D equals the space $\mathbb{F}^Y$ of all functions on $Y = supp(D)$ and, obviously, $|\partial_G(D)| = |\partial_G(Y)|$. Thus,

$$I_\circ \geq I_*.$$

The reverse inequality does not always hold:

there are finitely generated groups (with torsion), where the two profiles are not equivalent; for example $I_*(r)$ can be bounded while $r/I_\circ(r)$ grows arbitrarily (depending on Γ) slowly (see 8.1).

It is possible, albeit unlikely, that $I_*(r) \sim I_\circ(r)$ for groups without torsion. (This is similar to the *Kaplansky conjecture* saying that the group algebras of torsionless groups have no divisors of zero).

Sublinearity and Monotonicity of I_* of Group Algebras. Clearly,

$$(+)_* \qquad I_*(r_1 + r_2) \leq I_*(r_1) + I_*(r_2)$$

for infinite groups Γ; therefore,

the profile of a group algebra of an infinite group may only increase under a group extension.

Also we shall prove in 3.5 that

I_* decreases under surjective group homomorphisms when the target group is left orderable.

It remains unclear to what extend this remains valid in general but we shall see in 7.2 that some *entropic isoperimetric profile* is always monotone decreasing under surjective group homomorphisms of infinite groups.

Remark. The profile I_* is *residually monotone*: if $\Gamma_i \subset \Gamma$ is a decreasing sequence of normal subgroups, such that $\cap_i \Gamma_i = \{id\}$, then $I_*(r;\Gamma) \geq$

$lim_{i\to\infty} I_*(r;\Gamma/\Gamma_i)$, since the push forward homomorphism of the group algebra A of Γ to the algebra A_i of Γ/Γ_i is injective on each finite dimensional subspace $D \subset A$ for large $i = i(D)$. (This remains true for the algebra of l_1-functions on Γ).

About dependence of I_* on $\mathbb{F}$. The action of $G \ni id$ acting on r functions $X \to \mathbb{F}$ is described by the following infinite system of linear equations in unknowns $f_{jx\gamma} = f_j(y(x))$, indexed by $(x \in X, y \in G, j = 1, 2, ..., r)$,

$$LIN \qquad f_{j\gamma(x)id} = f_{jx\gamma}$$

The bound $I_*(r) \le a$ means that there is a linear space D^+ of solutions of LIN in functions with finite supports, such that $|D^+| \le r + a$ while the map from D^+ to the space of r-tuples of functions on x given by $f_{jx\gamma} \mapsto f_j(x)$ has rank r.

Thus, by the Lefschetz principle,

if $\mathbb{F}$ is algebraically closed, then the profile I_* for $\mathbb{F}$-functions with finite supports on any Γ-space X depends only on the characteristics of $\mathbb{F}$

(I_* may depend on $\mathbb{F}$, e.g. for finite groups, but the extent of this dependence is unclear.)

1.6 The boundaries $|\partial|^{max}$, $|\partial|_\mu$ and $|\partial|_{G^{*m}}$.

One defines $|\partial|^{max}$ and $|\partial|_\mu$ as in the combinatorial case with the corresponding notation $I_*^{max}(r;\Gamma, G)$ and $I_*(r;\Gamma,\mu)$ for the isoperimetric profiles and one observes that

$$(\partial_{G^{\cdot m}}). \qquad \frac{1}{m}|\partial|^{max}_{G^{\cdot m}}(D)| \le |\partial|^{max}_G(D)$$

Thus,

$$I_*^{max}(r;G) \ge \frac{1}{m} I_*^{max}(r;G^{\cdot m}),$$

where the triangle inequality applies to the metric on the set (Grassmannian) of subspaces D in L defined by

$$|D_1 - D_2| =_{def} \frac{1}{2}(|D_1/D_2| + |D_2/D_1|).$$

Furthermore,

$$|\bigcap_{h\in H} h(D)| \ge |Y| - |H| \cdot |\partial|^{max}_G(D).$$

for H as in the Intersection Inequality from 1.4 and finite dimensional subspaces D in linear Γ-spaces.

Remarks. (a) There is a larger set of operators, denoted $G^{\star m} \supset G^{\cdot m}$, which is generated by m operations of additions as well as compositions of linear maps $g \in G$. Namely, start from

$$G^{\star 1} = \bigcup_{c \in \mathbb{F}} cG$$

and then set

$$G^{\star m} = \bigcup_{m_1+m_2 \leq m} ((G^{\star m_1} \cdot G^{\star m_2}) \cup (G^{\star m_1} + G^{\star m_2})).$$

Here again,

$$I_*^{max}(r; G) \geq \frac{1}{m} I_*^{max}(r; G^{\star m}).$$

and

$$|\partial|_{\mu_0}(D) \geq \frac{1}{m} |\partial|_{\mu}(D),$$

for a probability measure μ_0 on G depending on a given probability measure μ on $G^{\star m}$.

(b) *Boundary defined with Non-invertible Operators*. The G-boundary is supposed to measure the degree of (non)invariance of a subspace D under the operators $g \in G$ and if (some of) the $g \in G$ are non-injective on D one may need to include into the G-boundary the kernels of these g.

1.7 I_* For Decaying Functions.

Let the underlying field $\mathbb{F}$ come with a norm, e.g. $\mathbb{F}$ equals the field of real, $\mathbb{R}$, complex $\mathbb{C}$, or p-adic $\mathbb{Q}_p$ numbers. Then every *decaying* $\mathbb{F}$-function f on a countable set X defines an indexing of X by positive integers $i \in \mathbb{N}_+$, for which the sequence of norms, $s(i) = ||f(x_i)||$ is (non-strictly) monotone decreasing.

If all functions f in a finite linear space D of functions on a set X (e.g. on $X = \Gamma$ decay super-exponentially i.e. the corresponding sequences satisfy $C^i s(i) \to 0$ for all $C > 0$ and $i \to \infty$, then one can replace (approximate) D by a linear space D_0 of functions with finite supports on X such that $|D_0| = |D|$ and $|\partial_G(D_0)| = |\partial_G(D)|$ for a given finite set G of transformations of X.

Sketch of the Proof. The determinants of the $(k \times k)$-minors of the matrix corresponding to the system LIN in 1.5 are integers (exponentially) bounded by $|G|^k$. The solutions of the system LIN in the variables f_x, now indexed by $i \leftrightarrow x_i$, can be obtained by extending the solutions of the sub-systems for $f_{j\gamma}(x_i)$, $j \leq j_N \to \infty$ and such extension amounts to solving some non-homogeneous system associated with LIN, where the non-homogeneity takes care of the $|D| = r$ condition by prescribing the values of the solutions (functions) at certain points in X. The rate of decay (as well as of growth) of such extensions can not be (much) faster than $|G|^{j_N}$ and then the existence of a faster decaying solution implies that for a finitely supported one.

(The "rate of decay" on linear subspaces D is seen with the help of suitable bases in D, where the *super*exponentiality is apparently needed. We leave the details to the reader, since we do not use such decaying solutions in this paper).

Questions. Does the isoperimetric profile of the space of decaying (i.e. converging to zero at infinity) functions with values in an ultrametric field always equal (or, at least, is equivalent) to I_* for functions with finite supports? Does a similar equality (or at least, equivalence) hold for the Hilbert spaces of square summable real and complex valued functions?

We shall prove the equality for the free Abelian groups and the equivalence for the *non-virtually nilpotent polycyclic groups* in 6.2.

Non-decaying functions, Representations, Finite Groups, etc. We are mostly concerned in this paper with the *regular representation of an infinite group* Γ where we try to "linearize" combinatorics of finite subsets in Γ.

Other infinite dimensional, say irreducible, representations L of Γ, e.g. realized by spaces of (non-decaying) functions on Γ, may have "good Isoperimetry" for "a non-combinatorial reason", e.g. if they do not contain, in some weak sense, "bad" sub-representations.

Also $|\partial(D)|/|D|$ may be bounded from below in terms of the distance from $|D|$ to "bad" invariant subspaces. e.g. those of dimensions close to $|D|$,where many examples come from (families of) finite groups, such as $GL(k;\mathbb{F}_p)$ where suitable Følner functions, probably, behave "nicely" for $k, p \to \infty$.

1.8 Non-Amenability of Groups and I_*-Linearity Theorems by Elek and Bartholdi.

A countable group is called *non-amenable* if it satisfies the *linear isoperimetric inequality* with some finite subset $G \subset \Gamma$ (where one may assume that G generates Γ if the group is finitely generated), i.e. if the function $I_\circ(r)$ is the maximal possible up to the asymptotic equivalence: $I_\circ(r) \sim r$, which amounts here to $I_\circ(r) \geq cr$ for $c > 0$; otherwise the group is called amenable.

The basic examples of non-amenable groups are the free groups on two or more generators and the groups containing such free groups as subgroups. Also, there are many non-amenable groups without free subgroups, e.g. non-amenable torsion groups.

On the other hand all solvable (e.g. Abelian groups) are amenable.

Remark. All known infinite amenable groups, in particular the groups of subexponential growth, either contain arbitrarily large finite subgroups or they have "many" commuting elements but no general result of this kind is known. Here is a test question.

Let $Z(y_0) \subset \Gamma$ for a given $y_0 \in \Gamma$ consist of those $y \in \Gamma$, where some powers $y_0^{m_0}$ and y^{m_1} commute. Does every infinite amenable group Γ contain a non-identity element $y_0 \in \Gamma$, such that $|Z(y_0) \cap G^{\cdot n}|/n \to \infty$ for $n \to \infty$, where $G \subset \Gamma$ is a finite generating subset?

An action of an algebra A on a linear space L is called *non-amenable* or *I_*-linear*, if its isoperimetric profile $I_*(r)$ defined with some finite subset of operators $G \subset A$ is equivalent to r. If L equals the algebra itself then the algebra is called I_*-linear or non-amenable. (Unlike the group theoretic case, it is unclear what is the true property that divides algebras into "amenable" and "non-amenable" categories; the less obliging terminology seems appropriate.)

It is obvious that if Γ is amenable then so is each of its linear actions and the converse is known in the following cases.

$I_*(F_2)$- Linearity Theorem. *The group algebra of the free group F_2, and hence, of every group containing F_2 is I_*-linear for all fields $\mathbb{F}$.* (Gabor Elek [7]; see 2.3. for a strengthening of this result).

l_2- Non-amenability Theorem. *The action of every non-amenable group Γ on the complex Hilbert space $l_2(\Gamma)$ is I_*-linear.* (Gabor Elek [6]; see 2.2 for the case of l_p , $p \neq 2$).

I_*-linearity Theorem. *The group algebras of all non-amenable groups over all fields $\mathbb{F}$ are I_*-linear.* (Laurent Bartholdi see [2] and see 3.6 for a combinatorial rendition of Bartholdi's proof; if $char(\mathbb{F}) = 0$, this follows from the I_2-theorem.)

Question. Is the action of a non-amenable group Γ (e.g. for $\Gamma = F_2$) on the space of decaying $\mathbb{F}$-valued functions on Γ, where $\mathbb{F}$ is a normed field (e. g. the field of real numbers or the field of formal power series over a finite field), I_*-linear?

1.9 The Growth Functions $G_\circ$ and the Følner Functions $F_\circ$ of Groups and F_* of Algebras.

The growth function $G_\circ(n), n = 1, 2, ...,$ of a group Γ relative to a given generating set $G \subset \Gamma$ is the number of elements in Γ representable by the words of length$\leq n$ in the letters $g \in G$. That is $G_\circ(n) = |G^{\cdot n}|$ for

$$G^{\cdot n} =_{def} \underbrace{G \cdot G \cdot ... \cdot G}_{n} \subset \Gamma.$$

The growth function is at most exponential for finite sets G, since $|G^{\cdot n}| \leq |G|^n$.

The Følner function $F_\circ = F_\circ(n), n = 1, 2, ...,$ of (Γ, G) is equal, by definition, to the minimal cardinality of a (Følner) subset $Y_n \subset \Gamma$, such that $|\partial_G(Y_n)| \leq \frac{1}{n}|Y|$.

This can be expressed entirely in terms of $I_\circ$: define, for an *arbitrary* function I, *the Følner function associated to I* by the rule:

the value of the function F_I at n equals the minimal N, such that $I(N) \leq N/n$ and observe that $F_I(n) = F_\circ(n; \Gamma)$ for $I(r) = I_\circ(r; \Gamma)$. In other words, F_I is the inverse function to the minimal monotone increasing majorant $M(r)$ of the function $r/I(r)$. Thus, returning from F back to I, gives us $r/M(r) \leq I(r)$ instead of $I(r)$ where the equality takes place if and only if if $r/I(r)$ is monotone increasing. (We disregard the minor ambiguity which arises for *non-strictly* monotone functions.)

Unlike the isoperimetric profiles $I(r)$ that grow at most linearly and often are concave functions, the Følner functions of infinite groups grow faster than linearly and are rather convex than concave, where the rate of their growth translates into how slowly the corresponding isoperimetric profiles deviate from non-constant linear functions.

One may regard the map $Y \mapsto \partial Y$ as an "operator" on the "space of sets" with the "$|...|$-norm" and think of the corresponding function $M(r) = \sup_{|Y| \leq r} |Y|/|\partial Y|$ as the "norm" of the "inverse operator" ∂^{-1} where this "norm" is not a number but rather a function depending on $r = |Y|$, since ∂ is a "non-linear operator".

It is also helpful to think of the ratio $n = |Y|/|\partial(Y)|$ as a linear size (diameter) of Y and the Følner function as the volume (of the minimal Y with given diameter,) depending on the diameter.

It is clear that $F_{cI}(n) = F_I(cn)$ for all $c > 0$. Thus, for example, the $G^{\cdot m}$-inequality $I_\circ(r; \Gamma, G) \geq \frac{1}{m} I_\circ(r; \Gamma, G^{\cdot m}$ (see 1.4) translates to

$$(n/m), \qquad F_\circ(n; \Gamma, G) \geq F_\circ(n/m; \Gamma, G^{\cdot m}).$$

and the (DIS)-inequality from 1.4 applied to the minimal m for which $|G^{\cdot m}| \geq 2|Y|$, yields

Couhlon-Saloff-Coste Isoperimetric Inequality.

$$(F_\circ \succeq G_\circ) \qquad F_\circ(n;\Gamma,G) \geq \frac{1}{2}G_\circ(\frac{n}{2};\Gamma,G).$$

This inequality (that improves upon an earlier result by Varopoulos [28] with a rather elaborate proof exploiting random walks in groups) implies, for instance, that

the groups of exponential growth have at least exponential Følner functions.

But unlike $G_\circ(n)$, the Følner functions $F_\circ(n)$, of some (amenable!) groups may grow (much) faster than exponentially as was conjectured by A. Vershik in the late 60s and confirmed in [27] and in [9]. (We meet such groups in the later sections.)

The useful notion of equivalence between two Følner functions is different from that for I:

we write $F \prec F'$ if $F(n) \leq F'(Cn)$ for some constant $C > 0$ and $F \sim F'$ if $F \prec F'$ as well as $F' \prec F$ and we apply the same notion of equivalence to the growth functions.

Examples (a) The Coulon-Saloff-Coste inequality gives the correct asymptotic for $F_\circ^{max}(n) \sim n^k$

zero boundary. This leads (see 8.1) to examples of "very large" groups where $F_*(n;\Gamma) \sim n$.

On Growth of Algebras. Let L be a linear space with a set G of bijective operators acting on it, e.g. the group algebra of Γ. Then the growth function of L, G is defined with some vector $l \in A$ as

$$G_*(n, L, l, G) =_{def} |G^{\cdot n} * l|$$

where $G^{\cdot n} * l$ denotes the linear span of the orbit $G^{\cdot n}(l) \subset L$.

If the group algebra A of Γ generated by G acts *freely* or *without torsion* on L, i.e. $a(l) = 0$ only if $a = 0$ and $l = 0$, then this growth function equals that of Γ. In general, it may be smaller than $G_\circ(n;\Gamma, G)$ and one may define $G_*^{inf}(n, L, G)$ as the infimum of $G_*(n; L, l, G)$ over all $l \neq 0$ in L.

It is unknown (Kaplansky conjecture) if the freedom condition is satisfied by the group algebras of the groups without torsion and it is unclear how much $G_*(n; L, l, G)$ may depend on l in general.

Question. When is $G_*(n, L, l, G)$ asymptotically equivalent to $G_*^{inf}(n, L, G)$?

Observe that the (obvious) linear algebraic counterpart of the (DIS)-inequality (see 1.4) reads as follows,

if an algebraic family H of linear operators a acting on a linear space X over an algebraically closed field has no zero vector in a given linear subspace $Y \subset X$, i.e. $a(y) \neq 0$ for $a \neq 0$ and $y \neq 0$, then

$$|\partial|_H^{max}(Y) \geq dim(Y)(1 - dim(Y)/dim(H)).$$

However, this does not match, at least not in an obvious way, the (G^{*m})-inequality for the linear actions.

It remains unclear if there is a bound on the growth of the algebra by the Følner function, say for the *torsion free* (i.e. without divisors of zero) algebras.

1.10 Multivariable Følner Functions, $P_\circ$-functions and the Loomis-Whitney Inequality.

The *multivariable* Følner function $F_\circ(n_i; X, G_i)$ for several sets of transformations G_i acting on X is defined as the maximal N such that every $Y \subset X$ with

$$|\partial_{G_i}(Y)| \leq \frac{1}{n_i}|Y|$$

has

$$|Y| \geq N$$

and similarly, one defines multivariable F_* for linear actions.

These functions reduce to the ordinary $F_\circ^{max_i}(n)$ and $F_*^{max_i}(n)$ at $n_1 = n_2 = ... = n$, where max_i refers to $|\partial| = max_i|\partial_{G_i}|$.

The *$P_\circ$-function for Orbit Partitions.* Let X be acted upon by groups Γ_i, $i = 1, 2, .., k$, and denote by $\Gamma_\circ$ the intersection $\cap_i\Gamma_i$. Consider subsets $Y \subset X$ that have finite (or finite measure) images $Y/\Gamma_\circ$ in the quotient space $X/\Gamma_\circ$ and define $P_\circ(n_i; X/\Gamma_i)$ as the minimal N such that the inequalities

$$|Y/\Gamma_i| \leq \frac{1}{n_i}|Y/\Gamma_\circ|, i = 1, 2, ..., k, \text{ imply } |Y/\Gamma_\circ| \geq N.$$

Observe that this $P_\circ$-function, unlike $I_\circ$ and/or $F_\circ$, depends only on the Γ_i-orbit *partitions* structure in X, rather than on the full Γ-structure in X.

If $\Gamma_\circ = \{id\}$, while the the Γ_i-orbits are all infinite, then $P_\circ(n_i) \leq F_\circ(n_i)$, since each *infinite* Γ_i orbits that meets a *finite* subset $Y \subset X$ contributes a point to the Γ_i-boundary of Y.

Basic Example: Loomis-Whitney Theorem. (see 5.7). The $P_\circ$-function for the family of the k partitions of $\mathbb{R}^k$ by the lines parallel to the k coordinate axes is,

$$P_\circ(n_i) = \prod_i n_i.$$

In other words,

among all subsets $Y \subset \mathbb{R}^k$ with given measures of the projections to the k coordinate hyperplanes, the maximal measure is achieved by the rectangular solids (and all subsets obtained from them by measurable transformations of $\mathbb{R}^k$ preserving the coordinate line partitions).

It follows that

the multivariable Følner function of the free Abelian group $\mathbb{Z}^k$ with respect to its k generators $g_i \in \mathbb{Z}^k$ (i.e. $G_i = \{g_i\}$) is

$F_\circ(n_1, n_2, ..., n_k) = n_1 \cdot n_2 \cdot .... \cdot n_k$.

Remark. We shall explain in 5.3 (this was pointed out to me by Noga Alon) how [LW] follows from the Shannon inequality for the entropies of random variables, where the latter, in the case of $\mathbb{R}^k$, is (essentially) equivalent to the *logarithmic Sobolev inequality*.

1.11 Filtered Boundary $\mathcal{F}\partial$ and P_*-Functions.

$F_*(n_i)$ and $P_*(n_i)$ for linear actions. The definition of the multivariable Følner function F_* for groups Γ_i acting on a linear space L is identical to that of $F_\circ$, while P_* needs a replacement of $|Y/\Gamma|$ by a suitable Γ-*rank (dimension)*, denoted $|D : \Gamma|$ for linear subspaces $D \subset L$.

There are several candidates for P_* corresponding to different notions of the rank of a linear Γ-space. We start with a notion of a boundary associated to the Krull dimension.

$\mathcal{F}\partial$-boundary. Given a group Γ (or, more generally, an algebra A) acting on a linear space L, consider a Γ-invariant ascending filtration $\mathcal{F} = \{0 = L_0 \subset L_1 \subset L_2 \subset ... \subset L\}$. Then, for a given linear subspace $D \subset L$ consider the subspaces $D_i = D \cap L_i$ and let $\partial_{\mathcal{F}}(D)$ denote the number of those $i = 1, 2, ...$ for which the subspace $D_i/D_{i-1} \subset L_i/L_{i-1}$ is *not* Γ-invariant. Clearly, this boundary minorizes the Σ-boundary $\partial_{\Sigma H}(D)$ for an arbitrary generating subset H in Γ.

Then we define

$$\mathcal{F}\partial_\Gamma(D) = \sup_{\mathcal{F}} \partial_{\mathcal{F}}(D)$$

for all Γ-invariant filtrations $\mathcal{F}$ in L and, given groups Γ_i acting on L, one defines the "filtered" isoperimetric profile associated to the boundaries $|\mathcal{F}\partial_{\Gamma_i}(D)|$ and

the corresponding multivariable Følner function. (The corresponding boundary in the combinatorial framework of Γ acting on X counts the number of the Γ-orbits meeting a $Y \subset X$ that are not contained in Y.)

Let the group Γ be amenable and $\mathcal{H}$ be an exhaustion by an increasing sequence of (Følner) subsets $H_j \subset \Gamma$ with $|\partial H_j|/|H_j| \to \infty$. If two finite dimensional subspaces $D_1, D_2 \subset L$, have equal Γ spans $\Gamma * D_{1,2} \subset L$, then their H_j-spans $H_j * D_{1,2}$ have the same asymptotic growth,

$$|H_j * D_1|/|H_j * D_2| \to 1$$

for $j \to \infty$.

On the other hand, if $\Gamma * D$ admits a Γ-invariant filtration L_i where L_i/L_{i-1} is Γ-generated by a k_i-dimensional subspace $\underline{D}_i \subset L_i/L_{i-1}$ for all $i = 1, 2, ...$, then

$$\limsup_{j\to\infty} |H_j * D|/|H_j| \leq \sum_i k_i.$$

Furthermore, if the group Γ is infinite, while $|D| < \infty$, then the filtration $\mathcal{F}$ obtained by taking the Γ-span of a filtration $0 = D_0 \subset D_1 \subset D_2 \subset ... \subset D$ with $|D_i/D_{i-1}| = 1$, satisfies

$$|\mathcal{F}\partial_\Gamma(D)| \geq \limsup_{j\to\infty} |H_j * D|/|H_j|.$$

All this motivates the definition of the *asymptotic rank (dimension)*

$$|D : \Gamma|_{\mathcal{H}} = |D : \Gamma|_{\{H_j\}} = \limsup_{j\to\infty} |D_j|/|H_j|,$$

where $|D| = rank(D)$ and $|H_j| = card(H_j)$. Then one defines

$$|D : \Gamma| = \sup_{\mathcal{H}} |D : \Gamma|_{\mathcal{H}}$$

and $P_*(n_i; \Gamma_i) = P_*^{sup}(n_i; \Gamma_i)$ with $\Gamma_i \subset \Gamma$ as in the case of $P_\circ$, but now with $|D : \Gamma_i|$ instead of $|Y/\Gamma_i|$.

Remark. Instead of "sup lim sup" one may use other "generalized limits" that give non-ambiguous definitions , see [6], [7], [14]. For example, one may consider $\liminf_{|H|\to\infty} |H * D|/|H|$ over all finite subsets $H \subset \Gamma$, where this leads to a meaningful (albeit not apparently useful) definition of $|D : \Gamma|$ for *all* not necessarily amenable groups Γ.

If $H \subset \Gamma$ is connected in the G-Cayley graph Γ for a generating set $G \subset \Gamma$, then $|H * D| \leq |\partial|_G^{max}(D)| \cdot |H|$; therefore,

$$P_*(n_i; \Gamma_i) \leq F_*(n_i; \Gamma_i, G_i)$$

for infinite Γ_i and arbitrary finite generating subsets $G_i \subset \Gamma_i$.

Examples. (a) If $\Gamma = \mathbb{Z}^k$, then (obviously) $|D : \Gamma|$ equals the dimension of the linear span of D over the field of rational functions in k variables that is the field of fractions of the group ring of $\mathbb{Z}^k$.

(b) If the action of Γ on L is *free* i.e. no non-trivial finite linear combination $\sum_\gamma c_\gamma(a)$ vanishes, e.g. if L equals A, the group algebra of Γ over some field, where A has no divisors of zero, then, obviously, $|D_i| \geq |H_i|$ and thus $|D : \Gamma| \geq 1$ for all non-zero $D \subset A$.

l_2-rank and $P_*^{l_2}$. If L equals the Hilbert space of l_2-functions on an X with a free action of Γ, then an alternative to the asymptotic rank is furnished by the *von Neumann l_2-dimension* $|D:\Gamma|_{l_2}$ defined for all closed Γ-invariant subspaces; if $D \subset L$ is not Γ-invariant, the notion of l_2-dimension applies to the Γ-span $\Gamma * D$ of D that is the minimal closed Γ-invariant subspace containing D,

$$|D:\Gamma| = |D:\Gamma|_{l_2} =_{def} |\Gamma * D:\Gamma|_{l_2}.$$

Clearly, $|D:\Gamma|_{l_2} \leq |D:\Gamma|_{\{H_j\}}$, whenever the two ranks are simultaneously defined. Furthermore, if D has a finite support in X, then (it is also well known and obvious) $|D:\Gamma|_{l_2} = |D:\Gamma|_{\{H_j\}}$ for an arbitrary Følner exhaustion H_j. (Thus, one does not have to worry here about the existence of the limit in the definition of $|D:\Gamma_i|$)

If one tries to straightforwardly define the $P_* = P_*^{l_2}$-function with $|D:\Gamma|_{l_2}$, one gets it identically zero already for the coordinate subgroups $\Gamma_i = \mathbb{Z}$, $i = 1, 2, ..., k$, acting on $\Gamma = \mathbb{Z}^k$ but, whatever the definition of $P_*^{l_2}$, it depends on the Hilbert structure of L, while the above P_* needs only the linear algebraic data and $P_*^{l_2} \leq P_*^{inf}$ whenever the two P-functions are simultaneously defined. Furthermore, if the space D of functions on X in the definition of $P_*^{l_2}$ has finite support, then $P_*^{l_2} = P_*^{inf} = P_*^{sup}$, where "$inf$" and "$sup$" refer to the "inf lim inf" and "sup lim sup" definitions of the asymptotic rank.

Comparison between the Graph Metrics and the Lattices of Subsets and of Linear Subspaces. The lattice of the finite subsets Y in a set X is fully determined by the *marked* edge graph of the X-cube $Z = 2^X$ of all finite subsets in X, where the vertices corresponds to the finite subsets $Y \subset X$ and where one marks the vertex corresponding to the empty set. For example the distance between two subsets equals one half of the minimal edge length of the paths between the corresponding vertices.

Similarly, the lattice of finite dimensional linear subspaces D in a linear space L is determined by the 1-skeleton of the corresponding spherical building that is the graph, where the vertices correspond to the finite dimensional linear subspaces $D \subset X$, where the two vertices D_1 and D_2 are joined by an edge if and only if one of the two subspaces contains the other with codimension 1, and where one marks the vertex corresponding to the 0-space.

The evaluation of $P_*(n_i)$ as well as of the isoperimetric profile $I_*(r;\Gamma, G)$ may be more difficult than that of their combinatorial $\circ$-counterparts (essentially) because the lattice of linear subspaces is not distributive: the intersection of a D with $span(D_1, D_2)$ can be strictly greater then the span of the intersections $D \cap D_1$ and $D \cap D_2$ (while $Y \cap (Y_1 \cup Y_2) = (Y \cap Y_1) \cup (Y \cap Y_2)$ for all subsets of an X).

1.12 Summary of Results.

1. We shall prove in 2.2, generalizing Elek's l_2-theorem, that

the action of every non-amenable group Γ on the space $l_p(\Gamma)$ of p-summable real functions is I_*-linear for all $1 \leq p < \infty$.

We define in 2.3 a class of groups Γ that includes the free groups F_k for $k \geq 2$ and many (but not all) non-amenable groups that contain no F_2, such that their group algebras are *$\mathcal{F}$-non-amenable (or $I_{\mathcal{F}}$-linear)* in the following sense.

There exists a finite family of subgroups $\Gamma_i \subset \Gamma$ in each group Γ from this class, such that every linear subspace D in the group algebra of Γ over every field $\mathbb{F}$ has

$$max_i|\mathcal{F}\partial_{\Gamma_i}(D)| \geq \varepsilon|D|$$

for some $\varepsilon = \varepsilon(\Gamma) > 0$.

2. We show in 3.2 (elaborating upon a remark by Dima Grigorev who pointed out to me how the polynomial Brunn-Minkowski inequality reduces to the combinatorial one with an order on the set of monomials) that

If Γ admits a left invariant order, then the linear algebraic profile equals the combinatorial one,

$$I_*(r;\Gamma) = I_\circ(r;\Gamma)$$

and prove some related results.

3. We combine in 6.2 the Coulon-Saloff-Coste argument with the l_2-Loomis-Whitney inequality and prove, for example, that

the Følner function $F_*(n)$ of the Hilbert space $l_2(\Gamma)$ of square summable functions on every polycyclic non-virtually nilpotent group Γ has exponential growth. (For the spaces of functions with finite supports this follows directly from the Coulhon-Saloff-Coste-inequality and **2**).

4. We shall apply the entropic formalism (that mimics the martingale method in isoperimetry) to the group algebras of normal extensions and thus, prove, for example, that

Grigorchuk groups (that may have subexponential growth $G_\circ \prec exp(n^\alpha)$ for $0 < \alpha < 1$ and may be pure torsion) satisfy

$$F_*(n) \succ exp(n^\beta) \text{ for some } \beta > 0,$$

(see 7.1,7.2).

Also we prove in 7.2 that

the (twice or more) iterated wreath products Γ of infinite groups, e.g. of (pure torsion amenable) Aleshin-Grigorchuk groups, have

$$F_*(n) \succ exp(exp(n)).$$

5. We shall exhibit in 8.1, 8.2 a class of amenable groups Γ, where

the combinatorial Følner functions $F_\circ(n;\Gamma)$ may grow arbitrarily fast

(these Γ are extensions of locally finite groups as in the last remark in section 3 of [9]) and, yet,

all these Γ have bounded linear algebraic profiles,

$$I_*(r;\Gamma) \leq const = const(\Gamma).$$

On the other hand we produce

orderable amenable groups Γ where both, combinatorial and linear algebraic Følner functions, grow arbitrarily fast.

(These Γ are extensions of locally nilpotent groups, rather than of locally finite groups that were suggested in [9]. Possibly, one can also render orderable

the construction of groups with *subexponential growth* from [10] and make not only $F_\circ$, as it is done in [10], but also F_* grow arbitrarily fast.)

6. We present in 9.1-9.3 a translation of the combinatorial argument from [9] to the language of partitions that improves the lower bounds from [9] on Følner functions of the *iterated wreath products* and related classes of groups.

7. We shall prove in 10.1-10.3 some Brunn-Minkovski type inequalities. for discrete groups and their group algebras.

Our presentation is self contained and a significant part of the paper is expository.

Acknowledgment. What follows is an outgrowth of a section in my (still unfinished) paper [16] turned into a separate article by a suggestion by Slava Grigorchuk. A part of this paper was written during my visit to the mathematics department of the Northwestern University in a stimulating and hospitable atmosphere. Also I want to thank Tullio Ceccherini-Silberstein, Christophe Pittet and the anonymous referees who generously spent their time in reading the paper, pointed out a variety of errors and made many useful suggestions.

2 Non-Amenability of Groups and Strengthened non-Amenability of Group Algebras.

2.1 l_2-Non-Amenability of the Complex Group Algebras of non-Amenable groups.

Elek non-Amenability Theorem. *If Γ is non-amenable, then the isoperimetric profile of the group algebra of Γ over $\mathbb{F} = \mathbb{C}$, and, hence over any field of characteristic zero, is linear:*

$$rank((G * D)) - rank(D) \geq \varepsilon \cdot rank(D)$$

. for all $G \subset \Gamma$ generating Γ, all finite dimensional linear subspaces $D \subset A$ and some constant $\varepsilon = \varepsilon(\Gamma, G) > 0$.

This is shown in [6] in the following stronger form.

l_2-Theorem. *Let Γ be a non-amenable group and $G \subset \Gamma$ be a finite generating set. Then there exists a positive constant ε such that for every N-dimensional linear subspace D in the complex Hilbert space $l_2(\Gamma)$ there exists $\gamma \in G$ such that $dim(\gamma(D) \cap D) \leq (1-\varepsilon)dim(D)$, where $\gamma(D)$ refers to the left action of Γ on $l_2(\Gamma)$.*

Proof. If Γ is non-amenable, then, according to one of the definitions of (non)amenability,

there is no almost invariant probability measure on Γ: every real positive l_1-function f on Γ satisfies

$$max_{\gamma \in G} \parallel \gamma(f) - f \parallel_{l_1} \geq \parallel \varepsilon_0 f(\gamma) \parallel_{l_1}$$

for some positive ε_0 depending on Γ and G .

Denote by $e_\gamma \in l_2 = l_2(\Gamma)$ for $\gamma \in \Gamma$ the (delta) function that equals 1 at γ and vanishes at all other points in Γ and let $f_D = f_D(\gamma)$ denote the square of the l_2-norm of the orthogonal projection of e_γ to D regarded as a function on Γ.

Since $e_\gamma \in l_2 = l_2(\Gamma)$ for all $\gamma \in \Gamma$ make an orthonormal basis in l_2, one has $\| f_D \|_{l_1} = dim(D)$ for all finite dimensional $D \in l_2$ and $\| f_D - f_D \|_{l_1} \leq dim((D+D')/(D \cap D'))$. In fact, if a subspace $D_+ \subset l_2$ splits into the orthogonal sum of D_1 and D_2 then $f_{D_+} = f_{D_1} + f_{D_2}$ by Pythagorean theorem.

Therefore, $\| \gamma(f_D) - f_D \|_{l_1} \leq 2dim((D + \gamma(D))/D))$ for all $\gamma \in \Gamma$ and the proof follows.

2.2 Geometric Proof that Applies to $l_p(\Gamma)$.

Given a finite dimensional Banach space D we denote by μ_D the normalized measure on the unit sphere S_D of D induced from the Lebesgue measure on D: the measure of a subset $U \subset S_D$ equals the measure of the cone over U in the unit ball in D divided by the Lebesgue measure of this ball.

Given two finite dimensional subspaces in a Banach space $(L, ||...||)$, say $D \subset L$ and $D' \subset L$, we denote by μ and μ' the push-forwards of the measures μ_D and μ_D to the unit sphere $S_L \supset S_D, S_D$ and consider all Borel maps ξ from $S_D \subset S_L$ to $S_D \subset S_L$ sending μ to μ'.

Define the distance between the two measures by

$$\delta(\mu, \mu') = inf_\xi \int_{S_D} ||s - \xi(s)||d\mu(s).$$

Say that L is *asymptotically M(easure)D(imension)-consistent* if for every two sequences of finite dimensional subspaces with dimensions converging to ∞ for $i \to \infty$, say $E_i \subset L$ and $E_i' \subset E$, the convergence $dim(E_i)/dim(E_i \cap E_i') \to 1$ for $i \to \infty$, implies the convergence $\delta(\mu_i, \mu_i') \to 0$.

This consistency is classically known for l_2 (by the Maxwell distribution law for the ideal gas) and equally obvious for all uniformly convex Banach spaces, such as l_p for $1 < p < \infty$, that these spaces are MD-consistent. (See [18], for instance).

If an isometric group action on an asymptotically MD-consistent space L has an almost fixed point on the Grassmannian $\mathcal{G}^*(L)$ with the normalized metric $|... - ...|$ (see 1.5) then there exists an almost invariant probability measure on the unit sphere S in L for the metric δ.

Another relevant property of $l_p(\Gamma)$, is the existence of a Γ-equivariant Lipschitz map (playing the role of the domination map in 3.1,3.2) from the unit sphere $S_{l_p} \subset l_2(\Gamma)$ to the space $S_1^+ \subset l_1$ of the probability measures on Γ: each l_p-function $f(\gamma)$ goes to $f^p(\gamma)$, where this map (a radial projection in the logarithmic coordinates, compare (c) in 5.4) is invariant under *all* permutations of the basic vectors.

Thus every almost invariant measure μ on S_{l_p} gives us such a measure μ_* on the positive "quadrant" $S_{l_1}^+ \subset S_{l_1}$, that is the space of probability measures on Γ. Since $S_{l_1}^+$ is a closed *convex* subset in $l_1(\Gamma) \supset S_{l_1}$, each measure on $S_{l_1}^+$ has a center of mass and this center, say $\mu_* \in S_{l_1}^+$, is a probability measure on Γ itself. Clearly, μ_* is almost invariant for the l_1-metric whenever the measure is μ almost invariant for δ . QED.

Remark. The first (Pythagorean) proof of Elek's theorem, shows that there is no Hilbert-Schmidt operator on $l_2(\Gamma)$ that almost commutes with the action of Γ, while the geometric argument establishes the "no almost fixed point"

property on the Grassmannian of finite dimensional linear subspaces with the metric induced by the (measure transportation) *Monge-Kantorovich metric*.

Questions. Let X be a countable set, L some linear space of $\mathbb{F}$-valued functions on X invariant under the group Γ of transformations of X (e.g. the group of *all* bijective transformations) and let Gr_d be the space of all d dimensional subspaces $D \subset L$ with the metric $|D_1 - D_2| =_{def} \frac{1}{d}|D_1/(D_1 \cap D_2|)$.

What is the infimum of the Lipschitz constants of the Γ-equivariant maps from Gr_d to the space $\mathcal{P}(X)$ of the probability measures on X for specific "natural" spaces L?

What are the Lipschitz constants for equivariant maps between the Grassmannians Gr_d associated to different spaces L (with, possibly, different $\mathbb{F}$) of functions on X?

Such maps to $\mathcal{P}(X)$ play the key role in the proof of the l_*-linearity theorem by Bartholdi; we shall present a version of his construction in 3.6.

Two Comments Made by an Anonymous Referee. (1) It might be interesting to (further, M.G.) geometrize the study of linear isoperimetric profiles á la Voisculescu, so that one considers not the linear span of translates of a finite dimensional subspace but rather the smallest dimension of a subspace approximately containing all of these translates granted that the ambient space is equipeed with a norm. This would seem to tie in with Bekka's notion of amenability for unitary representations and it would provide a useful analytic tool for the study of isoperimetry in spaces other than l_2 and group algebras while at the same time directly connecting to the linear theory in these special cases, which are the primary objects of study in this paper.

(2) The approximate invariance of a Følner set can be expressed in a dual way via approximately even coverings with a certain multiplicity. This leads to a recursive extraction procedure to the quasitiling theorem of Ornstain and Weiss according to which the Følner set can be approximately tiled by translates of finitely many tiles at different scales. Might the type of isoperometry technology elaborated upon in the paper be used, via some duality principle, to study the problem of the number of different scales required to quasitile Følner sets in variuos groups (about which little seems to be known)

2.3 $\mathcal{F}$- Non-Amenability of the Group Algebras of Groups with Acyclic Families of Subgroups.

(A) Topological Argument for Non-amenability of F_k. The free Group $\Gamma = F_k$ on $k \geq 2$ generators freely acts on the infinite regular $2k$-valent tree T_k. Every finite subtree with the vertex set $Y \subset X$ consisting of r vertices y_i, $i = 1, 2, ..., r$, of degrees d_i satisfies, by the Euler-Poincare formula,

$$r - \frac{1}{2}\sum_i d_i = 1.$$

Hence,

$$|\partial^{joint}(Y)| = \sum_i (2k - d_i) = 2rk - 2r + 2$$

and

$$l_\circ(r;\Gamma) = r(2k-2) + 2.$$

It follows that F_k for $k \geq 2$ is non-amenable and, by the extension property, that the groups containing an F_2 as a subgroup are non-amenable.

We shall return to the topological aspects of isoperimetry in [16] but now let us indicate another approach for estimating $I_\circ$ that applies, for example, to Burnside groups.

(B) Slicing Argument. Look at how the orbits (cosets) of the k generating cyclic subgroups Γ_i in $\Gamma = F_k$ meet and "slice" a finite subset $Y_0 \subset \Gamma$, say for $\Gamma = F_2$.

Since every finite (sub)tree contains at least one leaf (end-vertex) y, there exists, a subgroup, among the generating two, such that some of its orbits (cosets) in Γ meets Y_0 only at y. (In general, there are $k-1$ of such subgroups at every leaf) . We remove from Y the intersection $S_1 = S_1(Y_0)$ of such an orbit with Y_0 , then remove a similar $S_2 = S_2(Y_1)$ from $Y_1 = Y_0 \setminus S_1$, etc., until we exhaust all of Y_0. Thus we obtain $|Y_0|$ orbits non-trivially meeting Y_0; therefore there are at least $|Y_0|/2$ orbits of one of these subgroups, say of $\Lambda \subset \Gamma$, that meet Y_0 and

$$|\partial_\Lambda(Y)| \geq \frac{1}{2}|Y|;$$

thus the max-boundary of Y in Γ is also bounded from below by $\frac{1}{2}|Y|$.

(More generally, if $k \geq 2$, then $|\partial|^{max}(Y) \geq \frac{k-1}{k}|Y|$ by the slicing argument.)

This may seem no better than the above bound for $|\partial^{joint}(Y)|$ but the slicing argument itself carries more power. For example, it implies

(C) $\mathcal{F}$ and P_*-non-Amenabilty of the Group Algebra of $\Gamma = F_2$.

Let D_0 be a linear space of functions on Γ with finite support $Y_0 = supp(D_0)$. Then the restriction homomorphism h_1 from D_0 to (the space of linear functions on) the above slice $S_0 \subset Y_0$ clearly has rank 1.

Let $D_1 = ker(h_1) \subset D_0$ and let h_2 be a similar homomorphism from D_1 to such a slice $S_2 \subset Y_1 = supp(D_1)$, etc. Thus we obtain, for one of these two cyclic subgroups, say for $\Lambda \subset \Gamma$,

an increasing chain of Λ-invariant subsets, $Z_0 \subset Z_1 \subset ... \subset Z_n$ for $n \geq |D_0|/2$, such that the kernels $D'_i \subset D_0$ of the restriction homomorphisms from D_0 to (the spaces of functions on) Z'_i strictly(!) decrease with i.

It follows that the filtered boundary and the asymptotic rank (see 1.11) satisfy

$$|D : \Lambda| \geq |\mathcal{F}\partial_\Lambda(D)| \geq n \geq |D_0|/2.$$

Indeed, take a splitting $D_0 = \oplus_i \Delta_i$ compatible with the filtration D'_i of D_0 and observe that the spans $span(\Lambda(\Delta_i))$ are linearly independent, since $supp(\Lambda(D)) = \Lambda(supp(D))$ for all linear spaces D of functions on Γ.

This implies the required linear lower bound on $|\mathcal{F}\partial_\Lambda(D)|$ and thus, such a bound on the asymptotic rank $|D : \Lambda|$ that is the dimension of the span of D over the field of fractions of the (commutative!) group algebra of Λ.

More generally, one has

(D) $\mathcal{F}$-non-Amenability in the Presence of Divergent Families of Subgroups. *Let X be a Γ-space and let $\Gamma_i \subset \Gamma$ $i = 1, 2, ..., k < \infty$ be a 1-divergent family of subgroups, which means, by definition, that for every finite subset $Y \subset X$ there is an orbit $S \subset X$ with $|S| > k$ of some Γ_i which satisfies*

$$|S \cap Y| = 1.$$

Then the action of Γ *on the space of* $\mathbb{F}$*-valued functions on* X *with finite supports is* $l_{\mathcal{F}}$*-linear for all fields* $\mathbb{F}$*.*

Proof. There is an i such that by successively making the function from D vanish on Γ_i-orbits S_m we obtain a strictly descending chain $D_m \subset D$ of length $\geq |D|/k$, where there are at least $|D|/k^2$ subspaces D_m with $|D_{m-1}/D_m| < |S_m|$. Since every D_{m-1} contains a function supported on a single point in S_m, the corresponding quotients $|D_{m-1}/D_m|$ are not Γ_i-invariant.

(E) Corollary: Acyclicity $\Rightarrow$ $l_{\mathcal{F}}$-linearity. *Let* Γ *admit an infinite sequence of* Γ_j*,* $j = 1, 2, ...,$*, with cardinalities* $|\Gamma_j| > k$*, where* k *equals the number of di erent subgroups among* Γ_j*, such that*

$$\gamma_{j_1} \cdot \gamma_{j_1+1} \cdot ... \cdot \gamma_{j_2-1} \cdot \gamma_{j_2} \neq id$$

for $id \neq \gamma_j \in \Gamma_j$ *and all* $j_1 \leq j_2$*. (Such sequences are called acyclic.) Then the group algebra of* Γ *over any field is* $l_{\mathcal{F}}$*-linear.*

Proof. If a subset $Y \subset \Gamma$ has $|Y \cap S| \geq 2$ for all orbits (cosets) of all Γ_j that meet Y, then Y contains *infinitely many* γ of the form $\gamma_0, \gamma_{j_0} \cdot \gamma_{j_1}, \gamma_{j_0} \cdot \gamma_{j_1} \cdot \gamma_{j_1+1}, ...$ for $\gamma_0 \in Y$ and $\gamma_j \in \Gamma_j$ for $j \geq 1$.

Examples. (a) If Γ equals a free product, $\Gamma = \Gamma_1 * \Gamma_2$, then the sequence $\Gamma_1, \Gamma_2, \Gamma_1, \Gamma_2, ...$ is acyclic and the above applies to infinite Γ_1 and Γ_2. (This argument yields the non-amenabilty of Γ itself for $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$ as everybody knows).

(b). Consider a *square free* sequence $i_j \in I = \{1, 2, ..., k \geq 3\}$, i.e. where there are no subsequences $i_j, i_{j+1}, ..., i_{j+2l}$, where $i_{j+s} = i_{j+l+s}$ for $s = 0, 1, ...l$. Then take a free product $\Gamma_0 = \Gamma_1 * \Gamma_2 * ... * \Gamma_k$, enumerate all $\gamma \in \Gamma$ by $m = 1, 2, ...$, and add to Γ torsion relations $\gamma_m^{n_m} = id$ one after another, omitting those where the corresponding γ_m was already torsion in the preceding factor group.

The hyperbolic small cancellation tells us that the groups Γ_i inject into the resulting factor group Γ for torsionless Γ_i and fast growing n_m and that the sequence of subgroups Γ_{i_j} is acyclic in the above sense for square free sequences i_j.

Moreover, (this was explained to me by Azer Akhmedov, Ashot Minasyan and Dima Sonkin) this property is known for *free Burnside* p-torsion groups Γ_i and all $n_m = p$ for large enough p.

The Burnside groups $B(p, 2)$ on 2 generators contain $B(p, k)$ for all $k = 6$; thus, they contain the p-torsion quotients of $B(p, 2) * B(p, 2) * B(p, 2)$; therefore,

(F) *The groups* $B(p, k)$ *are* $l_{\mathcal{F}}$*-linear for all su ciently large* p *and* $k \geq 2$.

Remarks. (a) Random quotients Γ of all *resilient* (see [15]), e.g. of Burnside, groups are not acyclic in the above sense (where some of them are "Olshanskiy monsters" with all their proper subgroups isomorphic to $\mathbb{Z}_p$, as was pointed out to me by Dima Sonkin, but being a monster does not by itself exclude acyclicity) and it is unclear which of them possess some "strengthened non-amenability" controlled by the combinatorics of orbit partitions and/or of invariant filtrations in the linear algebraic setting. (The ordinary non-amenability of such groups follows by the Grigorchuk criterion and also can be derived from Kazhdan's T-property in some cases.)

(b) We emphasized the $I_{\mathcal{F}}$-linearity(= $\mathcal{F}$-non-amenability) rather than the (weaker property of) I_*-linearity(=linear algebraic non-amenability), since the latter is valid by the Elek-Bartholdi theorem for *all* non-amenable groups Γ.

3 Isoperimetry in the Group Algebras of Ordered Groups.

We show in this section that the isoperimetric inequalities in the *(left) orderable* groups Γ pass from subsets in Γ to linear subspaces in the group algebra of Γ over any field.

3.1 Triangular Systems and Bases.

Let $X = (X, \preceq)$ be an ordered set, e.g. $X = \{1, 2, ..., n\}$, take a function f on X with values in some field (any set with a distinguished element called 0 will do for the following definition) and denote by $x_f^{min} \in X$ the minimal $x \in X$ where $f(x) \neq 0$, where we assume, to guarantee the existence of such x_f^{min}, that the function f has a finite support $S = S_f \subset X$ or, at least S_f is bounded from below, i.e. S_f contains at most finitely many elements that are $\preceq x_0$ for every $x_0 \in X$. We also agree that $x_{f=0}^{min}$ equals the empty set.

Given a linear space D of functions on X with the supports uniformly bounded from below, assign to it the union of the points x_f^{min} for all $f \in D$, denoted

$$D \mapsto Y_D^{min} = \bigcup_{f \in D} \{x_f^{min}\} \subset X$$

and observe that

the cardinality of Y_D^{min} equals the rank of D,

$$|Y_D^{min}| = |D|.$$

Example. If $X = \{1, 2, ..., n\}$ then Y_D^{min} equals the set of the non-zero diagonal entries in a triangular matrix with n columns and $|D|$ rows where the rows make a basis in D.

3.2 Domination by Ordering and Derivation of the Linear Algebraic Isoperimetric Inequality from the Combinatorial One.

Isoperimetric Domination. Say that the isoperimetry of a linear space L acted upon by a group Γ *dominates* the isoperimetry of some set X (non-linearly) acted by Γ, if there exits a Γ-equivariant (domination) map from the set $\mathcal{G}^*$ of finite dimensional subspaces $D \subset L$ to the set 2^X of subsets $Y \subset X$, say $D \mapsto Y_D$, such that

$|Y_D| = |D|$ for all D

and

$$(* \succ \circ) \qquad |span(D_i)| \geq |\cup_i Y_{D_i}|$$

for all finite collections of subspaces D_i.

Clearly,

the map $D \mapsto Y_D^{min}$ based on a Γ-invariant order in X is an isoperimetric domination.

It is also clear that

the existence of an isoperimetric domination makes $I_*(r; L, \Gamma) \geq I_\circ(r; X, \Gamma)$.

This leads to the following

Equivalence $\circ \Leftrightarrow *$ with Order. *Let an action of Γ on X preserve some order on X. Then the linear algebraic isoperimetric profile I_* of (the space of functions with finite supports on) X with respect to an arbitrary subset $G \subset \Gamma$ majorizes the combinatorial profile $I_\circ$ of X, and since, obviously, $I_* \leq I_\circ$, the two profiles are equal,*

$$(\circ \Leftrightarrow *) \qquad I_*(r; X, \Gamma) = I_\circ(r; X, \Gamma).$$

Futhermore, the P-functions of X with respect to (the family of the orbit partitions of) arbitrary groups Γ_i acting on X satisfy

$$P_*(n_i) = P_\circ(n_i).$$

Remark.(a) Every domination map $\mathcal{G}^* \to 2^X$ is *distance decreasing* for the Γ-invariant metrics in $\mathcal{G}^*$ and in 2^X defined in 1.1, 1.4 and 1.5; this serves as good as the inequality $|span(D_i)| \geq |\cup_i Y_{D_i}|$ if we work with the I^{max}-profiles, (compare I_p-non amenability in 2.2), while the profiles defined with the joint boundary (see 1.1) need more of the (lattice) structure in the spaces $\mathcal{G}^*$ and in 2^X.

3.3 Examples of Orderable Groups and Corollaries to The Equivalence $\circ \Leftrightarrow *$.

Left orderable groups constitute an ample class of groups starting with $\Gamma = \mathbb{Z}$. They are built with the following constructions.

(1) *Extensions*. An extension Γ of a left orderable group Γ_0 by a left orderable Γ_1 is left orderable by the lexicographic ordering construction. In particular

torsionless polycyclic groups are orderable

and thus their isoperimetry extends to their group algebras.

(2) *Limits and Subgroups*. Inductive and projective limits of orderable groups are, obviously, orderable and subgroups of orderable groups are orderable. It follows that

If every non-trivial finitely generated subgroup in Γ admits a surjective homomorphism to $\mathbb{Z}$ then Γ is left orderable.

In particular, the following groups are orderable,

(a) free groups F_k, (b) wreath products Γ of orderable groups, (c) $\mathbb{Z}$-extension of torsion free locally nilpotent groups by $\mathbb{Z}$.

Thus the Equivalence $\circ \Leftrightarrow *$ with order gives us yet another proof of the Elek theorem on non-amenability of the group algebras of F_k for $k \geq 2$, and shows together with 8.2 that

the group algebra over a given field of a finitely generated amenable group may have arbitrarily fast growing Følner function.

(3) Finite groups, and hence, groups with non-trivial torsion, admit no left invariant order. In fact, if a linear space D consists of functions on Γ that are constant on the cosets of a finite subgroup $\Gamma_0 \subset \Gamma$ and $G \supset \Gamma_0$, then, obviously, $I_*(|D|) < I_\circ(|D|)$.

(4) Amenable groups are orderable if and only if all their non-trivial (i.e. $\neq \{id\}$) infinite subgroups admit epimorphisms to $\mathbb{Z}$ [22]. The "if" part mentioned earlier is valid for all groups and is rather obvious.)

Thus, many virtually polycyclic (e.g. virtually free Abelian) groups are not orderable; Yet,

if Γ is commensurable to a left orderable group, then I_* is equivalent to $I_\circ$ for Γ.

In fact, if $\Gamma_1 \subset \Gamma$ has finite index, or if Γ admits an *epi*morphism on Γ_1 with a *finite* kernel , then the isoperimetric profiles of the two groups as well as of their group algebras are equivalent.

(5) The group $homeo_+(\mathbb{R})$ of orientation preserving homeomorphisms of the real line is left orderable and every countable left orderable group embeds into $homeo_+(\mathbb{R})$. (This is well known and easy.) For example, the universal covering of $SL_2(\mathbb{R})$ is left orderable and also

the (finitely generated) Thompson group T (dyadic homeomorphisms of an interval) is left orderable; thus, the (yet unknown) profile $I_\circ$ of T is equal to I_* of T.

3.4 I_*-Profiles of Groups with Orderable Subgroups.

Let a finite subset $G_0 \subset \Gamma$ generate an orderable group $\Gamma_0 \subset \Gamma$, where $G_0 \subset G^{\cdot m}$ for a finite generating subset $G \subset \Gamma$. Then the inequality $G^{\cdot m}$ from 1.4 combined with the equivalence $\circ \Leftrightarrow *$ shows that

$$(*) \qquad I_*^{max}(r;\Gamma,G) \geq \frac{1}{m} I_\circ^{max}(r;\Gamma_0,G_0).$$

Examples.(a) Observe that the *relative growth* of $\Gamma_0 \subset \Gamma$, denoted $G_{\circ\circ}(n) = |G^{\cdot n} \cap \Gamma_)|$ may be faster than the growth of Γ_0 with respect to any finite generating subset in Γ_0 if Γ_0 is "sufficiently distorted" in Γ. Then the following inequality, that derived from the above along with the Displacement Inequality (see 1.4), may serve better than just $F_*(n,\Gamma) \geq F_*(n,\Gamma_0)$,

$$F_*^{max}(n;\Gamma,G) \geq \frac{1}{2} G_{\circ\circ}(\frac{n}{2}).$$

However, I see no example, with a finitely generated Γ_0, where one can not do better by other means.

(b) If Γ_0 is infinitely generated, e.g, being the restricted Cartesian power of some finitely generated group Γ_1, then one can use measures μ on Γ_0 spread over large sets of "independent" elements in Γ_0. Such situation arises, for example for the *wreath products* Γ of orderable finitely generated groups Γ_1 with finitely generated (not necessarily orderable) Γ_2, where one can apply the Loomis-Whitney inequality and obtain the lower bound

$$F_*(n;\Gamma) \succ exp(G_\circ(n;\Gamma_2).$$

But, in truth, this holds for all (not necessarily orderable) groups Γ_1 (see 7.2) and if Γ_1 *is not* pure torsion, such an inequality is also valid for real l_2-functions and for decaying functions on Γ with values in ultra metric fields (see 6.2).

3.5 I_*-Profiles of Groups with Orderable Factor Groups.

Consider an exact sequence,

$$1 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 1$$

and let $G_i \subset \Gamma_i$, $i = 1, 2$, be finite symmetric generating subsets in Γ_i containing the identity elements, where G_2 also denotes some lift of G_2 to Γ.

Orderable Extension Inequalities. *Let Γ_2 be a left orderable group. Then the isoperimetric profile of the group algebra A of Γ is bounded from below by that of Γ_2,*

$$I_*(r;\Gamma, G) \geq I_*(r;\Gamma_2, G_2) = I_\circ(r;\Gamma_2, G_2)$$

for $G = G_1 \cup G_2 \subset \Gamma$.

Proof. Denote by h the homomorphism $\Gamma \to \Gamma_2$, let $A_{\preceq\gamma_2} \subset A$, where $\gamma_2 \in \Gamma_2$, consist of the functions a on Γ such that all γ in the support $supp(a) \subset \Gamma$ satisfy $h(\gamma) \preceq \gamma_2$ for a given left order on Γ_2 and define $A_{\prec\gamma_2} \subset A_{\preceq\gamma_2}$ with the strict inequality $h(\gamma) \prec \gamma_2$.

Given a linear subspace $D \subset A$, let

$$r = r(\gamma_2) =_{def} rank(D \cap A_{\preceq\gamma_2})/(D \cap A_{\prec\gamma_2})$$

and observe that

The G_2-boundary of the function r on Γ_2 (i.e. the boundary of the subgraph of r in $\Gamma_2 \times \mathbb{Z}_+$) is bounded by the G_2-boundary of D,

$$|\partial_{G_2}(r)| \leq |\partial_{G_2}(D)|;$$

and the bound on $I_*(\Gamma)$ by $I_\circ(\Gamma_2)$ follows as in 3.2.

Remark. We shall isolate in 5.11 a class of $\bullet$*-convex* functions F and prove that

$$F_*(n_1, n_2;\Gamma, G_1, G_2)) \geq F_*(n_1;\Gamma_i, G_1) \cdot F_\circ(n_2;\Gamma_2, G_2),$$

when the functions $F_*(n_1;\Gamma_i, G_1) \cdot F_\circ(n_2;\Gamma_2, G_2)$ are $\bullet$*-convex* .

3.6 Proof of Bartholdi's I_*-Linearity Theorem via Orderings.

The set $\mathcal{O}(X)$ of all orders O on a countable set X equals the projective limit of the orders on the finite subsets Y in X; thus, it carries the structure of a compact topological space. The natural action of the group Γ of all bijective transformations of X is continuous on $\mathcal{O}(X)$, where it is free and transitive for finite X. (If X is a finite set, then $\mathcal{O}(X)$ equals the set of the top dimensional simplices in the first barycentric subdivision of the regular n-simplex for $n =$

$|X|-1$.) It is non-transitive for infinite X as there are non-isomorphic orders and it is not free: the isotropy subgroup of a point O in $\mathcal{O}(X)$ consists of the transformations preserving the order O.

There is a unique Γ-invariant probability measure, say dO on $\mathcal{O}(X)$, that is the projective limit of the uniform measures on $\mathcal{O}(Y)$ for the finite subsets $Y \subset X$.

Given an order O we invoke the (triangular basis) map from the set of all finite dimensional linear spaces of $\mathbb{F}$-functions on X to the set of finite subsets subsets in X,

$$D \mapsto Y = Y_D^{min} = Y_D^{min}(O)$$

associated to this O, and, following [2], let $\mu_Y = \mu_D^O$ be the measure on X which assigns the unit weights to all points in $Y \subset X$ and which is zero outside Y. Clearly,

$$(=) \qquad |\mu_D^O| = |D|$$

and

$$(\leq) \qquad D_1 \subset D_2 \Rightarrow \mu_{D_1}^O(x) \leq \mu_{D_2}^O(x)$$

for all $x \in X$.

Then we take the average over all orders O,

$$\mu_D =_{def} \int_{\mathcal{O}} \mu_D^O dO$$

and observe that the measure μ_D on X, being an average, inherits the relations $(=)$ and $(\leq)$ from μ_D^O; furthermore, since the measure dO is Γ-invariant the map $D \mapsto \mu_D$ (unlike $D \mapsto \mu_D^O$ for a fixed order O) is Γ-equivariant.

The relations $(=)$ and $(\leq)$ immediately imply (compare [2]) that $||\mu_{D_1} - \mu_{D_2}||_{l_1} = |D_1| - |D_2|$ whenever $D_1 \subset D_2$ and, hence, the map $D \mapsto \mu_D$ is 1-Lipschitz for the graph metric (see 1.11) in the space of D's (that is equaivalent to the metric defined in 1.6) and the l_1-metric in the space of measures; then the implication non-amenability $\Rightarrow$ l_*-linearity follows as in 2.2.

Remarks. (a) The above argument is a minor modification/simplification of the original one in [2], where the author defines the set $M_D \subset l_1(X) \subset l_2(X)$ of the measures μ_Y on X associated to the subsets $Y \subset X$, such that the restriction map $D \mapsto D|Y$ is isomorphic, and then averages with the measure ds on M_D which is induced by the (multivalued) Gauss spherical (support) map of M_D to the unit sphere in $l_2(X)$.

(b) The (compact topological) Γ-spaces of orders on countable groups Γ seem to have interesting dynamics, where, in particular, the presence of "dynamically small" invariant subsets and/or measures (e.g. with vanishing Kolmogorov entropy) may yield "good isoperimetry" in the linear spaces of functions on Γ.

4 Families of Measurable Partitions and their Invariants.

A Borel measurable (not necessarily measure preserving) map between Borel measure spaces, $P : (X, \lambda) \to (X', \lambda')$ gives rise to a *partition* of X into *slices* that are the pullbacks of points in X'. If the map is surjective, we ascribe the notation P to this partition, identify X' with the quotient space X/P and denote the slice of P through $x \in X$ by $P(x) \subset X$ and using the same notation for $P(x) \in X/P$ as well as $P^{-1}(x'), x' = P(x) \in X/P$.

These slices carry (almost everywhere defined and a.e. unique) *Fubini measure*, denoted λ/λ' and also $dP(x)$, such that

$$\int_X f(x)d\lambda = \int_{X/P} d\lambda' \int_{P(x)} f(x)dP(x)$$

. for all measurable functions f on X.

Coordinate Lines and Planes Partitions. Every Cartesian product space,

$$X = ... \times X_i \times ..., i \in I,$$

comes along with an I-family ("web") of partitions, each into the i-th "coordinate lines", that are the copies of X_i passing through the points in X. (These are the true parallel coordinate lines for the Euclidean spaces $X = \mathbb{R}^{|I|} = ... \times \mathbb{R} \times ...$ with the coordinates indexed by $i \in I$.) Moreover, X supports the "web" of 2^I partitions corresponding to the "planes" indexed by subsets $J \subset I$ and representing $\times_{i \in J} X_i$.

Induced Partitions. If $Y \subset X$ is a measurable subset, then the intersections of Y with slices of a partition P of X are called *P-slices (" threads") of Y*. Thus we obtain a partition $P|Y$ of Y, where we disregard the empty intersections (or rather those of measure zero).

Orbit Partitions. If X is acted by a group Γ then it is partitioned into the orbits of the action, where this partition is measurable for (measurably) proper actions.

In most our examples we deal with countable, also called *discrete*, spaces where all partitions are measurable. If such a space X is *unitary*, i.e. all atoms (points) have unit weights, then every partition P equals the orbit partition of the group of the transformations of X preserving the slices of P.

Remark. The only significant invariant of a measurable partition is the distribution of the values of the function $\lambda(P(x))$ on X (or, equivalently, of the function $\lambda(P^{-1}(x'))$ on $X' = X/P$). In particular, all partitions of discrete unitary spaces into countably many countable slices are mutually isomorphic. On the contrary, "measurable webs" that are *families* of partitions are vastly different and, typically, very rigid.

Example. The partitions of the 3-dimensional Heisenberg group H into the orbits of the three standard generating subgroups uniquely determine the Lie groups structure in H (this is easy); probably, the same remains true for all families of orbits partitions of groups except for the obvious "exceptionally soft" cases (such as Z^k generated by k copies of $\mathbb{Z}$).

In the following sections we introduce several invariants of *families of partitions* starting with the well known picture of the isoperimetry in the free Abelian groups serving as a motivating example.

4.1 Four Proofs of the Isoperimetric Inequality for $\mathbb{Z}^k$.

$\mathbb{Z}^k$-Inequality. *The max-Følner function of the free Abelian group $\mathbb{Z}^k$ with the standard k generators equals n^k.*

The non-trivial part of the claim, the inequality

$$F_\circ^{max}(n;\mathbb{Z}^k) \geq n^k,$$

follows by induction with either (**a**), (**b**) or (**c**) below.

Lemma. *Let a countable set X be acted upon by groups Γ_1 and Γ_2 with given finite generating subsets and Γ be the group generated by Γ_1 and Γ_2.*

(a) If the Γ_1- and Γ_2-orbits are transversal, i.e. intersect at single points, and if the Γ_2-orbits are infinite,

or

(b) if the transformations from Γ_2 send Γ_1-orbit to Γ_1-orbits and the orbits of the resulting action of Γ_2 on the quotient space X/Γ_1 are all infinite, then

$$F_\circ^{max}(n;X;\Gamma) \geq n \cdot F_\circ^{max}(n;X;\Gamma_1).$$

(c) If Γ_2 sends Γ_1-orbits to Γ_1-orbits as in (b) and all Γ_1-orbits are infinite, then

$$F_\circ^{max}(n;X;\Gamma) \geq n \cdot F_\circ^{max}(n;X/\Gamma_1;\Gamma_2).$$

Proof of (a) . Take a $Y \subset X$ and look at the two partitions of it where the slices are the intersections of Y with the Γ_1- and Γ_2- orbits. If

$$|\partial_{\Gamma_1}(Y)| \leq \frac{1}{n_1}|Y|,$$

then there is a Γ_1-slice of Y, say S_1, such that

$$|\partial_{\Gamma_1}(S_1)| \leq \frac{1}{n_1}|S_1|$$

and by the definition of the Følner function for (X,Γ_1),

$$|S_1| \geq F_\circ^{max}(n_1;X,\Gamma_1).$$

Since the Γ_2-orbits are infinite, each non-empty Γ_2-slice of Y contributes at least one point to the Γ_2-boundary and the number of the Γ_2-slices that meet S_1 is $\geq |S_1|$ due to the transversality of the two partitions. If, furthermore,

$$|\partial_{\Gamma_2}(Y)| \leq \frac{1}{n_2}|Y|,$$

then

$$|Y| \geq n_2 \cdot |\partial_{\Gamma_2}(Y)| \geq n_2 \cdot |S_1| \geq n_2 \cdot F_*^{max}(n_1;X;\Gamma_1)$$

and the required bound on $F_\circ^{max}(n;\Gamma)$ follows if we take $n_1 = n_2 = n$.

Proof of (b). Consider the subsets $\underline{Y}_m \subset X/\Gamma_1$ of those Γ_1 -orbits that meet Y at $\geq m$ points, where $m = 1,2,...,|S_1|$ for the above S_1, observe that

$$|\partial_{\Gamma_2}(Y)| \geq \sum_{m=1,2,\ldots,|S_1|} |\partial_{\Gamma_2}(\underline{Y}_m)| \geq |S_1|.$$

and conclude the proof as in (**a**).

Proof of (c). Since

$$|Y| = \sum_m |\underline{Y}_m|,$$

and

$$|\partial_{\Gamma_2}(Y)| \geq \sum_m |\partial_{\Gamma_2}(\underline{Y}_m)|,$$

the cardinality of $\underline{Y}_1$ satisfies

$$|\underline{Y}_1| = max_m |\underline{Y}_m| \geq F_\circ(n_2; X/\Gamma_2, \Gamma_2).$$

Since each Γ_1-slice of Y contributes a point to the Γ_1-boundary of Y, the subset $\underline{Y}_1 \subset X/\Gamma_1$, that is the set of the Γ_1-slices of Y, is bounded by

$$|\underline{Y}_1| \leq |\partial\Gamma_1(Y)| \leq \frac{1}{n_1}|Y|$$

and the proof follows.

Multivariable Følner Function of $\mathbb{Z}^k$. Recall that the *multivariable* Følner function $F_\circ(n_i; X, G_i)$ for several sets of transformations G_i acting on X is defined as the maximal N such that every $Y \subset X$ with

$$|\partial_{G_i}(Y)| \leq \frac{1}{n_i}|Y|$$

has

$$|Y| \geq N.$$

Multivariable Isoperimetric Inequality for $\mathbb{Z}^k$.

$$F_\circ(n_1, ..., n_k; \mathbb{Z}^k, g_1, ..., g_k) = n_1 \cdot ... \cdot n_k$$

for every set of generators $g_1, ..., g_k$ in $\mathbb{Z}^k$.

Proof. Since the orbits S of all k subgroups $\Gamma_i = \mathbb{Z}$ in $\mathbb{Z}^k$ generated by g_i are infinite, every S that meets a finite set $Y \subset \mathbb{Z}^k$ contributes at least one point to its boundary; thus, $|\partial_{g_i}(Y)|$ is bounded from below by the number m_i of the Γ_i-orbits S that meet Y.

These numbers satisfy $\prod_i m_i \geq |Y|^{k-1}$ by the Loomis-Whitney inequality (see 1.10); hence, $F_\circ \geq \prod n_i$, while the opposite inequality is obvious. QED.

About F_*. It is unclear how much of (**a**),(**b**) and (**c**) remain valid for linear actions. Yet, we shall see in 6.2 that (**b**) holds for Γ acting on the space of functions on X with finite supports if, for example, there is an element of infinite order in Γ_2 freely acting on X/Γ_1. Also we prove in 7.2 a (coarse entropic) version of (**b**) for F_* in the general case.

4.2 Følner Transform $F_\odot$ and Schwartz Symmetrization.

Given a family $\mathcal{P}$ of measurable partitions P_i, define the *Følner transform (composition)* $F_\odot : (F_i) \mapsto F = F(n_i)$, where $F_i = F_i(n)$, $n \in [0, \infty)$, are monotone increasing positive functions, as follows. Take $I_i = I_{F_i}$, for $I_{F_i}(r) = rJ_i(r)$,

where $J_i(r)$ stands for the inverse function of $F_i(n)$ (i.e. $F = F_I$ for this I compare 1.9), take the integral over Y/P_i of the values of I_i at the Fubini measures of all P_i-slices S of Y and denote

$$|\partial|_{I_i}(Y) = \int_{Y/P_i} I_i(|S|).$$

Finally, let $F(n_i) = F_{\odot}(F_i(n_i))$ be the maximal N, such that every Y with $|\partial|_{I_i}(Y) \leq \frac{1}{n_i}|Y|$ has $|Y| \geq N$.

Example. If a (discrete or locally compact) group Γ is partitioned into the orbits of (closed) subgroups $\Gamma_i \subset \Gamma$, then the multivariable Følner function of Γ with respect to given generating subsets $G_i \subset \Gamma_i$, obviously, satisfies (with the obvious extention of the notations to the locally compact non-discrete case).

$$F_{\circ}(n_i; \Gamma, G_i) \geq F_{\odot}(F_i(n_i; \Gamma_i, G_i)).$$

$P_{\circ}$-Functions and the Følner Transform . The $P_{\circ}$-function defined in 1.10 clearly, equals the Følner transform of the functions $F(n_i) = n_i$.

Schwartz Symmetrization. If $X = X_1 \times X_2$ with the two coordinate partitions then the corresponding Følner transform (composition) is denoted

$$F_1 \bullet F_2 =_{def} F_{\odot}(F_1, F_2; X_1 \times X_2).$$

It is obvious that $F_1 \bullet F_2 \leq F_1 \cdot F_2$.

Let us model the general product space by $X = \mathbb{R}^2_+$, where we consider closed *monotone* subsets $Y \subset X$ which means that the intersections of Y with the vertical lines, $x_1 \times \mathbb{R}_+$ are segments $x_1 \times [0, x_2 = f_1(x_1)]$ and the intersections with the horizontal ones, $\mathbb{R}_+ \times x_2$, are segments $[0, x_1 = f_2(x_1)] \times x_2$, where f_1 and f_2 are mutually inverse monotone decreasing functions with the common graph serving as the boundary of Y.

Let $I_i(r) = I_{F_i}(r)$, for $i = 1, 2$, and $0 \leq r < \infty$, be "the isoperimetric profiles" corresponding to F_i, set

$$\mathcal{I}_i(Y) = \int_0^{\infty} I_i(f_i(r))dr$$

and define $F(n_1, n_2)$ as the infimum of the measures μ of monotone subsets $Y \subset \mathbb{R}^2_+$ that satisfy

$$\mathcal{I}_i(Y)/\mu(Y) \leq n_i.$$

It is easy to see (we shall not use it in sequel) that

if I_i are positive concave monotone increasing functions, then this F equals the Følner transform of the functions F_i corresponding to I_i,

$$F = F_1 \bullet F_2.$$

For example, if the extremal $Y = Y(n_1, n_2)$ are rectangular for all $n_1, n_2, \geq 0$, (i.e. $Y = [0, x_1] \times [0, x_2] \subset \mathbb{R}_+$) then the the Følner functions F_i corresponding to I_i satisfy $F_1 \bullet F_2 = F_1 \cdot F_2$.

In general, the extremal Y are bounded by broken lines consisting of ***finitely many*** vertical and horizontal segments under mild regularity/genericity assumptions on I_i and thus the problem of evaluation $F = F_1 \bullet F_2 = F_1 \cdot F_2$ reduces to a finite dimensional variational problem.

Similar considerations also apply to the Cartesian products of more than two spaces. but this does yield any explicit formula for the Følner transform, nor does this deliver a transparent criterion for the equality $F_1 \bullet F_2 = F_1 \cdot F_2$.

On the brighter side, we shall prove in 5.11 this equality, under the assumption of convexity of $logF$ ($\frac{1}{}$

$$|Y| \geq E_\circ(L_i).$$

(Compare Lemma 1 in [9].)

Example: $E_\circ$ for Cartesian Products. Let X be the Cartesian product, $X = \times_{i\in I} X_i$ with the product measure, and let P_i be the coordinate "line" partitions with the respective slices isomorphic to (the copies of) X_i in X. Then, obviously,

$$(E\Pi) \qquad E_\circ(L_i) = \prod_{i\in I} L_i.$$

Let us generalize this to the coordinate "plane" partitions with the slices (isomorphic to the copies of) $\times_{i\in J} X_i$ for (all) subsets $J \in I$.

Define *a partition of unity on I* as an assignment of a non-negative number α_J to each $J \subset I$, such that $\sum \alpha_J \chi_J = 1$, where $\chi_J : I \to \{0,1\} \subset \mathbb{R}$ denote the characteristic (indicator) functions of the subsets $J \subset I$.

Coordinate $E_\circ$-Inequalities . *The $E_\circ$-function of the family of the partitions P_J is bounded from below by,*

$$(E\Pi^\alpha) \qquad E_\circ(L_J) \geq \prod_{J\subset I} L_J^{\alpha_J}$$

for all partitions of unity of I..

Proof. The equality $(E\Pi)$ shows that the minima $l(J)$ of the *log*s of the measures of P_J-slices of Y are sub-additive

$$l(J_1) + l(J_2) \leq l(J_1 \cup J_2)$$

for all pairs of disjoint subsets $J_1, J_2 \subset I$. Then $(E\Pi^\alpha)$ follows by the inclusion-exclusion principle for sub-additive set functions.

Questions. Can one evaluate $E_\circ$ for standard groups (Γ, G) with families of cyclic subgroups $C_g \subset \Gamma$ generated by $g \in G$? For example, what is $E_\circ(\Gamma, C_g)$ for the free nilpotent and solvable group Γ on a given generating set G with a prescribed nilpotency (solvability) degree? Does the function $E_\circ$ change much if one augments the collection C_g with extra (cyclic) subgroups?

Let $\mathbb{F}$ be a finite field, Γ be the special linear group and let Γ_i be some "standard" subgroups, e.g. the k subgroups $SL_{k-1}(\mathbb{F})$ positioned inside $SL_k(\mathbb{F})$ in the k obvious ways.

What is the expansion function for these? Can one prove the isoperimetric inequality for $\Gamma = SL_k(\mathbb{F})$ by some slicing argument? (This would bring us closer to the T-property for $SL_k(\mathbb{R})$.)

Can one meaningfully bound from below the cardinality of $G^{\cdot m} \subset SL_k(\mathbb{F})$, say for $m = 2$, for subsets $G \subset SL_k(\mathbb{F})$ with $|G| \geq \varepsilon \cdot |SL_k(\mathbb{F})|$, where ε is somewhat greater than $\frac{1}{2k}$?

4.4 Slice Removal and Slice Decomposition.

Decomposition Lemma. *Let $Y \subset X$ have the measure $< E_\circ(L_i)$, $i \in I$ for given partitions P_i of X and numbers L_i. Then Y can be decomposed into*

disjoint measurable subsets $Y_i \subset Y$, such that all P_i-slices of Y_i have measures $\leq L_i$ for all $i \in I$.

Proof. Keep removing slices S_i from Y of measures $\leq L_i$.

Corollary: Slicing Inequality for F_Σ^{incr}.

Let $F_i(n)$ and $I_i(r)$ be some "Følner" functions and the corresponding "profiles", and let the "boundary" of an $Y \subset X$ be understood as the sum of the I_i-"boundaries"

$$|\partial|_\Sigma(Y) = \sum_i |\partial|_{I_I}(Y).$$

If the functions $I_i(r)$ are monotone increasing, then the "Følner" function" of (X, P_i) associated to $|\partial|_\Sigma$ (a kind of a diagonal Følner transform of F_i) satisfies

$$(F_\Sigma) \qquad F_\Sigma(n; X, P_i) \geq F_{incr}(n) \geq E_\circ(L_i = F_i(n)),$$

where $F_{incr}(n)$ is a function for which the corresponding $I(r)$ is monotone increasing.

Proof. Decompose a given $Y \subset X$ into the union of Y_i and observe that $|\partial|_{I_i}(Y_i) \leq |\partial|_{I_i}(Y)$ by the monotonicity of I_i. Adding these inequalities and recalling the definitions of F_i and I_i we obtain the required lower bound on $|\partial|_\Sigma(Y)$.

Remark. This inequality is not sharp. It can be slightly improved by taking into account, by how much the measure of Y diminishes as we remove subslices from it; but this makes the inequality only marginally better.

Group-Subgroups Example. *Let Γ be a group and Γ_i be subgroups in Γ with generating subsets $G_i \subset \Gamma_i$, and let $I_i(r)$ be some monotone minorants of the isoperimetric profiles $I_\circ(r; \Gamma_i, G_i)$. Then there is a monotone function $I(r)$ minorizing the isoperimetric profile of Γ defined with $|\partial|_\Sigma = \sum_i |\partial|_{G_i}$, such that the Følner function associated to this I is bounded from below by $E_\circ(L_i = F_i(n))$ for F_i associated to I_i and with $E_\circ$ referring to the partitions of Γ) into the Γ_i-orbits.*

$\mathbb{Z}^k$-subexample. The above applies to the free Abelian group $\mathbb{Z}^k$ with the standard generators $g_1, g_2, ..., g_k$ and the corresponding "coordinate lines" partitions into the orbits of the cyclic subgroups generated by g_i; but the resulting inequality is *only* asymptotically sharp.

Similarly, every Cartesian product space $X = (X, \times_i \Gamma_i) = \times_i (X_i, \Gamma_i)$ satisfies,

$$(\Pi^{incr}) \qquad F_\circ^{inc,\Sigma}(n; X) \geq \prod_i F_\circ^{incr}(n; X, \Gamma_i).$$

4.5 Lower Bounds on $F_\odot$ by $E_\circ$.

Given a partition P of a measure space X and a subset $Y \subset X$, denote by $sup|Y \cap P|$ the supremum of the Fubini measures of the slices of P intersected with Y and by $inf|Y \cap P|$ the infimum of these measures taken over the slices that have *strictly positive* intersection measures with Y.

Then, for each number $\kappa > 0$, consider subsets $Y' = Y'(\kappa) \subset Y$, such that $sup|Y' \cap P| < \kappa$ and denote by $|P^{<\kappa}(Y)|$ the supremum of the measures of these Y'.

One can bound $|P^{<\kappa}(Y)|$ in terms of the distribution of the (Fubini) measures of the P-slices of Y as follows:

If the union $Y_{<\lambda} \subset Y$ of the slices of measures $< \lambda$ satisfies $|Y_{<\lambda}| \leq \varepsilon|Y|$ for some $\lambda \geq \kappa$ and $\varepsilon \geq 0$, then

$$[c\lambda\kappa] \qquad |P^{<\kappa}(Y)| \leq \varepsilon|Y| + c(1-\varepsilon)|Y| \text{ for } c = \kappa/\lambda.$$

In fact, the intersection of every $Y' = Y'(\kappa)$ with $Y_{<\lambda}$ has measure $\leq \varepsilon_\lambda|Y|$ for $\varepsilon_\lambda = |Y_{<\lambda}|/|Y|$ while the intersection of Y' with the complement $Y_{\geq\lambda} = Y \setminus Y_{<\lambda}$ has measure $\leq c(1-\varepsilon_\lambda)|Y|$. This implies $[c\lambda\kappa]$ with ε_λ instead of ε and, hence, with ε itself, since $\varepsilon \leq \varepsilon_\lambda$ and $c \leq 1$.

Slice Removal Lemma. (Compare Lemma 1 in [9].) *Let P_i, $i \in I$, be partitions of a measure space X, let $Y \subset X$ and let κ_i be positive numbers such that*

$$\sum_{i\in I} |P_i^{<\kappa_i}(Y)| < |Y|.$$

Then there exists a subset $Y_\odot \subset Y$ of positive measure, such that

$$inf|Y_\odot \cap P_i| \geq \kappa_i \text{ for all } i \in I.$$

Proof. Let $Y_\odot$ be a maximal subset in Y with

$$inf|Y_\odot \cap P_i| \geq \kappa_i \text{ for all } i \in I,$$

i.e. having all P_i-slices of measure$\geq \kappa_i$ for all i. (Such a maximal $Y_\odot$ is, in fact, unique, since the union of two subsets with P_i-slices $\geq \kappa_i$ also has the slices $\geq \kappa_i$). Then every subset in the complement, say $Z \subset Y \setminus Y_\odot$, contains a point $x \in Z$, such that the slice $S = S(x,i)$ of some P_i passing through x has $|S \cap Z| < \kappa_i$. Therefore, $Y \setminus Y_\circ$ decomposes (as in 2.3 **B**) into the union of subsets Z_i, $i \in I$, where each Z_i has $sup|Z_i \cap P| < \kappa_i$, i.e. it has all P_i- slices of measures $< \kappa_i$. Thus

$$|Y_\odot| \geq |Y| - \sum_{i\in I} |P_i^{<\kappa_i}(Y)| > 0$$

and the proof follows.

On the Average Measures of Slices. Denote by $Y_i = Y(\kappa_i) \subset Y$ the union of the P_i-slices of Y of measures $\geq \kappa_i$ and observe that $\cap_i Y_i \supset Y_\odot$; thus, the inequality $\sum_i P_i^{\kappa_i}(Y) < |Y|$ guarantees that the intersection $\cap_i Y_i$ is non-empty. In fact, one only needs for this the total bound on $\delta_i = 1 - |Y_i|/|Y|$, namely $\sum_i \delta_i < 1$, but the latter does not provide any lower bound on the cardinalities of *all* slices of $\cap_i Y_i$. However, if the sum $\sum_i \delta_i$ is small, then the P_i-slices of $\cap_i Y_i$ are "large on the average" in the following sense.

Let dx_i denote the measure on the quotient space $X_i = X/P_i$ associated to the partition P_i of X and let us regard the Fubini measures of the P_i-slices

of subsets $Y \subset X$ as functions on X_i denoted $|S(x_i)|(Y)$, $x_i \in X_i$. Then $\int_{X_i} |S(x_i)|(Y)dx_i = |Y|$ for all $Y \subset X$ by the definition of the Fubini measures and the average measure of the non-empty (or, rather, of postive measure) P_i-slices of Y is (defined as)

$$\left(\int_{X_i} |S(x_i)|(Y)dx_i\right) /|Y/P_i| = |Y|/|Y/P_i|.$$

Since the non-empty slices of every Y_i have measures$\geq \kappa_i$,

$$|Y_i|/|Y_i/P_i| = \left(\int_{X_i} |S(x_i)|(Y_i)dx_i\right) /|Y_i/P_i| \geq \kappa_i$$

and the average measure of the non-empty P_i-slices of the intersection $\cap_i Y_i$ is bounded from below for all i by

$$[\cap_i/P_i)] \qquad |\cap_i Y_i)|/|(\cap_i Y_i)/P_i| \geq |\cap_i Y_i|/|Y/P_i| \geq \kappa_i(1 - \sum_i \delta_i).$$

Evaluation of $F_\odot$ by $E_\circ$. (Compare [9]) *The Følner transform for every family of partitions P_i is bounded from below by the $E_\circ$-function,*

$$(F_\odot \succ E_\circ) \qquad F(n_i) = F_\odot(F_i(n_i); P_i) \geq E_\circ(c_i F_i(\varepsilon_i n_i)).$$

where $0 \leq c_i, \varepsilon_i \leq 1$ are arbitrary numbers satisfying

$$(*) \qquad \sum_i (\varepsilon_i + c_i(1 - \varepsilon_i)) < 1.$$

Proof. If $|\partial|_{I_i}(Y) \leq |Y|/n_i$, then the measure of the subset $Y_{<\lambda_i} \subset Y$ that is the union of the P_i-slices of Y with measures $< \lambda_i = F_i(\varepsilon_i n_i)$ satisfies $|Y_{<\lambda_i}| \leq \varepsilon_i |Y|$ according to the definition of F_i. Then, according to $[c\lambda\kappa]$,

$$|P_i^{<\kappa_i}(Y)| \leq (\varepsilon_i + c_i(1 - \varepsilon_i))|Y|$$

and the slice removal lemma yields the required bound

$$F(n_i) \geq |Y| \geq |Y_\odot| \geq E_\circ(\kappa_i = c_i F_i(\varepsilon_i n_i)).$$

Product Example. *The multivariable Følner function F of the Cartesian product X of Γ-spaces X_i, $i = 1, 2. ..., k$, is bounded by the Følner functions F_i of X_i by*

$$F(n_i) \geq \prod_i c_i F_i(\varepsilon_i n_i),$$

provided $\sum_i (\varepsilon_i + c_i(1 - \varepsilon_i))$ $)n \quad \lambda \qquad F$

$$F_\circ(n; X_1) \le F_\circ(n; \Gamma \times \Gamma_1) \le F_\circ(n; \Gamma) \cdot F_\circ(n; \Gamma_1).$$

Question. What is the (exact if possible) relation between the isoperimetric profiles of a group Γ and of this Γ regarded as the homogeneous space under the right/left action of $\Gamma \times \Gamma$?

5 Entropy.

Let (X, λ) be a Borel measure space, where λ is regarded as a *background measure* and where we use the notation $|Y| = |Y|_\lambda = \lambda(Y)$ for all $Y \subset X$.

The basic examples are given by countable spaces (X, λ) with the *unitary* measures, where all atoms have unit weights (thus, $|Y| = card(Y)$) and by the Euclidean spaces with the Lebesgue or with the Gaussian measures λ.

Consider measures $\mu = f(x)\lambda$ for (non-strictly) positive measurable functions f on X and first define the entropy of such a μ where $f(x)$ is *constant* on its (essential) support $S = supp(f) \subset X$ by

$$ent_\lambda(\mu) = log|S| = log(\mu(S)) - \mu(S)^{-1} \int_S \log(f) d\mu,$$

where, observe, $f \equiv \mu(S)/|S|$.

Then, for a general $\mu = f\lambda$, let $|\mu|_\varepsilon = \lambda_\varepsilon(\mu)$ denote the infimum of the λ-measures of the subsets $S_\varepsilon \subset X$ with $\mu(S_\varepsilon) \ge (1-\varepsilon)\mu(X)$, where we assume that μ has finite total mass, $|\mu| =_{def} |X|_\mu = \mu(X) < \infty$.

Take the Cartesian (tensorial) powers $(X^N, \lambda^N = \lambda^{\otimes N}, \mu^N = \mu^{\otimes N})$ and with $\lambda^{\otimes N}$ for the background measures on X^N. Set

$$ent(\mu^N - [\varepsilon]) = \liminf_{N\to\infty} \frac{1}{N} log|\mu^N|_\varepsilon)$$

and

$$ent(\mu) = ent_\lambda(\mu) = \lim_{\varepsilon\to 0} ent(\mu^N - [\varepsilon]).$$

Observe that, this entropy is invariant under scaling of μ, that is $ent(c \cdot \mu) = ent(\mu)$, while $ent_{c\lambda} = ent_\lambda + log(c)|\mu|$.

If μ is a probability measure with a λ-measurable density function $f = d\mu/d\lambda$ and with the support denoted $S \subset X$, then $ent_\lambda(\mu) \le log\lambda(S)$ with equality (only) for $\mu = \lambda(S)$. On the other hand, $ent_\lambda(\mu) \ge log(\sup_{x\in S} f(x)) - 1$.

We shall use the above definition only for *log-LLN-measures* μ, i.e. where $\mu = f\lambda$ for a λ-measurable function f, such that $log(f)$ satisfies

The Law of Large Numbers. *The $\mu^{\otimes N}$ measure of the subset $Y(\varepsilon, N) \subset X^N$ of the points $y \in X^N$, where*

$$\frac{1}{N}|log(f^{\otimes N}(y)) - \int_S log(f) d\mu| \ge \varepsilon$$

satisfies

$$(LLN) \qquad \mu^{\otimes N}(Y(\varepsilon, N)) \to 0 \text{ for } N \to \infty.$$

. For example, LLN holds if $|log(f)|$ is bounded on the support S of μ.

If μ is not log-LLN, one can LLN-regularize it, i.e. approximate by log-LLN-measures: just cut off the part of S where $|log(f)|$ approaches infinity and then define the regularized entropy with such an approximation.

Cartesian Additivity of the Entropy. Observe that LLN ensures the additivity of the entropy under the Cartesian product of measure spaces and yields the celebrated

Boltzman Formula. *All log-LLN-measures μ satisfy,*

$$ent_\lambda(\mu) = log|\mu| - |\mu|^{-1}\int_S log(f)d\mu = log|\mu| - |\mu|^{-1}\int_S f\,log(f)d\lambda$$

(for $|\mu|$ denoting the total mass $\mu(X) = \mu(S)$).

In other words,

the μ-average of $log(f)$ plus $ent_\lambda(\mu)$ equals the log of the total mass of μ.

In particular, the entropy of a *probability* measure μ is expressed by the Boltzman integral,

$$ent(\mu) = \int_S log\frac{1}{f}d\mu = \int_S f\,log\frac{1}{f}d\lambda.$$

This formula is customary taken for the definition of the entropy without assuming LLN, but only the convergence of the Boltzman integral, possibly to $\pm\infty$. This definition is equivalent to the above "regularized entropy" but in all our applications we can (and do) assume that μ is log-LLN.

5.1 Hölder Inequality via Tensorisation.

We introduce below the *Gibbs tensorisation trick* and then use it for the proof of the *Shannon inequalities* relating the entropy of a measure and its pushforards under the maps (partitions) in a given family.

Hölder Inequality. *The log of the integral*

$$\int_X \prod_{i\in I} f_i(x)^{\beta_i}dx$$

is a convex function of $\overline{\beta} = \{\beta_i\} \in \mathbb{R}^I$ for arbitrary positive functions f_i on X.

Proof. The inequality

$$log\left(\int_X \prod_{i\in I} f_i(x)^{\alpha_i\beta_i}\right)dx \le \sum_{i\in I}\alpha_i log\left(\int f_i(x)^{\beta_i}\right) = log\prod_{i\in I}\left(\int f_i(x)^{\beta_i}\right)^{\alpha_i}$$

for $\sum_i \alpha_i = 1$, $\alpha_i \ge 0$ is (trivially) true if the functions $f_i(x)$ are constant on the intersection $S \subset X$ of their supports (with the equality for functions with a common support $S \subset X$, where all f_i are constant) and the general case reduces to this by the law of large numbers via the tensorisation.

This argument also shows by how much the inequality deviates from equality.

Denote $\mu_i = g_i dx$ for $g_i = f_i^{\beta_i}$ and let $\mu = (\prod_{i\in I} g_i^{\alpha_i})dx$. Then

$$log \int \prod_{i\in I} g_i^{\alpha_i} dx \leq \sum \alpha_i ent_{\mu_i}(\mu) \leq \sum \alpha_i log(\mu_i(X)) = log \prod_{i\in I} \left(\int_X g_i dx \right)^{\alpha_i}.$$

The Hölder Inequality can be equivalently stated as follows

let v be a measure on the linear space X, and let Y be the linear dual to X. Then the function

$$\Psi(y) = \Psi_\nu(y) = log \left(\int_X exp\langle x, y\rangle dv \right)$$

is convex on Y, where the entropy of a measure with the density function $exp(\phi(x))$, $x \in \mathbb{R}$ equals the derivative $\psi'(y = 1)$ for $\psi(y) = \int_{\mathbb{R}} exp\langle x, y\rangle dx$ by the Boltzman formula.

This appears in the Gibbsian thermodynamics as the *concavity of the entropy of the ideal gas* and represents a tiny instance of Boltzmann's and Gibbs' ideas (see [20]).

Remarks.(a) The information theoretic rendition of the Gibbs argument is often presented as a chat between Alice and Bob. (See [11] and references therein.)

(b) The differential $D\Psi : Y \to X$ *injectively* sends Y to X, where *the closure of the image equals the convex hull of the support of μ.*

Thus, if $X = Y = \mathbb{R}^n$, then the volume of this hull equals the integral of the determinant of the Hessian of the (convex!) function Ψ, where the $\mathbb{R}_+$-valued map

$$\Psi \mapsto M(\Psi) =_{def} \int_Y det(Hess(\Psi(y)))dy$$

obeys non-trivial convexity relations: the Minkovski inequality, $M^{\frac{1}{n}}(\Psi_1+\Psi_2) \geq M^{\frac{1}{n}}(\Psi_1) + M^{\frac{1}{n}}(\Psi_2)$, and *the Alexandrov-Fenchel-Hodge inequality*. (See [13] for a survey and references).

5.2 Entropic Profiles and Stable $E_\circ$ Functions of Families of Partitions.

Given a finite mass measure μ on a Borel measure space $X = (X, \lambda)$ with a family of partitions P_i, we denote by $\mu_i = \mu/P_i =$ the pushforward of μ to $X_i = X/P_i$, call this the *P_i-reduction of μ*, and write

$$ent(\mu/P_i) = ent(\mu_i) = ent_{\lambda_i}(\mu_i)$$

for the background measures λ_i in X_i.

For example, if μ equals the restriction of the background measure λ on X to a subset $Y \subset X$, then the value of the density function of μ_i with respect to the background measure λ_i on X/P_i at each point $x_i \in X/P_i$ equals the Fubini mass of the corresponding P_i-slice of Y.

Denote by μ_{x_i} $x_i \in X_i = X/P_i$ the measure $f dP(x)$ on the slice $P^{-1}(x_i)$ for the background Fubini measure $dP(x)$ on this slice and $f = d\mu/d\lambda$ and let ent_{x_i} be the entropy of μ_{x_i} with respect to $dP(x)$ on this slice. Define *the entropy of (X, μ) over X_i*, also denoted $ent(P_i)$ as the average

$$ent(P_i) = \mu(X)^{-1} \int_{X_i} ent_{x_i} d\mu_i.$$

It is obvious (but significant) that

Entropy is additive.

$$ent+ \qquad ent(P) + ent(\mu/P) = ent(\mu).$$

Finite Example. Let P be a partition of X, a finite set with the unitary atoms and take a subset $Y \subset X$. Denote by $|P(y)|$ the cardinality of the P-slice of Y through $y \in Y$, and observe with the Boltzman (and Shannon in the finite case) formula that

$$ent(P|Y) = log \prod_{y \in Y} |P(y)|^{\frac{|P(y)|}{|Y|}}.$$

Entropic Profile. Consider a family $\mathcal{P}$ of partions P_i, $i \in I$, of X, where we usually assume that the single slice partition, corresponding to the map of X to a single point, is among our P. Every LLN measure μ on X defines the point $e(\mu) = \{ent(P_i)\} \in \mathbb{R}^I$; the set $ENT(\mathcal{P})$ of these points for all μ is called *the entropic profile of* $\mathcal{P}$. In what follows we shall evaluate the *conical convex hull* of $ENT(\mathcal{P}) \subset \mathbb{R}^I$ in the simple cases.

The definition of the entropy and the slice removal lemma from 4.4 imply the following

Sliced Tensorisation Lemma. *Given a finite family $\mathcal{P}$ of partitions P_i, $i \in I$, of X and an LLN measure μ on X, there exists, for every $\varepsilon > 0$, an integer $N_0 = N_0(\varepsilon, \mu, \mathcal{P})$ and a subset $Y = Y_N$ in the Cartesian power X^N, for every $N \geq N_0$, such that*

$$Y \subset supp(\mu^{\otimes N}), \text{ where } \mu^{\otimes N}(Y) \geq (1-\varepsilon)\mu^{\otimes N}(X),$$

and the Fubini measures $\varphi_i^N = \lambda^{\otimes N}/\lambda_i^{\otimes N}$ of the P_i^N-slices of Y satisfy,

$$N(ent(P_i^N) - \varepsilon) \leq log(\varphi_i^N(P_i^N(y) \cap Y)) \leq N(ent(P_i^N) + \varepsilon)$$

for all $y \in Y$ and all $i \in I$.

Next, observe that the $E_\circ$-functions of Cartesian powers of partitions P_i of X, satisfy,

$$E_\circ(n_i^{N_1}; P_i^{N_1}) \cdot E_\circ(n_i^{N_2}; P_i^{N_2}) \geq E_\circ(n_i^{N_1+N_2}; P_i^{N_1+N_2})$$

and define

$$E_\infty(n_i; P_i) = \lim_{N \to \infty} (E_\circ(n_i^N; P_i^N))^{\frac{1}{N}}.$$

The above lemma implies the following

Shannon E_∞-Inequality. *Let $\mathcal{P} = \{P_i\}$ be a finite family of partitions on X and μ a measure of finite mass on $X = (X, \lambda)$. Then the entropies $ent(P_i) = ent(\mu) - ent(\mu/P_{P_i}$ with respect to μ satisfy,*

$$ent(\mu) \geq log E_\infty(exp(ent(P_i)); \mathcal{P}).$$

Remark on Hölder. The tensorisation lemma also implies the Hölder version of the above inequality.

Let $f_i \geq 0$ be measurable functions on $X_i = X/P_i$, let

$$|f_i|_p = \left(\int_{supp(f_i)} f_i^p \right)^{\frac{1}{p}}$$

and let $|\Pi_{\mathcal{P}} f_i|_1$ denote the integral of the product of the pullbacks of f_i to X. Then

$$|\Pi_{\mathcal{P}} f_i|_1 \geq E_\infty(|\Pi_{\mathcal{P}} f_i|_1 / |f_i|_{p_i}; \mathcal{P})$$

for all $\{p_i\} \in \mathbb{R}_+^I$.

If all P_i are single slice partitions, this reduces to the Hölder inequality from 5.1 with positive p_i (and with no entropic correction term).

5.3 Shannon Inequalities for the Coordinate Line and Plane Partitions.

Let $(X, \lambda) = \times_i (X_i, \lambda_i)$, $i = 1, 2, ..., k$. Then the partitions P_i of X into the "coordinate lines" with the slices isomorphic to X_i and corresponding to the projections $P_i : X \to X_{\hat{i}} = (X_{\hat{i}}, \lambda_{\hat{i}}) = \times_{j \in I \setminus \{i\}} (X_j, \lambda_j)$ satisfy

Sh_1
$$ent(\mu) \geq \sum_i ent(P_i),$$

or, equivalently,

$$ent(\mu) \leq \frac{1}{k-1} \sum_i ent(\mu / P_i)$$

for all measures μ on X. Furthermore, the partitions P_J of X into the fibers of the projections $X = X_I \to X_{I \setminus J} = \times_{i \in I \setminus J} X_i$ (with "J-plane" slices representing $X_J = \times_{i \in J} X_i$) satisfy

Sh_α
$$ent(\mu) \geq \sum_{J \subset I} \alpha_J \cdot ent(P_J)$$

for all partitions of unity α_J of I (see 4.3).

Proof. Here, obviously, $E_\infty = E_\circ$ and the above applies.

Loomis-Whitney Inequality. This is an upper bound on $|Y| = \lambda(Y)$ for subsets $Y \subset X$ in terms of the background measures of Y/P_i, (assuming these are measurable) written as if it were a lower bound,

$$|Y| \geq \prod_i |Y| (\lambda_{\hat{i}}(Y/P_i))^{-1}.$$

This follows from the Shannon Inequality, since $ent_{\lambda_{\hat{i}}}(\mu_i)) \leq \lambda_{\hat{i}}(supp(\mu_i))$ and $ent(\mu) = log(\lambda(Y))$ for $\mu = \lambda|Y$.

Similarly one derives the *Shearer Inequality* that is the bound on $log|Y|$ by $log|Y| - log(\lambda_{\hat{J}}(Y/P_J)$ substituting $ent(P_J)$ in Sh_α. (The role of the entropy in such inequalities was pointed out to me by Noga Alon.)

If X_i are countable sets with the atoms of unit weights, then the Shannon inequality for subsets $Y \subset X = \times_i X_i$ with the restricted product unitary measures reads,

Combinatorial Shannon Inequality for the Coordinate Line Partitions. *Let $|P_i(y)|$, $y \in Y$, denote the cardinality of the P_i-slice of Y through y. Then the geometric means*

$$|MP_i| = \left(\prod_{y \in Y} |P_i(y)| \right)^{\frac{1}{|Y|}}$$

satisfy

$$\prod_i |MP_i| \leq |Y|.$$

Harper Inequality. The Shannon inequality, when applied to the vertex set X of the edge graph of a Euclidean n-cube with the edges for slices, says that

the vertex and the edge numbers of every subgraph Y in the cubical graph satisfy,

$$N_{vert} \geq 4^{N_{edg}/N_{vert}}. \tag{4^N}$$

For example, if all vertices in Y have the valency (degree) at least d, then $|Y| \geq 2^d$.

Another corollary of the combinatorial Shannon inequality is the following (well known) relation between the three numbers: the cardinality $|Y|$, the number N of the slices of Y with respect to all P_i and the sum C of the cardinalities of all these slices.

A^B-Inequality. *Let $A = C/N$ and $B = C/|Y|$. Then*

$$|Y| \geq A^B.$$

Proof. Since the function s^s is log-convex, $log(s^s)'' = 1/s$,

$$A^B \leq \prod_S |S|^{\frac{|S|}{|Y|}} \leq |Y|,$$

where the product is taken over all slices S of the partitions P_i.

5.4 Strict Concavity of the Entropy and Refined Shannon Inequalities.

A probability measure μ on $X_1 \times X_2$ can be regarded as a family of probability measures μ_{x_1} on X_2 parametrized by $x_1 \in X_1$, where the density $f_{x_1}(x_2)$ of (almost) every measure μ_{x_1} on X_2 equals the restriction of the density of μ to $x_1 \times X_2 \subset X_1 \times X_2$ divided by $p(x_1) = \int_{X_2} f_{x_1}(x_2) d\lambda_2$.

The Shannon inequality written as $ent(\mu/P_2) \geq ent(P_1)(= ent(\mu) - ent(\mu/P_1))$ says that the entropy is a *concave* function on the space of probability measures on X_2, since the measure μ_2 on X_2, that is the pushforward of μ, equals the

$p(x_1)$-weighted convex combination of the probability measures μ_{x_1}, while the entropy is (defined as) the corresponding convex combination of the entropies of μ_{x_1}.

In fact, the entropy is *strictly* concave as follows from the Boltzmann formula and the strict convexity of the function $t \cdot log(t)$. (This is the common way for deriving the Shannon inequality). Then the quantity $ent(\mu) - ent(P_1) - ent(P_2) \geq 0$ tells us how far μ is from *equilibrium*, i.e. a probability measure μ' on $X_1 \times X_2$, for which the probability measures μ'_{x_1} on X_2 are mutually equal for all $x_1 \in X_1$, or equivalently all μ'_{x_2} on X_1 are equal.

Here is another characteristic of (non-)equilibrium for measures μ on product spaces $X = \times_i X_i$, $i \in I$.

The index set $I \sqcup I$, (disjoint union of I with itself) and, hence, the Cartesian power X^2 of X, is naturally acted by the *Mendelian recombination group* $\mathbb{Z}_2^I = (\mathbb{Z}/2\mathbb{Z})^I$ generated by $|I|$ coordinate involutions on $I \sqcup I$ and/or on $X_i \times X_i$ for all $i \in I$. By strict convexity, a measure μ on X is at equilibrium, where (by definition if you wish) all Shannon inequalities Sh_α becomes equalities, if and only if the measure $\mu^{\otimes 2}$ on X^2 is invariant under $\mathbb{Z}_2^I$ and (where, observe, $\mu^{\otimes 2}$ is invariant under the diagonal involution on X^2 for all μ on X.)

We introduce *the entropic displacement* of $\mu^{\otimes 2}$ by z,

$$|\mu^{\otimes 2} - z(\mu^{\otimes 2})|_{ent} =_{def} ent(\frac{1}{2}(\mu^{\otimes 2} + z(\mu^{\otimes 2})) - \frac{1}{2}(ent(\mu^{\otimes 2}) + ent(z(\mu^{\otimes 2}))) \geq 0$$

and then identify involutions $z \in \mathbb{Z}_2^I$ with subsets $J \subset I$ by

$$z \leftrightarrow J = J(z) = supp(z) \subset I$$

where the *support* of z is defined by $z(i) \neq i$.

The composition of involutions corresponds to the symmetric difference of subsets that we denote $J_1 \cdot J_2 =_{def} (J_1 \cup J_2) \setminus (J_1 \cap J_2)$. We also abbreviate by writing

$$|J|_{ent}(\mu) = |\mu^{\otimes 2} - z(J)(\mu^{\otimes 2})|_{ent},$$

where, observe, $|J|_{ent}(\mu) = |J^\perp|_{ent}(\mu)$, for $J^\perp = I \setminus J$.

A measure μ on X satisfies the *equality*

$$ent(P_J) + ent(P_{J\ \ }) = ent(\mu)$$

if and only if $\mu^{\otimes 2}$ is $z(J)$- (or,equivalently $z(J^\perp)$)-invariant; this is also equivalent to

$$|J|_{ent}(\mu) = 0.$$

Since the entropy is strictly concave, the function $|J|_{ent}(\mu)$ of $J \subset I$ satisfies some *triangle-type inequalities*,

$$(\Delta) \qquad |J_1 \cdot J_2|_{ent}(\mu) \leq \Delta_\mu(|J_1|_{ent}(\mu), |J_2|_{ent}(\mu)),$$

where $\Delta_\mu(0,0) = 0$ for all μ and $\Delta_\mu(a,b)$ is uniformly continuous in (a,b) with the *modulus of continuity δ depending on μ*. Moreover, δ is uniformly bounded on certain (compact in a suitable sense) classes of measures μ.

For example, if the density function f of μ satisfies

$$\int_X |log(f(x))|d\mu \leq const < \infty,$$

then δ is bounded by some universal δ_{const} as a simple continuity argument shows.

This is useful, for instance, if $log(f(x)) \leq 0$, e.g. if X is a discrete space with unitary atoms, where (Δ) becomes a relation between the entropies of P_J depending *only* on $ent(\mu)$,

$$(\Delta) \qquad |J_1 \cdot J_2|_{ent}(\mu) \leq \Delta_{ent(\mu)}(|J_1|_{ent}(\mu), |J_2|_{ent}(\mu)),$$

for some function $\Delta_e(a, b)$ that is continuous in a, b and e and such that $\Delta_e(0, 0) = 0$.

Remarks. (a) All this is, apparently, well known but I could not find a reference; nor do I know a specific sufficiently "elegant" $\Delta_e(a, b)$. I guess, there are sharp "mixed symmetric mean inequalities" for measures on $\times X_i$ similar to the classical Muirhead's inequalities, such as the mixed discriminant inequality of Alexandrov (that is $GL(k)$- rather than just S_k-symmetric).

(b) The above generalizes to the Cartesian powers X^N with the Cartesian products of I-copies of the permutation group S_N acting on it. The resulting inequalities become, in a sense, asymptotically sharp for $N \to \infty$ due to the law of large numbers (applied to convolution of measures on the spaces of measures).

A possible framework for this is suggested by the *Mendelian dynamics* on the space of measures on the disjoint union $X^{\sqcup} = \sqcup_N X^N$ of the the Cartesian powers of $X = \times_{i \in I} X_i$, where, observe, $X^N = \times_i X_i^N$, and where this equality is defined (only) up to the action of the Cartesian power of the permutation group, $(S_N)^I$ acting on X^N. Every partition of unity α_J where all α are rational with the common denominator, say $\alpha_J = n_J/d$, defines a bijection (equality) $\times_J X_J^{n_J} = X^d$ for $X_J = \times_{i \in J} X_i$ and similarly the equalities $\times_J X_{J^N}^{n_J} = X^{dN}$ for all N.

The *Mendel map* F sends measures μ_N on X^N to measures $F(\mu_N)$ on the tensor products of the J^N-reductions of μ_N to X_{J^N} for all subsets $J \subset I$. This F is accompanied by the power map $\mu_N \mapsto \mu^{\otimes d}$ that should be thought of as "multiplication by the scalar $= d$" in the some kind of log of the tensor algebra. Then the "ideal dth root" $F^{\otimes \frac{1}{d}}$ of the Mendel map apparently (I did not check this) exponentially contracts (in a suitable metric) the space of "graded measures" $\{\mu_N\}$ to the subspace of the *equilibrium* (i.e. tensor product) measures (as in the

Mendel-Robbins-Geiringer theorem, (see [17] and references therein) and satisfies $\frac{1}{d} ent(F(\mu_N)) \geq ent(\mu)$ in the agreement with the Shannon-Shearer inequality.

Question. What is a comprehensive formalism that would fully reflect the $(S_N)^I$-symmetry along with the N-grading and that would also embrace the Brascamp-Lieb and the Lieb-Ruskai inequalities below?

(c) The Hessian of the entropy on the space (affine simplex) M of probability measures, being positive definite by the Shannon inequality, defines a (non- complete) Riemannian metric on M, called *Fisher-Rao-Kramer (Antonelli- Strobeck, Svirezhev-Shahshahani, Karlquist) metric*, (also underlying the *Einstein-Onsager*

relations as was explained to me by Alexander Gorban) that, by the Boltzmann formula, equals the spherical metric induced by the *log*-radial (compare 2.2) projection $(p_i) \mapsto (\sqrt{p_i})$ of the unit n-simplex $\Delta_n \subset \mathbb{R}^{n+1}$ of probability measures to the unit sphere $S^n \subset \mathbb{R}^{n+1}$.

The meaning of $(p_i) \mapsto (\sqrt{p_i})$ becomes clearer with the *the moment map*

$$\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}/\mathbb{T}^{n+1} = \mathbb{R}_+^{n+1} \supset \Delta_n$$

for the coordinate-wise (Hamiltonian!) action of the torus $\mathbb{T}^{n+1}$ on $\mathbb{C}^{n+1}$, where the Riemannian quotient metric on $\mathbb{R}_+^{n+1} = \mathbb{C}^{n+1}/\mathbb{T}^{n+1}$ identifies with the Hessian of $\sum p_i log(p_i)$ on $\mathbb{R}_+^{n+1} \supset \Delta_n$.

Questions. Is there a geometric picture where this unitary symmetry of the Fisher metric is seen simultaneously with the tensorisation property (definition) of the entropy?

What are other "entropy type" functions associated to Kähler manifolds with isometric actions of compact groups?

5.5 On Sliced Spaces.

A *sliced set* is a set X with a distinguished family $\mathcal{S} \subset 2^X$ of subsets $S \subset X$, called *slices*. More generally "a slicing" of a measure space X is given by a measurable map $\tilde{X} \to X$, where $\tilde{X}$ is a measure space with a distinguished partition $\tilde{P}$, where the map $\tilde{X} \to X$ is one-to-one on (almost) every slice of $\tilde{P}$. (Alternatively, a sliced structure is a measure on 2^X with some regularity to it.)

Here is a generalization of the combinatorial Shannon inequality for pairs of partitions to sliced spaces.

No-Δ Inequality. *Let X be sliced by subsets $S \in \mathcal{S} \subset 2^X$, such that,*

(a) every point of X is contained in at most two slices;

(b) every two non-equal slices intersect at a single point if at all.

If $\mathcal{S}$ contains no triple of pairwise intersecting slices, then every finite subset Y in X satisfies the combinatorial Shannon inequality,

$$|Y|^{|Y|} \geq \prod_{S \in \mathcal{S}} |Y \cap S|^{|Y \cap S|}.$$

Proof. Consider the 2-*chains* $(y_0, S_1, y_1, S_2, y_2)$ in Y, where the pairs of points (y_{i-1}, y_i) are contained in the slices S_i for $i = 1, 2$ and where $S_1 \neq S_2$.

The geometric mean G_0 of the number of such chains over $y_0 \in Y$, i.e, the geometric mean of the number $N_0(y)$ of the chains *starting at* $y \in Y$ is $\geq$ than the geometric mean G_1 of the number $N_1(y_1) = |S_1| \cdot |S_2|$ of the chains with the *middle point* $y_1 \in Y$ by the geometric-arithmetic mean inequality.

Since the number of the chains between every pair of points (y_0, y_2) in X is ≤ 2 by our assumptions on $\mathcal{S}$, we get our inequality,

$$|Y| \geq \left(\prod_y N_0(y)\right)^{\frac{1}{|Y|}} = G_0 \geq G_1 = \left(\prod_y N_1(y)\right)^{\frac{1}{|Y|}} = \left(\prod_{S \in \mathcal{S}} |Y \cap S|^{|Y \cap S|}\right)^{\frac{1}{|Y|}}.$$

Questions. What of this kind would generalize Shannon for more than two partitions? What are further "chain uniqueness" and/or "no short cycle" conditions leading to further inequalities? In particular, if a certain co-entropic

inequality holds for all subsets Y of cardinality $\leq N$, for large N depending on the type of the inequality, does it then hold for all finite subsets Y? Is Gibbs' tensorisation helpful for such questions?

Notice in this regard that the *entropic profile* of a sliced space is monotone decreasing under slice-wise injective maps $\psi : X_1 \to X_2$. In fact, a simple computation shows that the decrease of the entropy under

aerage entropic decreases of the slices of the two spaces for

X_1,

ent

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5.6 Fiber Products, Fubini Symbols and Shannon Expansion Inequalities.

Call (typically, surjective) measurable maps $X \to Y$ *partitions of X over Y* with the slices in X thought of as the fibers. Given two spaces X_1 and X_2 partitioned over Y, denote by $X_1 \times_{/Y} X_2 \subset X_1$ their *fiber product*, that is the set of the pairs $(x_1, x_2) \in X_1 \times X_2$ that lie over the same point in Y.

The set $X_1 \times_{/Y} X_2 \subset X_1$ carries a unique *Fubini* measure, such that its natural projections to X_1 and X_2 are measurable, and consistent with the Fubini measures in the slices.

The space $X_1 \times_{/Y} X_2 \subset X_1$, comes along with two *fiber* maps:

the first one is *induced from* P_1, i.e. is given by the map $(P_1, Id_2) : X_1 \times_{/Y} X_2 \to X_1 \times X_2$, where all slices isomorphically go to to slices of P_1, and the second one is similarly induced from P_2.

Given a family of partitions of a space X, or, better, a small category $\mathcal{C}$ of measure spaces X_i (where we do not distinguish any X at this point) and measurable maps $P_{ij} : X_i \to X_j$, we compose the above operations and thus generate a small category $\mathcal{F} = \mathcal{F}(\mathcal{C})$ of measurable spaces and maps that is built inductively, starting from $\mathcal{C}$ by taking fiber products for new objects and the fiber maps for the morphisms.

We always include into this category the identity map of each space, corresponding to the partition into points as well the constant map, where the only slice is the space itself.

Examples. (1) If there is a single partition $P_1 : X \to X_1$ of X then all one essentially can do is to take the *iterated graph of* P_1 denoted $X^k/P_1 = X^k/X_1$ that is the set of points in X^k that go to the same slice in X under the k projections $X^k \to X$. This X^k/X_1 comes along with k (isomorphic but different) partitions, each of them is indiced from P_1.

(2) Given a sequence of partitions P_i, $i = 1, 2, ..., k$, of X, define the *space $Z(k)$ of k-chains* of points $x_0, x_2, x_2, ..., x_k$, where every two consecutive points x_{i-1}, x_i lie in some slice of P_i. This space is constructed by taking the consecutive fiber products of the graphs $X \times_{P_i} X$ over X and thus it carries the Fubini measure on it. This $Z(k)$ is mapped to $X \times X$ by sending the two end points of chains of points back to X but this map is not necessarily measurable.

For example, if $X = \Gamma$ is a locally compact group and P_i are partitions into the orbits of closed subgroups $\Gamma_i \subset \Gamma$, then this map is measurable if the group product map $\times_i \Gamma_i \to \Gamma$ is open and countable-to-one.

The fiber product is neither a commutative nor an associative operation: a subset $Y \subset \Pi = \times_i X_i^{k_i}$ can be decomposed into a fiber product in many different ways (if at all). However,

the measures on Y coming from these decompositions are all equal by the Fubini theorem and can be described with the following

Fubini Symbols. Let $\mathcal{A} = \mathcal{A}(\mathcal{F}) = \mathcal{A}(\mathcal{C})$ be the (Grothendieck) Abelian group generated by the symbols $[X]$ for all objects X in $\mathcal{F}$ with the relations

$$\textbf{(F)} \qquad [X_1 \times_{X_3} X_2] = [X_1] + [X_2] - [X_3].$$

for all fiber products in $\mathcal{F}$. Clearly, $\mathcal{A}$ is generated by (the symbols of) the objects in $\mathcal{F}$.

Example. The space Z of the k-chains of partitions of X over X_i has $[Z] = [X] + \sum_i([X] - [X_i])$. In particular, the k-iterated graph Z of X over Y has $[Z] = [X] + k([X] - [Y])$.

Expansion Coe cients of Coordinate Maps. Consider two objects Z_1 to Z_2 in $\mathcal{F}$ realized by subsets in the Cartesian products of X_i, say in $\times_i X_i^{k_i}$ and $\times X_i^{l_i}$ where $l_i \leq k_i$ and assume that some coordinate projection $p : \times_i X_i^{k_i} \to \times X_i^{l_i}$ sends Z_1 to Z_2 .

Example. The above space of k-chains $c = \{x_0, x_2, x_2, ..., x_k\}$ in X goes thus to $X \times X$ by $c \mapsto (x_0, x_k)$.

In general, such *coordinate maps* p between Z's are not measurable and the pushforward λ_* of the background measure λ_{Z_1} from Z_1 to Z_2 may be everywhere infinite even for measurable p.

We are concerned with the case when λ_* is locally finite, that is $\lambda_* = \delta(z_2)\lambda_{Z_2}$ for a measurable function δ on Z_2 and we say in this case that p is Δ-*expanding* for a positive constant Δ if $\delta(z_2) \leq \Delta^{-1}$ (almost) everywhere on Z_2. Whenever such a map exists we write, the corresponding

log-inequality,

$$[Z_2] \geq [Z_1] + log(\Delta).$$

Examples. (a) A map between countable spaces with unitary atoms is expanding, i.e. 1-expanding, if and only if it is injective.

(b) An injective smooth equidimensional map between Euclidean spaces is Δ-expanding if its Jacobian $\geq \Delta$.

Shannon Expansion Theorem. *Let $\mathcal{C}$ be a small category of Borel measure spaces with the maximal object X, where all other objects X_i, $i \in I$, are quotients X/P_i for partitions P_i of X. Let $p; Z_1 \to Z_2$ be a Δ_p-expanding map in the fiber-product category $\mathcal{F}$ generated by $\mathcal{C}$ and write the corresponding log-inequality in the symbols $[X]$ and $[X_i]$ (that generate the group $\mathcal{A}$),*

$$k[X] + \sum_i k_i[X_i] \geq l_p$$

for the (positive and/or negative) integers k and k_i defined by (the Fubini symbols of) (Z_1, Z_2)) and $l_p = log\Delta_p$. Then every measure μ on X satisfies

$$(ExSh)_p \qquad k \cdot ent(\mu) + \sum_i k_i \cdot ent(\mu_i) \geq l_p$$

for the pushforward (reduction) measures $\mu_i = \mu/P_i$ on X_i.

Proof. The claim is obvious if μ equals the restriction of λ to a Borel subset $Y \subset X$ with $\lambda(Y) < \infty$ and such that the P_i-slices of Y have constant Fubini measure for every partition P_i, since the Fubini measures are multiplicative for fiber products. Then the general case follows by the Gibbs tensorisation argument as earlier. QED

5.7 Weighted Loomis-Whitney-Shearer and Brascamp-Lieb Inequalities.

Each $(\mathrm{ExSh})_p$ is a linear inequality in the variables e and e_i representing the entropies, where the Shannon expansion theorem says that

the closed convex hull $H_{ent} = H_{ent}(X, P_i)$ of the set of the vectors $(ent(\mu), ent(\mu_i))$ in the Euclidean space $E_+ = \mathbb{R}^{|I|+1}$ for μ running over all measures μ is contained in the intersection H_{ex} of the closed half spaces defined by all $(ExSh)_p$. Moreover, since the expansion condition passes to the Cartesian powers (X^N, P_i^N), the (stabilized and possibly enlarged) closed convex hull $H_{ent}^{st} \subset \mathbb{R}^{|I|+1}$ of the vectors $\frac{1}{N}(ent(\mu), ent(\mu/P_i^N))$ for all measures μ on X^N, is also contained in H_{ex}.

Observe that the convex subsets $H_{ent}^{st} \subset H_{ex} \subset \mathbb{R}^{|I|+1}$ are unbounded and only in rare cases $H_{ent}^{st} \subset H_{ex}$. On the other hand, the two sets are often *parallel at infinity*: if a linear form is bounded from above on H_{ent}^{st} then it is also bounded on H_{ex}.

Here is the simplest (ExSh).

1. Shannon and Loomis-Whitney Inequalities for Expanding Families of Partitions. *If the (two ends) map from the space $Z(k)$ of chains to $X \times X$ for partitions $P_1, ...P_i, ..., P_k$ (see above Example 2) is expanding, then P_i satisfy the Shannon inequality*

$$\sum_i ent(P_i) \leq ent(\mu)$$

for all measures μ on X and, therefore, the P_i-projections of every $Y \subset X$ satisfy the Loomis-Whitney inequality,

$$\prod_i \frac{\lambda(Y)}{\lambda_i(P_i(Y))} \geq \lambda(Y).$$

This inequality is sharp, i.e. $H_{ent}^{st} = H_{ex}$, provided the spaces X/P_i are not monoatomic.

Example. The "expanding" assumption is (obviously) satisfied for the orbits (cosets) partitions of a simply connected nilpotent Lie group $X = \Gamma$ by the orbits (cosets) of connected subgroups Γ_i if the Lie algebras of Γ_i are linearly independent and span the Lie algebra of Γ, provided the product map $\times\Gamma_i \to \Gamma$ has the Jacobian $= 1$ at the origin and the same is true for unipotent algebraic groups over the locally compact groups.

But the issuing Shannon and Loomis-Whitney inequalities are non-sharp for non-commuting subgroups Γ_i (i.e. $\leq$ can be replaced by $<$), where one expects significantly stronger inequalities (due to non-commutativity) for the measures that are "sufficiently uniformly spread" over Γ.

2. Shannon and Shearer Inequalities for Weighted Coordinate Partitions. Let P_J be the coordinate plane partitions of $X = X_I = \times_{i\in I} X_i$ into the slices that are the fibers of the projections $X \to X_{I\setminus J} = \times_{i\in I\setminus J} X_i$, $J \subset I$, where the background measures λ_J on $X_J = \times_{i\in J} X_i$ are not assumed to be product measures but rather satisfy the following assumption.

The measures λ_J on $X_J = \times_{i\in J} X_i$ are greater than the λ_I-Fubini measures on the P_J-slices: the (bijective!) coordinate projection of every P_J -slice $S \subset X$ to X_J is expanding with respect to the λ_I-Fubini measure on S and λ_J on X_J.

This condition implies that all bijective coordinate maps between the fiber products of X_J are expanding and the proof follows by the inclusion-exclusion principle. Thus

the coordinate inequalities from 5.3 remain valid under the above expansion assumption.

In particular, one has

Shearer-Loomis-Whitney Projection Inequality. *The measures of the projections $P_J(Y) \subset X_J$ satisfy*

$$(P_\circ) \qquad \prod_{J \subset I} (\lambda_J(Y))^{\alpha_J} \geq \lambda_I(Y)$$

for all $Y \subset X$ and all partitions of unity α_J.

3. Euclidean Brascamp-Lieb Inequalities. Let P_i, $i = 1, 2, ..., n$, be partitions of $\mathbb{R}^k$ into parallel *non-coordinate* affine subspaces. Then

the entropies of P_i with respect to an arbitrary probability measure μ on $\mathbb{R}^k$ satisfy the inequality

$$\left(\sum_i k_i \cdot ent(P_i) \right) - ent(\mu) \geq l$$

for some real constants $k_i \geq 0$ and $-\infty < l < \infty$ (depending on P_i but not on μ, where one may always normalize to $l = 0$ by rescaling the background measure λ be $exp(-l)$) if and only if they satisfy this inequality (with these very k_i and l) for all Gaussian measures ν on $\mathbb{R}^k$. This is usually stated in the (equivalent) Hölder form (see [12] and references therein).

The Shannon Expansion theorem is weaker in the Euclidean case: it only delivers qualitative form of these inequalities, saying, in effect, that

the intersections of H^{st}_{ent} and H_{ex}, with the (convex) subset $e = -1, e_i \geq 0 \subset \mathbb{R}^k$ are parallel at infinity.

Sketch of the proof. All fiber products are Euclidean spaces and the coordinate projections are all linear; those which are bijective are Δ-expanding for various Δ and these deliver a set of (ExSh)-inequalities that are, by simple linear algebra, equivalent, up to l's, to the Brascamp-Lieb inequalities.

This also applies to families of partitions into orbits of locally compact Abelian groups and nilpotent (and locally compact unipotent) Lie groups by families of subgroups, but , as in the above Loomis-Whitney case, this is less satisfactory (apart from being non-sharp even in the Abelian case) since the effects of non-commutativity (potentially, strengthening such inequalities) are neglected.

One can slightly improve such inequalities similarly by adding to our category maps Π permuting the coordinates of product spaces, where the additional terms in the Shannon inequalities come from non-invariance of measures on (and /or subsets in) such products.

These Π may generate infinite groups of transformations of such products that admit no finite invariants measures.

Example. Let P_i, $i = 1, 2, 3$ be the partitions of $\mathbb{R}^2$ into the three families of parallel lines in general position. Then the group generated by the three involutions of $\mathbb{R}^4$ corresponding to the three splitting of $\mathbb{R}^2$ into the product of lines, contains a free subgroup. (But the resulting improvement of the Shannon inequality, even in this simple case, remains unsharp unlike the corresponding Brascamp-Lieb inequality.)

Riemannian Remark. The Δ-expansion condition makes sense for the exponential maps for Riemannian manifolds X, where, for example, it is satisfied

with $\Delta = 1$ by the complete simply connected manifolds with the sectional curvatures $K \leq 0$.

The geometric counterpart to the Loomis-Whitney inequality is a bound on the volume $|Y|$ of a $Y \subset X$ by some function of the (Liouville) measure $|S|$ of the set S of geodesics that meet Y in X. If Y is convex, i.e. it meets every geodesics over a (connected and simply connected) segment, then $|S| = |\partial(Y)|$ with a suitably normalized Liouville measure and $|S| \leq |\partial(Y)|$ in general).

Non-sharp bounds follow by the Fubini theorem (as in the Coulhon-Saloff-Coste case) but a sharp inequality is available only for $X = \mathbb{R}^n$ by Almgren's variational argument [1] that applies, to Riemannian manyfolds X, (say, complete with bounded geometry, for safety) such that

the integral mean curvature M of the "exposed" locally convex part E of the boundary of every Y admits a lower bound $M \geq I'_X(|\partial(Y)|)$, where a point $y \in \partial(Y)$ is called *exposed* if the set C_y of the geodesic rays s that start at y and meet Y at some $y' \neq y$, contains no pair $(s, -s)$, and where the lower bound on M depends on the function I'– a kind of a derivative of the isoperimetric profile.

If $X = \mathbb{R}^n$, then such bound on M is obtained by taking the convex hull H of Y, where the total Gauss curvature of $\partial(H)$, (that is independent of $H \subset \mathbb{R}^n$) minorizes the (obviously "homogenized") mean curvature M, since the Gauss curvature of $\partial(H)$ vanish outside $E = \partial(Y) \cap \partial(H)$. Thus,

among all subsets $Y \subset \mathbb{R}^k$ of given k-volume, the round balls minimize the (Liouville) measure of the set of straight lines that intersect Y. (If Y are convex, this follows from the ordinary isoperimetric inequality).

5.8 Linearized Shearer-Loomis-Whitney Inequalities.

Let X_i, $i \in I$, be vector spaces over some field and denote by X_J, $J \subset I$, the tensor product of X_i over J, i.e. $X_J = \otimes_{i \in J} X_i$. Define the *$J$-reduction* $Y_J \subset X_J$ of a linear subspace $Y_I \subset X_I$ as the minimal subspace in X_J, such that $Y_J \otimes X_{I \setminus J}$ contains Y_I.

To see it better, take the tensor product $Z_I = X_I \otimes Y'$ for some linear space Y' take a vector $z = Z_I$ and let $z_J : X'_{I \setminus J} \otimes Y' \to X_J$ be the homomorphism corresponding to z under the canonical isomorphism $X_I \otimes Y' = Hom(X'_{I \setminus J} \otimes Y', X_J)$, where $X'_{I \setminus J}$ denotes the linear dual of $X_{I \setminus J}$.

Then the J-reduction of the image $Y \subset X_I$ of z_I equals the image of z_J in X_J; thus, $rank(Y_J) = rank(z_J)$.

Tensorial Reduction Inequality. *The ranks of the J-reductions of every linear subspace $Y \subset X_I = \otimes_{i \in I} X_i$ satisfy*

$$(P_\otimes) \qquad \prod_{J \subset I} (rank(Y_J))^{\alpha_i} \geq rank(Y)$$

for an arbitrary partition of unity $\{\alpha_J\}$ on I.

Proof. Take bases B_i in each X_i, let $B_I = \otimes_{i \in I} B_i$ be the corresponding basis in X_I and project Y to a coordinate plane P along the complementary coordinate plane. Clearly, $|J|_T \leq |J|_Y$ for all $Y \subset X_I$ and all $J \subset I$. Since there is a P, such that the coordinate projection $Y \to P$ is bijective, the above $(P_\circ)$ applies and yields $(P_\otimes)$.

Remark. Given a $k+1$-linear form $y = y(x_0, ..., x_k)$, each splitting J of the variables x_i into two groups defines a bilinear form y_J (on the tensor product corresponding to the splitting) and $(P_\otimes)$ becomes an inequality between the ranks of the forms y_J.

A particular instance of this (where one may have stronger inequalities) is the cohomology (sub)algebra X of an (algebraic) manifold V, where $y(x_i)$ equals the value of the cup-product of x_i on the fundamental class of V. (See [16] for topological applications of such inequalities.)

5.9 Orbit and Lieb-Ruskai Inequalities for Representations of Compact Groups.

Consider a finite dimensional linear representation Z of a group Γ with a given family of subgroups Γ_J, $J \in \mathcal{J}$, take a vector $z \in Z$ and let $|\Gamma_J(z)|$ denote the rank of linear span of the Γ_J-orbit of z.

If Γ is the Cartesian product, $\Gamma = \times_i \Gamma_i$, $i \in I$, and $\Gamma_J = \times_{i \in J} \Gamma_i$ we have the following

Linearized Shearer Inequality. *If the representations of Γ and all Γ_J are semisimple (over the algebraic closure of the ground field), then, for an arbitrary partition of unity $\{\alpha_J\}$ on I, the rank of the linear span of the Γ-orbit of every point $z \in Z$ satisfies,*

$$(P_*) \qquad |\Gamma(z)| \leq$$

the orthogonal projection onto a subspace $Y \subset Z$ and $\mu = dim(Y)^{-1}P_Y$, then, by definition, $ent(\mu) = log(dim(Y))$.

Then, in general, one passes to the tensorial powers $(Z^{\otimes N}, \mu^{\otimes N})$, takes the ε-approximation (in the trace norm) of $\mu^{\otimes N}$ by a "measure" μ_ε^N associated to a projection on a subspace $Y_N \subset Z^{\otimes N}$ of maximal (depending on ε dimension) and set

$$ent(\mu) =_{def} \lim_{N\to\infty\varepsilon\to 0} \frac{1}{N} ent(\mu_\varepsilon^N).$$

The law of large numbers (for the spectral measure of μ) shows that $ent(\mu)$ equals the entropy of the Boltzman entropy of the spectral measure, where the background measure (corresponding to the background Hilbert structure on Z) is that of a discrete space with all atoms of unit weight.

Given such a "measure" on a tensor product of Hilbert spaces, $L = L_1 \otimes L_2$, one defines its reductions μ_i on L_i, $i = 1, 2$, say μ_1, by taking the traces of the corresponding self-adjoint operators over L_2. Equivalently, one can average μ over a compact group Γ_2 unitarily and irreducibly acting on L_2, and thus, on L, where the averaged "measure", say μ_1' on L, can be uniquely written as $\mu_1' = \mu_1 \otimes \lambda_2$ for λ_2 corresponding to the scalar operator on L_2 with $trace = 1$ that is $|L_2|^{-1} \cdot id$.

Lieb-Ruskai Strong Superadditivity.. *Let μ correspond to a positive self-adjoint operator with $trace(\mu) = 1$ ("probability measure") on the tensor product of three finite dimensional Hilbert spaces, $L = L_1 \otimes L_2 \otimes L_3$. Then the entropies of its reductions to $L_1 \otimes L_2$, $L_2 \otimes L_3$ and L_2 satisfy,*

$$ent(\mu_{12}) + ent(\mu_{23}) \geq ent(\mu_2) + ent(\mu = \mu_{123}).$$

(See [26], [5] and references therein).

This inequality, (trivially, by the inclusion-exclusion principle) implies the (weighted with partitions of unity) Shannon type inequalituies for tensor products of (more than three) Hilbert spaces and thus for Hilbert spaces acted upon by Cartesian products of compact groups. This sharpens the above reduction inequalities when the underlying field $\mathbb{F} = \mathbb{C}$. Furthermore, this generalizes to the actions of infinite groups Γ_i, on $l_2(\Gamma = \times_i \Gamma_i)$, where the entropy is defined via the von Neumann Γ_i-dimensions ([23]). But this does not seem (at least, at the first glanc) to yield Loomis-Whitney projection (reduction) inequalities (nor isoperimetric inequalities) for finite dimensional subspaces $D \subset l_2(\Gamma)$ for non-Abelian Γ_i.

Questions. (a) The quantum entropy is a *concave* function on the space M_n of positive selfadjiont operators on $\mathbb{R}^n$ (by the superadditivity of the quantum entropy proved by Landford prior to the Lieb-Ruskai theorem) and the geometry of the corresponding "quantum " (Fisher-)Kubo-Mori-Bogolubov metric and similar Bures-Uhlmann and Hasegawa-Nagaoka-Pets metrics (see [24]) along with the symmetric Riemannian metric on $M_n = GL_n/O_n$, probably, tell you something else about (relations between the quantum entropies of) various reductions. Furthermore, the *relative entropy* defined on the pairs of positive selfadjoint operators by $Trace(\mu log(\mu) - \mu log(\nu))$ is also concave by the Lieb-Ruskai theorem that expresses the positivity of the metric given by the Hessian of this $Trace$. (See [21] for the computation of curvatures of such metrics.)

Is there a "natural" (symplectic/Kählerian?) definition of these metrics that reveals their geometry and makes transparent the entropy inequalities?

(b) Do the Euclidean Brascamp-Lieb inequalities remain valid for the quantum entropy of operators (say, with continuous compactly supported kernels $K(x_1, x_2)$) on the L_2-Hilbert space $L_2(\mathbb{R}^n, dx)$ (for the Lebesgue and/or for the Gauss measure dx on $\mathbb{R}^n$) with respect to the reductions in the tensorial decompositions $L_2(\mathbb{R}^n) = L_2(X_1) \otimes L_2(X_2)$ for the (not necessarily coordinate) orthogonal decompositions $\mathbb{R}^n = X_1 \bigoplus X_2$?

5.10 Reduction Inequality for Orderable Groups Γ.

Let $\Gamma = \times_i \Gamma_i$, $i \in I$, take a linear space B of $\mathbb{F}$-functions on Γ with finite supports for some field $\mathbb{F}$ and recall the asymptotic $\Gamma_{J^\setminus}$-dimensions $|D : \Gamma_{J^\setminus}|$ (see 1.10) for $\Gamma_{J^\setminus} = \times_{i \in I \setminus J} \Gamma_i$. By combining the Shearer-Loomis-Whitney projection inequality $(P_\circ)$ with the equivalence $\circ \Leftrightarrow *$ with order we conclude that

If Γ is a left orderable amenable group, then

$$(P_*^\alpha) \qquad rank(B) \leq \prod_{J \subset I} |D : \Gamma_{J^\setminus}|^{\alpha_J}$$

for all partitions of unity $\{\alpha_J\}$ on I and all linear (sub)spaces B of functions on Γ with finite supports.

Furtermore, if the product map $\times_i \Gamma_i \to \Gamma$, for amenable subgroups in an orderable (not necessarily amenable) Γ is one-to-one, then

$$(P_*) \qquad (rank(D))^{k-1} \leq \prod_{i=1,2,\ldots,k} |D : \Gamma_i|$$

for all linear spaces D of functions on Γ with finite supports.

Remarks. (a) The above generalizes to other types of "expanding" families of subgroups, e.g. in Abelian Γ with the use of the corresponding Brascamp-Lieb kind of inequalities. These, however, are not so good as far as the Brascamp-Lieb constant I is concerned, albeit the discrete version, delivered by the Shannon Expansion Inequality, is sharp, for the trivial reason; the presence of the δ-measures. A true generalization/linearization of the Brascamp-Lieb inequality for (discrete) groups Γ and their group algebras A needs additional invariants of measures in Γ and/or linear subspaces in A that would rule out their concentration on small subsets.

(b) If $\Gamma\mathbb{Z}^I$, then (P_*) applies to linear subspaces D in the algebra of regular functions on Zariski open subsets in $\mathbb{F}^I$, where the J-reduction of D becomes the restriction D_J of (the functions in) D to a generic J-plane in $\mathbb{F}^k$, where $|D_J| = |D : \Gamma_{J^\setminus}|$ and the so interpreted (P_*) makes sense and remains valid for the algebra of formal (and convergent, for normed fields) power series at a point in $\mathbb{F}^I$. Furthermore, there are similar inequalities for the restrictions of D to generic members of non-plane families of algebraic (analytic) subvarieties in $\mathbb{F}^I$.

5.11 $\bullet$-Convexity and the Entropic Følner Functions $F_\bullet$.

A positive monotone increasing function $F(n)$, $n > n_0 > 0$, is called *$\bullet$-convex* if $logF(\frac{1}{\varepsilon})$, $0 < \varepsilon \leq 1/n_0$, is convex, that is equivalent to the convexity of $J(exp(l))$, where $\frac{1}{J}$ is the inverse function of F.

Examples. The functions $const \cdot n^k$ are $\bullet$-convex for all $k \geq 1$ and $const > 0$. Also the functions $const \cdot exp(n^a)$ are $\bullet$-convex for $a \geq 0$.

Every positive monotone increasing function $F(n)$ admits a unique maximal $\bullet$-convex minorant $F_\bullet(n) \leq F(n)$ and this $F_\bullet$, as it is easy to see, has growth (roughly) comparable to that of F,

$$F_\bullet(n) \geq \min_\alpha \max(\alpha F(n), F(n)^{1-\alpha})$$

for $0 \leq \alpha \leq 1$.

Given a positive function ("isoperimetric profile") $I(r)$, $r \geq 0$ define the associated

entropic Følner function, denoted $F_\bullet(n)$ as the above maximal $\bullet$-convex minorant of the ordinary Følner function $F(n) = F_I(n)$. Clearly, the inverse function to $F_\bullet$ equals the minimal monotone increasing majorant $M(r)$ of $r/I(r)$ for which $\frac{1}{M(exp(l))}$ is convex.

If X is a Γ-space with the orbit partition P and the ordinary Følner function F then $F_\bullet$ minorizes (not only the cardinalities of the subsets $Y \subset X$ with $|\partial(Y)| \leq n$, but also) the entropy of the partition $P|Y$, of Y:

If $|\partial(Y)|/|Y| \leq 1/n$, then

$$ent(P|Y) =_{def} |Y|^{-1} \sum_{s \in P|Y} r_s log(r_s) \geq log(F_\bullet(n))$$

for $r_s = |s \cap Y|$, where the sum is taken over the P-slices s of $P|Y$

Proof. The boundary of Y is bounded from below, according to the definition of $I(r)$, by

$$|\partial Y| \geq \sum_s I(r_s)/|Y| = \sum_s \frac{r_s}{\sum_s r_s} J(r_s)$$

for $J(r) = I(r)/r$. If $J(exp(l))$ is convex, then

$$\sum_s \frac{r_s}{\sum_s r_s} J(r_s) \geq J\left(exp \frac{\sum_s r_s log(r_s)}{\sum_s r_s}\right) = J(exp(ent(P|Y)))$$

and the proof follows from the definition of the Følner function $F = F_\circ$ of the action.

The above equally applies to general partitions P (with $\sum_s I(r_s)/|Y|$ taken for the definition of the "boundary") and shows that the Følner transform of $\bullet$-convex functions for families of partitions P_i of an X is governed by the lower bounds on the entropy of subsets Y (recall that $ent(Y) =_{def} log|Y|$ for discrete spaces) by the entropies of the partitions. In particular, if

$$ent(Y) \geq \sum_i \kappa_i ent(P_i|Y) + log(\beta)$$

for all $Y \subset X$, then the Følner transform $F = F(n_i)$ of $\bullet$-convex functions F_i satisfies

$$F(n_i) \geq \beta F_i(n_i)^{\kappa_i}.$$

Thus, for example,

the Shannon inequalities yield (often sharp) lower bounds on the multivariable entropic Følner functions of Cartesian products $X = \times_i X_i$ of Γ_i-spaces in terms of the entropic Følner functions of individual X_i and/or of $X_J = \times_{i \in J} X_i$, $J \subset I$.

This applies to the ordinary Følner functions if these are $\bullet$-convex or (well) approximated by $\bullet$-convex functions. The simplest instance of this is the (multivariable) isoperimetric inequalities for free Abelian groups where the (ordinary) Følner functions *are* $\bullet$-convex.

Similarly,

the Shannon inequalities provide lower bounds on the Følner functions of discrete groups Γ with families of subgroups Γ_i, $i = 1, 2, ..., k$, where the product map $\times_i \Gamma_i \to \Gamma$ is injective

and

the Brascamp-Lieb inequalities give similar bounds on the multivariable Følner function of $\Gamma = \mathbb{R}^k$ with a given family of linear subspaces $\Gamma_i \subset \Gamma$, where one may use the Følner functions F_i defined with the ordinary Euclidean boundaries of subsets in Γ_i.

By combining the above with the Schwartz symmetrization for normal extensions,

$$1 \longrightarrow \Gamma_1 \longrightarrow \Gamma \longrightarrow \Gamma_2 \longrightarrow 1$$

we conclude that

the entropic Følner function of Γ is bounded from below by the product of these of Γ_i, $i = 1, 2,$.

Finally, we recall $\circ \Leftrightarrow *$ (see 3.2) and obtain a similar inequality for the linear algebraic $F_*(\Gamma)$ for the space of functions with finite supports on Γ,

If the group Γ_2 is orderable and the Følner functions $F_*(n_1, \Gamma_1)$ and $F_\circ(n_2, \Gamma_2)$ are minorized by $\bullet$-convex functions $F_i(n_i)$, $i = 1, 2$, then

$$F_*(n_1, n_2; \Gamma) \geq F_1 \cdot F_2.$$

5.12 Entropic Poincaré Inequalities.

Consider a measure $\mu = f(x)\lambda$ on a Γ-space X, where λ is an invariant measure on X, e.g. $X = \Gamma$ with the Haar measure, and let $Y_\mu = \{f(x) \leq m\} \subset X \times \mathbb{R}_+$ be the subgraph of f.

We define $|\partial(\mu)| = |\partial(Y_\mu)|$ for the obvious action of Γ on $X \times \mathbb{R}_+$ (for whichever notion of the boundary for subsets) and then apply the above entropic inequality to the partition of Y_μ to the levels $f(x) = m$, $m \in \mathbb{R}_+$. Since the entropy of this partition is $\leq ent(\mu)$ by the Shannon inequality, we see that

if the Følner function of X is minorized by a $\bullet$-convex function $F(n)$, then

$$ent(\mu) \geq F(|\mu|/|\partial(\mu)|)$$

for all finite measures μ on X (where, recall, $|\mu| =_{def} \mu(X) = |Y_\mu|$ and $ent(\mu) = log|\mu| - |\mu|^{-1} \int_S log \frac{1}{f} dx$ for the support $S \subset X$ of μ.

Remarks.(a) The above *log*-Poincaré (Sobolev) inequality in the $\mathbb{R}^k$ case holds, in the sharp form, for the Euclidean boundary, (see [12]).

(b) Measures μ on X, rather than subsets $Y \subset X$, make a natural domain of definition for the Følner transform and the entropic Følner functions. In fact, everything in the previous section (trivially) generalizes to measures, where some of the resulting *log*-Poincaré inequalities become sharp in this context. For example, the extremal measures for the *log*-Poincare inequalities associated to the Euclidean Brascamp-Lieb inequalities are the Gaussian ones. (See [12] and references therein).

6 Lower Bounds on F_* for Groups with Large Orderable Subgroups.

We bound from below the linear algebraic algebraic Følner functions of groups Γ by applying the projection inequalities to certain infinitely generated orderable (e.g. free Abelian) subgroups in Γ.

6.1 Expanding Subgroups Inequality.

Let Γ be a group with a finite generating subset $G \subset \Gamma$ and let $N = N(n)$ denote the maximal number of elements g 1

6.2 Distorted and Undistorted Actions on l_2 and on Decaying p-adic Functions.

Consider an action of the group $\Gamma = \mathbb{Z}^k$ on a linear space L over a field $\mathbb{F}$ and show that the following properties of such an action are equivalent

1. The action is free: if $\sum_\gamma c_\gamma(l) = 0$, then either all $c_\gamma \in \mathbb{F}$ or $l \in L$ are zero.

2. The action is *non-degenerate*: $|D : \Gamma| > 0$ for all non-zero subspaces $D \subset L$ i.e. if $H_i \subset \Gamma$ is an exhaustion of Γ by Følner sets then

$$\limsup_{i\to\infty} |H_i * D|/|H_i| > 0.$$

3. The action is *uniformly non-degenerate*:

$$inf_D \liminf_{i\to\infty} |H_i * D|/|H_i| > 0.$$

4. The spans of these H_i-orbits satisfy

$$\limsup_{i\to\infty} |H_i * D|/|H_i|^{\frac{k-1}{k}} = \infty.$$

5. The action of the group $\mathbb{F}$-algebra A of Γ on L does not factor through an algebra of transcendence degree $< k$.

To see the equivalence of these conditions, observe that every finitely generated module L over A equals the set of $\mathbb{F}$-sections of a coherent sheaf $\mathcal{L}$ over the torus $(\overline{\mathbb{F}}^\times)^k$, where obviously, 1–5 are satisfied if and only if $\mathcal{L}$ is *not* supported on a proper subvariety in $(\overline{\mathbb{F}}_\times)^k$.

Remark. It is clear that $1 \Rightarrow 3 \Rightarrow 2$ for all amenable groups Γ; furthermore, if the group algebra A of Γ contains no non-constant finite dimensional subalgebras (e.g. A has no zero divisors) then also $2 \Rightarrow 1$; but it is unclear what corresponds to 4 and 5 in the general case.

Examples: Functions with Fast Decay. (a) The action of $\Gamma = \mathbb{Z}^k$ on the Hilbert space $l_2(\Gamma)$ satisfies the conditions 1–5. and the same remains true for the space $l_2(X)$, where X is a set with a free action of Γ by the following theorem of Gabor Elek (see [8]).

If the group algebra of an amenable group Γ over $\mathbb{R}$ has no zero divisors, then the action of Γ on $l_2(\Gamma)$ is free.

Alternative Proof. Regard $\mathbb{C}$-valued (moderately growing) functions a on $\mathbb{Z}^k$ as Fourier transforms of distributions $\hat{a}$ on the torus $\mathbb{T}^k$: if this distribution has a Zariski dense support then the linear span L of a satisfies 1-5 as we just seen. In particular, this proves 1–5 for $l_2(\mathbb{Z}^k)$ since the $\hat{a} \in L_2(\mathbb{T}^k)$ for $a \in l_2(\mathbb{Z}^k$ and, being a measurable function, $\hat{a}$ has Zariski dense support.

(b) The spaces of decaying functions on $\mathbb{Z}^k$ with values in discrete valuation fields also satisfy 1-5, since they are representable by power series.

(c) *Counterexample for Decaying $\mathbb{C}$-valued Functions.* The Fourier transform a of a distribution $\hat{a}$ supported on an algebraic subvariety $V \subset \mathbb{T}^k$ of positive codimension is annihilated by a polynomial in the group algebra of $\mathbb{Z}^k$ that vanishes on V with a sufficient multiplicity; thus (a) is violated. But such an $a = a(n_i)$ may be a decaying function on $\mathbb{Z}^k$.

For instance, let $V \subset \mathbb{T}^k$ be a smooth subvariety in that is *nowhere infinitesimally flat* (if V is analytic this means it is not contained in a sub-torus of $\mathbb{T}^k$)

and $\hat{a}$ be given by a smooth measure on V. Then a does decay, being expressed by an oscillating integral on V.

The fastest rate of decay, of such an a is $O(\sum_{i=1,2,..,k}|n_i|)^{\frac{1-n}{2}}$: smooth hypersurfaces $V \subset \mathbb{T}^k$ with non-vanishing Gaussian curvature, e.g. round spheres, give such decay. (This was pointed out to me by Terence Tao.)

Question. Does a faster rate of decay, i.e. $o(\sum_{i=1,2,..,k}|n_i|)^{\frac{1-n}{2}}$, rules out distributions with Zariski non-dense supports?

Domination Lemma. *Let L be a space of functions on $\Gamma = \mathbb{Z}^k$ that is finitely generated as a $\mathbb{Z}^k$-module and that satisfies 1–5. Then the isoperimetry of L dominates (as defined in 3.2) that of Γ. Consequently, the linear spaces of the real l_2-functions and the fast decaying p-adic functions on Γ have the same isoperimetry as Γ itself.*

Proof. The linear space in question equivariantly embeds into the space of formal power series and the Domination by Ordering from 3.2 applies.

In particular, it follows from the lemma that the combinatorial projection inequalities for families of partitions of $\mathbb{Z}^k$, such as the Loomis-Whitney and Shearer inequalities, pass to the linear subspaces D in these spaces L of functions, where the cardinality of the image of the projection $Y \to \mathbb{Z}^k/\Gamma_i$ for a (free Abelian) subgroup $\Gamma_i \subset \mathbb{Z}^k$ is replaced by the dimension of the span of the Γ_i-orbit of D over the fraction field of the group ring of Γ_i.

Here are two applications.

1. Split Abelian Subgroup Inequality. *Let Γ be a group with a finite generating subset $G \subset \Gamma$ and let $N = N(n)$ denote the maximal number of elements $g_i \subset G^{\cdot n} \subset \Gamma$, such that g_i, $i = 1, 2, ..., N(n)$, generate a free Abelian subgroup of rank $N(n)$. Then the max-Følner functions F_* of $l_2(\Gamma)$ and of the spaces of decaying functions with values in ultrametric fields satisfy*

$$F_*(n;\Gamma, G) \geq \frac{1}{2}2^{N(\frac{n}{2})}.$$

Proof. Use the measure μ uniformly distributed on g_i and proceed as in the above Expanding Subgroups Inequality.

2. Distorted Abelian Subgroups Inequality. *Let Γ contain a finitely generated free Abelian subgroup Γ_0 and let $G \subset \Gamma$ be a generating subset containing the generators of Γ_0. Then the max-Følner functions F_* for the spaces of l_2-functions and of decaying "ultrametric" functions on Γ are bounded from below by the relative growth function $G_{\circ\circ}(n) = |\Gamma_0 \cap G^{\cdot n}|$ as follows,*

$$F_*(n;\Gamma, G) \geq \frac{1}{2}G_{**}(\frac{n}{2}).$$

Proof. Use the measure μ uniformly distributed on $\Gamma_0 \cap G^{\circ n}$ and apply the linearized Displacement Inequality from with the G^{*n}-inequality as in the Coulhon-Saloff-Coste argument in 1.4.

These **1** and **2** can be applied, e.g. to solvable groups as these have many Abelian subgroups.

Example. Every non-virtually nilpotent polycyclic group contains an Abelian subgroup with exponentially growing $G_{\circ\circ}$ (this is well known and easy); therefore

the l_2 and "ultrametric" Følner functions $F_*(n)$ grow exponentially for all non-virtually nilpotent polycyclic groups.

Questions Can one replace "polycyclic" by "solvable without torsion" with a suitable class of μ generalizing those in **1** and **2**?

Is the "free Abelian" requirement truly necessary in **1** and **2** or something like "orderable" would suffice?

Remark. The linearized L(oomis)-W(hitney)-S(hearer) inequality, that is an upper bound on the dimension of a linear space D, say, of (germs of) analytic functions on $\mathbb{R}^k$ (or any other class of functions with similar genericity properties) by the dimensions of the restrictions of D to generic members of parallel families of the coordinate planes, can be proven by induction on k and $dim(D)$ as follows.

Take a generic hyperplane $H \subset \mathbb{R}^k$ parallel to the coordinate hyperplane $\mathbb{R}^{k-1} \subset \mathbb{R}^k$, let $D|H$ be the restriction of D to H and $D_{/H} \subset D$ be the subspace of functions that vanish on H. Then the LWS inequalities for $D|H$ and $D_{/H}$ yield that for D by the Minkovski inequality $(\prod_j (a_j + b_j))^{\frac{1}{l}} \geq (\prod_j a_j)^{\frac{1}{l}} + (\prod_j b_j)^{\frac{1}{l}}$. (There are similar inequalities for non-coordinate "webs of subvarieties" that we shall not discuss here.)

6.3 Isoperimetry with "Large" $H \subset \Gamma$.

The above suggest regarding (various) boundaries $|\partial|_H(Y)$ as something depending on the size of H as well as of Y.

Here are two examples for $\Gamma = \mathbb{Z}$ and a subset $H \subset \mathbb{Z}$.

(1) *Let μ be the probability measure uniformly distributed on H. Then all finite subsets $Y \subset \mathbb{Z}$ satisfy*

$$(\partial_\mu \succ min). \qquad |\partial_\mu(Y)| = def \frac{1}{H} \sum_{h \in H} |\partial_h(Y)| \geq min\left(\frac{1}{2}|Y|, \frac{1}{8}|H|\right)$$

To show this, consider two cases.

Case 1. Let $|H| \geq 2|Y|$. Then the proof follows from the Displacement inequality in 1.4.

Case 2. Divide H into positive and negative parts, $H = H_+ \cup H_-$, and let $Y_- \subset |Y|$ be the first (for the natural order) k_- elements in Y for $k_- = |H_-|$ and $Y_+ \subset |Y|$ be the last $k_+ = |H_+|$ elements. Clearly,

$$|\partial_{H_+}(Y_+)| \leq |\partial_{H_+}(Y)| \text{ and } |\partial_{H_-}(Y_-)| \leq |\partial_{H_-}(Y)|$$

and the proof reduces to Case 1.

Remarks. (a) The inequality $(\partial_\mu \succ min)$ yields a lower bound on the *max*-profile of $\mathbb{Z}$ with respect to H, but the *exact* evaluation of $I_\circ^{max}(r; \mathbb{Z}, H)$, even for $H = \{1, 2, ..., k\}$, is not apparent and seems to pertain to the combinatorial number theory.

(b) The inequality $(\partial_\mu \succ min)$, obviously, remains true for all *biorderable* groups Γ.

(c) Let $H_i \subset \Gamma$, $i = 1, 2, ...k$, be subsets, such that the product map $(h_1, h_2, ..., h_k) \mapsto h_1 \cdot h_2 \cdot ... \cdot h_k$ of the Cartesian product $H_1 \times H_2 \times ... \times H_k$ to Γ is one-to-one. Then, by the $G^{\cdot m}$-inequality,

$$\sum_i |\partial_{\mu_i}(Y)| \geq |\partial_\mu(Y)|,$$

where μ_i are the uniform probability measures on H_i and μ is such measure on the product $H = H_1 \cdot H_2 \cdot ... \cdot H_k \subset \Gamma$. Since $|H| = \prod_i |H_i|$ under the "one-to-one" assumption, one gets a good lower bound on $\sum_i |\partial_{\mu_i}(Y)|$ for biorderable (and sometimes more general) groups Γ.

(2) *If $H \subset \mathbb{Z}$ admits no nontrivial relation*

$$\sum_{h \in H} n_h \cdot h = 0$$

where $|n_h| \leq N$ for a given $N \geq 0$ and all $h \in H$, then every subset $Y \subset \mathbb{Z}$ of cardinality $|Y| \leq N$ satisfies

$$|Y|^{k-1} \leq \prod_{h \in H} |\partial_h(Y)|.$$

Proof. Let C_i be the (Cayley) graph on the vertex set Y, where the edges are the pairs $(y, y + h_i) \in Y \times Y$ (for the group composition in $\mathbb{Z}$ depicted by "+").

The number of the connected component in C_i, that is the number l_i of the maximal arithmetic h_i-progressions in Y, equals $|\partial_{h_i}(Y)|$. Furthermore, the family of the k partitions P_i of Y into the components of C_i is expanding in the sense of 5.7 due to the absence of the above relations. Hence, $|Y|^{k-1} \leq \prod_i l_i$ by the Loomis-Whitney inequality.

(In general, the inequality $|Y|^{k-1} > \prod_i l_i$ implies, for given positive integers N_i, either the existence, for some i, of an arithmetic h_i-progression in Y of length N_i, or a nontrivial relation $\sum_i n_i h_i = 0$ with $|n_i| < N_i$.)

Example If $H = \{h_i\}$, where $|h_{i+1}| > N \sum_{j \leq i} |h_j|$, then the " no relation" condition is obviously satisfied.

The above easily generalizes to the sparse subsets H in all Γ, where one may have a much larger $|\partial_H|$ with relatively small H for *non-Abelian* groups Γ.

On the other hand, the *exact* evaluation of the *max*-profile is not apparent already for $\Gamma = \mathbb{Z}$ and $H = \{1, 2, ..., k\}$ (where the problem pertains to the combinatorial number theory.)

This suggests

Dual Isoperimetric Problem: Find, for every $r = 1, 2, ...$ and $0 \leq \alpha \leq 1$, a "minimal" subset $H \subset \Gamma$, such that

$$\frac{|\partial|_H^{max}(Y)}{|Y|} \geq \alpha$$

for all $Y \subset \Gamma$ with $|Y| \leq r$, where "minimal", may refer to the cardinality of Y, or to the radius of the minimal ball containing H, or to another notion of a size of a subset in Γ.

It seems that most (all?) proofs of lower bounds on the isoperimetric profiles of groups Γ also deliver, under a close inspection, "small" H giving "large" *max*-boundaries to all $Y \subset \Gamma$.

7 Linear Algebraic Entropic Inequalities for Normal Group Extensions.

We prove in this section coarse product inequalities for a suitably defined "filtered" (compare 1.11) entropic Følner functions of linear actions that apply, for example, to pure torsion groups where other techniques are not available.

7.1 Vertical Families $\mathcal{L}$ of Linear Subspaces and $\mathcal{L}$-Entropy.

Let L be a linear space over some field with a distinguished class $\mathcal{L}$ of what we call *vertical* linear subspaces in L, where we assume this class to be closed under finite and infinite intersections. If $D \subset L$ is a linear subspace, then we denote by $\mathcal{L}|D$ the *induced* class of subspaces in D that are the intersection of D with the subspaces from $\mathcal{L}$.

Tensor Product Example. Consider linear spaces L_{hor} and L_{vert} and take $L = L_{hor} \otimes L_{vert}$, where $\mathcal{L}$ consists of the subspaces $L'_{hor} \otimes L_{vert}$ for all linear subspaces $L'_{hor} \subset L_{hor}$. Clearly, $\mathcal{L}$ consists of all subspaces invariant under the group of linear transformations of L_{vert} naturally acting on L.

Γ_{vert}-Examples. Let L equals some Γ-invariant space of functions on a group Γ and $\mathcal{L}$ consists of all Γ_{vert}-invariant subspaces in L for some normal subgroup $\Gamma_{vert} \subset \Gamma$.

The quotient group Γ_{hor} does not, in general, act on L but it does act on $\mathcal{L}$ by *projective* transformations where each individual transformation comes from a linear transformation of L.

This class $\mathcal{L}$ is not always closed under complements: if $L_1 \subset L_2$ for $L_1, L_2, \in \mathcal{L}$, then the quotient space L_2/L_1 is not always Γ_0-equivariantly isomorphic to anybody in $\mathcal{L}$.

Here are two particular classes $\mathcal{L}$ where the complement always exists.

1. *Support Saturated class.* A subspace $L_1 \subset L$ is called *support saturated* if it contains all functions from L which have supports contained in the support of L_1.

If L consists of the functions with finite supports in X, then

the class of all Γ_{vert}-invariant support saturated subspaces admits complements corresponding to taking complements of Γ_{vert} invariant subsets in X.

2. *l_2-Spaces.* If the underlying field is $\mathbb{R}$ or $\mathbb{C}$ and $L = l_2(X)$, then

the class of all closed Γ_{vert}-invariant subspaces also admit (now just orthogonal) complements.

$\mathcal{L}$-Length and $\mathcal{L}$-Entropy. Take a finite dimensional linear subspace $D \subset L$ and consider all its *$\mathcal{L}$-"partitions"*, i.e. decompositions into sums (spans) of *non-zero* independent subspaces $D = D_1 \oplus D_2 \oplus ... \oplus D_i \oplus ... \oplus D_N$, such that the associated filtrations are $\mathcal{L}$-vertical:

$$D_1 \oplus D_2 \oplus ... \oplus D_i \in \mathcal{L}|D \text{ for all } i = 1, 2, ...N.$$

Define the *$\mathcal{L}$-length* of D as the maximal N, for which such "partition" exists and the *$\mathcal{L}$-entropy* $ent_{\mathcal{L}}(D)$ as the supremum of the entropies of the finite measure spaces made of N atoms of weights $d_i = |D_i| = rank(D_i)$ over all $\mathcal{L}$-"partitions" of D for all $N = 1, 2, ...$

Clearly,

$$|D| \geq length_{\mathcal{L}}(D) \geq exp(ent_{\mathcal{L}}(D))$$

for all $\mathcal{L}$ and D.

Example. If $\mathcal{L}$ consists of all linear subspaces in L then $\mathcal{L}$-length of D equals $|D|$ and $ent_{\mathcal{L}}(D) = log|D|$ but if $\mathcal{L} = \{L'_{vert} \otimes L_{hor}\}$ for an infinite dimensional space $L_{vert} \supset L'_{vert}$ then generic finite dimensional subspaces $D \subset L = L_1 \otimes L_2$ have $ent_{\mathcal{L}}(D) = 0$.

$\mathcal{L}$-Boundary. Given a finite dimensional linear subspace, define the $g|\mathcal{L}$-boundary of D for a linear transformation g of L preserving $\mathcal{L}$ by

$$|\partial_{g|\mathcal{L}}|(D) = sup_{L\ \in \mathcal{L}}|D \cap L'/g^{-1}(D)|.$$

Clearly, this equals the ordinary boundary $|\partial_g(D)|$ if $D \in \mathcal{L}$; in general,

$$|\partial_{g|\mathcal{L}}|(D) \leq |\partial_g(D)|.$$

Logarithmic Filtration Lemma. *Let G be a set of transformations of L, such that $\mathcal{L}$ is G-invariant, yet it contains no non-zero G-invariant subspace. If a finite dimensional subspace D is contained in some $L'_D \in \mathcal{L}$ and*

$$|\partial_{g|\mathcal{L}}|(D) \leq \varepsilon|D|,$$

for all $g \in G$, then there is a descending sequence of subspaces $D = D_0 \supset D_1 \supset D_2 \supset ... \supset D_i \supset ... \supset 0$, where each $D_i = D \cap L'_i$ for $L'_i \in \mathcal{L}$ and such that

$$(2+\varepsilon) \qquad |D/D_i| \leq 2|D/D_{i-1}| + \varepsilon$$

for all $i = 1, 2, ...$

Proof. Take a sequence γ_i of invertible linear transformations of L, where $\gamma_0 = id$, where $\gamma_i \cdot \gamma^{-1} \in G$ for all i and where every transformation in the (linear) transformation group generated by G appears as some γ_i in the sequence. Then the sequence

$$D'_i = D \cap \gamma_0(L'_D) \cap \gamma_1(L'_D), ..., \gamma_i(L').$$

Remark. If $\mathcal{L}$ is associated with a normal subgroup Γ_{vert} in Γ generated by G, then the inequality $(2+\varepsilon)$ looks abysmally weak compared to the corresponding (equally obvious) combinatorial intersection inequality (compare 1.4)

$$(\varepsilon) \qquad |Y \setminus Y_i| \leq \varepsilon$$

where $Y_i = Y \cap Y'_i$ for Γ_{vert}-invariant subsets in Γ. It is unclear to me if/when 2 can be removed from $(2+\varepsilon)$ for linear actions.

$\frac{1}{4}$-Entropy Corollary. *If*

$$|\partial|_G^{max}$$

Remark. The corresponding (also obvious) bound on the length of D,

$$length_{\mathcal{L}}(D) \geq log_3(\varepsilon) - 2$$

looks somewhat better, but ($ent \geq \frac{1}{4}$), albeit independent of ε, better behaves under induction as we shall see below.

7.2 $\mathcal{L}$-Entropic Følner Functions.

Given a countable group with a (usually generating) subset $G \subset \Gamma$, consider all countable overgroups $\tilde{\Gamma} \to \Gamma \to 1$ with the kernel denoted $\Gamma_{vert} \subset \tilde{\Gamma}$ and let $\tilde{L}$ stands either for the space of $\mathbb{F}$-valued functions on $\tilde{\Gamma}$ with finite supports on $\tilde{\Gamma}$ for a given field $\mathbb{F}$, or for the space of the complex l_2-functions on $\tilde{\Gamma}$.

Denote by $\tilde{\mathcal{L}}$ the family of Γ_{vert}-invariant spaces $\tilde{L}'$ of functions on $\tilde{\Gamma}$ with either finite projections of their supports to Γ, or of l_2-spaces with finite von Neumann Γ_{vert}-dimensions.

Define the *internal* $G|\mathcal{L}$-boundary of a $\tilde{D} \subset \tilde{L}$ by

$$|\partial_{\tilde{G}|\mathcal{L}}|(D) = \sup_{L \in \mathcal{L}} (|D \cap L' \cap_{g \in \tilde{G}} \gamma^{-1}(D)|,$$

where $\tilde{G}$ denotes some lift of G to $\tilde{\Gamma}$ and where, observe, the definition is independent of a lift and $|\partial_{\tilde{G}|\mathcal{L}}|(D) \geq |\partial^{max}_{\tilde{G}|\mathcal{L}}|(D)$.

Define the entropic Følner function $\tilde{F}_\bullet(n) = \tilde{F}_\bullet(n; \Gamma, G)$ as the maximal $\bullet$-convex function, such that

$$exp(ent_{\mathcal{L}}(\tilde{D})) \geq \tilde{F}_\bullet(n)$$

for all finite dimensional subspaces $\tilde{D} \subset \tilde{L}$ satisfying

$$|\partial_{\tilde{G}|\tilde{\mathcal{L}}}|(D) \leq \frac{1}{n}|D|.$$

for all overgroups $\tilde{\Gamma}$ and all finite dimensional subspaces $\tilde{D} \in \tilde{L}$.

If there are several G_i, let

$$\tilde{F}^{max_i}_\bullet(n) = \tilde{F}_\bullet(n_i; \Gamma, G_i)$$

be the maximal $\bullet$-convex function, such that

$$exp(ent_{\updownarrow}(\tilde{D})) \geq \tilde{F}_\bullet(n_i)$$

for all finite dimensional subspaces $\tilde{D} \subset \tilde{L}$ satisfying

$$|\partial_{\tilde{G}_i|\tilde{\mathcal{L}}}|(D) \leq \frac{1}{n_i}|D|.$$

for all overgroups $\tilde{\Gamma}$ and all finite dimensional subspaces $\tilde{D} \in \tilde{L}$, where the $\bullet$-convexity is understood as the convexity of $log(F(1/\varepsilon_i))$ as a function of the vector-variable $\{\varepsilon_i\}$.

Super-multiplicativity Lemma. *Let*

$$1 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 1$$

and let $G_1 = G \cap \Gamma_1$ *and* $G_2 \subset \Gamma_2$ *be the projection of a given* $G \in \Gamma$ *to* Γ_2*.*

Then

$$\tilde{F}_{\bullet}^{max_{1,2}}(n_1, n_2; \Gamma, G_1, G_2) \geq \tilde{F}_{\bullet}(n_1; \Gamma_1, G_1) \cdot \tilde{F}_{\bullet}(n_2; \Gamma_2, G_2).$$

Proof. The $\tilde{\mathcal{L}}$-"partition" of a given $D \subset \tilde{L}$ is constructed in two steps.

First we let $\tilde{\mathcal{L}}_1 \subset \tilde{\mathcal{L}}$ be the the subclass of subspaces that are invariant under Γ_1, or rather under the pullback of Γ_1 to $\tilde{\Gamma}$ and take the $\tilde{\mathcal{L}}_1$-"partition", $D = \oplus_i D_i$ of D of the maximal entropy that is bounded from below with the second Følner function by $log(\tilde{F}_{\bullet}(n; \Gamma_2, G_2))$.

Then we partition each D_i with $\mathcal{L}$. Since the entropy is additive (see section 5) and the boundary $\partial_{G|\mathcal{L}}$ is "slice-wise"-additive the resulting entropy is bounded from below by the sum of the log's of the first and the second Følner functions by the $\bullet$-convexity of the first one as in 5.11.

Remark. This argument, unlike the one in 5.11, does not use the Shannon inequality.

By combining this Lemma with the above $\frac{1}{4}$-Corollary, we arrive at the following.

Chain Extension Inequality . *Let* $1 = \Gamma_0 \subset \Gamma_1 \subset ... \subset \Gamma_i \subset ... \subset \Gamma_N \subset \Gamma$ *be a normal sequence, i.e. each* Γ_i *is normal in* Γ_{i+1}*, and let* $G_i \subset \Gamma_i$*, be subsets, such that the image of* G_i *in* Γ_i/Γ_{i-1} *generates an infinite subgroup for all* $i = 1, 2, ...$ *Then*

$$\tilde{F}_{\bullet}^{max_i}(n_i; \Gamma, G_i) \geq exp(\frac{N}{4})$$

for $n_i \geq 4$*.*

Since the $\tilde{F}$ minorizes the ordinary linear algebraic Følner function F_* (for functions with bounded supports and l_2-functions respectively), we conclude as in 6.1, by applying the above to Cartesian products of infinite groups, that

the l_2*-Følner function* F_{l_2} *as well as* F_* *for finitely supported functions with values in an arbitrary field of the wreath product* Γ *of* Γ_1 *and* Γ_2 *are bounded from below in terms of the growth function of* Γ_2 *by*

$$F_{l_2}(n), F_*(n) \succ exp(\frac{1}{4} G_\circ(n; \Gamma_2)),$$

provided the group Γ_1 *is infinite (finitely generated).*

There are amenable pure torsion groups, e.g. the wreath products of Grigorchuk groups, with growth $G_\circ(n) \sim 2^n$ and where $F_*(n)$ and $F_{l_2}(n)$ are $\succ 2^{2^n}$.

Questions. Do the higher iterates of wreath products of the Grigorchuk groups have faster growing F_* and F_{l_2}. Are there amenable groups (torsion or no torsion) with $F_{l_2}(n)$ growing faster than 2^{2^n}? What are the Følner functions of the iterated wreath products acting on the spaces of decaying functions with values in ultrametric fields?

(If Γ_1 contains an element of infinite order, then the Følner function of $\Gamma = \Gamma_1^{\wr \Gamma_2}$ for decaying functions grows $\succ \varepsilon(1+\varepsilon)^{G_\circ(n;\Gamma_2)}$, $\varepsilon > 0.1$ by 6.1).

Application to Grigorchuk Groups. If a finitely generated group Γ contains a subgroup commensurable to $\Gamma \times \Gamma$ (e.g. a suitable Grigorchuk group), then, by the super-multiplicativity,

$$\tilde{F}(n;\Gamma) \geq \varepsilon_\Gamma \cdot (\tilde{F}(\varepsilon_\Gamma n;\Gamma))^2$$

for some strictly positive ε_Γ. It follows that

the Følner functions F_* and F_{l_2} of such Γ grow at least as fast as $exp(n^\alpha)$ for some $\alpha = \alpha_\Gamma > 0$.

Isoperimetry in Vertical Grassmannians. Let us regard a class $\mathcal{L}$ as a subspace in the Grassmannian $Gr(L)$ of linear subspaces L' of a linear space L. The notion of isoperimetry makes sense for a projective action of a group Γ on $\mathcal{L}$, whenever there is some notion of "dimension" or "rank" for linear subspaces $L' \in \mathcal{L}$.

Such a notion is provided in the l_2-enviroment by the von Neumann Γ_{vert}-dimension, when all $L' \in \mathcal{L}$ are acted upon by a group Γ_{vert} as in the above examples. Futhermore, if Γ_{vert} is amenable, there is the asymptotic rank defined in 1.10.

It seems, such projective actions on sub-Grassmannians constitute a correct general setting for the "linearized isoperimetry". For example, the above entropic argument applies here and delivers the corresponding isoperimetric inequalities. But I have not looked into the subject matter beyond this point.

8 Examples of Group Algebras with Fast and with Slow Growing Følner functions.

We start with a bound on the isoperimetric profiles and Følner functions of certain group algebras and then construct groups where the linear algebraic profiles grow much slower than the combinatorial ones. Also we construct groups with arbitrarily fast growing Følner functions of their group algebras.

8.1 An Upper Bound on I_* for Split Extensions.

Consider an exact sequence of groups

$$1 \to \Theta \to \Gamma \to \Gamma_0 \to 1,$$

where Γ (and, hence, Γ_0) is finitely generated.

If the sequence splits by some embedding $\Gamma_0 \to \Gamma$ and the group Θ is locally finite (i.e. every finitely generated subgroup in Θ is finite) then the isoperimetric profile $I_*(r;A)$ of the group algebra A of Γ is equivalent to the profile $I_*(r;A_0)$ of the group algebra A_0 of Γ_0 with an arbitrary coe cient field $\mathbb{F}$. In particular, if $\Gamma_0 = \mathbb{Z}$, then $I_*(r;A)$ is bounded.

Proof. Choose a symmetric finite generating set $G = G_0 \cup G_1 \subset \Gamma$, where $G_0 \subset \Gamma_0 \subset \Gamma$ and $G_1 \subset \Theta$. Take a linear space $D_0 \subset A_0$ of $\mathbb{F}$-valued functions on Γ_0 with supports in a finite subset $Y_0 \subset \Gamma_0$. Denote by $\Theta_1 \subset \Theta$ the (finite!) subgroup generated by yg_1y^{-1} for all $y \in Y_0$ and $g_1 \in G_1$.

Let $a_1 \in A$ be an $\mathbb{F}$-valued function on Γ that equals 1 on $\Theta_1 \subset \Theta \subset \Gamma$ and that vanishes everywhere else on Γ. Observe that a_1 is invariant under all above yg_1y^{-1}.

The embedding $\Gamma_0 \to \Gamma$ makes Γ_0 and hence its group algebra A_0 act on functions on Γ; we denote by D the linear span of the functions $d * a_1$ on Γ for

by the $\mathcal{L}$-length inequality in 7.1.

Questions. Let an infinite finitely generated group Γ have I_* bounded. Is it then a $\mathbb{Z}$-extension of a locally finite group?

Does any of the following conditions help?

(1) Γ is a torsion group;

(2) Γ contains an infinite finitely generated subgroup of infinite index;

(3) Γ contains an element of infinite order;

(4) Γ has no torsion.

If Γ has no torsion, does then the polynomial bound on the Følner function, $F_*(r) \preceq r^k$ for some $k > 1$, imply that Γ is virtually nilpotent?

8.2 Amenable Groups with Fast Growing Følner Functions.

Let F_X denote the free group on a set X of generators and let $[Y]_k \subset F_X$ be the set of the commutators $[...[[x_1, x_2], x_3]...x_k]$ for all k-long sequences $x_i \in Y$.

Denote by $F_X(p)$ the quotient group of F_x by the relation $x^p = 1$ for some prime number p and all $x \in X$ (that is the free product of X copies of $\mathbb{Z}_p$) and observe that

1. *If an element in $F_X(p)$ is representable by a word $y_1^{i_1} \cdot y_2^{i_2} \cdot ... \cdot y_k^{i_k}$, where the neighboring $y_i \subset Y$ are not equal and where no i_j is divisible by p, then the image of w under the composed (surjective) homomorphism $h : F_X \to F_Y(p)/[Y]_k$,*

$$F_X \to F_Y \to F_Y(p) \to F_Y(p)/[Y]_k$$

is $\neq 1$.

Here is the (standard)

Proof. Assume $X = Y$ and let W_k be the $\mathbb{F}_p$-linear space (formally) spanned by the irreducible words $w = y_1^{j_1} \cdot y_2^{j_2} \cdot ... \cdot y_l^{j_l} \subset F_X$, where all $j \geq 0$ and the total degree of w satisfies $deg(w) = j_1 + j_2 + ... + j_l \leq k$.

Let A_y be the (nilpotent) linear operators on L_k defined by their values at $w \in L_k$ as follows, $A(w) = yw$ for $deg(w) < k$ and $A(w) = 0$ otherwise.

The map $y \mapsto 1 - A_y$ extends to a homomorphisms of $F_Y(p)/[Y]_k$ to the group of linear transformations of W_k, where each y^{-1} goes to the *finite* sum $1 + A_y + A_y^2 + ... + A_y^k$, since $A_y^{k+1} = 0$. Then the non-equality $h(w) \neq 1$ follows from non-vanishing of the product operator $A_{y_1}^{j_1} \cdot A_{y_2}^{j_2} \cdot ... \cdot Ay_l{}^{j_l}$.

This k-acyclicity/divergence of $F_Y(p)/[Y]_k$ implies the following lower bound on the Følner function of the (finite for $p < \infty$) group $F_Y(p)/[Y]_k$ for the (joint, see 1.1) boundary ∂_Y, where we regard Y as a (generating) subset in $F_Y(p)/[Y]_k$,

$$F_\circ(n) \geq ((|Y| - 1)(q - 1))^{\alpha k}$$

for $q = min(n, p)$ and some $\alpha \geq 1/4$.

Remark. One can do better including estimates on the $E_\circ$ and $P_\circ$ functions of the group $N(k) = F_Y(p)/[Y]_k$ with respect to the cyclic subgroups generated by $y \in Y$, while we shall need below only the (rough but uniform) bound,

$$(1_k) \qquad F_\circ(n) = F_\circ(n; N(k), Y) \geq k \text{ for all } p \geq 3 \text{ and } |Y| \geq 2$$

for all $n \geq 1$, that is stated below in a form that is convenient for making our examples.

2. Let $k(Y)$ be a function (rather than a number) on the subsets $Y \subset X$ with values $1, 2, 3...., \infty$, such that $Y_1 \subset Y_2 \Rightarrow k(Y_1) \leq k(Y_2)$, and let $k_2(Y)(\leq k(Y))$ be the maximum of $k(Y')$ over all subsets $Y' \subset Y$ of cardinality $|Y'| = 2$. Let $N(k)$ be the factor group of $F_X(p)$ obtained with the above (commutator) relations $[Y_{k(Y)}]$ for all $Y \subset Y$, where $[Y_{k(Y)}]$ for $k(Y) = \infty$ means "no relation". Then

$$F_\circ(n) = F_\circ(n; N(k), Y) \geq k_2(Y)$$

for all $Y \subset X$ with $|Y| \geq 2$, all $p \geq 3$ and $n \geq 1$.

Next, let X be acted by a group Γ with a given finite generating set $G \subset \Gamma$ and let the function $k(Y)$ be Γ-invariant.

Then Γ acts by automorphisms on $N(k)$ and thus, diagonally, on $N(k) \times X$. The group $N(k)$ also acts on the product $N(k) \times X$ by translations on the first factor and the group generated by the two actions is denoted by $N(k) \ltimes \Gamma$. If the action of Γ on X is transitive, then the group $N(k) \ltimes \Gamma$ is finitely generated, namely, by the subset $\{x_0\} \cup G \subset N \ltimes \Gamma$ for an arbitrary $x_0 \in X \subset N \ltimes \Gamma$,

3. Let $Diam_G(Y)$ be the minimal number m, such that every point $y_1 \in Y$, can be moved to any other point $y_2 \in Y$ by some $y \in G^{\cdot m}$, i.e. $y(y_1) = y_2$, where we assume that G is symmetric (under $y \longleftrightarrow y^{-1}$) and contains id. Denote by $k_2(m)$ the supremum of $k_2(Y)$ over all $Y \subset X$ with $Diam_G(Y) = m$.

The Følner function of the action of $N(k) \ltimes \Gamma, \{x_0\}$ on $N(k) \times X$ with the generating subset $\{x_0\} \cup G \subset N(k) \ltimes \Gamma, \{x_0\}$satisfies

$$F_\circ(mn; N(k) \times X) \geq k_2(m)$$

for $n \geq 1$ and all $m = 2, 3, ...$ In particular, if $k(Y)$ depends only on $m = Diam_G(Y)$, then $k_2 = k$ and

$$F_\circ(mn; N(k) \times X) \geq k(m).$$

Proof. Combine the above with (n/m) in 1.9.

This shows that one can make $F(n)$ grow arbitrarily fast with a fast growing function $k(m)$ and, at the same time, have $N(k)$ locally nilpotent, thus, $N(k) \ltimes \Gamma$ amenable, if Γ is amenable.

Here is the summary.

Example: Amenable Groups with $F_\circ \geq \kappa(n)$. *Let $X = \Gamma$ be an amenable group with a finite generating subset $G \subset \Gamma$ and let $\kappa(n)$, n=1,2,..., be an arbitrary monotone increasing $\mathbb{Z}_+$-valued function. Then the group $N(k) \ltimes \Gamma$ for $k(Y) = \kappa(Diam(Y)^2)$, $Y \subset \Gamma$, is finitely generated amenable and*

$$F_\circ(n; N(k) \ltimes \Gamma, G) \geq \kappa(n).$$

for all $n \geq 1$.

Examples with the Linear Algebraic Følner functions F_*. If $p = \infty$ then the group $N(k)$ is orderable and the above yields

the lower bound

$$F_*(n; N(k) \ltimes \Gamma) \geq \kappa(n)$$

for the (action of $N(k) \ltimes \Gamma$ on) functions with compact supports on $N(k) \ltimes \Gamma$ with values in an arbitrary field (where one does not need Γ to be orderable).

On the other hand,

if $p \leq \infty$, then

$$F_*(n; N(k) \ltimes \Gamma) \leq F_\circ(n; \Gamma)$$

while $F_\circ(n; N(k) \ltimes \Gamma)$ can grow as fast as one wishes.

Remark and Questions. (a) The existence of groups $N \ltimes \mathbb{Z}$ with arbitrarily fast growing $F_\circ$, where N is a locally finite group, is indicated in [9] and groups of *intermediate growth* with arbitrarily fast growing $F_\circ$ are constructed in [10].

(b) It is not hard to give an upper bound on $F_\circ$ for the above group and observe that

$$F_*(n; N(k) \ltimes \Gamma) \sim \kappa(n)$$

for the functions $\kappa(n)$ with sufficiently fast growing derivatives.

(c) Can one achieve the above with a construction (invariant under all permutation of X) with $k(Y)$ depending on the cardinality, rather than the diameter of Y? (Possibly, this can be done with the existence theorem for non-Abelian free subgroups in groups with universal commutator relations, [3]).

(d) Is there an universal bound on the asymptotic growth of the Følner functions of *finitely presented* amenable groups by a recursive (primitively recursive?) function? (Maybe there is such a bound in every given recursive class of presentations?). Or, at another extreme, are there finitely presented amenable groups with so fast growing $F_\circ(n)$, such that their amenability is unprovable in Arithmetic? (An enticing possibility would be this situation for the Thompson group).

9 $E_\circ$-Functions and Isoperimetric Inequalities for Wreath Powers of Partitions and Groups .

The isoperimetric inequalities in the previous section are similar to those for the wreath product of groups that were obtained earlier in an asymptotically sharp form in [9] with an (implicate) use of $E_\circ$-functions. We shall present below a translation of the argument from [9] to the language of partitions that provides a *quantitative* (better constants) improvement of the isoperimetric inequalities for iterated wreath products of groups and a *qualitative* (asymptotic behavior) improvement for infinitely iterated wreath products.

9.1 Wreath Powers of Sets.

Let Z be a discrete set with a distinguished family $\mathcal{S}$ of subsets (slices) $S \subset Z$, and define the *$\wr$-space* associated to $\mathcal{S}$, denoted $Z_\wr$, that is the set of of the pairs (S, z) for $S \in \mathcal{S}$ and $z \in S$.

The simplest (and the largest) such $Z_\wr$ is obtained with $\mathcal{S} = 2^Z$, the family of all subsets in Z.

The set $Z_\wr$ naturally carries two partitions P_1 and P_2: the slices of P_1 are obtained by varying z in the pair (S, z) within S, while the slices of P_2 consist of the pairs (S, z) with fixed z and all $S \ni z$.

Basic example. Let $Z = X^I$ with $\mathcal{S}$ being the family of the "coordinate lines". In this case we write $Z_\wr = Z \times I = X^{\wr I}$ and denote the two corresponding partitions of $X^{\wr I}$ by $P_1 = P_X$, where all slices are copies of X, and by $P_2 = P_I$, where the slices are copies of I.

Recall that this $\mathcal{S}$ is *Shannon*: the geometric mean of the cardinalities of the $\mathcal{S}$-slices of every $Y \subset Z$ is bounded by the cardinality of Y,

$$\prod_S |S \cap Y|^{\frac{1}{|Y|}} \leq |Y|,$$

where the product is taken over the slices $S \in \mathcal{S}$ with non-empty intersections $S \cap Y$.

Evaluation of $E_\circ$, $P_\circ$ and $F_\odot$ for $Z_\wr$. *If $\mathcal{S}$ is Shannon, then*

$$(E_\circ) \qquad E_\circ(L_1, L_2; P_1, P_2) = L_2 \cdot L_1^{L_2},$$

$$(P_\circ) \qquad P_\circ(n_1, n_2; Z_\wr/P_1, Z_\wr/P_2) = n_2 \cdot n_1^{n_2}$$

and

$$(F^{\wr F}) \qquad F_\odot(F_1(n_1), F_2(n_2)) \geq d_2 F_2(\varepsilon_2 n_2$$

for the "isoperimetric profiles" I_i associated with F_i and let $Y_i \subset Y$, $i = 1, 2$, be the union of the P_i-slices of cardinalities $\geq F_i((1-\varepsilon_i)n_i)$. Then the intersection $Y_\bullet = Y_1 \cap Y_2$ is non-empty and has "large average measures" of P_1- and P_2-slices according to $[\cap_i/P_i]$ in 4.5. Then (A^B) applies with $A = d_1 F_1(\varepsilon_1 n_1))$ and $B = d_2 F_2(\varepsilon_2 n_2)$.

Finally we turn to $F_\bullet$ and observe that the inequality

$$\sum_{S \in P_2} I_2(|S \cap Y|) \leq \frac{1}{n_2}|Y|$$

implies that the number of the P_2-slices of Y is bounded by

$$|P_2 \cap Y| \leq \frac{1}{n_2}|Y|$$

and the proof follows by the $\bullet$-convexity of $F_\circ(n; \Gamma_1)$.

Remarks. (a) The evaluation of the $E_\circ$ function for $X^{\wr I}$ (and for $Z\wr$ associated to an expanding family of partitions P_i in general) can be seen directly by counting P_i-chains in Z, (see, [9] where it is done, in the language of hypergraphs rather than partitions, for wreath products of groups and graphs).

(b) There are further constructions and partitions associated to power spaces X^I, e.g the $\wr$-spaces associated to the "plane" rather' than "line" partitions. Then, one can take Cartesian powers of $Z\wr$ that are products of the original P_i. Evaluation of the invariants the resulting families of partitions remains an open problem.

(c) It is unclear, in general, if there is a Shannon type inequality for *measures* μ on (rather than subsets in) $Z_\wr$, except for the case of the atoms in each P_2-slice S having equal weights, denoted μ_S, where the Shannon inequality relates the entropy of P_1 and the measure of P_2 with the weights $S \mapsto \mu_S$,

$$(\ ) \qquad ent_\mu(X) \geq ent_\mu(Z_\wr) = \frac{\mu(Z_\wr)}{\sum_{S \in P_2} \mu_S} + log \frac{\mu(Z_\wr)}{\sum_{S \in P_2} \mu_S}.$$

But it is doubtful that the concepts of "measure" and/or "entropy" adequately reflect the combinatorics of such families of partitions P_i .

9.2 Følner Functions for Iterated Wreath Product of Groups.

Given groups Γ_1 and Γ_2, denote (with a minor abuse of notation) by $\Gamma_1^{\Gamma_2}$ the *restricted* Cartesian power group consisting of the functions $\theta : \Gamma_2 \to \Gamma_1$ with finite supports (i.e. equal $id \in \Gamma_1$ away from a finite subset in Γ_2) and observe that both Γ_1 and Γ_2 naturally act on $\Gamma_1^{\Gamma_2}$, where Γ_1 is implemented in $\Gamma_1^{\Gamma_2}$ by the functions trivial (i.e. $= id \in \Gamma_1$) away from $id \in \Gamma_2$. The group generated by these two actions is called the *wreath product* of Γ_1 and Γ_2, that we denote by $\Gamma_1^{\wr\Gamma_2}$. (The customary notation for the wreath product is $\Gamma_1 \wr \Gamma_2$ and/or $\Gamma_2 \wr \Gamma_1$).

There is a natural exact sequence

$$1 \to \Gamma_1^{\Gamma_2} \to \Gamma_1^{\wr\Gamma_2} \to \Gamma_2 \to 1,$$

where $\Gamma_1^{\Gamma_2}$ is given the coordinate-wise group structure (as the Cartesian product of Γ_2 copies of Γ_1) and where Γ_1 is embedded into $\Gamma_1^{\Gamma_2}$ and, hence, into $\Gamma_1^{\wr\Gamma_2}$,

by functions $\theta : \Gamma_2 \to \Gamma_1$ such that $\theta(\gamma_2) = id \in \Gamma_1$ for $\gamma_2 \neq id \in \Gamma_2$. This sequence is split by the obvious embedding $\Gamma_2 \to \Gamma_1^{\wr\Gamma_2}$.

Observe that the group $\Gamma_1^{\wr\Gamma_2}$ is finitely generated if Γ_1 and Γ_2 are finitely generated, while the subgroup $\Gamma_1^{\Gamma_2} \subset \Gamma_1^{\wr\Gamma_2}$ is infinitely generated for infinite groups Γ_2.

The wreath product is a binary operation on groups, now written $\Gamma_1 \wr \Gamma_2$ for $\Gamma_1^{\wr\Gamma_2}$, that is neither commutative nor associative: the composed wreath product of k groups is determined by the way we put the brackets. Here are three (most symmetric) examples for $k = 8$,

$$.(.(.(... \qquad \Gamma_1 \wr (\Gamma_2 \wr (\Gamma_3 \wr (\Gamma_4 \wr (\Gamma_5 \wr (\Gamma_6 \wr (\Gamma_7 \wr \Gamma_8)))))),$$

$$.).).).... \qquad ((((((\Gamma_1 \wr \Gamma_2) \wr \Gamma_3) \wr \Gamma_4) \wr \Gamma_5) \wr \Gamma_6) \wr \Gamma_7) \wr \Gamma_8$$

and

$$...(()()).... \qquad ((\Gamma_1 \wr \Gamma_2) \wr (\Gamma_3 \wr \Gamma_4)) \wr ((\Gamma_5 \wr \Gamma_6) \wr (\Gamma_7 \wr \Gamma_8))$$

The $E_\circ$-functions (obviously) follow the composition rule for partitioning slices and since the $E_\circ$-function of the simple wreath product equals the numerical $\wr$-*product*, i.e. $L_1 \wr L_2 =_{def} L_2 \cdot L_1^{L_2}$. the $E_\circ$-functions of the iterated wreath products relative to the subgroups Γ_i equal the corresponding composed $\wr$-product with the same bracketing as for the corresponding Γ, e.g.

$$E(L_1, L_2, L_3...) = L_1 \wr (L_2(\wr(L_3 \wr ....)))...$$

for $.(.(.(...$

Now we invoke the lower bound of the Følner functions by $E_\circ$-functions (see 4.5) and obtain

Erschler Inequality for Wreath Products. *The multivariable Følner function of an iterated wreath product of Γ_i is bounded from below by the correspondingly composed $\wr$-product of $c_i F_i(\varepsilon_i n_i)$, where F_i are the Følner functions of Γ_i and c_i, ε_i are arbitrary positive constants satisfying $\sum_i(\varepsilon_i + c_i(1-\varepsilon_i)) < 1$.*

Remark. The above $(F^{\wr F})$ serves slightly better for the simple products,

$$F_\circ(n_1, n_2; \Gamma_1^{\wr\Gamma_2}) \geq d_2 F_2(\varepsilon_2 n)(d_1 F_1(\varepsilon_1 n))^{d_2 F_2(\varepsilon_2 n)},$$

where,

$$d_i = \frac{1 - \varepsilon_1 - \varepsilon_2}{1 - \varepsilon_i}$$

for arbitrary positive numbers ε_i, $i = 1, 2$ satisfying $\varepsilon_1 + \varepsilon_2 < 1$.

Similarly, $(F_\bullet)$ implies that

$$F_\bullet(n_1, n_2; \Gamma_1^{\wr\mathbb{Z}}) = n_2 F_\bullet(n_1, \Gamma_1)^{n_2}.$$

Question. Is there a sharp bound on the Følner functions of the (composed) wreath products more general than $(F_\bullet)$?

Remarks. (a) The question of evaluating the Følner functions for wreath products was raised by A.Vershik in [29]. C. Pittet and L. Saloff-Coste, found (see [27]) a lower bound on the Følner function $F_\circ$ of the wreath products of Abelian by finite groups (i.e. $Fin^{\wr Ab}$); their inequality, albeit asymptotically non-sharp, provided the first examples of $F_\circ$ with super-exponential growth.

The Vershik question was answered in [9] with the inequality $(F^{\wr F})$, where the iterated case followed by induction. (The constants c_i and ε_i in $c_i F_i(\varepsilon_i n_i)$ in [9] are smaller than those imposed by the above $\sum_i(\varepsilon_i + c_i(1-\varepsilon_i)) < 1$ and the inductive argument makes them dependent on the shape of bracketing with the best result for (()...()) where the induction goes from i to $2i$. We see the relevance of the constants below.)

(b) The isoperimetric inequalities for iterated wreath products of *finite* groups Γ and graphs (see [9]) and their modifications provide examples of graphs with expander-type isoperimetry (see [19]).

(c) *Wreath Products over Homogeneous Spaces.* The above inequalities for groups straightforwardly generalize (see [9]) to homogeneous spaces, e.g. to $X_2 = \Gamma_2/\Gamma_2'$ and the group $\Gamma \supset \Gamma_1^{X_2}$ generated by the restricted Cartesian power group $\Gamma_1^{X_2}$, where Γ_2 naturally acts on $\Gamma_1^{X_2}$:

the max-Følner function $F_\circ(n;\Gamma/\Gamma_2')$ of the action of Γ on $\Gamma/\Gamma_2' = \Gamma_1^{\wr X_2}$ as well as the max-Følner of the group Γ itself are bounded in terms of those for Γ_1 and the action of Γ_2 on X by

$$F_\circ(n;\Gamma) \geq F_\circ(n;\Gamma/\Gamma_2') \geq F_2 \cdot F_1^{F_2}$$

for $F_1(n) = \frac{1}{2}F(\frac{n}{3};\Gamma_1)$ and $F_2(n) = \frac{1}{2}F(\frac{n}{3};X_2)$.

(d) The lower bound by $(L_1^{\wr L_2})$ for $E_\circ$-function of the wreath powers (of sets) combined with 4.4 yields a proof of the following version of $F^{\wr F}$ for the Følner functions associated to $I_\circ^{incr}$, the maximal monotone increasing minorant of $I_\circ$,

F^{incr} -Inequality.

$$F_\circ^{incr}(\Gamma_1^{\wr\Gamma_2}, n) \geq F_\circ^{incr}(\Gamma_2, \frac{n}{2})\left(F_\circ^{incr}(\Gamma_1, \frac{n}{2})\right)^{F_\circ^{incr}(\Gamma_2, \frac{n}{2})}.$$

This inequality F^{incr} is slightly sharper than $F^{\wr F}$ if Γ_i are (necessarily infinite) groups with monotone increasing profiles $I_\circ(r)$. But $F^{\wr F}$, meaningfully applies to finite groups where the inequality F^{incr} is vacuous.

Isoperimetric Inequality for Groups with Distorted Wreath Product Subgroups.. Take an iterated wreath product Γ of Γ_i and let α be an endomorphism of Γ induced by injective endomorpisms α_i of Γ_i. Then Γ_i can be distorted in the group $\Gamma^\alpha = \Gamma \times_\alpha \mathbb{Z}$.

For instance, if Γ_i are free Abelian and $\alpha_i(\gamma_i) = 2\gamma_i$ for all i and $\gamma_i \in \Gamma_i$, then $\Gamma_i \subset \Gamma^\alpha$ are exponentially distorted and, thus,

$$F_\circ(n, \Gamma^\alpha) \sim F_\circ(exp(n), \Gamma).$$

9.3 Infinite $\wr$- and $\uparrow$-Products.

Probably, the Cartesian products and the wreath products are the only 2-variable functorial operations, obtained by adding universal relations to free products of pairs of groups, in the category of *amenable* groups, but there are further "products" in the category of ordered groups obtained as follows.

Denote by $\Gamma^{\wr_n} = \Gamma^{(\wr_n}$ the n-th wreath iteration .(.(.(... with all $\Gamma_i = \Gamma_1$, let I be an arbitrary countable ordered set and assign to each n-element subset $J \subset I$ the group $\Gamma_1^{\wr_J} = \Gamma_1^{\wr_n}$. This assignment is obviously functorial for inclusions of subsets and we can define the inductive limit for J exhausting I, denoted $\Gamma_1^{\wr_I}$.

If I is infinite then this group is infinitely generated; but if we take some finitely generated group Γ_2 of order preserving transformations transitively acting on I, then the group generated by $\Gamma_1^{\wr_I}$ and Γ_2 is finitely generated. If we take an ordered group Γ_2 for I, we denote the resulting group Γ by $\Gamma_1^{\uparrow\Gamma_2}$ (in agreement with Knuth arrow notation) and regard it as the "first order wreath product" (we regard $\wr$ as the zero order product) of the two groups. (See [9] for the case $I = \mathbb{Z}$).

We iterate this n times and, by passing from $\Gamma_1^{\uparrow_n} = \Gamma_1^{(\uparrow_n}$ to the inductive limit, arrive at next product $\Gamma = \Gamma_1^{\uparrow\uparrow\Gamma_2}$, where we need both groups to be ordered. Similarly we define the $\uparrow\uparrow\uparrow$-product ,etc.

The above construction equally applies to the right bracketing .).).)... but it does not (seem to) work as it stands for the ...(()())...-bracketing: we need a finitely generated order preserving group Γ_2 transitive on all n-tuples of points in I. There is a finitely generated group with this property, the Richard Thompson group, but this group is unknown to be amenable or non-amenable. (We are keen on getting large amenable group by such procedures). On the other hand the *finitely generated* group obtained in such a way is rather amusing; for instance, it contains the finite iterated wreath products with all positions of the brackets and the same applies to $\uparrow\uparrow$ ...-products of any given order.

Questions. (a) Do the $E_\circ$-functions (and/or the Følner functions) of the partitions of the above Γ into the orbits of Γ_i equal (asymptotically equivalent to) the corresponding "arrowings" of the function $n_1 \wr n_2 = n_2 n_1^{n_2}$?

Here is a (little) step in this direction.

Isoperimetric Inequality for $\Gamma = \Gamma_1 \uparrow \Gamma_2$. Let $\Gamma^k \subset \Gamma$ be the subgroup generated by the union of $G_1^{\gamma_i} \subset \Gamma$, $i = 1, 2, ..., k$, for some k elements in Γ_2 of the G_2-length $\leq l$, where G_i are given generating subsets in $\Gamma_i \subset \Gamma$, $i = 1, 2$.

The max-Følner function of the group Γ^k, that isomorphic to the left-iterated i.e. ...$\wr(\wr(...$, bracketed wreath product of Γ_1 with itself, is bounded from below by the k-iterated $\wr$-composition of $F_k(n) = 1/2kF_\circ(n/2k; \Gamma_1, G_1)$, denoted $\wr_k F_k(n)$, where the max-boundary refers to all $k|G_1|$ generators in $\Gamma_k \subset \Gamma$.

Thus, the Følner function of $(\Gamma, G_1 \cup G_2)$ is bounded from below by $F_k(n/3l)$ (compare with 6.1).

Take γ_i from $G_2^l \subset \Gamma_2$, such that $l \cdot |G_2^l| \sim n/10$ for the appropriate $l = l(n)$, and regard the cardinality $|G_2^l|$ as the function $\phi(n)$. For example, if the growth function $G_\circ(l) = |G_2^l|$ of Γ_2 equals l^m then $\phi(n) \sim n^{\frac{m}{m+1}}$ and if Γ_2 has exponential growth, then $\phi(n) \sim n/log(n)$.

Then, by the argument in in 6.1

the Følner function $F_\circ(n; \Gamma)$ *is bounded from below by* $\wr_{\varphi(n)} 2^n$ *that is equivalent to the* $\phi(n)$*-iterated exponential function* $exp(exp(...(exp(2))...))$.

Questions. (a) Apparently, one has a poor understanding of the combinatorial structure of the orbit partitions of such groups as $\Gamma_1 \uparrow_i \Gamma_2$, where the attractive feature of this structure is being universal, defined for arbitrary ordered sets, rather than groups, and where the necessity of the order is due to the fact that the $\wr$-product, a two-variable functor on sets, is non-commutative.

If $i = 2$, there are two partitions P_1 and P_2 of $\Gamma_1 \uparrow \Gamma_2$, where the combina-

torics is faithfully encoded by the bipartite graph on the vertex set of the slices of P_i with the edges corresponding to the pairs of intersecting slices.

What is the combinatorics of such graphs? For example, what are the spectra (of the combinatorial Laplacians) of such graphs constructed with *finite* ordered sets?

Does the graph theoretic picture reveal anything or it is better to think in terms of partitions?

What does correspond to the concepts "measure" and/or "entropy" that would lead to something like Shannon inequalities?

Do the groups like $\Gamma_1 \uparrow_i \Gamma_2$ embed (preferably functorially) into *finitely presented ordered* groups Γ (similar to Thomson group $\Gamma \supset \mathbb{Z} \wr \mathbb{Z}$), where one also wants Γ to be amenable for amenable Γ_i?

Are there other "interesting " milti-variable operations (functors?) in the category of finitely generated groups that preserve the classes of amenable and/or of orderable groups? What are the functors from the category of finite ordered sets to the category of such operations on groups?

An example of a one variable operation is $\Gamma \rightsquigarrow \Gamma' \supset \Gamma$, where Γ' consists of the bijective maps $\Gamma \to \Gamma$ which are equal to left translations in Γ away from a finite subset. (If $G \subset \Gamma$ generates Γ then Γ' is generated by G and the permutations of G; probably, $F_\circ(\Gamma') \sim (F_\circ(\Gamma))!$.)

There are similar extensions of Γ with "simple" bijective maps $\Gamma \to \Gamma$, e.g. the maps that are translations on a subgroup $\Gamma_0 \subset \Gamma$ and that fix $\gamma \in \Gamma \setminus \Gamma_0$, but making such an extension functorially is problematic.

(b) It seems impossible to continue the $\uparrow$ hierarchy in the category of ordered groups. For example, it is unclear if all groups $\mathbb{Z}^{(\uparrow_k}$ embed into an amenable group on two generators. (In general, one does not seem to know which countable families of finitely generated amenable groups embed into a single finitely generated amenable group, while the answer is an easy "yes for all" in the class of *all* groups by the small cancellation theory for relatively hyperbolic groups.)

Yet, one can go several steps up if the background group Γ_2 admits an infinite strictly increasing family of subgroups say $\Gamma_3 \subset \Gamma_4 \subset ... \subset \Gamma_k \subset ....$ where, similarly to the k-commutator relations from in 8.2, one adds the relations defining the kth arrowing operation to each Γ_k. Then the resulting group Γ (that can be arranged with three generators) contains all $\mathbb{Z}^{(\uparrow_k}$ and, probably, its Følner function grows (at least) as fast as the Ackermann function.

It is unclear up to which ordinals one can go in a similar manner keeping groups finitely generated. (Every elementary amenable group Γ built by normal extensions and inductive limits is characterized by the ordinal that expresses the minimal "number" of steps in such description of Γ; it seems unknown if all countable ordinals correspond to finitely generated Γ). Apparently, one needs a group theoretic realization of more powerful (implicit) recursion schemes starting from groups with some "self similarity", e.g. finitely generated (even better finitely presented) amenable groups Γ containing (better, isomorphic to) $\Gamma^{\wr\Gamma}$; but such groups are yet to be found (or shown not to exist).

10 Brunn-Minkowski inequalities.

Recall that the Euclidean Brunn-Minkowski inequality provides the following lower bound on the product (called *Minkowski sum* and usually denoted "+"

rather than "$\cdot$" in the Abelian case) of measurable subsets $Y_1, ..., Y_m$ in $\mathbb{R}^k$,

$$BM^{\frac{1}{k}} \qquad |Y_1 \cdot Y_2 \cdot ... \cdot Y_m|^{\frac{1}{k}} \geq |Y_1|^{\frac{1}{k}} + |Y_2|^{\frac{1}{k}} + ... + |Y_m|^{\frac{1}{k}}.$$

(See the survey article by Gardner [12].)

Remark on Nilpotent Groups. The classical proof of $\mathrm{BM}^{\frac{1}{k}}$ extends to the nilpotent case and yields $\mathrm{BM}^{\frac{1}{k}}$ in the simply connected nilpotent (and, probably, solvable) Lie groups Γ of dimension k.

10.1 Asymptotic Brunn-Minkowski inequalities for Discrete Groups.

The inequality $\mathrm{BM}^{\frac{1}{k}}$, unlike the isoperimetric inequality, does not directly pass to co-compact discrete subgroups Γ as it is seen, for example, for subsets Y_1 and Y_2 in $\Gamma = \mathbb{Z}^2$ that are both contained in a cyclic subgroup. To correct for this one may stabilize the Y's by taking their powers

$$Y^N = \underbrace{Y \cdot Y \cdot ... \cdot Y}_{N} \subset \Gamma.$$

(Such stabilization in the context of polynomial rings was suggested to me by Andrei Suslin.)

If $\Gamma = \mathbb{Z}^k$ then the majority of points in the power set Y^N of every finite subset $Y \subset \mathbb{Z}^k$ are obtained by intersecting the subgroup $\Gamma_Y \subset \Gamma$ generated by Y with a convex subset $C_N \subset \mathbb{R}^k \supset \mathbb{Z}^k$, where the subsets $\frac{1}{N}C_N$ converge in $\mathbb{R}^k$ for $N \to \infty$.

Then $\mathrm{BM}^{\frac{1}{k}}$ in $\mathbb{R}^k$ implies the following asymptotic inequality in $\mathbb{Z}^k$,

$$BM_{\varepsilon}^{\frac{1}{k}} \qquad (|Y_1^N \cdot Y_2^N \cdot ... \cdot Y_m^N|)^{\frac{1}{k}} - (|Y_1^N|^{\frac{1}{k}} + |Y_2^N|^{\frac{1}{k}} + ... + |Y_m^N|^{\frac{1}{k}}) \geq -\varepsilon_N N^k$$

where, $\varepsilon_N \to 0$ for $N \to \infty$.

Question. What happens for finite N? Is there a meaningful bound on $|\varepsilon_N|$ independent of Y's ? Are there BM-type inequalities between "monomials" $|Y_1^{N_1} \cdot Y_2^{N_2} \cdot ... \cdot Y_m^{N_m}|$?

Naively, one expects that BM-type lower bounds on $|Y_i^N|$, say $|Y_i^2| \geq 2^k |Y_i|$, may yield such a bound on $|Y_1 \cdot Y_2 \cdot ... \cdot Y_m|$ but it seems rather subtle already for $\mathbb{Z}$.

Limit form of $BM_{\varepsilon}^{\frac{1}{k}}$ for $N \to \infty$. Since $\mathbb{Z}^k$ is commutative, so is the Minkowski sum (written here as product) and $Y_1^N \cdot Y_2^N = (Y_1 \cdot Y_2)^N$. Using this and $\mathrm{BM}_{\varepsilon}^{\frac{1}{k}}$, one concludes that, for every finite $Y \subset \mathbb{Z}^k$ there exists the limit,

$$|Y|_\infty =_{def} lim_{N \to \infty} \frac{1}{N^k} |Y^N|,$$

and the inequality $\mathrm{BM}_{\varepsilon}^{\frac{1}{k}}$ takes a more compact form,

$$BM_{\infty}^{\frac{1}{k}} \qquad |Y_1 \cdot Y_2 \cdot ... \cdot Y_m|_\infty^{\frac{1}{k}} \geq |Y_1|_\infty^{\frac{1}{k}} + |Y_2|_\infty^{\frac{1}{k}} + ... + |Y_m|_\infty^{\frac{1}{k}}.$$

Remark

measures and the cardinalities of subsets by the entropies of these measures, but I have not worked out any example of a BM-inequality in this context.

10.2 Linearized BM-inequalities for Biorderable Groups.

Given an algebra A over some field $\mathbb{F}$ and linear subspaces $D_1, D_2 \subset A$ we denote by $D_1 * D_2 \subset A$ the $\mathbb{F}$-linear span of the products $d_1 d_2$ for all $d_1 \in D_1$ and $d_2 \in D_2$, and we reiterate the above discussion with the D-monomials

$$|D_{i_1}^{N_1} * D_{i_2}^{N_2} * ... * D_{i_n}^{N_n}| \text{ instead of } |Y_{i_1}^{N_1} \cdot Y_{i_2}^{N_2} \cdot ... \cdot Y_{i_n}^{N_n}|,$$

where $|...|$ in the D-context stands for the $\mathbb{F}$-rank of a linear space rather than cardinality as in the Y-situation.

We are most interested in the case of A being the group algebra of some Γ where we expect the D-monomials satisfy BM-type inequalities similar to those for Y-monomials and we shall prove this below for *biorderable* groups:

a group Γ is called biorderable if it admits an order invariant under left and right translations.

Examples.

$BM^{\frac{1}{k}}_{lin}$ $\qquad |D_1 * D_2 * ... * D_m|^{\frac{1}{k}}_{\infty} \geq |D_1|^{\frac{1}{k}}_{\infty} + |D_2|^{\frac{1}{k}}_{\infty} + ... + |D_m|^{\frac{1}{k}}_{\infty}.$

Remark. The Euclidean BM-inequality, as well as a more powerful *Alexandrov-Fenchel inequality for mixed volumes* for convex sets, can be proved by an algebra-geometric (or by an essentially equivalent complex analytic) argument due to Hovanski and Teissier (see [13] and references therein), where BM is reduced to a similar inequality between the ranks of the spaces $H^0(\mathcal{L}_i)$ of regular sections of positive line bundles $\mathcal{L}_i$, $i = 1, 2, ..., m$, (over toric varieties associated to the sets of the integer points of convex polyhedra) and the tensor products of $\mathcal{L}_i$. Probably, $\mathrm{BM}^{\frac{1}{k}}_{lin}$ (and also Alexandrov-Fenchel) remains valid, under suitable assumptions, for linear subspaces $D_i \subset H^0(\mathcal{L}_i)$ and it would be also interesting to find something similar for non-commutative nilpotent groups Γ.

10.3 BM-Inequality for Fast Decaying Functions on Γ.

Since the construction of the triangular bases needs only the one-sided bound on the support of functions on $\Gamma = \mathbb{Z}^k$, the above argument shows that

the inequality $BM^{\frac{1}{k}}_{lin}$ holds in the (commutative) algebra A_∞ of formal power series in k variables over any field.

It follows that

($\mathbf{1}_{alg}$) *The inequality $BM^{\frac{1}{k}}_{lin}$ holds true in the algebra of regular functions on an arbitrary irreducible k-dimensional algebraic variety,*

($\mathbf{2}_{\mathbb{C}}$) *$BM^{\frac{1}{k}}_{lin}$ holds for finite dimensional linear spaces D_i in the algebra A of $\mathbb{C}$-valued function on $\mathbb{Z}^k$ with exponential decay at infinity (relative to the word length in $\mathbb{Z}^k$), where the product in A is given by the convolution.*

In fact, such functions are Fourier transforms of *real analytic* functions f on the torus $\mathbb{T}^k$ the above applies to the Taylor expansions $\tau_t(f)$ of f at some point $t \in T$.

Remarks. (a) The super-polynomial rate of decay is not sufficient for $\mathrm{BM}^{\frac{1}{k}}_{lin}$ in ($\mathbf{2}_{\mathbb{C}}$), since the ring of C^∞-function on $\mathbb{T}^k$ has zero divisors.

(b) One needs significantly less from an algebra A of smooth functions than analyticity in order to have $\mathrm{BM}^{\frac{1}{k}}_{lin}$. For example, if the Taylor map $\tau_t : A \to A_\infty$ for $f \mapsto \tau_t(f)$ is injective, i.e. if $\tau(f) \neq 0$ unless $f = 0$ (at least in a neighborhood of $t \in T$), then $\mathrm{BM}^{\frac{1}{k}}_{lin}$ holds in A.

Moreover, $\mathrm{BM}^{\frac{1}{k}}_{lin}$ holds in algebras A of *quasi-analytic* functions, where the germs of function at a point t can be graded by ordinal numbers but I do not know the most general condition on the (relative or absolute) decay rate that would ensure the injectivity of τ (extended to infinite ordinals) and/or the inequality $\mathrm{BM}^{\frac{1}{k}}_{lin}$.

Remark. Linear spaces D of $\mathbb{C}$-valued functions on a general group Γ with sufficiently fast (superexponential ?) decay are likely to satisfy the same type of BM-inequalities as spaces of functions with finite supports, but the approximation argument from 1.7 does not seem to apply in the present case.

$\mathbf{3}_{\mathbb{Q}_p}$ *The inequality $BM^{\frac{1}{k}}_{lin}$ holds true for the decaying $\mathbb{F}$-valued function $f : \Gamma \to \mathbb{F}$ for an arbitrary discrete valuation field $\mathbb{F}$, e.g. for $\mathbb{F} = \mathbb{Q}_p$.*

Proof. Let $f(x)$ be a non-zero *analytic function* that is a power series,

$$f(x) = \sum_{-\infty<i<+\infty} a_i x^i \text{ for } a_i \in \mathbb{F}$$

with decaying coefficients, $||a_i|| \to 0$ for $i \to \pm\infty$, and let $m(f)$ denotes the number of the coefficients a_i with the maximal norm, i.e. with $||a_i|| = ||f|| =_{def} max_i ||a_i||$.

If $f = (1-x)^d g(x)$ for an analytic function g, then $||g(x)|| \leq ||f(x)||$ and $d \leq m(f) - 1$.

In fact, the coefficients b_i of $(\sum_i a_i x^i)/(1-x)$ are the partial sums,

$$b_i = \sum_{j \leq i} a_j$$

and if $b_j \to 0$ for $j \to +\infty$, then $m(f(x)/(1-x)) \leq m(f) - 1$; thus, at most $m(f) - 1$ divisions are possible in the rings of analytic functions, since $||(1-x)^d|| \leq ||(1-x)||^d = 1$ and $||(1-x)^d g(x)|| \leq ||g(x)||$.

This implies the (trivial special case of)

Division Theorem. *Each non-zero analytic function f admits a unique product decomposition $f(x) = (1-x)^d g(x)$, where $g(x)$ is an analytic function with $g(1) = \sum_i b_i \neq 0$.*

Next, consider a central extension

$$1 \to \mathbb{Z} \to \Gamma \to \Gamma_1 \to 1$$

and observe (this was explained to me by Ofer Gabor) that this division automatically extends to the algebra $\mathcal{A}(\Gamma)$ of "$\mathbb{F}$-analytic functions" $f = \sum_{\gamma \in \Gamma} a(\gamma)$ with decaying coefficients $a(\gamma)$, where the role of $x \in \mathcal{A}$ is played by one of the two generators $+1$ or -1 in $\mathbb{Z} \subset \Gamma \subset \mathcal{A}$, say by $+1$:

every non-zero $f \in \mathcal{A}(\Gamma)$ is uniquely decomposable as $f = (1-x)^d g$, where $g \in \mathcal{A}(\Gamma)$ is such that its image in $\mathcal{A}(\Gamma_1)$ under the push forward homomorphism does not vanish.

This implies, in particular, by induction on k that the algebra $\mathcal{A}(\mathbb{Z}^k)$ embeds (by the counterpart of the above τ-homomorphism) into the algebra of formal power series and the proof of $\mathbf{3}_{\mathbb{Q}_p}$ follows.

Remark on non-Commutative Groups Γ. Let $I \subset A$ denote the *augmentation ideal* (generated by $1 - g_j$ for some generators $g_j \in \Gamma$) of the group algebra A of Γ and let A_i denote the quotients A/I^i for the powers I^i of I.

These $\to A_i \to A_{i-1} \to ... \to A_1 = \mathbb{F}$ make a projective system of nilpotent algebras that are finite dimensional for finitely generated Γ, where the projective limit of this system is denoted by A_∞ and is equipped with the usual topology. If $\Gamma = \mathbb{Z}^k$, then A_∞ equals the space of formal power series (in k commuting variables) at the point $t = \mathbf{1} = (1, 1, ..., 1)$ in the torus $\mathbb{T}^k$.

More generally, if the intersection of I^i equals zero (e.g. if Γ is a torsion free nilpotent or a free group), then A_∞ is regarded as the formal completion of A. The above division argument shows that the algebra $\mathcal{A}(\Gamma)$ of $\mathbb{F}$-analytic functions embeds to A_∞ and a similar conclusion holds for Archimedean fields (e.g. $\mathbb{R}$ and $\mathbb{C}$) for "analytic" functions with a suitably defined radius of convergence > 1. Thus the isoperimetric problems in $\mathcal{A}(\Gamma)$ reduce to these in A_∞.

Question. When does the algebra A_∞ satisfy the same isoperimetric and BM inequalities as A itself? Is it true in the case where the intersection of

the powers of the ideal I equal zero? (This "nil-intersection" condition is quite strong: if the underlying field $\mathbb{F}$ has characteristic zero it is close to the group being residually nilpotent and for a finite field it looks even more restrictive.)

11 Bibliography.

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