150 Years of Riemann Hypothesis.

Julio C. Andrade

University of Bristol
Department of Mathematics

April 9, 2009 / IMPA

Outline

Synopsis of Riemann's Paper.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.
(On the number of primes less than a given magnitude)

What Riemann proves?

Riemann begins with Euler's observation that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1} \qquad (s > 1)$$

But he lets $s = \sigma + it$ be complex.

He denotes the common value by $\zeta(s)$ and proves:

- $\zeta(s)$ has an **analytic continuation** to $\mathbb{C}$, except for a simple pole at $s = 1$. The only zeros in $\sigma < 0$ are simple zeros at $s = -2, -4, -6, \ldots$
- $\zeta(s)$ has a **functional equation**

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$$

What Riemann Claims?

- $\zeta(s)$ has infinitely many **nontrivial** zeros $\rho = \beta + i\gamma$ in the "critical strip" $0 \le \sigma \le 1$.

- If $N(T)$ denotes the number of nontrivial zeros $\rho = \beta + i\gamma$ with ordinates $0 < \gamma \le T$, then as $T \to \infty$,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

- The function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is entire and has the **product formula**

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right).$$

Here ρ runs over the nontrivial zeros of $\zeta(s)$.

What Riemann Claims?

- **explicit formula**
  Let $\Lambda(n) = \log p$ if $n = p^k$ and 0 otherwise. Then

$$\psi(x) = \sum_{n \le x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)}.$$

  (Riemann states this for $\pi(x) = \sum_{p \le x} 1$ instead).

Note from this one can see why the Prime Number Theorem,

$$\psi(x) \sim x$$

might be true.

Riemann also makes his famous conjecture.

Conjecture (The Riemann Hypothesis)

All the zeros $\rho = \beta + i\gamma$ *in the critical strip lie on the line* $\sigma = 1/2$.

Outline

Hadamard 1893

Hadamard developed the theory of entire functions (Hadamard product formula) and proved the product formula for

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

$$\xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right) = \xi(0)\prod_{\mathrm{Im}\ \rho>0}\left(1-\frac{s+s^2}{\rho(1-\rho)}\right).$$

To do this he proved the estimate

$$N(T) \ll T\log T,$$

which is weaker than Riemann's assertion about $N(T)$.

von Mangoldt 1895

von Mangoldt proved Riemann's explicit formula for $\pi(x)$ and

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)}.$$

Hadamard and de la Vallée Poussin 1896

Hadamard and de la Valleé Poussin independently proved the asymptotic form of the Prime Number Theorem, namely

$$\psi(x) \sim x$$

To do this, they both needed to prove that

$$\zeta(1 + it) \neq 0$$

de la Vallée Poussin 1899

de la Vallée Poussin proved the Prime Number Theorem with a remainder term:

$$\psi(x) = x + O(xe^{-\sqrt{c_1 \log x}}).$$

This required him to prove that there is a *zero-free* region

$$\sigma < 1 - \frac{c_0}{\log t}$$

von Mangoldt 1905

von Mangoldt proved Riemann's formula for the counting function of the zeros

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

von Koch 1905

von Koch showed that the Riemann Hypothesis implies the Prime Number Theorem with a “small” remainder term

$$\mathrm{RH} \Rightarrow \psi(x) = x + O(x^{1/2} \log^2 x)$$

Outline

$\zeta(s)$ in the critical strip

The critical strip is the most important (and mysterious) region for $\zeta(s)$.

By the functional equation, it suffices to focus on $1/2 \leq \sigma \leq 1$.

A natural question is: how large can $\zeta(s)$ be as t grows?

This is important because

- the growth of an analytic function and the distribution of its zeros are intimately connected.
- the distribution of primes depends on it.
- answers to other arithmetical questions depend on it.

Relation between growth and zeros:

Theorem (Jensen's Formula.)

Let $f(z)$ be analytic for $|z| \leq R$ and $f(0) \neq 0$. If $z_1, z_2, ..., z_n$ are all the zeros of $f(z)$ inside $|z| \leq R$, then

$$\log\left(\frac{R^n}{|z_1 z_2 \cdots z_n|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|.$$

Estimates at the edge of the strip

Upper bounds for $\zeta(s)$ near $\sigma = 1$ allow one to widen the zero-free region.

This leads to improvements in the remainder term for the PNT.

For instance, we saw that de la Vallée Poussin showed that

$$\zeta(\sigma + it) \ll \log t \qquad \text{in} \quad \sigma \geq 1 - \frac{c_0}{\log t},$$

and this implied that the O-term in the PNT is $\ll xe^{-\sqrt{c_1 \log x}}$.

Littlewood 1922

$$\zeta(\sigma+it) \ll \frac{\log t}{\log\log t} \quad \text{and no zeros in} \quad \sigma \geq 1 - \frac{c\log\log t}{\log t}$$

$$\Rightarrow O\text{–term in PNT} \ll xe^{-c\sqrt{\log x \log\log x}}$$

The idea is to approximate

$$\zeta(\sigma+it) \approx \sum_{1}^{N} \frac{1}{n^{\sigma+it}}$$

then use Weyl's method to estimate the exponential sums

$$\sum_{a}^{b} n^{-it} = \sum_{a}^{b} e^{if(n)}.$$

Estimates at the edge of the strip

Vinogradov and Korobov 1958 (independently)

$$\zeta(\sigma + it) \ll \log^{2/3} t \quad \text{and no zeros in} \quad \sigma \geq 1 - \frac{c}{\log^{2/3} t}$$

$$\Rightarrow O\text{–term in PNT} \ll x e^{-c \log^{3/5-\epsilon} x}$$

Where Littlewood used Weyl's method to estimate the exponential sums

$$\sum_{a}^{b} n^{-it},$$

Vinogradov and Korobov used Vinogradov's method.

Estimates at the edge of the strip

Here is a summary:

$$\zeta(1+it) \ll \log t \qquad \text{(de la Vallée Poussin)}$$

$$\zeta(1+it) \ll \frac{\log t}{\log\log t} \qquad \text{(Littlewood-Weyl)}$$

$$\zeta(1+it) \ll \log^{2/3} t \qquad \text{(Vinogradov-Korobov)}$$

What should the truth be? One can show that

$$(1+o(1))e^{\gamma}\log\log t \leq_{\mathrm{i.o.}} |\zeta(1+it)| \leq_{\mathrm{RH}} 2(1+o(1))e^{\gamma}\log\log t.$$

Estimates inside the strip

Definition (Lindelöf 1908)

For a fixed σ let $\mu(\sigma)$ denote the lower bound of the numbers μ such that

$$\zeta(\sigma + it) \ll (1 + |t|)^{\mu}.$$

- $\zeta(s)$ bounded for $\sigma > 1 \Rightarrow \mu(\sigma) = 0$ for $\sigma > 1$.
- $|\zeta(s)| \sim (|t|/2\pi)^{1/2-\sigma}|\zeta(1-s)| \Rightarrow \mu(\sigma) = 1/2 - \sigma + \mu(1-\sigma)$.
- In particular, $\mu(\sigma) = 1/2 - \sigma$ for $\sigma < 0$.

Lindelöf's μ–function

Lindelöf proved that $\mu(\sigma)$ is

- continuous
- nonincreasing
- convex

The prove of these properties are in the same circle of ideas as the proof of the Phragmen-Lindelöf theorems.

It follows that $\mu(1/2) \leq 1/4$, that is,

$$\zeta(1/2 + it) \ll |t|^{1/4+\epsilon}.$$

This is a so called *convexity bound*.

Breaking convexity

Using Weyl's method of estimating exponential sums, Hardy and Littlewood showed that

$$\zeta(1/2 + it) \ll |t|^{1/6+\epsilon}.$$

The best results for $\mu(\sigma)$ since have come from exponential sum methods: van der Corput, Vinogradov, Kolesnik, Bombieri–Iwaniec, Huxley–Watt.

Huxley and Watt show that $\mu(\sigma) < 9/56$.

Conjecture (Lindelöf)

$\mu(\sigma) = 0$ for $\sigma \geq 1/2$. That is, $\zeta(1/2 + it) \ll |t|^{\epsilon}$ for t large.

What we expect the order to be

The LH says that for large $|t|$

$$\log|\zeta(1/2+it)| \leq \epsilon \log|t|.$$

It is also known that

$$\sqrt{c\frac{\log t}{\log\log t}} \leq_{\mathrm{i.o.}} \log|\zeta(1/2+it)| \ll_{\mathrm{RH}} \frac{\log t}{\log\log t}.$$

Which bound, the upper or the lower, is closest to the truth is one of the important open questions.

Outline

Mean value theorems

Averages such as $\int_0^T |\zeta(\sigma + it)|^{2k} dt$ have been another main focus of research because

- averages as well as pointwise upper bounds tell us about zeros and have other applications.
- mean values are easier to prove than point wise bounds.
- the techniques developed to treat them have proved important in other contexts.

Mean value theorems

Landau 1908

$$\int_0^T |\zeta(\sigma+it)|^2 dt \sim \zeta(2\sigma)T \qquad (\sigma > 1/2 \ \text{ fixed}).$$

Hardy–Littlewood 1918

$$\int_0^T |\zeta(1/2+it)|^2 dt \sim T\log T.$$

For this H–L developed the **approximate functional equation**

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} + \chi(s)\sum_{n\le y}\frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{1/2-\sigma}y^{\sigma-1}), \quad (\text{for } 0<\sigma<1)$$

which has proved an extremely important tool ever since.

Mean value theorems

Hardy–Littlewood 1918

$$\int_0^T |\zeta(\sigma+it)|^4 dt \sim \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} T \qquad (\sigma > 1/2 \ \text{fixed}).$$

Ingham 1926

$$\int_0^T |\zeta(1/2+it)|^4 dt \sim \frac{T}{2\pi^2} \log^4 T.$$

This was done by using an approximate functional equation for $\zeta^2(s)$. When k is a positive integer Ramachandra showed that

$$\int_0^T |\zeta(1/2+it)|^{2k} dt \ll T \log^{k^2} T.$$

This is believed to be the correct upper bound as well.

Mean value theorems

This suggests the problem of determining constants C_k such that

$$\int_0^T |\zeta(1/2+it)|^{2k} dt \sim C_k T \log^{k^2} T.$$

Conrey–Ghosh suggested that

$$C_k = \frac{a_k g_k}{\Gamma(k^2+1)},$$

where

$$a_k = \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \sum_{r=0}^{\infty} \frac{d_k^2(p^r)}{p^r} \right)$$

and g_k is an integer.

Moments of the Riemann Zeta Function

In

$$\int_0^T |\zeta(1/2+it)|^{2k} dt \sim \frac{a_k g_k}{\Gamma(k^2+1)} T \log^{k^2} T,$$

- $g_1 = 1$ and $g_2 = 2$ are known.
- Conrey and Ghosh conjectured that $g_3 = 42$.
- Conrey and Gonek conjectured that $g_4 = 24024$.
- Keating and Snaith used random matrix theory to conjecture the value of g_k for every value of $k > -1/2$.
- K. Soundararajan has recently shown that on RH

$$\int_0^T |\zeta(1/2+it)|^{2k} dt \ll T \log^{k^2+\epsilon} T.$$

Outline

Zero-density estimates

Let $N(\sigma, T)$ denote the number of zeros of $\zeta(s)$ with abscissae to the right of σ and ordinates between 0 and T.

Zero–density estimates are bounds for $N(\sigma, T)$ when $\sigma > 1/2$.

Bohr and Landau 1912 showed that for each fixed $\sigma > 1/2$,

$$N(\sigma, T) \ll T.$$

Since

$$N(T) \sim (T/2\pi) \log T,$$

this says the proportion of zeros to the right of $\sigma > 1/2$ tends to 0 as $T \to \infty$.

Zero-density estimates

Bohr and Landau used Jensen's formula and

$$\int_0^T |\zeta(\sigma + it)|^2 dt \ll T \qquad (\sigma > 1/2 \quad \text{fixed})$$

to prove this.

Today we have much better zero-density estimates of the form $N(\sigma, T) \ll T^{\theta(\sigma)}$ with $\theta(\sigma)$ strictly less than 1.

The conjecture that $N(\sigma, T) \ll T^{2(1-\sigma)} \log T$ is called the *Density Hypothesis*.

We have two results concerning the Density Hypothesis:

- RH implies the Density Hypothesis.
- LH implies $N(\sigma, T) \ll T^{2(1-\sigma)+\epsilon}$.

Outline

The distribution of a–values of $\zeta(s)$

What can we say about the distribution of non-zero values, a, of the Riemann zeta-function?

A lovely theory due mostly to H. Bohr developed around this question.

Here are two results.

First, the curve $f(t) = \zeta(\sigma + it)$ $\quad (1/2 < \sigma \leq 1$ fixed, $t \in \mathbb{R})$ is dense in $\mathbb{C}$. The idea is to

- show that $\zeta(\sigma + it) \approx \prod_{p \leq N}(1 - p^{-\sigma - it})^{-1}$ for most t.
- use Kronecker's theorem to find a t so that the numbers p^{-it} point in such a way that $\prod_{p \leq N}(1 - p^{-\sigma - it})^{-1} \approx a$.

The distribution of a–values of $\zeta(s)$

As a second result, let $N_a(\sigma_1, \sigma_2, T)$ be the number of solutions of $\zeta(s) = a$ in the rectangular area $\sigma_1 \leq \sigma \leq \sigma_2$, $0 \leq t \leq T$.

Suppose that $1/2 < \sigma_1 < \sigma_2 \leq 1$.

Then there exists a positive constant $c(\sigma_1, \sigma_2)$ such that

$$N_a(\sigma_1, \sigma_2, T) \sim c(\sigma_1, \sigma_2) T.$$

Notice that this is quite different from the case $a = 0$, because modern zero-density estimates imply

$$N_a(\sigma_1, \sigma_2, T) \ll T^{\theta} \quad (\theta < 1).$$

Outline

Number of zeros on the line as $T \to \infty$

Let $N_0(T) = \#\{1/2 + i\gamma \mid \zeta(1/2 + i\gamma) = 0, 0 < \gamma < T\}$ denote the number of zeros on the critical line up to height T.

The important estimates were

Hardy 1914 $N_0(T) \to \infty \quad (\text{as } T \to \infty)$

Hardy–Littlewood 1921 $N_0(T) > cT$

Selberg 1942 $N_0(T) > cN(T)$

Levinson 1974 $N_0(T) > \frac{1}{3}N(T)$

Conrey 1989 $N_0(T) > \frac{2}{5}N(T)$

These all rely heavily on mean value estimates.

Hardy's idea

One can write the functional equation as $\zeta(s) = \chi(s)\zeta(1-s)$, or as

$$\chi^{-1/2}(s)\zeta(s) = \chi^{1/2}(s)\zeta(1-s).$$

Then

$$Z(t) = \chi^{-1/2}(1/2 + it)\zeta(1/2 + it)$$

has the same zeros as $\zeta(s)$ on $\sigma = 1/2$ and is real.

If $Z(t)$ had no zeros for $t \geq T_0$, the integrals

$$\left| \int_{T_0}^{T} Z(t)dt \right| \quad \text{and} \quad \int_{T_0}^{T} |Z(t)|dt$$

would be the same size as $T \to \infty$. But they are not.

Outline

Numerical calculations of zeros

Riemann 1859 calculation of a few zeros.

Gram 1903 The zeros up to 50 (the first 15) are on the line and simple.

Backlund 1912 The zeros up to 200 are on the line

Hutchison 1925 The zeros up to 300 are on the line

Titchmarsh, Turing, Lehman, Brent, van de Lune, te Riele, Odlyzko, Wedeniwski, ...

Gourdon-Demichel 2004 The first 10^{13} (ten trillion) zeros are on the line. Moreover, billions of zeros near the 10^{24} zero are on the line.

Outline

Pair correlation

A major theme of research over the last 35 years has been to understand the distribution of the zeros on the critical line *assuming* that the Riemann Hypothesis is true.

- In 1974 F. Dyson and H. Montgomery conjectured that the zeros of the Riemann zeta function are distributed like the eigenvalues of random Hermitian matrices.

- From 1980 on Odlyzko did a vast amount of numerical calculation that strongly supported Montgomery's conjecture.

New mean value theorems

Gonek and then Conrey, Ghosh, and Gonek proved a number of discrete mean value theorems of the type

$$\sum_{0<\gamma\leq T} |\zeta(\rho+i\alpha)|^2 \quad \text{and} \quad \sum_{0<\gamma\leq T} |\zeta'(\rho)M_N(\rho)|^2,$$

where $\rho = 1/2 + i\gamma$ runs over the zeros.

Assuming RH and sometimes GLH and GRH, Conrey, Ghosh, and Gonek used these to prove that

- there are large and small gaps between consecutive zeros.
- over 70% of the zeros are simple.

Random matrix models

A major development was Keating and Snaith's modeling of $\zeta(s)$ by the characteristic polynomials of random Hermitian matrices.

- It allowed them to determine the constants g_k in
$$\int_0^T |\zeta(1/2+it)|^{2k} dt \sim \frac{a_k g_k}{\Gamma(k^2+1)} T \log^{k^2} T.$$
- It has had applications to elliptic curves, for example.
- Hughes, Keating, O'Connell used it to conjecture the discrete means
$$\sum_{0<\gamma\leq T} |\zeta'(\rho)|^{2k}$$
- Mezzadri used it to study the distribution of the zeros of $\zeta'(s)$.

Lower order terms and ratios

The Keating–Snaith results led to the quest for the lower order terms in the asymptotic expansion of the moments.

This resulted in the discovery of new heuristics for the moments not involving RMT.

It also led to heuristics for very general moment questions (the so called "ratios conjecture").

(Conrey, Farmer, Keating, Rubenstein, Snaith, Zirnbauer, DukKhiem and others.)

A hybrid formula

The Keating-Snaith model finds the moment constants g_k, but the arithmetical factors a_k have to be inserted after the fact.

This led to the problem of finding a model for zeta incorporating characteristic polynomials and arithmetical information.

Gonek, Hughes, Keating found an unconditonal hybrid formula for $\zeta(s)$.

It says (roughly) that

$$\zeta(s) = \prod_{p \leq X} (1 - p^{-s})^{-1} \prod_{|s-\rho| \leq 1/\log X} (1 - X^{(\rho - s)e^{\gamma}})$$

A heuristic calculation of moments using this leads to g_k and a_k appearing naturally.
It also explains why the constant in the moment splits as $\frac{a_k g_k}{\Gamma(k^2+1)}$.

The order of $\zeta(s)$ again

Finally, the hybrid formula has led to convincing answers on the deep question of the exact order of $\zeta(s)$ in the critical strip.

Recall that

$$(1 + o(1))e^{\gamma} \log\log t \leq_{\mathrm{i.o}} |\zeta(1 + it)| \leq_{\mathrm{RH}} 2(1 + o(1))e^{\gamma} \log\log t,$$

so that a factor of 2 is in question.

Arguments from the hybrid model suggest that the 2 should be dropped.

The order of $\zeta(s)$ again

On the 1/2–line itself recall that

$$\sqrt{c\frac{\log t}{\log\log t}} \leq_{\mathrm{i.o}} \log|\zeta(1/2+it)| \ll_{\mathrm{RH}} \frac{\log t}{\log\log t}.$$

Here Farmer, Gonek, and Hughes have shown that the hybrid formula suggests

$$\sqrt{1/2}(1+o(1)) \leq_{\mathrm{i.o}} \frac{\log|\zeta(1/2+it)|}{\sqrt{\log t \log\log t}} \leq_{\mathrm{RH}} \sqrt{1/2}(1+o(1)).$$

Outline

New Subjects and Conclusion.

For the next 150 years we can wait a strong interaction between the Riemann Zeta Function and the following subjects:

- Quantum Chaos.
- Noncommutative Geometry.
- Motives.
- String Theory and Duality.
- The zoo of Zeta Functions.
- Quantum Fields and Motives in Number Theory.
- Fractals and Quasicrystals.
- A better understanding of the proofs of A. Weil and P. Deligne for the Riemann hypothesis on the Function Fields.

Conclusion

THANK YOU FOR YOUR ATTENTION.