MASS-LIKE INVARIANTS FOR ASYMPTOTICALLY
HYPERBOLIC METRICS

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Abstract. In this article, we classify the set of asymptotic mass-like invariants for asymptotically hyperbolic metrics. It turns out that the mass is just one example (yet probably the most important one) among the family of invariants we find. These invariants can be described in terms of wave harmonic polynomials in Minkowski space. We associate them to geometric differential operators the way that the mass is associated to the scalar curvature operator.

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1. Introduction

Asymptotically hyperbolic manifolds are, broadly speaking, non-compact Riemannian manifolds having an end on which the geometry approaches that of hyperbolic space. Such manifolds have attracted a lot of attention for the last decades, in theoretical physics as well as in geometry. One one hand, they appear naturally in general relativity, in the description of isolated gravitational systems in a universe with negative cosmological constant (see for example [3]), modeled on Anti-de Sitter spacetime. One can also see them arising as asymptotically umbilical hypersurfaces in asymptotically Minkowski spacetimes, although this case will not be discussed further here.

In geometry on the other hand, the constraints imposed on the behaviour at infinity of asymptotically hyperbolic manifolds (as well as for asymptotically Euclidean manifolds), although they may a priori allow considerable freedom, offer actually interesting geometric properties such as rigidity, when a suitable positivity assumption of a curvature tensor is made. The first achievements in this directions were scalar curvature rigidity results, for which the scalar curvature is required to be equal or greater than the one of the hyperbolic space of same dimension. Under completeness and strong enough decay assumptions as well as a further topological (spin) assumption, the manifold has then to be isometric to hyperbolic space, see [2, 28]. These results were subsequently studied with weaker decay assumptions, the decay of the metric being quantified by the so-called “mass”. This notion is reminiscent of the ADM mass for an asymptotically Euclidean manifold, and “positive mass theorems”, with their rigid counterparts in the case of vanishing zero mass, have also been obtained [1, 10, 32].

As well as for asymptotically Euclidean manifolds, one of the difficulties when studying the mass is its apparent dependence on the choice of an asymptotic chart. This problem appears in both the approach by Wang [32] and the approach by Chruściel and Herzlich [10]. In the last work it is however established that the mass does not depend on a given chart provided it satisfies the appropriate asymptotic decay conditions. The fundamental result at the origin of this fact was derived first by Chruściel and Nagy in [11, Theorem 3.3]. It states in particular that the transition diffeomorphism between any two asymptotic charts admits a principal part which is an hyperbolic isometry, and a correction part that decays to zero on approach to infinity. The decay of the correction part is sufficiently fast so that it does not alter the expression of the components of the mass vector, themselves computed at infinity. The essential difference with the asymptotically Euclidean case comes rather from the nature of the mass. Whereas it is fully encoded in a single number (the ADM mass) in the asymptotically euclidean case, it is rather a vector in the hyperbolic case, which transforms equivariantly under the action of hyperbolic isometries. When considering the Minkowski quadratic form evaluated on this vector that one gets a genuine asymptotic invariant of the asymptotically hyperbolic manifold. This difference of nature of asymptotic invariants can be explained by the existence of a much bigger conformal infinity for the hyperbolic space (a codimension 1 sphere at infinity) than for the Euclidean space (a single point at infinity). With the large conformal infinity of hyperbolic space one could guess that there should be other geometric invariants at infinity.

The aim of this paper is to find other such asymptotic invariants and to provide a complete list of them for asymptotically hyperbolic manifolds whose Riemannian
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metric decays towards the hyperbolic metric at any arbitrary rate. Beyond the “relativistic case” that involves the mass for a precise rate of decay, this investigation is also motivated by the study of asymptotically hyperbolic Poincaré-Einstein metrics, where asymptotic invariants do appear, see for example [13] and references therein. Note however that the renormalized volume introduced by Graham in [17] does not fall into our classification since it depends on the whole geometry of the manifold not only on its asymptotics.

The paper is structured as follows. In Section 2, the relevant definitions are introduced and followed by the description of the group of asymptotic isometries, whose elements are seen as transition maps between two asymptotic charts in which the metric has a given decay rate towards the reference hyperbolic metric. This analysis culminates in Theorem 2.12 which shows, in the spirit of [11, Theorem 3.3], that such a diffeomorphism is essentially an isometry of the hyperbolic space (namely an element of the Lie group $O^+(n,1)$) together with a correction given by a diffeomorphism asymptotic to the identity. As a consequence, we show that the group $O^+(n,1)$ acts naturally on (germs of) asymptotically hyperbolic metrics. In Sections 3 and 4, we investigate asymptotic invariants for asymptotically hyperbolic metrics in the sense of Wang. In this case, the metric is required to take the form of an asymptotic expansion in which the dominant error term, the mass-aspect tensor, can appear at any decay rate. For this purpose, we start by looking for the expression of the action of $O^+(n,1)$ on mass-aspect tensors, using the analysis of Section 2. The “invariants” are then actually intertwining maps between the space of mass-aspect tensors and some finite dimensional representation of the same group. The relevant representation theory for the corresponding Lie algebra $so(n,1)$ is indicated in Appendix A. In Section 5 we return to the weaker definition of asymptotically hyperbolic metrics from Chruściel and Herzlich [10]. Following Michel’s work [27], we show that the invariants previously obtained actually stem from geometric differential operator, similar to the way the mass stems from the scalar curvature operator. We argue that we in this way obtain all of invariants under such asymptotic assumptions.

This can be compared with Herzlich’s recent study of the asymptotically Euclidean case [21], where asymptotic invariants stemming from a class of admissible curvature operators turn out to be nothing but the ADM mass, up to a constant factor.

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2. Preliminaries and basic definitions

In this section we give the basic definitions that will be used throughout the paper.

2.1. The hyperbolic space. The reference Riemannian manifold in the context under consideration is the hyperbolic space. Unless stated otherwise, we will always use the Poincaré ball model where hyperbolic space $\mathbb{H}^n$ is be described as the unit
ball $B_1(0)$ centered at the origin in $\mathbb{R}^n$ equipped with the metric
\[ b := \rho^{-2} \delta, \]
where $\delta$ denotes the Euclidean metric and
\[ \rho := \frac{1 - |x|^2}{2}. \]
Several other models exist to describe this space but this one seems the most convenient in our context. The interested reader can consult for example [9], [30] or [7]. The hyperbolic distance in the ball model is given by
\[ \cosh d_b(x, y) = 1 + \frac{|x - y|^2}{2 \rho(x) \rho(y)}, \]
for $x, y \in B_1(0)$.

Since the isometry group of the hyperbolic space will play a prominent role in what follows, we will discuss it briefly now. Let $\mathbb{R}^{n,1}$ denote Minkowski space, that is $\mathbb{R}^{n+1}$ equipped with the quadratic form
\[ \eta := -(dX^0)^2 + \sum_{k=1}^n (dX^k)^2. \]
We denote by $v_0, v_1, \ldots, v_n$ the standard basis of $\mathbb{R}^{n,1}$ which is the pre-dual basis of $(dX^0, dX^1, \ldots, dX^n)$. Hyperbolic space can be embedded into the Minkowski space as the unit hyperboloid,
\[ \mathbb{H}^n = \left\{(X^0, X^1, \ldots, X^n) \in \mathbb{R}^{n,1} \middle| -(X^0)^2 + \sum_{k=1}^n (X^k)^2 = -1, X^0 > 0 \right\}. \]
The orthogonal group $O(n,1)$ consists of linear maps preserving the quadratic form $\eta$. It has 4 connected components. Indeed, an element $A \in O(n,1)$ can have determinant $\pm 1$ and the scalar product $\eta(v_0, Av_0)$ can be either negative meaning that $v_0$ and $Av_0$ point in the same direction ($A$ is then said to be future preserving), or positive (and $A$ is then called future reversing). The two components mapping hyperbolic space to itself are the future preserving ones which we together denote by $O^+_{\uparrow}(n,1)$. The connected component of the identity (that is, the future preserving isometries with positive determinant) is denoted by $SO^+_{\uparrow}(n,1)$.

The group $O^+_{\uparrow}(n,1)$ is the group of isometries of hyperbolic space while $SO^+_{\uparrow}(n,1)$ is its subgroup of direct isometries. This subgroup coincides with the connected component of the identity of $O(n,1)$, see for example [29, Chapter 5, Section 3.10] for details.

The ball model is obtained from the hyperboloid model via a stereographic projection $p$ to the plane $X^0 = 0$ with respect to the point $(-1,0,\ldots,0)$:
\[ p(X^0, X^1, \ldots, X^n) := \frac{1}{1 + X^0}(X^1, \ldots, X^n). \]
Note that there is no 0-coordinate in the right hand side. The inverse of $p$ is given by
\[ p^{-1}(x^1, \ldots, x^n) = \left(\frac{1 + |x|^2}{1 - |x|^2}, \frac{2x^1}{1 - |x|^2}, \ldots, \frac{2x^n}{1 - |x|^2}\right), \]
where $|x|^2 = (x^1)^2 + \cdots + (x^n)^2$.

If $A \in O^+_{\uparrow}(n,1)$ is an isometry of the hyperboloid, we transfer it to an isometry of the ball model acting as $pAp^{-1}$. In what follows we will mainly restrict ourselves
to elements belonging to the subgroup $SO_t(n, 1)$ and consider two types of such elements.

1. Rotations $R$ in some $X^iX^j$-plane ($0 < i, j \leq n$) and by some angle $\theta$,

$$R^\theta_{ij}(X^0, \ldots, X^i, \ldots, X^j, \ldots, X^n)$$

$$= (X^0, \ldots, \cos(\theta)X^i - \sin(\theta)X^j, \ldots, \sin(\theta)X^i + \cos(\theta)X^j, \ldots, X^n).$$

We denote with a script letter the corresponding infinitesimal generator,

$$r_{ij} = dX^i v_j - v_i dX^j.$$

Note that rotations commute with $p$, so $pR^\theta_{ij}p^{-1}$ reads

$$pR^\theta_{ij}p^{-1}(x^1, \ldots, x^i, \ldots, x^j, \ldots, x^n)$$

$$= (x^1, \ldots, \cos(\theta)x^i - \sin(\theta)x^j, \ldots, \sin(\theta)x^i + \cos(\theta)x^j, \ldots, x^n).$$

The derivative of $pR^\theta_{ij}p^{-1}$ with respect to $\theta$ at $\theta = 0$ is the rotation vector field $t_{ij}$, where

$$t_{ij} := x^i \partial_j - x^j \partial_i.$$  \hspace{1cm} (3)

2. Lorentz boosts $A$ in the direction $x^i$ ($0 < i \leq n$) with a parameter $s \in \mathbb{R}$:

$$A^s_i(x^0, \ldots, x^i, \ldots, x^n)$$

$$= (\cosh(s)x^0 + \sinh(s)x^i, \ldots, \sinh(s)x^0 + \cosh(s)x^i, \ldots, x^n).$$

The corresponding infinitesimal generator is given by

$$a_i = dX^0 v_i + dX^i v_0.$$ 

As before, we express the corresponding isometry of the ball model as

$$pA^s_i p^{-1}(x^1, \ldots, x^i, \ldots, x^n)$$

$$= \frac{1}{D} \left( x^1, \ldots, \cosh(s)x^i + \sinh(s)\frac{1 + |x|^2}{2}, \ldots, x^n \right),$$  \hspace{1cm} (4)

where

$$D := 1 - \frac{|x|^2}{2} + \frac{1 + |x|^2}{2} \cosh(s) + x^i \sinh(s).$$

The derivative of $pA_i^s p^{-1}$ with respect to $s$ at $s = 0$ is the boost vector field $a_i$, where

$$a_i := \frac{1 + |x|^2}{2} \partial_i - x^i x^0 \partial_a.$$  \hspace{1cm} (5)

In this article we will use the convention that upper case latin letters denote elements in the Lie group $O_t(n, 1)$, while lower case latin letters will be used for elements in the Lie algebra $a(n, 1)$ and Fraktur letters will denote the corresponding vector fields on the ball $B_1(0) \subset \mathbb{R}^n$ or on the sphere $S_1(0)$.

2.2. Asymptotically hyperbolic metrics. We continue by defining asymptotically hyperbolic manifolds. Several definitions exist in the litterature. We refer the reader to [15, 20, 24] for an overview. We choose here to use the simplest such definition to avoid technical complications. The relevance of this choice will be discussed later.
Definition 2.1. Let \( M^n \) be a compact manifold with boundary \( \partial M \cong S^{n-1} \). We choose a collar neighborhood \( \Omega \) of \( \partial M \) in \( M \), a diffeomorphism \( \Psi : \Omega \to B_1(0) \setminus \overline{B}_{1-\varepsilon}(0) \) from \( \Omega \) to the standard annulus and a positive integer \( k \). Then a metric \( g \) on \( M \) is said to be \textit{asymptotically hyperbolic} of order \( k \) (with respect to \( \Psi \)) if the metric

\[
\bar{g} := \rho^2 \Psi_* g
\]

which is a priori defined only on \( B_1(0) \setminus \overline{B}_{1-\varepsilon}(0) \) extends to a smooth metric on \( B_1(0) \setminus \overline{B}_{1-\varepsilon}(0) \) such that

\[
|\bar{g} - \delta|_b = O \left( \rho^k \right).
\]

Let \( r = r(x) \) denote the distance from the origin in the ball model of \( bH^n \). From Formula (2) we have

\[
\rho(x) = \frac{1}{\cosh(r)} + 1.
\]

As a consequence, the estimate for the decay of the metric can be rewritten in a more intrinsic way as

\[
|g - b|_b = O \left( e^{-kr} \right),
\]

and a simple argument using the triangle inequality shows that replacing \( r \) by the distance function from any given point \( p_0 \in B_1(0) \) gives an equivalent decay estimate.

Since we are interested in the asymptotic behavior of an asymptotically hyperbolic metric, it is sufficient to restrict our attention to germs of such metrics. An open subset \( U \subset B_1(0) \) is called a \textit{neighborhood of infinity} if for some \( \varepsilon > 0 \) \( B_1(0) \setminus \overline{B}_{1-\varepsilon}(0) \subset U \). We denote by \( \mathcal{N}_\infty \) the set of neighborhoods of infinity. Given an element \( U \in \mathcal{N}_\infty \) we define \( \overline{U} := U \cup S_1(0) \), that is as the set of points in \( U \) together with the points at infinity.

Definition 2.2. For any positive integer \( k \), we consider the set \( G^0_k \) consisting of pairs \( (U, g) \) where \( U \) is a neighborhood of infinity and \( g \) is a metric on \( U \) such that \( \bar{g} := \rho^2 g \) extends to a smooth metric on \( \overline{U} \) satisfying

\[
|g - b|_b = |\bar{g} - \delta|_b = O \left( \rho^k \right).
\]

The relation \( \sim \) is defined by \( (U_1, g_1) \sim (U_2, g_2) \) if there is a \( U_3 \in \mathcal{N}_\infty \) such that \( U_3 \subset U_1 \cap U_2 \) and \( g_1 = g_2 \) on \( U_3 \). This is an equivalence relation on \( G^0_k \) and we define the \textit{stalk of asymptotically hyperbolic metrics of order} \( k \) as \( G_k := G^0_k / \sim \).

We will abuse notation, and blur the distinction between an element \( g \in G_k \) and the metric in an element \( (U, g) \in G^0_k \) representing \( g \). Such an element \( g \in G_k \) will be called an \textit{asymptotically hyperbolic germ}. This terminology is modeled on the standard terminology, see for example [19, Chapter 2] or [8, Chapter 1]. We will now introduce the set of asymptotic isometries of an asymptotically hyperbolic metric.

Definition 2.3. Given a positive integer \( k \) and \( g \in G_k \), we define the set \( I^k(g) \) of asymptotic isometries of \( g \) as the stalk of diffeomorphisms \( \Psi : U \to V \) where \( U \) and \( V \) are neighborhoods of infinity such that \( \Psi_* g \in G_k \).

It will be proven in Lemma 2.11 that \( I^k(g) \) does not depend on the choice of \( g \in G_k \). As a corollary we will see that \( I^k(g) \) is a group under composition of maps.

We also introduce a particular class of germs of asymptotically hyperbolic metrics. This class will play an important role in what follows.
Definition 2.4. Let \( g \) be a germ of asymptotically hyperbolic metric of order \( k \). We say that \( g \) is transverse if there exists \((U, g)\) representing \( g \) such that
\[
g_{ij}x^i = b_{ij}x^i
\]
on \( U \). Or equivalently,
\[
g(X, \cdot) = b(X, \cdot)
\]
on \( U \), where \( X := x^i \partial_i \) is the dilation vector field on \( \mathbb{R}^n \). We denote by \( G^T_k \) the set of transverse asymptotically hyperbolic germs of order \( k \).

A similar condition appears in the context of asymptotically hyperbolic Einstein metrics (see for example [13]) and in Wang’s approach to the mass [32].

Note that the transversality condition can also be stated in terms of the dual of the metric as
\[
g(d|\!\!\!|x|\!\!\!|^2, \cdot) = b(d|\!\!\!|x|\!\!\!|^2, \cdot)
\]
on \( U \), or equivalently in terms of the covectors \( dx^i \) as
\[
\overline{g}(d\rho, dx^i) = -x^i
\]
on \( U \). In particular,
\[
\begin{align*}
g(d|\!\!\!|x|\!\!\!|^2, d|\!\!\!|x|\!\!\!|^2) &= b(d|\!\!\!|x|\!\!\!|^2, d|\!\!\!|x|\!\!\!|^2) \\
g(d\rho, d\rho) &= b(d\rho, d\rho) \\
\overline{g}(d\rho, d\rho) &= \delta(d\rho, d\rho) = |x|^2,
\end{align*}
\]
and
\[
|d\rho|^2_{\overline{g}} = 1 - 2\rho. \tag{8}
\]

Next we define diffeomorphisms asymptotic to the identity.

Definition 2.5. Given a positive integer \( k \) and a neighborhood of infinity \( U \), we say that a diffeomorphism \( \Psi : \tilde{U} \to \tilde{\Psi} \tilde{U} \) is asymptotic to the identity of order \( k \) if
\[
|\Psi(x) - x|_g = O(\rho^k).
\]
We denote by \( I^k_0 \) the group of germs of diffeomorphisms asymptotic to the identity.

Lemma 2.6. Given \( g \in G_k \), there is an inclusion \( I^{k+1}_0 \subset I^k(g) \).

Proof. Note that \( I^{k+1}_0 \) is a group under composition. Thus we can check that for all \( \Psi \in I^{k+1}_0 \), \( \Psi^* g \in G_k \) since this is equivalent to \( (\Psi^{-1})_* g \in G_k \). We work in component notation:
\[
\rho^2(x)(\Psi^* g)(x)_{ij} - \delta_{ij} = \rho^2(x)g(\Psi(x))_{kl}\partial_i \Psi^k \partial_j \Psi^l - \delta_{ij}
\]
\[
= \frac{\rho^2(\Psi(x))}{\rho^2(\Psi(x))} \overline{g}(\Psi(x))_{kl}\partial_i \Psi^k \partial_j \Psi^l - \delta_{ij}.
\]
By a standard trick we prove that the function \( \frac{e^{\nu}}{\rho} \) is smooth near \( S_1(0) \). Write

\[
\Psi^j(x) = \Psi^j \left( \frac{x}{|x|} \right) - \int_{|x|}^{1} \frac{d}{ds} \Psi^j \left( \frac{s}{|x|} \right) ds
\]

\[
= \frac{x^j}{|x|} - \int_{|x|}^{1} \partial_j \Psi^i \left( \frac{s}{|x|} \right) \frac{x^i}{|x|} ds
\]

\[
= \frac{x^i}{|x|} - (1 - |x|) \int_0^1 \partial_j \Psi^i \left( \lambda \left(1 + (1 - \lambda) |x| \right) \frac{x}{|x|} \right) \frac{x^j}{|x|} d\lambda
\]

\[
= \frac{x^i}{|x|} - (1 - |x|) \int_0^1 \partial_j \Psi^i \left( \lambda \left(1 + (1 - \lambda) |x| \right) \frac{x}{|x|} \right) - \delta_j \frac{x^j}{|x|} d\lambda
\]

where we set \( s = \lambda + (1 - \lambda) |x| \). The vector field \( E^i \) is smooth near \( S_1(0) \) and has components satisfying \( E^i(x) = O(\rho^k) \). Thus,

\[
\frac{\rho \circ \Psi}{\rho} (x) = \frac{1 - |\Psi(x)|^2}{1 - |x|^2}
\]

\[
= 1 + 2 \frac{(1 - |x|) E^i(x)}{1 - |x|^2} - \frac{(1 - |x|)^2 |E(x)|^2}{1 - |x|^2}
\]

\[
= 1 + 2 \frac{x^i E^i(x)}{1 + |x|} - \frac{1 - |x|}{1 + |x|} |E(x)|^2.
\]

From this we conclude that \( \frac{e^{\nu}}{\rho} \) is smooth and satisfies

\[
\frac{\rho \circ \Psi}{\rho} = 1 + O(\rho^k).
\]

It is a simple calculation to check that \( \overline{g}(\Psi(x))_{ij} \partial_i \Psi^k \partial_j \Psi^l - \delta_{ij} = O(\rho^k) \). The lemma follows by multiplying these estimates. \( \square \)

The following proposition is a variant of [16, Lemma 5.2].

**Proposition 2.7.** Given an element \( g \in G_k \), there exists a unique \( \Theta \in I_k^{k+1} \) such that \( \Theta \circ g \) is transverse. Further, \( \Theta \circ g \in G_k \). We call \( \Theta \) the adjustment diffeomorphism associated to the metric \( g \).

**Proof.** We are going to construct new coordinates at infinity that satisfy the transversality condition for the metric \( g \). These new coordinates, which we denote by \( (x'_1, \ldots, x'_n) \) will provide the required diffeomorphism as

\[
\Theta(x_1, \ldots, x_n) = (x'_1, \ldots, x'_n).
\]

We first set

\[
\rho' = \frac{1 - |x'|^2}{2}.
\]

Since we want \( \rho' \) to be close to \( \rho \) in a sense to be specified later, we set \( \rho' = \rho e^{\nu} \). We define \( \tilde{g} := \rho^2 g = e^{2\nu} g \). We first impose an analog of Condition (8) upon \( \tilde{g} \) and \( \rho' \),

\[
|d\rho'|^2 = 1 - 2\rho'.
\]
We rewrite this as an equation for $v$,

$$1 - 2\rho e^v = e^{-2v}|\rho e^v d\nu + e^v d\rho|_\mathcal{F}^2$$

$$= |\rho d\nu + d\rho|_\mathcal{F}^2$$

$$= \rho^2|d\nu|^2 + 2\rho \mathcal{G}(d\nu, d\rho) + |d\rho|_\mathcal{F}^2,$$

or

$$2\mathcal{G}(d\nu, d\rho) + |d\rho|^2_\mathcal{F} = \frac{1 - 2\rho - |d\rho|^2_\mathcal{F}}{\rho} + 2 - 2e^v.$$  \hspace{1cm} (10)

Since $|\mathcal{G} - \delta|_\delta = O(\rho^k)$, we have

$$|d\rho|^2_\mathcal{F} = 1 - 2\rho + O(\rho^k),$$

so that

$$\frac{1 - 2\rho - |d\rho|^2_\mathcal{F}}{\rho} = O(\rho^{k-1}).$$ \hspace{1cm} (11)

Equation (10) is a first order partial differential equation for $v$. The relevant theory for such equations can be found in [12, Chapter 2] or in [26, Theorem 22.39]. In particular, the condition $|d\rho|^2_\mathcal{F} \equiv 1$ on $S_1(0)$ ensures that there exists a unique solution $v$ in a neighborhood of $S_1(0)$ such that $v = 0$ on $S_1(0)$.

From Estimate (11) it follows that $v = O(\rho^k)$. We have now determined the function $\rho' = e^v \rho$ is now known.

We introduce the analog of Equation (7) for the coordinates $x'^i$,

$$\tilde{g}(d\rho', dx'^i) = -x'^i,$$

with the boundary condition $x'^i = x^i$ on $S_1(0)$.

As for the previous equation, existence and uniqueness of a smooth solution in a neighborhood of $S_1(0)$ is guaranteed by classical results. From a simple calculation it follows that

$$\tilde{g}(d\rho', dx'^i) = -x'^i + O(\rho^k).$$

This implies that $x'^i - x^i = O(\rho^{k+1})$ which means that $\Theta \in I^{k+1}_0$. It remains to check that $\rho' = \frac{1-|x'^i|^2}{2}$ Setting $\rho'' = \frac{1-|x'^i|^2}{2}$ we have

$$\tilde{g}(d\rho', d\rho'') = -x'^i \tilde{g}(d\rho', dx'^i)$$

$$= \sum_i (x'^i)^2$$

$$= 1 - 2\rho''.$$

This equation is a non-characteristic first order partial differential equation for the function $\rho''$. Since $x'^i$ coincides with $x^i$ on $S_1(0)$ we have that $\rho'' = 0 = \rho'$ on $S_1(0)$. It follows that $\rho'' = \rho'$ in a neighborhood of $S_1(0)$.

Uniqueness is easy to prove. Assume that two such diffeomorphisms $\Theta$ and $\Theta'$ exist. Then, considering $\Psi = \Theta' \circ \Theta^{-1}$, we have to prove that for a transverse metric $g$ the only element $\Psi \in I^{k+1}_0$ such that $\Psi_* g$ is also transverse is the identity. Assume that the diffeomorphism $\Psi$ is given in coordinates by

$$\Psi(x^1, \ldots, x^n) = (x'^1, \ldots, x'^n),$$

where $x'^i - x^i = o(\rho)$. As before we set

$$\rho = \frac{1-|x|^2}{2} \quad \text{and} \quad \rho' = \frac{1-|x'|^2}{2}.$$
From the assumption on the coordinates, we have that $\rho' = \rho + o(\rho)$. Computing as for Equation (10), we get that $v := \log \rho'$ vanishes on $S_1(0)$ and satisfies

$$2(dv, d\rho)_g + \rho|dv|^2 = 2 - 2e^v,$$

where we have used the fact that $|d\rho|^2_g = 1 - 2\rho$ since $g$ is transverse. The solution to this equation being unique, we must have $v \equiv 0$ or equivalently $\rho \equiv \rho'$. From the condition (7), we deduce that the coordinates $x^i$ and $x'^i$ both satisfy the transport equation

$$\langle dv, dx^i \rangle_g = -x^i.$$

Since they coincide on $S_1(0)$ they coincide in a neighborhood of $S_1(0)$.

The proof that $\Psi^*g \in G_k$ is contained in the lemma 2.6 above. \hfill \square

**Remark 2.8.** Note here that in all previous proofs of the existence of geodesic normal coordinates (see for example [25, Lemma 5.1]) there is a loss of regularity of one derivative. This is one reason why we chose to restrict our study to smooth conformally compact manifolds. However, the proof of Proposition 2.15 may be used to circumvent this difficulty.

**Remark 2.9.** The dependence of this adjustment diffeomorphism $\Theta$ with respect to the germ $g$ will be studied in Section 2.4. The first non-trivial term in the asymptotic expansion of $\Theta$ will appear in Proposition 2.17.

Before stating the main result of this section, we prove two important lemmas. The first one is a rephrasing of [10, Theorem 6.1]. The proof is based on the theory developed in [4–6]. In order to keep this section reasonably short, we refer the reader to [5] for the definition of an essential set and the relevant results.

**Lemma 2.10.** For $k$ a positive integer let $g \in G_k$ be an asymptotically hyperbolic metric and let $\Psi \in I^k(g)$ be an asymptotic isometry. If $\psi : U \to V$ represents the germ $\Psi$, then $\psi$ extends to a smooth diffeomorphism $\tilde{\psi} : \tilde{U} \to \tilde{V}$.

**Proof.** From Proposition 2.7 it follows that we can pull back the metrics $g$ and $\Psi^*g$ by elements in $I^k_{b+1}$ so that they satisfy the transversality condition. Since this corresponds to composing the diffeomorphism $\psi$ on the left and on the right with elements of $I^k_{b+1}$ which are smooth up to the boundary, we can assume without loss of generality that $g$ and $\Psi^*g$ are transverse.

From [14, Lemma 2.5.11] we know that the set $K = \{\rho \geq \varepsilon\}$ is an essential set for both $g$ and $\psi^*g$ provided $\varepsilon > 0$ is small enough.

Equivalently, $K_1 := K$ and $K_2 := \psi(K)$ are essential sets for $g$. Further, since the metrics $g$ and $\psi^*g$ are $C^\infty$-conformally compact, their sectional curvatures satisfy

$$\sec^g = -1 + O(\rho) \quad \text{and} \quad \sec^{\psi^*g} = -1 + O(\rho).$$

With some more effort one checks that

$$|\nabla^g R^g|_g = -1 + O(\rho) \quad \text{and} \quad |\nabla^{\psi^*g} R^{\psi^*g}|_{\psi^*g} = -1 + O(\rho).$$

The transversality condition imposes that the distance from $K_1$ (resp. $K_2$) with respect to the background hyperbolic metric and $g$ (resp. $\psi^*g$) agree. For points $y$

\footnote{The notion of an essential set is a priori defined only for complete manifolds. Here we can simply fill the region of $B_1(0)$ where $g$ (resp. $\Psi^*g$) is not defined by an arbitrary Riemannian metric. The argument of [14, Lemma 2.5.11] depends only on the metric outside some compact set.}
lying on the boundary of \( \{ \rho \geq \varepsilon \} \), i.e. such that \( \rho(y) = \varepsilon \), we have \( |y| = \sqrt{1 - 2\varepsilon} \).

We also remark that the orthogonal projection \( \pi(x) \) of a point \( x \) onto \( \Sigma = \rho^{-1}(\{ \varepsilon \}) \) with respect to the metrics \( b, g \) and \( \psi^* g \) all coincide with the Euclidean orthogonal projection onto \( \Sigma \) because \( \Sigma \) is a round sphere centered at the origin. Hence,

\[
|x - \pi(x)|^2 = (|x| - \sqrt{1 - 2\varepsilon})^2.
\]

From Equation (2), the distance from \( K \) to any point \( x \) lying outside \( K \) is given by

\[
\cosh d(x, K) = 1 + \frac{(|x| - \sqrt{1 - 2\varepsilon})^2}{2\varepsilon \rho(x)}.
\]

A rather straightforward calculation shows that \( e^{-d(x,K)} \) can be expressed as an analytic function of \( \rho \) such that

\[
e^{-d(x,K)} \sim \frac{\varepsilon}{(1 - \sqrt{1 - 2\varepsilon})^2} \rho(x).
\]

Thus we can replace the conformal factor \( e^{-d(x,K)} \) in \([5] \) by \( \rho \) and the results of this article still apply. It follows from Equations (14) and (12), (13) that both \( g \) and \( \Psi^* g \) fulfill the conditions of \([5, Theorem A] \) with \( \alpha = 1 \). In particular, the \( C^{1,\alpha}\)-structure of manifold with boundary obtained by geodesic conformal compactification being unique (\( 0 < \alpha < 1 \)), we conclude that \( \psi \) extends to a \( C^{1,\alpha}\) diffeomorphism \( \bar{\psi} : \tilde{U} \to \tilde{V} \). The end of the proof then follows from a standard trick using the Christoffel symbols.

Let \( \bar{\Gamma} \) denote the Christoffel symbols of the metric \( \bar{g} = \rho^2 g \) in the coordinate system \( (x^1, \ldots, x^n) \). Similarly, let \( \tilde{\Gamma} \) denote the Christoffel symbols of the metric \( \tilde{g} := \rho^2 \psi^* g \). By assumption, all the components of \( \bar{\Gamma} \) and \( \tilde{\Gamma} \) are smooth functions. The transformation law for the Christoffel symbols reads

\[
\frac{\partial^2 \psi^k}{\partial x^i \partial x^j} = \frac{\partial \psi^l}{\partial x^i} \frac{\partial \psi^m}{\partial x^j} \bar{\Gamma}^k_{lm} - \bar{\Gamma}^k_{ij} \frac{\partial \psi^l}{\partial x^i} \frac{\partial \psi^m}{\partial x^j}.
\]

Since \( \psi \) is a \( C^1 \) diffeomorphism up to the boundary, the previous formula immediately shows that \( \psi \) is actually a \( C^2 \) diffeomorphism since the righthand side only involves first order derivatives of \( \psi \) (hence \( C^\infty \) functions) together with \( C^\infty \) functions (the Christoffel symbols). The process can be iterated to conclude that \( \psi \) is actually smooth up to the boundary. \( \square \)

**Lemma 2.11.** Given a metric \( g \) in \( G_k \), any element \( \Psi \in I^k(g) \) decomposes uniquely as \( \Psi = A \circ \Psi_0 \) (resp. \( \Psi = \Psi_0 \circ A \)), where \( A \) is an isometry of the hyperbolic metric \( b \) and \( \Psi \in \Psi_0^{k+1} \) (resp. \( \Psi' \in \Psi_0^{k+1} \)). Conversely, any element of the form \( A \circ \Psi_0 \) (resp. \( \Psi_0 \circ A \)) belongs to \( I^k(g) \). In particular, the set \( I^k(g) \) does not depend on the choice of \( g \in G_k \).

**Proof.** We first prove that if \( A \) is an isometry of the metric \( b \) then \( A \) belongs to \( I^k(g) \) for any metric \( g \in G_k \). Any such element can be written as the composition of an element of \( O(n) \) and a Lorentz boost \( pA^* p^{-1} \). Hence, looking at the formulas given in Section 2.1, we immediately see that \( A \) is actually a smooth diffeomorphism of \( \mathcal{B}_1(0) \). The metric \( \rho_2 A_* g \) can be rewritten as follows:

\[
\bar{g} = \frac{\rho^2}{\rho \circ A^{-1}} A_* g.
\]
Arguing as in the proof of Lemma 2.6, we have that the function \( \frac{\rho^2}{\rho \circ \Psi^{-1}} \) is smooth. The metric \( A \tilde{g} \) being obviously smooth we have that \( \tilde{g} \) is a smooth metric on some \( \tilde{U}, U \in \mathcal{N}_\infty \). The condition (6) for the metric \( A_s g \) is readily checked since \( r(A(x)) = d_0(A(x), 0) = d_0(x, A^{-1}(0)) \) (see the remark following Equation (6)).

Thus, for any \( g \in G_k \), given \( A \in O(1, n) \) and \( \Psi_0 \in I_0^{k+1} \) we have \((\Psi_0)_*g \in G_k \), so from the previous analysis \((A \circ \Psi_0)_*g = A_s(\Psi_0)_*g \in G_k \). A similar argument shows that \((\Psi_0 \circ A)_*g \in G_k \).

Next, given \( \Psi \in I^k(g) \), we will find the element \( A \). It follows from Lemma 2.10 that \( \Psi \) extends to a smooth diffeomorphism up to the boundary \( S_1(0) \). We claim that \( \Psi \) induces a conformal diffeomorphism on \( S_1(0) \). The metric \( \tilde{g} := \rho^2 \Psi_* g \) can be rewritten as

\[
\tilde{g} = \frac{\rho^2}{(\rho \circ \Psi^{-1})^2} \Psi_* \tilde{g}.
\]

Arguing as in the proof of Lemma 2.6, we have that the function \( \frac{\rho^2}{(\rho \circ \Psi^{-1})^2} \) is smooth on some \( \tilde{U} \) where \( U \in \mathcal{N}_\infty \). Restricting to \( S_1(0) \), since both \( \tilde{g} \) and \( \tilde{g} \) restrict to the round metric \( \sigma \), we have that \( \Psi \) induces a conformal isometry of \( S^{n-1} \). It follows from Liouville’s theorem (see \([7, \text{Chapter A.3}]\) that the restriction of \( \Psi \) to \( S_1(0) \) coincide with the restriction of a unique isometry \( A \) of the ball model of the hyperbolic space.

Considering \( A^{-1} \circ \Psi \) (resp. \( \Psi \circ A^{-1} \)) we are left with proving that an element \( \Psi \in I^k(g) \) such that \( \Psi \) induces the identity on \( S_1(0) \) belongs to \( I_0^{k+1} \). Composing \( \Psi \) on the left and on the right by elements of \( I_0^{k+1} \) we can assume that both metrics \( g \) and \( \Psi_* g \) are transverse. The lemma will follow if we can prove that, under these assumptions, \( \Psi \) is the identity.

Following the proof of Proposition 2.7, it suffices to show that the ratio \( \frac{\rho \circ \Psi}{\rho} \) tends to \( 1 \) at infinity. We set \( \Xi := \Psi^{-1} \) and write

\[
\Xi(x) = x^i + \rho \xi^i(x)
\]

together with \( g = \rho^2 g = \delta + \mathcal{E} \) with \( |\mathcal{E}| = O(\mu^k) \). Note that

\[
\partial_i \Xi^k = \delta_i^k + (\partial_i \rho) \xi^k + \rho \partial_i \xi^k = \delta_i^k - x^i \xi^k + O(\rho).
\]

The condition \( \partial^2 \Xi(x) \) can be written in coordinates as

\[
\rho^{-2} \Xi(x) \partial^2 \Xi(x) (\partial_k \xi^i + \mathcal{E}_{kl}(\Xi(x)) \partial_i \xi^k \partial_j \xi^l = \rho^{-2} \delta_{ij} + O(\mu^{k-2}),
\]

\[
\rho^{-2} \Xi(x) \partial^2 \Xi(x) (\delta^k_i - x^i \xi^k) (\delta^l_j - x^j \xi^l) = \rho^{-2} (\Xi(x)) \rho^2 \delta_{ij} + O(\rho)
\]

\[
\rho^{-2} \Xi(x) \partial^2 \Xi(x) (\delta_{ij} - x^i \xi^j - x^j \xi^i + x^i x^j |\xi|^2 = \rho^2 (\Xi(x)) \rho^2 |\xi|^2 \delta_{ij} + O(\rho).
\]

The limit of \( \frac{\rho \circ \Psi}{\rho} \) on \( S_1(0) \) is then obtained by looking at the last equality contracted twice with any vector \( V \) orthogonal to the position vector \( x \) \( (V^i x^i = 0) \):

\[
\delta_{ij} V^i V^j \frac{\rho^2 (\Xi(x))}{\rho^2} \delta_{ij} V^i V^j + O(\rho).
\]

Hence,

\[
\frac{\rho \circ \Psi}{\rho} \bigg|_{S_1(0)} \equiv 1.
\]

This concludes the proof of the lemma. \( \square \)
One of the consequences of Lemma 2.11 is that the set $I^k(g)$ is independent of
the reference metric $g \in G_k$ since any of its elements can be written as $A \circ \Psi_0$ where
$A$ is an isometry of $b$ and $\Psi_0 \in I_0^{k+1}$. It can also be seen that $I^k(g)$ is actually a
group under composition. Indeed, since $I^k(g)$ is independent of the metric $g \in G_k$,
given $\Psi, \Psi_1, \Psi_2 \in I^k(g)$, we have $(\Psi_2) \circ g = G_k$ and
\[(\Psi_1 \circ \Psi_2) \circ g = (\Psi_1) \circ ((\Psi_2) \circ g) \in G_k,
\]
thus proving that $I^k(g)$ is stable under composition.

Combining all the previous results, we give here a variant of a result by Chruściel
and Herzlich [10], see also [11]. Another simplified proof (of a weaker result) appears
in [20].

**Theorem 2.12.** There is a short exact sequence of groups
\[
0 \to I_0^{k+1} \to I^k(g) \xrightarrow{\pi} O_1(n, 1) \to 0. \tag{15}
\]

**Proof.** It follows from Lemma 2.11 that $I_0^{k+1}$ embeds into $I^k(g)$ so the map $I_0^{k+1} \to I^k(g)$ is one-to-one On the other hand, given any element $\Theta \in I_0^{k+1}$ and any $\Psi \in I^k(g)$ we can decompose $\Psi = A \circ \Psi_0$ where $\Psi_0 \in I_0^{k+1}$ and $A \in O_1(n, 1)$. Then
\[\Psi \circ \Theta \circ \Psi^{-1} = A \circ \Psi_0 \circ \Theta \circ \Psi_0^{-1} \circ A^{-1}.
\]
Note that $\Psi_0 \circ \Theta \circ \Psi_0^{-1} \in I_0^{k+1}$ so there exists an element $\Theta_1 \in I_0^{k+1}$ such that $A \circ \Psi_0 \circ \Theta \circ \Psi_0^{-1} = \Theta_1 \circ A$. Consequently,
\[\Psi \circ \Theta \circ \Psi^{-1} = \Theta_1 \circ A \circ A^{-1} = \Theta_1 \in I_0^{k+1}.
\]
This prove that $I_0^{k+1}$ is a normal subgroup of $I^k(g)$. The quotient $I^k(g)/I_0^{k+1}$ is
clearly identified with $O_1(n, 1)$ from Lemma 2.11.

The sequence (15) actually splits. This can be seen by taking $g = b$. The
natural action of $O_1(n, 1)$ on $\mathbb{H}^n$ by isometry gives an embedding of $O_1(n, 1)$ into
$I^k(b)$ which is a right inverse for $\pi$. There is a much more general way to construct
such a splitting. This will be one of the main starting points of our analysis.

**Proposition 2.13.** There exists a unique action of $SO(n, 1)$ on the set $G_k^T$ such
that, for any $A \in O_1(n, 1)$ and $g \in G_k^T$, we have
\[A \cdot g = \tilde{A} \cdot g,
\]
where $\tilde{A} \in I^k$ is the unique element in $I^k(g)$ such that $\tilde{A} \cdot g$ is transverse and
$\pi(\tilde{A}) = A$.

The proof is immediate.

2.3. Jets of asymptotically hyperbolic metrics. In this article, we extend the
definition of the mass as given by Wang [32]. At this point it is important to notice
that the set of germs of asymptotically hyperbolic metrics (of order $k \in \mathbb{N}^*$) is an
affine space.

A priori, the natural definition for such an invariant $\Phi$ would be a map from $G_k$
to some $O_1(n, 1)$-set such that for any $g \in G_k$ and any $\Psi \in I^k(g)$,
\[\Phi(\Psi, g) = \pi(\Psi) \cdot \Phi(g).
\]

There is however an important caveat preventing from using such a general de-
definition. Namely, one wants to impose some continuity assumption for $\Phi$. However,
the stalk of asymptotically hyperbolic metrics has no natural topology. For this reason, we introduce the set of $l$-jets of asymptotically hyperbolic metrics:

**Definition 2.14.** For any $k, l \in \mathbb{N}^*$ with $l > k$, we define the set of $l$-jets of asymptotically hyperbolic metrics of order $k$ as

$$J^l_k := G_k / \sim_{l+1}$$

where $\sim_{l+1}$ is the following equivalence relation:

$$g_1 \sim_{l+1} g_2 \text{ iff } |g_1 - g_2|_b = O(\rho^{l+1}).$$

We denote by $\Pi^l_k$ the projection from $G_k$ to $J^l_k$.

A jet $j \in J^l_k$ is called transverse if there exists a germ $g \in G_k$ representing $j$ which is transverse. We denote by $T^l_k$ the set of transverse jets in $J^l_k$.

The topology on the set of $l$-jets can be described as follows. Passing to polar coordinates, an asymptotically hyperbolic metric $g$ defined in an open subset of the form $B_1(0) \setminus B_{1-\varepsilon}(0)$ can be viewed as a 1-parameter curve $\rho \in (0, \varepsilon') \mapsto g(\rho)$, where $g(\rho)$ is a smooth metric on the bundle $T\mathbb{R}^n|_{S(0)}$, namely the restriction of the tangent bundle of $\mathbb{R}^n$ to the unit sphere, with the further property that the map $\rho \mapsto \rho^2 g(\rho)$ extends (smoothly) up to $\rho = 0$. Let us denote by $E$ the bundle of symmetric 2-tensors on $T\mathbb{R}^n|_{S(0)}$. The Levi-Civita connection of $\mathbb{R}^n$ induces a connection on $E$ which allows us to define the standard Fréchet space topology on $\Gamma(E)$, the set of smooth sections of $E$.

The relevant theory for Fréchet spaces can be found in [18, 31].

In this terminology, an asymptotically hyperbolic germ can be thought as a germ at $\rho = 0$ of curves $\rho \mapsto g(\rho)$ defined on an interval of the form $(0, \varepsilon)$ such that $\mathcal{G}(\rho) := \rho^2 g(\rho)$ extends smoothly to the interval $[0, \varepsilon)$. In polar coordinates

$$b = \rho^{-2} \left( \frac{(d\rho)^2}{1 - 2\rho} + (1 - 2\rho)\sigma \right).$$

A metric $g$ is then transverse if $g - b$ is a 1-parameter family of sections of $S^2(S^{n-1})$ extended trivially in the $\rho$-direction.

The set of $l$-jets of asymptotically hyperbolic metrics thus get identified with the $l + 1$ copies of $\Gamma(E)$ via

$$g \mapsto (\mathcal{G}(0), \partial_\rho \mathcal{G}(0), \ldots, \partial^l_\rho \mathcal{G}(0)).$$

This allows us to endow $J^l_k$ with the product topology. We now relate this to the theory developed in Section 2.2.

**Proposition 2.15.** Given $g \in G_k$, we let $\Theta$ be the adjustment diffeomorphism corresponding to $g$ (see Proposition 2.7). For any $l > k$, the $(l + 1)$-jet of $\Theta$ is fully determined by the $l$-jet of $g$ and depends smoothly on it.

Thus the Taylor expansion of the diffeomorphism $\Theta$ can be obtained formally from the Taylor expansion of the metric $\mathcal{G}$. This proposition has an important corollary. Namely, if the $l$-jet of the metric $g$ is transverse, then $\Theta \in I_0^{l+2}$. The principal use of this remark will appear later.

**Proof of Proposition 2.15.** We shall show that setting $\Psi = \Theta^{-1}$, the $(l + 1)$-jet of $\Psi$ is determined by the $l$-jet of $g$. We assume that $\Psi \in I_0^{k+1}$ is such that $\Psi^* g$ is
transverse:
\[ \rho^2 \Psi^* (\rho^{-2} g)_{ij} x^i = x^j. \]

Introducing “polar” coordinates \((\rho, \varphi^i)\), where \((\varphi^\mu)\) is an arbitrary coordinate system on (some open subset of) the sphere \(S^{n-1}\), the transversality condition reads:
\[ \rho^2 \Psi^* (\rho^{-2} g)(\partial_{\rho^*}) = \rho^2 b(\partial_{\rho^*}) = \frac{d\rho}{1-2\rho}. \]

In the remaining of this proof, we shall use the index 0 to denote the \(\rho\) direction and Greek indices ranging from 1 to \(n-1\) to denote directions that are tangent to the sphere. Using these notations, the transversality condition can be rephrased as follows:
\[ \mathcal{G}_{ab}(\Psi(x)) \partial_0 \Psi^a(x) \partial_j \Psi^b(x) = \left( \frac{\rho(\Psi(x))}{\rho(x)} \right)^2 \delta_j^0 \frac{\delta_i^0}{1-2\rho}. \]

Since we assumed that \(|\Psi(x) - x| = O(\rho^k)\), with \(k \geq 1\), we can write (in polar coordinates)
\[ \Psi^0(x) = \rho(x) + \psi^0(x), \quad \Psi^\mu(x) = \varphi^\mu(x) + \psi^\mu(x), \]
with \(\psi^0(x), \psi^\mu(x) = O(\rho^{k+1})\). Introducing this into the transversality condition, we get:
\[ \left(1 + \frac{\psi^0}{\rho}\right)^2 \frac{\delta_j^0}{1-2\rho} = \mathcal{G}_{ab}(\Psi(x)) \partial_0 \Psi^a(x) \partial_j \Psi^b(x) + (\mathcal{G}_{ab}(\Psi(x)) - \mathcal{G}_{ab}) \partial_0 \Psi^a(x) \partial_j \Psi^b(x) \]

We rewrite this equation in the case \(j = 0\) and in the case \(j = \tau \neq 0\):
\[ (1 + \frac{\psi^0}{\rho})^2 \frac{1}{1-2\rho} = \frac{1}{1-2\rho} (1 + \partial_0 \psi^0)^2 + (1 - 2\rho) \sigma_{\mu\nu} \partial_0 \psi^\mu \partial_0 \psi^\nu + (\mathcal{G}_{ab}(\Psi(x)) - \mathcal{G}_{ab}) \partial_0 \Psi^a(x) \partial_j \Psi^b(x), \]

\[ 0 = \frac{1}{1-2\rho} (1 + \partial_0 \psi^0) \partial_\tau \psi^0 + (1 - 2\rho) \sigma_{\mu\nu} \partial_0 \psi^\mu (\delta_\tau^\sigma + \partial_\tau \psi^\sigma) + (\mathcal{G}_{ab}(\Psi(x)) - \mathcal{G}_{ab}) \partial_0 \Psi^a(x) \partial_\tau \Psi^b(x). \]

The first equation can be rewritten in the following form:
\[ \left( 2 + \frac{\psi^0}{\rho} + \partial_0 \psi^0 \right) \left( \frac{\psi^0}{\rho} - \partial_0 \psi^0 \right) = (1 - 2\rho)^2 \sigma_{\mu\nu} \partial_0 \psi^\mu \partial_0 \psi^\nu + (1 - 2\rho)(\mathcal{G}_{ab}(\Psi(x)) - \mathcal{G}_{ab}) \partial_0 \Psi^a(x) \partial_0 \Psi^b(x). \]

We next make a Taylor expansion of all the unknowns \(\psi^0, \psi^\mu\):
\[ \left\{ \begin{array}{l}
\psi^0 = \psi^0_0 \rho^2 + \cdots \psi^0_{l+1} \rho^{l+1} + O(\rho^{l+2}), \\
\psi^\mu = \psi^\mu_0 \rho^2 + \cdots \psi^\mu_{l+1} \rho^{l+1} + O(\rho^{l+2}).
\end{array} \right. \]

Note that the first two terms in the Taylor expansion disappear because we assumed that \(\psi^i = O(\rho^2)\). The proof now goes by induction on \(l\). The point is then that, having determined \(\psi^0\) and \(\psi^\mu\) up to order \(\rho^l\), we can determine the coefficient \(\psi^0_{l+1}\) by expanding Equation \((16a)\) up to order \(\rho^l\). The only place where \(\psi^0_{l+1}\) shows up in \((16a)\) is in
\[ \frac{\psi^0}{\rho} - \partial_0 \psi^0 = -\rho \psi^0_2 - 2\rho^2 \psi^0_3 - \cdots - l \rho^l \psi^0_{l+1} + O(\rho^{l+1}). \]
We can then determine $\psi^\mu_{l+1}$ by looking at Equation (16b). Here also, $\psi^\mu_{l+1}$ only shows up at one place, namely in $(1 - 2\rho)\sigma^\mu, \partial_0 \psi^\mu, \delta^\nu _{\tau}$.

It is important at each step to notice that to determine the coefficients $\psi^0_0$, $\psi^\mu_0$, and $\psi^\mu_{l+1}$, we only need the Taylor expansion of the coefficients $g^0_{ab}$ up to order $\rho^l$, and that the coefficients of the Taylor expansions (17) are actually polynomials in $\bar{g}_{ab}$ and its derivatives at $\rho = 0$.

The details of the proof being pretty messy, we leave them to the interested reader. □

Proposition 2.15 also implies that the “transversalization” operation via the adjustment diffeomorphism descends to jets:

**Proposition 2.16.** There exists a map

$$\theta : J^l_k \rightarrow T^l_k$$

which is such that the following diagram commutes:

$$\begin{array}{ccc}
G_k & \xrightarrow{\Theta_*} & G^T_k \\
\| & \searrow & \| \\
J^l_k & \xrightarrow{\theta} & T^l_k,
\end{array}$$

where $\Theta_*$ is, abusing the notation, the map associating to a given element $g \in G_k$ the element $\Theta_* g \in G^T_k$, where $\Theta \in I^0_k$ is the adjustment diffeomorphism defined in Proposition 2.7. Assuming that $l < 2k$, the mapping $\theta$ is affine.

**Proof.** The proof of the first part of the proposition follows directly from Proposition 2.15. The (still) non-trivial thing to prove is that $\theta$ is affine.

Let $g_0$ and $g_1$ be two elements of $G_k$. For $\lambda \in [0, 1]$, we set $g_\lambda = (1 - \lambda)g_0 + \lambda g_1$ and denote by $\Psi_\lambda$ the (unique) element in $I^l_{k+1}$ so that $\Psi_\lambda^* g_\lambda$ is transverse. We wish to show that

$$\Psi_\lambda^* g_\lambda = (1 - \lambda) \Psi_0^* g_0 + \lambda \Psi_1^* g_1,$$

at least at the level of $l$-jets. Since the righthand side is linear in $\lambda$, it is enough to show that

$$\rho^2 \frac{d^2}{d\lambda^2} \Psi_\lambda^* g_\lambda = 0,$$

once again at the level of $l$-jets. We are going to perform formal calculations, not worrying about the fact that $\Psi_\lambda$ depends in a $C^2$-manner on $\lambda$ but from the previous proposition, we know that the $l$-jet of $\Psi_\lambda$ depends smoothly on $\lambda$.

We will compute this second order derivative at some $\lambda_0 \in [0, 1]$. To simplify calculations, we can replace the metrics $g_1$ and $g_2$ by $\Psi_{\lambda_0}^* g_1$ and $\Psi_{\lambda_0}^* g_2$ and hence assume that $\Psi_{\lambda_0}$ is the identity and that $g_{\lambda_0}$ is transverse.
Using component notation, we compute:

\[ \rho^2 \frac{d^2}{d\lambda^2} (\Psi^*_\lambda g_\lambda)_{ij} \]
\[ = 2 \partial_\lambda \tilde{g}_{\lambda ij} \Psi^a + 2 \tilde{g}_{\lambda kj} \partial_\lambda \Psi^k \partial_\lambda \Psi^j - 4 \frac{\partial_\lambda \rho}{\rho} \Psi^a \tilde{g}_{\lambda ij} \]
\[ - 2 \frac{\partial_\lambda \rho}{\rho} \Psi^a \tilde{g}_{\lambda ij} \Psi_{\lambda k} \partial_\lambda \Psi^k + 2 \tilde{g}_{\lambda kj} \partial_\lambda \Psi^k \partial_\lambda \Psi^j + 6 \frac{\partial_\lambda \rho}{\rho} \Psi^a \Psi^b \tilde{g}_{\lambda ij} \partial_\lambda \Psi^{\lambda ij} \]
\[ + 4 \frac{\partial_\lambda \rho}{\rho} \partial_\lambda \tilde{g}_{\lambda ij} \partial_\lambda \Psi^k \partial_\lambda \Psi^j + 2 \partial_\lambda \tilde{g}_{\lambda kj} \Psi^a \partial_\lambda \Psi^k + 2 \partial_\lambda \tilde{g}_{\lambda ij} \Psi^a \Psi^b \partial_\lambda \Psi^{\lambda ij}. \]

Since \( \Psi^\prime, \Psi^\prime\prime \in I_0^{k+1} \), it follows by inspection of the decay order of each term that

\[ \rho^2 \frac{d^2}{d\lambda^2} (\Psi^*_\lambda g_\lambda)_{ij} = O(\rho^{2k}) \]

so

\[ \Pi_k^l \left( \frac{d^2}{d\lambda^2} \Psi^*_\lambda g_\lambda \right) = 0 \]

provided \( l < 2k \). \( \square \)

As an example, we give the first non-trivial term in the asymptotic expansion of \( \Theta'_g \):

**Proposition 2.17.** Let \( g \in G_k \) be any given metric. We denote by \( m \) the first non-trivial term in the asymptotic expansion of \( g \):

\[ g = b + \rho^{k-2} m + O(\rho^{k-1}). \]

Denoting by \( \Theta \) the adjustment diffeomorphism (see Proposition 2.7), the metric \( \bar{g} := \Theta'_g \) has the following asymptotic expansion:

\[ \bar{g} = b + \rho^{k-2} \bar{m} + O(\rho^{k-1}), \]

where \( \bar{m} \) is given by

\[ \bar{m}_{ij} = m_{ij} - m_{a x} a^x x_i - m_{a} a^a x_j + m_{a b} \frac{x^a x^b}{k} ((k-1)x_i x_j + \delta_{ij}). \]  

(19)

**Proof.** Calculations are fairly easy from Equations (16a) and (16b). Indeed, the first non-trivial term in the asymptotic expansion of \( \Psi \) is the one of order \( \rho^{k+1} \). They can be obtained looking at the terms of order \( O(\rho^k) \) of (16a)-(16b):

\[ \psi_{k+1}^0 = - \frac{1}{2k} m_{00}, \sigma_{\nu \mu} \psi^\nu = - \frac{m_{0 \nu}}{k + 1} \]

Hence, from the identity

\[ \rho^2 \bar{g} = \left( \frac{\rho}{\rho \circ \Psi} \right)^2 \Psi^* \bar{g}, \]

we find that

\[ \bar{m}_{00} = 0, \bar{m}_{0 \mu} = 0, \bar{m}_{\mu \nu} = m_{\mu \nu} + \frac{m_{0 \mu}}{k} \sigma_{\mu \nu}. \]
The relation between \( m \) and \( \tilde{m} \) can be condensed into the following:

\[
\tilde{m} = m - d\rho \otimes m(\cdot,\partial_{\rho}) - m(\partial_{\rho},\cdot) \otimes d\rho + \frac{m(\partial_{\rho},\partial_{\rho})}{k} (\delta + (k-1)d\rho \otimes d\rho).
\]

We now wish to return to Cartesian coordinates. To this end, it suffices to remark that they are related to the \((\rho,\varphi)\)-coordinates via

\[
x^i = \sqrt{1-2\rho F^i(\varphi)},
\]

where \((F^i)\) is a set of \(n\) given functions such that \(\sum (F^i)^2 \equiv 1\) meaning that \(F^i = x^i/|x|\). As a consequence,

\[
\frac{\partial}{\partial \rho} = \frac{\partial x^i}{\partial \rho} \frac{\partial}{\partial x^i} = \frac{-1}{\sqrt{1-2\rho}} F^i \frac{\partial}{\partial x^i} = \frac{-1}{|x|^2} x^i \frac{\partial}{\partial x^i}.
\]

Formula (19) follows.

**Remark 2.18.** Returning to the proof of Proposition 2.16, it is important to remark at this point that we can gain in the order up to which \(\theta\) is linear by restricting ourselves to metrics that are already transverse up to some high order. Indeed, from Proposition 2.15, if \(g \in J^l_k \cap T^l_k\), we have that the adjustment diffeomorphism \(\Theta\) is in \(I^{l+2}_0\). Hence, following the lines of the proof of Proposition 2.16, we have that

\[
\theta : J^l_k \cap T^l_k \rightarrow T^l_k
\]

is linear as long as \(l \leq k + l'\).

The following theorem will allow us to define asymptotic invariants:

**Theorem 2.19.** There exists a unique action of the group \(O_\uparrow(n,1)\) on \(T^l_k\) such that the projection \(\Pi^l_k : G^l_k \rightarrow T^l_k\) is a \(O_\uparrow(n,1)\)-equivariant map. Namely,

\[
\forall A \in O_\uparrow(n,1), \forall g \in G^l_k, \quad A \cdot (\Pi^l_k(g)) = \Pi^l_k(A \cdot g),
\]

where the action of \(O_\uparrow(n,1)\) on \(G^l_k\) was defined in Proposition 2.13. Further the action is linear and smooth as long as \(l \leq 2k\) and reduces in the case \(l = k\) to the pushforward action

\[
A \cdot g = A_\ast g,
\]

where \(A \in O_\uparrow(n,1)\) is any hyperbolic isometry.

**Proof.** The only non-trivial point in the proof is the fact that the action is linear for all \(l \leq 2k\). This is where Remark 2.18 turns out to be important. Indeed we shall see that, given \(g \in T^l_k\) and \(A \in O_\uparrow(n,1)\), we have that \(A_\ast g \in T^l_k\): transversality of the first non-trivial term in the asymptotic expansion of \(g\) is preserved under the (non-adjusted) action of the Lorentz group. From Remark 2.18 it then follows that \(g \mapsto \theta A_\ast g\) is affine for all \(l \leq 2k\).

We denote by \(P = x^i\partial_i\) the position vector and set \(g = b + \varepsilon\). Since \(A\) is an hyperbolic isometry, we have

\[
A_\ast g = b + A_\ast \varepsilon.
\]
Transversality of $b$ reads $e(P, \cdot) = 0$. So we need to check that

$$|\rho^2 (A_* e)(P, \cdot)|_\delta = |\rho^2 A_* (e(A^* P, \cdot))|_\delta = O(\rho^{k+1}).$$

This will follow by showing that $A^* P = \lambda(x, A) P + O(\rho)$ for some function $\lambda(x, A)$.

Before entering calculations, we note that this fact is natural from the point of view of hyperbolic geometry. Indeed, all (hyperbolic) geodesics intersect the boundary $\mathbb{S}^{n-1}$ orthogonally. The action of an hyperbolic isometry can be understood as a change of origin of the hyperbolic space.

The condition

$$A^* P = \lambda(x, A) P + O(\rho)$$

only needs to be checked for generators of the Lorentz group. It is obvious for rotations since $R^* P = P$ for any rotation $R$. The case of Lorentz boosts requires some more calculations.

Replacing $A$ by $A^{-1}$, Condition (20) is equivalent to

$$A_* P = \lambda(x, A^{-1}) P + O(\rho),$$

or to

$$dA(P)(x) = \lambda(A(x), A^{-1}) P(A(x)) + O(\rho).$$

We use Formula 4:

$$dA^*_\delta(x) = \frac{1}{D} (dx^1, \ldots, \cosh(s) dx^i + \sinh(s) x^j dx^j, \ldots, dx^n)$$

$$- \frac{1}{D^2} \left( x^1, \ldots, \cosh(s) x^i + \sinh(s) \frac{1 + |x|^2}{2}, \ldots, x^n \right) \left( (\cosh(s) - 1) x^j dx^j + \sinh(s) dx^i \right),$$

$$dA^*_\delta(P) = \left( \frac{1}{D} - \frac{(\cosh(s) - 1)|x|^2 + \sinh(s)x^i}{D^2} \right) \left( x^1, \ldots, \cosh(s) dx^i + \sinh(s) \frac{1 + |x|^2}{2}, \ldots, x^n \right) + O(\rho)$$

$$= \frac{1}{\cosh(s) + \sinh(s)x^i} P(A^*_\delta(x)) + O(\rho),$$

so the claim follows. □

2.4. Geometric invariants at infinity. We can now come to the main definition of this article:

**Definition 2.20.** A linear geometric invariant at infinity for the set of asymptotically hyperbolic metrics of order $k$ is an affine map $\Phi : G_k \to V$, where $V$ is a finite dimensional representation of the group $O_+(n, 1)$ such that:

1. $\Phi(b) = 0$,
2. for any $\Psi \in I_k^+$ and any $g \in G_k$, the following holds:

$$\Phi(\Psi \cdot g) = \pi(\Psi) \cdot \Phi(g).$$

Namely, $\Phi$ is an intertwining map.

3. there exists $l > k$ such that $\Phi$ factors through $T_k^l$:

$$G_k \xrightarrow{\Phi} V \xrightarrow{\pi} T_k^l \xrightarrow{\gamma} V.$$
where \( \tilde{\Pi}_k \) denotes the composition \( \Pi_k \circ \Theta_* \), \( \Theta_* \) being (abusing the notation) the map from \( G_k \) to \( G^T_k \) mapping an element \( g \) to the unique transverse element \( \theta_* g \in G^T_k \), \( \theta \in I^*_k \).

4. \( \varphi \) is a continuous affine map.

Note that at this point we could be more general assuming only that the space \( V \) is a \( SO(n,1) \)-set. This restriction is not relevant to some large extent, see [29, Chapter 7, Section 1.3].

The name invariant can also seem inappropriate since we do not get strictly speaking something independent of the chart chosen at infinity. The situation is the same for the classical mass. If one insists on having a true invariant, one then has to look at the \( O_t(n,1) \)-orbit to which \( \Phi(g) \) belongs. This relies on the classical invariant theory described well in [29, Chapter 11].

3. Action of the Lorentz group and algebra on mass-aspect tensors

In this section and all the remainder of the paper, we deal with a particular class of metrics \( g \) which are asymptotically hyperbolic of order \( k \in \mathbb{N}^* \) and moreover transverse (\( g \in G^T_k \)). We start by describing the action of the Lorentz group \( SO^+(n,1) \) on them, particularly on the mass-aspect tensors. We will then investigate intertwining maps under the action of the Lorentz Lie algebra \( \mathfrak{so}(n,1) \).

3.1. Action of the Lorentz group and algebra on mass-aspect tensors

We turn our attention to metrics \( g \in G^T_k \) that admit an asymptotic expansion in a neighborhood of infinity as follows:

\[
g = b + \rho^{k-2} m + O(\rho^{k-1}).
\]

where \( x^a(g_{ab} - b_{ab}) = 0 \). The coefficient \( m \) in the right-hand side, called the mass-aspect tensor, is a symmetric two-tensor on the sphere at infinity \( \mathbb{S}^{n-1} \). This can be reformulated by saying that it satisfies the transversality condition \( x^a m_{ab} = 0 \).

The mass-aspect tensor of a metric as above clearly depends on the choice of the (transverse) coordinate system chosen in the neighborhood of infinity. More precisely, the subsequent lemma gives the dependence upon the action of the identity component of the Lorentz group:

**Lemma 3.1.** Let \( g \in G^T_k \) be as above with a mass-aspect tensor \( m \) and \( A \in SO^+(n,1) \). Then the metric \( A \cdot g \) defined in Theorem 2.19 has the expression

\[
A \cdot g = b + \rho^{k-2} A \cdot m + O(\rho^{k-1})
\]

and the map \((A, m) \mapsto A \cdot m \) defines a \( SO^+(n,1) \)-action on the space of sections of symmetric two-tensors over the unit sphere \( S^2 \mathbb{S}^{n-1} \).

**Proof.** From Proposition 2.13, one has \( A \cdot g = \tilde{A} \ast g \) with \( \tilde{A} = \Theta \circ A \), where \( \Theta \) is the unique diffeomorphism in \( I^{k+1}_k \) of Proposition 2.7 associated to \( A \ast g \). Next, we can write

\[
A \ast g = b + \left( \frac{1 - |A^{-1}(x)|^2}{2} \right)^{k-2} A \ast m + O \left( (1 - |A^{-1}(x)|)^{k-1} \right).
\]

Let us denote \( \hat{x} := \frac{x}{|x|} \) for \( x \neq 0 \). Let us then introduce

\[
u[A](\hat{x}) := \lim_{|x| \to 1} \frac{1 - |A^{-1}(x)|^2}{1 - |x|^2}.
\]
The function \( u[A] \) is defined on \( \mathbb{S}^{n-1} \) and has the property to be positive and smooth on \( \mathbb{S}^{n-1} \). Then, we obtain
\[
A \cdot g = b + \rho^{k-2}(u[A](\hat{x}))^{k-2}A_\ast m + O(\rho^{k-1}).
\]

In fact, we note that the tensor \( A_\ast m \in S^2(\mathbb{S}^{n-1}) \) is transverse. This is due to the fact that \( A \) preserves the unit sphere over which \( m \) is defined, so that \( A_\ast m \) is itself in \( S^2(\mathbb{S}^{n-1}) \). This implies that the adjustment diffeomorphism \( \Theta \) leaves invariant the principal part of the expansion and may only modify the rest. Finally, we get
\[
\Theta_\ast A \cdot g = b + \rho^{k-2}(u[A](\hat{x}))^{k-2}A_\ast m + O(\rho^{k-1}).
\]

Thus, we identify the expression of \( A \cdot m \) as
\[
A \cdot m = (u[A](\hat{x}))^{k-2}A_\ast m.
\]

The fact that this is a group action on \( S^2(\mathbb{S}^{n-1}) \) is now an easy consequence of the group action property for the map \( (A \cdot g) \mapsto A \cdot g \) of Proposition 2.13.

One can deduce from the above result the corresponding action on the mass-aspect function, \( \text{tr}^\sigma m \), as well as on the volume form \( \text{tr}^\sigma m \, d\mu^\sigma \).

**Corollary 3.2.** We have
\[
A \cdot (\text{tr}^\sigma m) = (u[A])^k \text{tr}^\sigma m \circ A^{-1},
\]
and
\[
A \cdot (\text{tr}^\sigma m \, d\mu^\sigma) = (u[A])^{k+1-n}A_\ast (\text{tr}^\sigma m \, d\mu^\sigma).
\]

**Proof.** We begin by noting that
\[
A_\ast \sigma = u[A]^2 \sigma.
\]

Indeed, let us express the fact that \( A \cdot b = b \). This translates into
\[
\left( \frac{2}{1 - |A^{-1}(x)|^2} \right)^2 A_\ast \delta = \left( \frac{2}{1 - |x|^2} \right)^2 \delta
\]
which we can rewrite as
\[
A_\ast \sigma = A_\ast \delta|_{x|=1} = \lim_{|x| \to 1} \left( \frac{1 - |A^{-1}(x)|^2}{1 - |x|^2} \right)^2 \sigma,
\]
which gives the desired result considering the definition of \( u[A] \). To get the expression for the action on \( \text{tr}^\sigma m \), one can write
\[
A \cdot g = \left( \frac{2}{1 - |x|^2} \right)^2 \left[ \sigma + \left( \frac{1 - |x|^2}{2} \right)^k u[A]^{k-2}(\hat{x})A_\ast m + O((1 - |x|)^{k+1}) \right]
\]
so that the mass-aspect function of the metric \( A \cdot g \) reads
\[
A \cdot (\text{tr}^\sigma m) = u[A]^{k-2}(\hat{x}) \text{tr}^\sigma(A_\ast m).
\]

We handle this expression using the identity
\[
A_\ast (\text{tr}^\sigma m) = \text{tr}^{A_\ast \sigma}(A_\ast m) = u[A]^{-2} \text{tr}^\sigma(A_\ast m),
\]
which yields at last \( \text{tr}^\sigma(A_\ast m) = u[A]^2 \text{tr}^\sigma m \circ A^{-1} \) and the result follows. For the action on the volume form \( \text{tr}^\sigma m \, d\mu^\sigma \), we write
\[
A_\ast (\text{tr}^\sigma m \, d\mu^\sigma) = \text{tr}^\sigma m \circ A^{-1} \, d\mu^{A_\ast \sigma} = \text{tr}^\sigma m \circ A^{-1} \, u[A]^{n-1} d\mu^\sigma,
\]
and, using again the expression above for $A \cdot g$, we conclude that

$$A \cdot (\text{tr}^\sigma m \, d\mu^\sigma) = \text{tr}^\sigma (A_* m) u[A]^{k-2} d\mu^\sigma$$

which can then be rewritten as

$$u[A]^k \text{tr}^\sigma m \circ A^{-1} \, d\mu^\sigma = u[A]^{k+1-n} A_* (\text{tr}^\sigma m \, d\mu^\sigma),$$

as desired. □

In particular for $k = n - 1$, $\text{tr}^\sigma m \, d\mu^\sigma$ is an invariant under the action of the group $\text{Conf}(S^{n-1})$ of conformal diffeomorphisms over the sphere. We can in fact restate this in terms of the group $\text{Conf}(S^{n-1})$ acting on bundles of tensors with some conformal weight over $S^{n-1}$, see e.g. [13] for more about this terminology.

We now come to the associated Lie algebra action of $\mathfrak{so}(n,1)$ on $S^2(S^{n-1})$ that we will need further in this work. It is defined by

$$X \cdot m := \frac{\partial}{\partial s} (A^s \cdot m) \big|_{s=0},$$

for all $X \in \mathfrak{so}(n,1)$ and $m \in S^2(S^{n-1})$, where $(A^s)_{s \in \mathbb{R}}$ is the one-parameter subgroup of $SO^+(n,1)$ generated by $X$.

In the next proposition we compute this action for Lorentz boosts $a_i$ and for rotations $r_{ij}$, whose expressions are introduced in Section 2.

**Proposition 3.3.** Let $a_i$ be the Lorentz boost vector field as defined in (5) and $r_{ij}$ be the rotation vector field defined in (3). Then

$$a_i \cdot m = -\nabla_\sigma^a m + k x^i m,$$

$$r_{ij} \cdot m = -\nabla_{ij}^\sigma m - (m r_{ij} (\cdot) \cdot) + m (\cdot, r_{ij} (\cdot)),$$

where the expression of $r_{ij}$ acting as an operator on vector fields tangent to $S^{n-1}$ is given by

$$r_{ij}(U) = U^j \partial_j - U^j \partial_i = U^i a_j - U^j a_i \mod (x^a \partial_a).$$

**Proof.** From the expression of the boost hyperbolic isometries $A^s_i$ defined in Section 2, we compute

$$\frac{1 - |A^{-s}_i(x)|^2}{2} = \frac{1}{\cosh s \frac{1+|x|^2}{1-|x|^2} - \sinh s \frac{2x^i}{1-|x|^2} + 1}$$

which gives us

$$u[A^s_i](\hat{x}) = \frac{1}{\cosh s - x^i \sinh s}.$$

Hence, we have the expression for the action of $A^s_i$:

$$A^s_i \cdot m = \frac{1}{(\cosh s - x^i \sinh s)^k-2} (A^s_i)_* m.$$

We now compute the derivative of this expression with respect to $s$, and get

$$a_i \cdot m = \frac{\partial}{\partial s} \left( \frac{1}{(\cosh s - x^i \sinh s)^k-2} (A^s_i)_* m \right) \big|_{s=0} = -\mathcal{L}_{a_i} m + (k - 2)x^i m.$$

(24)
Next, we want to rewrite this using covariant derivatives instead of Lie derivatives. Since elements of $SO_t(n, 1)$ preserve the sphere at infinity $|x| = 1$, $a_i$ is tangent to it. We write
\[
(L_\nu m)(a_i, a_b) = a_i(m(a_i, a_b)) - m([a_i, a_b], a_b) - m(a_b, [a_i, a_b]) = (\nabla^\sigma a_i)m(a_i, a_b) + m(\nabla^\sigma a_i, a_b) + m(a_i, \nabla^\sigma a_i).
\]

We then write
\[
\nabla^\sigma_{\nu} a_i = \nabla^\delta_{\nu} a_i + \delta(a_i, a_i)\nu
\]
where $\nu = x/|x|$ is the unit outgoing normal of the $(n - 1)$-sphere in $\mathbb{R}^n$. The transversality property expresses as $t_\nu m = 0$. We just need to compute $\nabla^\delta_{\nu} a_i$ at $|x| = 1$:
\[
\nabla^\delta_{\nu} a_i = -x^i \partial_a + 2x^a x^c \partial_c - \delta^i_a x^c \partial_c.
\]

From the transversality property again, this yields
\[
m(\nabla^\sigma a_i, a_b) = -x^i m_{ab} = -x^i m(a_i, a_b)
\]
since we restrict ourselves on the sphere $|x| = 1$. Thus, we obtain for all $a, b$:
\[
(L_\nu m)(a_i, a_b) = (\nabla^\sigma a_i)m(a_i, a_b) - 2x^i m(a_i, a_b),
\]
or, more simply,
\[
L_\nu m = \nabla^\sigma m - 2x^i m.
\]
The right-hand side of (24) takes therefore the equivalent form
\[
-\nabla^\sigma_{\nu} m + kx^i m.
\]

This yields finally that
\[
a_i \cdot m = -\nabla^\sigma_{\nu} m + kx^i m,
\]
as desired.

We now derive the infinitesimal action for the one-parameter group of rotations $R^\theta_{ij}$. Since $r_{ij} = -[a_i, a_j]$, we have
\[
[r_{ij} \cdot m] = -[a_i, a_j] \cdot m.
\]

Then, a straightforward computation yields
\[
[a_i, a_j] \cdot m = [-\nabla^\sigma_{ij} + kx^i, -\nabla^\sigma_{ij} + kx^j]m = \nabla^\sigma_{[a_i, a_j]}m + R^\sigma(a_i, a_j)m.
\]

For $V$ a tangent vector to $S^{n-1}$ we have
\[
R^\sigma(a_i, a_j)V = \sigma(a_j, V)a_i - \sigma(a_i, V)a_j = dx^i (V)a_i - dx^j (V)a_j = -r_{ij}(V),
\]

Then we compute
\[
R^\sigma(a_i, a_j)m(U, V) = m(r_{ij}(U), V) + m(U, r_{ij} V).
\]
Thus we find
\[
(r_{ij} \cdot m)(U, V) = -\left(\nabla^\sigma_{r_{ij}} m(U, V) + m(r_{ij}(U), V) + m(U, r_{ij} V)\right).
\]

\(\Box\)
3.2. Intertwining operators. We denote by \( V \) an arbitrary finite dimensional representation of the Lorentz group. We recall the following standard definition from representation theory.

**Definition 3.4.** Let \( \Phi : S^2(S^{n-1}) \to V \) be a linear map. We call \( \Psi \) an intertwining operator if for any \( A \in SO(n, 1) \) and any \( m \in S^2(S^{n-1}) \) we have

\[
\Phi(A \cdot m) = A \cdot (\Phi(m)).
\]

On the Lie algebra level, this definition implies that for any element \( a \in \mathfrak{so}(n, 1) \) and any \( m \in S^2(S^{n-1}) \), we have

\[
\Phi(a \cdot m) = a \cdot (\Phi(m)).
\]

We define an the action of the Lie algebra \( \mathfrak{so}(n, 1) \) on linear maps \( S^2(S^{n-1}) \to V \) by

\[
(a \cdot \Phi)(m) := a \cdot (\Phi(m)) - \Phi(a \cdot m)
\]

for \( a \in \mathfrak{so}(n, 1) \) and \( \Phi : S^2(S^{n-1}) \to V \). Intertwining operators are then the fixed points for this action.

We choose a basis \( (v_\mu) \) of \( V \) and write \( \Phi = \sum_\mu \Phi^\mu v_\mu \). Then the components \( \Phi^\mu \) are linear forms on \( S^2(S^{n-1}) \) which we assume continuous with respect to the standard topology on \( S^2(S^{n-1}) \), that is the \( \Phi^\mu \) are distributions.

As usual in distribution theory, we keep the same notation for the distribution \( \Phi^\mu \) itself and for the distribution density valued in \( S^2(S^{n-1}) \), so that we may write

\[
\Phi(m) = \sum_\mu \int_{S^{n-1}} \langle \Phi^\mu, m \rangle d\mu^\sigma v_\mu,
\]

where the symbol \( \langle \cdot, \cdot \rangle \) denotes the inner product induced by the metric \( \sigma \) on \( S^2(S^{n-1}) \). The notation \( \Phi := \sum_\mu \Phi^\mu v_\mu \) will apply for both the distribution and the \( S^2(S^{n-1}) \)-valued function. We hope that it will be clear at any place in the sequel which object we are referring to through this symbol.

By dualizing the action of \( \mathfrak{so}(n, 1) \) we shall find the conditions that \( \Phi \) must satisfy to be an intertwining operator:

**Proposition 3.5.** If \( \Phi : S^2(S^{n-1}) \to V \) is intertwining for the action of \( \mathfrak{so}(n, 1) \) then the tensors \( \Phi^\mu \in S(S^{n-1}) \) satisfy

\[
\nabla_{a_i} \Phi + (k + 1 - n)x^i \Phi - \sum_\mu \Phi^\mu a_i \cdot v_\mu = 0 \quad (25)
\]

and

\[
\nabla_{r_{ij}} \Phi + \Phi(r_{ij}(\cdot, \cdot)) + \Phi(\cdot, r_{ij}(\cdot)) - \sum_\mu \Phi^\mu r_{ij} \cdot v_\mu = 0. \quad (26)
\]

Conversely, if (25) and (26) hold then the map \( \Phi : S(S^{n-1}) \to V \) is intertwining.

In the proof we will use the formula

\[
\int_{S^{n-1}} \langle \Phi^\mu, \nabla_X^\sigma m \rangle d\mu^\sigma = -\int_{S^{n-1}} \langle \nabla_X^\sigma \Phi^\mu + (\text{div}^\sigma X) \Phi^\mu, m \rangle d\mu^\sigma,
\]

which holds for any vector field \( X \). This comes from the identity

\[
0 = \int_{S^{n-1}} \text{div}^\sigma ((\Phi^\mu, m)X) d\mu^\sigma = \int_{S^{n-1}} ((\nabla_X^\sigma \Phi^\mu, m) + \langle \Phi^\mu, \nabla_X^\sigma m \rangle + \langle \Phi^\mu, m \rangle \text{div}^\sigma X) d\mu^\sigma.
\]

Note that \( a_i = \text{grad}^\sigma x^i \) so \( \text{div}^\sigma a_i = \Delta^\sigma x^i = -(n - 1) x^i \), while \( r_{ij} \) is a Killing vector field of \( (S^{n-1}, \sigma) \), hence \( \text{div}^\sigma r_{ij} = 0. \)
Proof. Assume $\Phi$ is an intertwining operator and $X_i \in \mathfrak{so}(n,1)$ is a boost. Using (22) we have

$$0 = (a_i, \Phi)(m)$$

$$= \sum_{\mu} \int_{S^{n-1}} (\Phi^\mu, m) \, d\mu^\sigma a_i \cdot v_\mu - \int_{S^{n-1}} \langle \Phi^\mu, a_i \cdot m \rangle \, d\mu^\sigma v_\mu$$

$$= \sum_{\mu} \int_{S^{n-1}} (\Phi^\mu, m) \, d\mu^\sigma a_i \cdot v_\mu + \int_{S^{n-1}} \langle \Phi^\mu, \nabla^\sigma_{a_i} m - kx^i m \rangle \, d\mu^\sigma v_\mu$$

$$= \sum_{\mu} \int_{S^{n-1}} (\Phi^\mu, m) \, d\mu^\sigma a_i \cdot v_\mu - \int_{S^{n-1}} \langle \nabla^\sigma_{a_i} \Phi^\mu + (\text{div}^\sigma a_i + kx^i) \Phi^\mu, m \rangle \, d\mu^\sigma v_\mu$$

$$= \sum_{\mu} \int_{S^{n-1}} (\Phi^\mu, m) \, d\mu^\sigma a_i \cdot v_\mu - \int_{S^{n-1}} \langle \nabla^\sigma_{a_i} \Phi^\mu - (n - 1 - k)x^i \Phi^\mu, m \rangle \, d\mu^\sigma v_\mu$$

for $i = 1, \ldots, n$. Since this holds for all $m \in S^2(S^{n-1})$ we find that

$$\nabla_{a_i} \Phi - (n - 1 - k)x^i \Phi - \sum_{\mu} \Phi^\mu a_i \cdot v_\mu = 0.$$

To compute the action of a rotation $r_{ij} \in \mathfrak{so}(n,1)$ we need the following formula. Here $\varepsilon_A$ denotes an orthonormal frame on $S^{n-1}$,

$$\langle \Phi, m(r_{ij}(\cdot), \cdot) \rangle = \sum_{A,B} \Phi(\varepsilon_A, \varepsilon_B) m(r_{ij}(\varepsilon_A), \varepsilon_B)$$

$$= \sum_{A,B,C} \Phi(\varepsilon_A, \varepsilon_B) m((r_{ij}(\varepsilon_A), \varepsilon_C) \varepsilon_C, \varepsilon_B)$$

$$= \sum_{A,B,C} \langle \varepsilon_A, r_{ij}(\varepsilon_C) \rangle \Phi(\varepsilon_A, \varepsilon_B) m(\varepsilon_C, \varepsilon_B)$$

$$= - \sum_{A,B,C} \Phi((\varepsilon_A, r_{ij}(\varepsilon_C)) \varepsilon_A, \varepsilon_B) m(\varepsilon_C, \varepsilon_B)$$

$$= - \sum_{B,C} \Phi(r_{ij}(\varepsilon_C), \varepsilon_B) m(\varepsilon_C, \varepsilon_B)$$

$$= - \langle \Phi(\cdot, r_{ij}(\cdot)), \cdot \rangle, m).$$

Hence (23) tells us that

$$0 = (r_{ij} \cdot \Phi)(m)$$

$$= \sum_{\mu} \int_{S^{n-1}} (\Phi^\mu, m) \, d\mu^\sigma r_{ij} \cdot v_\mu - \int_{S^{n-1}} \langle \Phi^\mu, r_{ij} \cdot m \rangle \, d\mu^\sigma v_\mu$$

$$= \sum_{\mu} \int_{S^{n-1}} (\Phi^\mu, m) \, d\mu^\sigma r_{ij} \cdot v_\mu - \int_{S^{n-1}} \langle \Phi^\mu, \nabla^\sigma_{r_{ij}} m + m(r_{ij}(\cdot), \sigma) + m(\cdot, r_{ij}(\cdot)) \rangle \, d\mu^\sigma v_\mu$$

$$= \sum_{\mu} \int_{S^{n-1}} (\Phi^\mu, m) \, d\mu^\sigma r_{ij} \cdot v_\mu - \int_{S^{n-1}} \langle \nabla^\sigma_{r_{ij}} \Phi^\mu + \Phi^\mu(r_{ij}(\cdot), \cdot) + \Phi^\mu(\cdot, r_{ij}(\cdot)), m \rangle \, d\mu^\sigma v_\mu.$$


and we conclude that
\[
\nabla r_{ij} \Phi + \Phi(r_{ij}(\cdot, \cdot)) + \Phi(\cdot, r_{ij}(\cdot)) - \sum_{\mu} \Phi^\mu r_{ij} \cdot v_\mu = 0.
\]

Since \( r_{ij} = -[a_i, a_j] \) we see that (26) follows from (25). Since boosts \( a_i \) and rotations \( r_{ij} \) form a basis of \( \mathfrak{so}(n,1) \) it is sufficient that (25) and (26) hold to conclude that \( \Phi \) is an intertwining map. \( \square \)

Since the vector fields \( a_i \) span the tangent space \( T_p S_{n-1} \) at each point \( p \) we conclude that the distributions \( \Phi^\mu \) are actually analytic maps.

4. Classification of mass-like invariants

In this section we will classify all maps \( \Phi : S^2(S_{n-1}) \to V \) which are intertwining for the Lie algebra action. We begin by looking at two simple examples.

4.1. Simple examples of mass-like invariants.

Example 4.1. As a first example, we take for \( V \) the trivial 1-dimensional representation of the Lorentz group. A basis consists of a single vector \( v_0 \neq 0 \) such that \( a \cdot v_0 = 0 \) for any \( a \in \mathfrak{so}(n,1) \). Set \( \Phi = \Phi_0 v_0 \). From Equation (25) evaluated at the point where \( a_i = 0 \) we get \( k = n - 1 \). All the rotations \( r_{AB} \), \( 2 \leq A < B \leq n \) vanish at the south pole \( p_0 = (-1, 0, \ldots, 0) \), so Equation (26) evaluated at this point yields
\[
\Phi_0(r_{AB}\cdot, \cdot) + \Phi_0(\cdot, r_{AB}\cdot) = 0.
\]
This means that \( \Phi_0 \) is a bilinear form which is invariant under the action of \( SO(n-1) \). The only such forms are proportional to the metric \( \sigma \). It follows that \( \Phi = \sigma v \) for some \( v \in V \) at the south pole. By rotational symmetry this extends to \( \Phi = \sigma v \) on all of \( \mathbb{S}^{n-1} \), where \( v : \mathbb{S}^{n-1} \to V \). From (25) it follows that \( v \) is constant, and \( \Phi = \sigma v \) for some \( v \in V \).

Conversely, \( \Phi = \sigma v \) obviously satisfies Equations (25) and (26) with \( k = n - 1 \).

Example 4.2. As a second example, we choose \( V = \mathbb{R}^{n,1} \) with the standard representation of the Lorentz group. Let \( (v_0, \ldots, v_n) \) an orthonormal basis of \( V \). In the case \( k = n \), X. Wang defines in [32] the energy-momentum vector of \( g \) as the following element of \( V \):
\[
\Phi(m) = \int_{S^{n-1}} \text{tr}^\sigma m \, d\mu^\sigma \, v_0 + \sum_i \int_{S^{n-1}} x^i \text{tr}^\sigma m \, d\mu^\sigma \, v_i,
\]
where the \( x^i \)'s are, again, the coordinate functions in \( \mathbb{R}^n \) restricted to \( \mathbb{S}^{n-1} \). Wang as well as Chruściel-Herzlich [10, 32] prove that the Lorentz group acts on such vectors so that the map \( \Phi \) is intertwining. Furthermore, they obtain that the Minkowski quadratic form evaluated on this vector yields an asymptotic invariant, namely the square of the asymptotically hyperbolic mass.

4.2. Strategy for the classification. We want to find all intertwining maps \( \Phi : S^2(S_{n-1}) \to V \). The strategy we will use is the following.

- The Lie algebra \( \mathfrak{so}(n,1) \) has a (parabolic) subalgebra \( \mathfrak{p} \) consisting of all elements whose associated vector field vanishes at the south pole \( p_0 = (-1, 0, \ldots, 0) \). This corresponds to the parabolic subgroup \( P \) of \( SO^+(n,1) \) fixing the south pole. We first study the necessary conditions on \( \Phi \) coming from Equations (25) and (26) with vector fields fixing the south pole. The
derivative terms in those equations vanish at the south pole, and we get a set of algebraic equations for \( \Phi(p_0) \).

- Having solved the equations at the south pole we transport \( \Phi(p_0) \) to the rest of \( S^{n-1} \) by using the equations corresponding to the remaining elements of \( \mathfrak{so}(n,1) \). We introduce stereographic coordinates at the south pole. The remaining equations then correspond to ordinary differential equations in the coordinate directions, and they can be explicitly integrated.

- Once we have identified candidates for intertwining operators in the stereographic coordinate system, we check that they extend smoothly to all of \( S^{n-1} \) and use Equation (25) to verify that they are solutions.

### 4.3. Elements fixing the south pole

The subalgebra \( \mathfrak{p} \) of \( \mathfrak{so}(n,1) \) fixing the south pole has a basis consisting of
- the infinitesimal boost \( a_1 \) in the \( x^1 \)-direction,
- infinitesimal translations at the north pole \( s_A := a_A + r_{1A} \), \( 2 \leq A \leq n \),
- infinitesimal rotations \( r_{AB} \) where \( 2 \leq A < B \leq n \).

**Proposition 4.3.** Suppose \( \Phi : S^2(S^{n-1}) \to V \) is an intertwining map. Then \( \Phi(p_0) \) satisfies the following equations

\[
\sum_\mu \Phi^\mu(p_0)(a_1 \cdot v_\mu) = (n - 1 - k)\Phi(p_0),
\]

\[
\sum_\mu \Phi^\mu(p_0)s_A \cdot v_\mu = 0,
\]

\[
\sum_\mu \Phi^\mu(p_0)r_{AB} \cdot v_\mu = \Phi(p_0)(r_{AB}(\cdot), \cdot) + \Phi(p_0)(\cdot, r_{AB}(\cdot)).
\]

**Proof.** Equation (25) for \( a_1 \) evaluated at the south pole \( p_0 = (-1, 0, \ldots, 0) \) gives us

\[
0 = \left( \nabla_{a_1} \Phi(p_0) - (n - 1 - k)x^1 \Phi(p_0) - \sum_\mu \Phi^\mu(p_0)a_1 \cdot v_\mu \right)_{p_0} = (n - 1 - k)\Phi(p_0) - \sum_\mu \Phi^\mu(p_0)a_1 \cdot v_\mu.
\]

For translations at the north pole

\( s_A = a_A + r_{1A} \),

Equations (25) and (26) tell us that

\[
0 = \nabla_{a_1} \Phi - (n - 1 - k)x^A \Phi - \sum_\mu \Phi^\mu a_A \cdot v_\mu + \nabla_{r_{1A}} \Phi + \Phi(r_{1A}(\cdot), \cdot) + \Phi(\cdot, r_{1A}(\cdot)) - \sum_\mu \Phi^\mu r_{1A} \cdot v_\mu
\]

\[
= \nabla_{s_A} \Phi - (n - 1 - k)x^A \Phi - \sum_\mu \Phi^\mu s_A \cdot v_\mu + \Phi(r_{1A}(\cdot), \cdot) + \Phi(\cdot, r_{1A}(\cdot)).
\]

At the south pole, the vector field \( s_A := a_A + r_{1A} \) vanishes and \( x^A = 0 \). Further, the tangent space at the south pole is spanned by \( \partial C \), \( 2 \leq C \leq n \), and

\( r_{1A}(\partial C) = dx^1(\partial C)\partial_A - dx^A(\partial C)\partial_1 = -\delta_C^A \partial_1 \),
so \( \Phi(r_1A(\cdot),\cdot) = 0 \). Together we find
\[
\sum_{\mu} \Phi^{\mu}(p_0)s_{A} \cdot v_{\mu} = 0.
\]
Finally, Equation (26) evaluated at the south pole tells us that
\[
0 = \left( \nabla_{r_{AB}} \Phi + \Phi(r_{AB}(\cdot),\cdot) + \Phi(\cdot, r_{AB}(\cdot)) - \sum_{\mu} \Phi^{\mu}r_{AB} \cdot v_{\mu} \right)_{p_0}
\]
\[
= \Phi(p_0)(r_{AB}(\cdot),\cdot) + \Phi(p_0)(\cdot, r_{AB}(\cdot)) - \sum_{\mu} \Phi^{\mu}(p_0)r_{AB} \cdot v_{\mu}.
\]
\[\square\]

4.4. **Stereographic coordinates at the south pole.** We choose the point \( N = (1,0,\ldots,0) \in \mathbb{R}^n \) as the North pole of the sphere \( S^{n-1} \) and introduce the stereographic projection \( \pi \) given by
\[
\pi : S^{n-1} \setminus \{N\} \rightarrow \mathbb{R}^{n-1} \quad x \mapsto (y^2, \ldots, y^n) = \frac{1}{1-x^1}(x^2, \ldots, x^n).
\]
The inverse map \( \pi^{-1} \) is given by
\[
\pi^{-1}(y^2, \ldots, y^n) = \frac{1}{|y|^2 + 1} (|y|^2 - 1, 2y^2, \ldots, 2y^n).
\]
For short, we use the notation \( \partial_B := \frac{\partial}{\partial y^B} \) for all \( 2 \leq B \leq n \). We compute the pushforward of the vector fields \( a_i \) and \( r_{ij} \) under \( \pi \),
\[\begin{align*}
\pi_*a_1 &= y^A \partial_A, \\
\pi_*a_B &= \frac{1+|y|^2}{2} \partial_B - y^By^C \partial_C, \\
\pi_*r_{1B} &= \frac{|y|^2-1}{2} \partial_B - y^By^C \partial_C, \\
\pi_*r_{AB} &= y^C \partial_B - y^B \partial_A.
\end{align*}\]
We introduced earlier the vector fields \( s_B := a_B + r_{1B} \) and remark that \( a_1, r_{AB} \) and \( s_B \) form a basis of the elements of the Lie algebra vanishing at the south pole. Let us also introduce the vector field
\[
t_B := a_B - r_{1B} \\
= (1-x^1)\partial_A + x^A \partial_1 - x^A x^i \partial_i.
\]
It has the property \( \pi_*t_B = \partial_B \) and thus will be referred to as an “infinitesimal translation at the south pole”. In particular, it does not vanish at the south pole \( (y = 0) \), and we will use it to propagate \( \Phi \) from its expression at \( p_0 \).
To do so, we use Proposition 3.5 to write the condition \( t_B \cdot \Phi = 0 \) as
\[
\nabla_{t_B} \Phi + (k + 1 - n)x^B \Phi \\
- \Phi(r_{1B}(\cdot),\cdot) - \Phi(\cdot, r_{1B}(\cdot)) - \sum_{\mu} \Phi^{\mu}t_B \cdot v_{\mu} = 0
\] (31)
and evaluate it on the vectors $t_A$ and $t_C$. We first compute
\[ r_{1B}(t_A) = dx^1(t_A)\partial_B - dx^B(t_A)\partial_1 = x^A(1 - x^1)\partial_B + ((x^1 - 1)\delta^A_B + x^A x^B)\partial_1 \]
\[ = x^A t_B - (1 - x^1)\delta^A_B a_1 \mod (x^i \partial_i), \]
\[ \nabla^\sigma_{t_B} t_A = \nabla_{t_B} t_A \mod (x^i \partial_i) \]
\[ = (1 - x^1) (\delta^A_B (\partial_1 - x^i \partial_i) - x^A \partial_B) - x^B B (1 - x^1) \partial_A \]
\[ - 2x^A x^B (\partial_1 - x^i \partial_i) \mod (x^i \partial_i) \]
\[ = (1 - x^1) \delta^A_B a_1 - x^A t_B - x^B t_A \mod (x^i \partial_i). \]

Hence,
\[ 0 = (t_B \cdot \Phi)(t_A, t_C) \]
\[ = \nabla^\sigma_{t_B} \Phi(t_A, t_C) + (k + 1 - n)x^B \Phi(t_A, t_C) \]
\[ - \Phi(r_{1B}(t_A), t_C) - \Phi(t_A, r_{1B}(t_C)) - \sum_\mu \Phi^\mu(t_A, t_C) t_B \cdot v_\mu \]
\[ = t_B (\Phi(t_A, t_C)) - \Phi(\nabla^\sigma_{t_B} t_A, t_C) - \Phi(t_A, \nabla_{t_B} t_C) + (k + 1 - n)x^B \Phi(t_A, t_C) \]
\[ - x^A \Phi(t_B, t_C) - x^C \Phi(t_A, t_B) + (1 - x^1) (\delta^A_B \Phi(a_1, t_C) + \delta^A_B \Phi(t_A, a_1)) \]
\[ - \sum_\mu \Phi^\mu(t_A, t_C) t_B \cdot v_\mu \]
\[ = t_B (\Phi(t_A, t_C)) + (k + 3 - n)x^B \Phi(t_A, t_C) - \sum_\mu \Phi^\mu(t_A, t_C) t_B \cdot v_\mu \]
\[ = t_B (\Phi(t_A, t_C)) + (k + 3 - n)x^B \Phi(t_A, t_C) - \sum_\mu \Phi^\mu(t_A, t_C) t_B \cdot v_\mu. \]

Written in stereographic coordinates, this reformulates as
\[ \partial_B \Phi_{AC} - 2(n - k - 3) \frac{y^B}{1 + |y|^2} \Phi_{AC} - \sum_\mu \Phi^\mu_{AC} t_B \cdot v_\mu = 0, \quad (32) \]

where, abusing the notation, we denoted
\[ \Phi_{AC} := \Phi(t_A, t_C). \]

Solving these ODE successively for each variable $y^B$ and using the fact that $[t_A, t_B] = 0$ for any $A, B$, one can gather the results and obtain
\[ \Phi(y) = (1 + |y|^2)^{n-k-3}\Phi(y = 0)^\mu e^{y^A t_A} v_\mu. \quad (33) \]

It should be noted at this point that $t_A \in \mathfrak{so}(n, 1)$ is nilpotent as an endomorphism of $\mathbb{R}^{n-1}$. Hence it is ad-nilpotent and so it has to be nilpotent in any irreducible representation $V$ of $\mathfrak{so}(n, 1)$. This means in particular that $\Phi$ has polynomial entries in stereographic coordinates.

The following proposition summarizes all the results we obtained so far:

**Proposition 4.4.** Let $V$ denote a finite dimensional representation of the Lorentz group and let $\Phi : S^2(\mathbb{S}^{n-1}) \to V$ be an intertwining operator. Then $\Phi$ satisfies
conditions (28), (29) and (30) at the South pole. Conversely, let \( \Phi(y = 0) \) satisfy (28), (29) and (30) and translate it on \( S^{n-1} \setminus \{N\} \) by Formula (33). Then \( \Phi \) is an intertwining operator.

Proof. The only point left to prove is that transporting an element satisfying (28), (29) and (30) at the South pole yields an intertwining operator. But, Equation (33) is equivalent to

\[
t_B \cdot \Phi = 0
\]

for all \( B \). The vector space \( t \) spanned by \( t_2, \ldots, t_n \) is a complementary subspace for \( p \). Let \( p \) denote the projection operator onto \( p \) parallel to \( t \). We write

\[
\begin{align*}
t_B \cdot (a \cdot \Phi) &= a \cdot (t_B \cdot \Phi) + [t_B, a] \cdot \Phi \\
&= \text{ad}(t_B)a \cdot \Phi \\
&= p(\text{ad}(t_B)a) \cdot \Phi.
\end{align*}
\]

Computing as for Equation (32), we get that this translates into:

\[
\partial_B(a \cdot \Phi)_{AC} - 2(n-k-3) \frac{y^B}{1 + |y|^2} (a \cdot \Phi)_{AC} - \sum_\mu (a \cdot \Phi)_\mu^t B \cdot v_\mu = -p(\text{ad}(t_B)a) \cdot \Phi
\]

This can be viewed as a set of \( n-1 \) equations for a \( p^* \otimes S^2(\mathbb{R}^{n-1}) \otimes V \)-valued object (namely \( a \mapsto a \cdot \Phi \)) and Conditions (28), (29) and (30) are equivalent to saying that this object is zero at the origin. From the uniqueness of the solution of this system, we conclude that \( a \cdot \Phi \equiv 0 \) on the whole \( \mathbb{R}^{n-1} \) for all \( a \in p \).

Regularity at the North pole is actually immediate since it suffices to remark that the same analysis can be pursued choosing e.g. \( N' = (0, \ldots, 0, 1) \) and \( S' = (0, \ldots, 0, -1) \) as new North and South poles. \( \square \)

4.5. The 3-dimensional case. In the case \( n = 3 \), we will establish the classification of the intertwining operators \( \Phi \). In this case, one has an isomorphism \( \mathfrak{so}(3,1) \otimes \mathbb{C} \simeq \mathfrak{sl}(2)_A \oplus \mathfrak{sl}(2)_B \). The precise splitting is written in Appendix A.2. We also use the fact that all finite dimensional irreducible representations of \( \mathfrak{so}(3,1) \) can be written \( V_A \otimes V_B \), where \( V_A \) (resp. \( V_B \)) is an irreducible representation of \( \mathfrak{sl}(2) \). We moreover agree to describe these representations as spaces of homogeneous polynomials of a given degree in two variables, following again Appendix A.2. The approach we follow in this section is rather pedestrian and serves as a motivation for the general case.

Assume that \( \Phi(p_0) \) lies in the representation space of type \((s_A, s_B)\) for some \( s_A, s_B \geq 0 \). In other words, the representation space is \( V = V^{s_A}_A \otimes V^{s_B}_B \), where \( V^{s_A}_A = \mathbb{C}_{s_A}[X_A, Y_A] \) (resp. \( V^{s_B}_B = \mathbb{C}_{s_B}[X_B, Y_B] \)). In order to find the expression of \( \Phi(p_0) \), we start by writing

\[
s_2 = a_2 + r_{12} = f_A + f_B \quad s_3 = a_3 + r_{13} = i(f_A - f_B).
\]

Applying (29), this yields

\[
f_A \cdot (\Phi(p_0)) = f_B \cdot (\Phi(p_0)) = 0.
\]

From the expression of the generators \((e, f, h)\) of \( \mathfrak{sl}(2) \) and their respective action on homogeneous polynomials, we obtain that

\[
\Phi(p_0) = \varphi X_A^{s_A} X_B^{s_B},
\]
where \( \varphi \) is a symmetric 2-tensor over \( \mathbb{R}^2 \).

On the other hand, Equation (28) yields
\[
a_1 \cdot (\Phi(p_0)) = (2 - k)\Phi(p_0),
\]
where \( a_1 \) coincides with \( \frac{1}{2}(h_A + h_B) \). From the expression of the action \( h \cdot P = -X\partial_X P + Y\partial_Y P \), we get \( a_1 \cdot (\Phi(p_0)) = -\frac{s_A + s_B}{2}\Phi(p_0) \), which gives finally the expression of the decay rate \( k \):
\[
k = 2 + \frac{s_A + s_B}{2}.
\]

Then, we use the formula (30), together with the expression \( r_{23} = \frac{1}{2}(h_A - h_B) \). At the south pole \( (y = 0) \), we note that \( r_{23}(\partial_2) = \partial_4 \) and \( r_{23}(\partial_3) = -\partial_2 \). Evaluating (30) in all possible pairs of vectors tangent to the sphere \( \mathbb{S}^2 \) at its south pole, we obtain the system
\[
\begin{align*}
\frac{1}{2i}(-s_A + s_B)\varphi_{22} &= 2\varphi_{23}, \\
\frac{1}{2i}(-s_A + s_B)\varphi_{23} &= \varphi_{33} - \varphi_{22}, \\
\frac{1}{2i}(-s_A + s_B)\varphi_{33} &= -2\varphi_{23}.
\end{align*}
\]

It is a simple exercise to solve this eigenvalue problem. We find three different cases,
- \( s_B = s_A, \varphi = dy^2 \otimes dy^2 + dy^3 \otimes dy^3 = \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) \),
- \( s_B = s_A + 4, \varphi = dy^2 \otimes dy^2 - dy^3 \otimes dy^3 - i(dy^2 \otimes dy^3 + dy^3 \otimes dy^2) = d\bar{z} \otimes dz, \)
- \( s_B = s_A - 4, \varphi = dy^2 \otimes dy^2 - dy^3 \otimes dy^3 + i(dy^2 \otimes dy^3 + dy^3 \otimes dy^2) = dz \otimes d\bar{z}, \)

where we set \( z = y^2 + iy^3 \).

To go one step further and get the expression of the intertwining map at every point of \( \mathbb{S}^2 \), we need to evaluate the exponential in Formula 33:

**Lemma 4.5.**
\[
e^{y^2t_2 + y^3t_3} \cdot (X_A^{s_A} X_B^{s_B}) = (X_A - \bar{z}Y_A)^{s_A} (X_B - zY_B)^{s_B}.
\]

**Proof.** Note first that \( t_2 \) and \( t_3 \) admit the following respective expressions \( t_2 = e_A + e_B \) and \( t_3 = -i(e_A - e_B) \). Thus, the term in the exponential reads
\[
y^2t_2 + y^3t_3 = \bar{z}e_A + z e_B,
\]
where again \( z \) is \( y^2 + iy^3 \). The fact that \( e_A \) and \( e_B \) commute implies that
\[
e^{y^2t_2 + y^3t_3} \cdot (X_A^{s_A} X_B^{s_B}) = (e^{\bar{z}e_A} \cdot X_A^{s_A}) (e^{z e_B} \cdot X_B^{s_B}).
\]

In this last formula, the dots \( \cdot \) are the \( SL(2, \mathbb{C}) \)-Lie group action on respectively \( \mathbb{C}_{s_A}[X_A, Y_A] \) and \( \mathbb{C}_{s_B}[X_B, Y_B] \), so that we can write
\[
e^{\bar{z}e_A} \cdot X_A^{s_A} = v_{s_A} \left( e^{-\bar{z}e_A} \left( \frac{X_A}{Y_A} \right) \right) = v_{s_A} \left( \frac{X_A - \bar{z}Y_A}{Y_A} \right) = (X_A - \bar{z}Y_A)^{s_A},
\]
where \( v_{s_A} \) stands for the polynomial of \( \mathbb{C}_{s_A}[X_A, Y_A] \) defined as \( v_{s_A} \left( \frac{X_A}{Y_A} \right) = X_A^{s_A} \).

We can apply the same treatment to the factor \( e^{ze_B} \cdot X_B^{s_B} \) in order to obtain the desired formula. \( \square \)
Theorem 4.6. For all \( k \geq 2 \), let \( \mathcal{H}_{k-2} \) denote the set of wave harmonic polynomials on \( \mathbb{R}^{3,1} \). This is a representation of \( SO(3,1) \) under 
\[
SO(3,1) \times \mathcal{H}_{k-2} \rightarrow \mathcal{H}_{k-2},
\]
\[\Phi(A, P) \mapsto P \circ A^{-1}.\]

The representation \( V := (k - 2, k - 2) \) of \( \mathfrak{so}(3,1) \) gets identified with the dual of \( \mathcal{H}_{k-2} \). In the \( n = 3 \) case, there is an intertwining map \( \Phi : S^2(\mathbb{S}^2) \rightarrow V \). For each such \( k \), this map has the following form:
\[
m \mapsto \Phi(m),
\]
where \( \Phi(m) \in (\mathcal{H}_{k-2})^* \) is defined as follows:
\[
\Phi(m)(P) = \int_{\mathbb{S}^2} P(x^0 = 1, x^1, x^2, x^3) \text{tr}^\sigma m \, d\mu^\sigma.
\]

Proof. Based on the above remarks, we can rewrite the expression of \( \Phi(y) \), transforming back to \( x^i \)-coordinates on the sphere \( \mathbb{S}^2 \) and get
\[
\Phi(y) = 2^{-k} \sigma \left[ Q_0 + x^1 Q_1 + x^2 Q_2 + x^3 Q_3 \right]^{k-2},
\]
where the polynomials \( Q_\alpha \), \( \alpha = 0, \ldots, 3 \) take the expression
\[
\begin{aligned}
Q_0 &= X_A X_B + Y_A Y_B, \\
Q_1 &= -X_A X_B + Y_A Y_B, \\
Q_2 &= -X_A Y_B - Y_A X_B, \\
Q_3 &= -i(X_A Y_B - Y_A X_B).
\end{aligned}
\]

One can check that the linear map defined by \( Q_\mu \rightarrow \partial_\mu \) for \( \mu = 0, \ldots, 3 \), where \( (\partial_0, \partial_1, \partial_2, \partial_3) \) is the standard basis of \( \mathbb{R}^{3,1} \) intertwines the action of the Lorentz group \( SO(3,1) \). The element \( \left[ \partial_0 + x^1 \partial_1 + x^2 \partial_2 + x^3 \partial_3 \right]^{k-2} \) is then an element of \( \text{Sym}^{k-2}(\mathbb{R}^{3,1}) \). Hence we can view \( \Phi(m) \) as an element of \( \text{Sym}^{k-2}(\mathbb{R}^{3,1}) \).

It actually turns out that the elements of the basis \( (\partial_\mu) \) of \( \mathbb{R}^{3,1} \) can be seen as derivations over \( \mathbb{C}_{k-2}[X^0, X^1, X^2, X^3] \) and the operator \( x^\mu \partial_\mu \) is a polarization operator, see [29, Section 2]. It then follows that for any polynomial \( P \in \mathbb{C}_{k-2}[X^0, X^1, X^2, X^3] \), we have
\[
\left[ \partial_0 + x^1 \partial_1 + x^2 \partial_2 + x^3 \partial_3 \right]^{k-2} P = (k - 2)! P(1, x^1, x^2, x^3).
\]

This way \( \Phi(m) \) appears as an element of \( (\mathbb{C}_{k-2}[X^0, X^1, X^2, X^3])^* \). Notice however that since \( (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \) on \( \mathbb{S}^2 \), if
\[
P = ((X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^3) Q,
\]
we have
\[
\Phi(m)(P) = \int_{\mathbb{S}^2} \left( 1 - (x^1)^2 - (x^2)^2 - (x^3)^3 \right) Q(1, x^1, x^2, x^3) \text{tr}^\sigma m \, d\mu^\sigma
\]
\[
= 0.
\]

From the decomposition
\[
\mathbb{C}_{k-2}[X^0, X^1, X^2, X^3] = \mathcal{H}_{k-2} \oplus ((X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^3) \mathbb{C}_{k-4}[X^0, X^1, X^2, X^3],
\]
it follows that \( \Phi(m) \) is genuinely an element of \( (\mathcal{H}_{k-2})^* \). \qed
5. Interpretation of mass-like invariants

The aim of this section is to associate to any intertwinning map \( \Phi : m \mapsto \Phi(m) \) found in Section 4 a geometric differential operator, \( F : g \mapsto F(g) \) defined on the space of metrics over \( M \) and tensor-valued over \( M \) following B. Michel’s framework [27], which we will recall below. For instance, the Chruściel-Herzlich mass vector and the scalar curvature operator \( g \mapsto \text{Scal}_g \) are associated in this framework.

5.1. A look at B. Michel’s theory. We present the framework of Michel [27] in our special case. Let \( F : \mathcal{M}(M) \rightarrow \Gamma(H) \) a differential operator defined on the space of metrics of \( M \) and valued in the space of sections \( \Gamma(H) \) of a tensor bundle \( H \) over \( M \). We assume moreover that \( F \) is invariant by diffeomorphisms, i.e. such that \( \varphi^*F(g) = F(\varphi^*g) \), and that \( F(b) \) is the zero section of \( H \). If \( g \in \mathcal{M}(M) \) is asymptotically hyperbolic of some order \( k \) (namely if \( (U, g) \in G^k_R \) for open neighborhoods \( U \) of infinity) and if \( e := g - b \), we define

\[
Q(e) := F(g) - DF|_b \cdot e.
\]

The quantity \( Q(e) \) is at least quadratic in \( e \) and we used the fact that \( F(b) = 0 \). Let us now take any section \( h \in \Gamma(H) \). We want to integrate the following over \( U \):

\[
\langle h, F(g) \rangle = \langle h, DF|_b \cdot e \rangle + \langle h, Q(e) \rangle,
\]

where the brackets \( \langle \cdot, \cdot \rangle \) refer to the lift of the metric \( b \) to \( \Gamma(H) \). We can rewrite this as

\[
\langle h, F(g) \rangle = \text{div}_F(h, e) + \langle DF|_b^* \cdot h, e \rangle + \langle h, Q(e) \rangle,
\]

where \( U_F(h, e) \) is a one-form called in [27] the charge integrand, and the \( ^* \) denotes the formal adjoint applied to an operator. Based on this formula, Michel observed that if we take \( h \) in the kernel of \( DF|_b^* \) and such that \( \langle h, F(g) \rangle \) and \( \langle h, Q(e) \rangle \) are integrable near infinity, then the limit

\[
m(g, h) := \lim_{R \rightarrow +\infty} \int_{S_R} U_F(h, e)(\nu_R) d\mu_{S_R}
\]

exists and is called the total charge of \( h \) with respect to \( e \) [27]. Here \( S_R \) is the coordinate sphere of radius \( R \) and \( \nu_R \) its outwards pointing normal (w.r.t. the metric \( b \)).

Moreover, he obtains that for any \( \Psi \in I^k(g) \) decomposed as \( \Psi = A \circ \Psi_0 \) along Lemma 2.11, we have

\[
\text{5.1: jet check ?}
\]

Remark 5.1. As mentioned in [27, Remark 2.6], the Lorentz group \( O_{\tau}(n, 1) \) acts by pushforward on the space \( \ker DF|_b^* \). Moreover, if a total charge functional \( h \mapsto m(g, h) \) is defined on a Lorentz-invariant (finite-dimensional) subspace of \( \ker DF|_b^* \), it is then an equivariant linear form under the Lorentz action.

Applying this to the scalar curvature operator, precisely \( F(g) := \text{Scal}_g + n(n-1) \), one recovers the Chruściel-Herzlich mass of [10]. Moreover, the analysis carried out by Michel using the above formalism enables one to recover the following result:

**Theorem 5.2** (Chruściel-Herzlich, Michel). Let \( \tau > n/2 \) and let \( g \in G_\tau \) be an asymptotically hyperbolic metric of order \( \tau \). Assume moreover that if \( e := g - b \),
we have also $|\nabla e|_b = O(e^{-\tau r})$ and $|\nabla^2 e|_b = O(e^{-\tau r})$, and that $e^r(\text{Scal}_g + n(n-1))$

is integrable in a neighborhood of infinity. Then the mass functional

$$f \mapsto \lim_{R \to \infty} \int_{S_R} \mathcal{U}(f, e)(\nu_R)dS_R$$

is well defined on the space of lapse functions $\text{ker } D\text{Scal}^*_b = \langle \cosh r, x^1 \sinh r, \ldots, x^n \sinh r \rangle$.

In particular, under Wang’s asymptotics [32], we recover the mass vector formula as in example 4.2 up to a constant factor. Indeed, for any such $f$ and $g \in G_n$, we have the expression for large $R$

$$\mathcal{U}(f, e)(\nu_R) = (3n-1)(nf + \partial_r f) \text{ tr}^\sigma m \text{ sh}^{-n} R + O(\text{sh}^{-(n+2)} R),$$

where one has to remember that $f = O(e^r)$ in any case. This comes from the expression

$$\mathcal{U}_{\text{Scal}}(f, e) := \mathcal{U}(f, e) = f[\text{div} e - d \text{tr } e] - e_v f e + (\text{tr } e)df.$$  

5.2. The strategy, for $n = 3$. A key idea is that the space of lapse functions defined above coincides precisely with the space $\mathcal{H}_1$ of homogeneous (trivially wave) polynomials of degree 1 in the variables $x^0, x^1, \ldots, x^n$ of Minkowski space.

We will now consider more generally for $k \geq 0$ the space $\mathcal{H}_{k-2}$ of wave homogeneous polynomials in the same variables with degree $k-2$ that we met in Section 4. But first, let us state the following result:

**Lemma 5.3.** Let $d > 0$. Let $\tilde{f} \in C^\infty(\mathbb{R}^{n+1})$ a smooth function solution to the wave equation

$$\Box_\eta \tilde{f} := -\frac{\partial^2 \tilde{f}}{\partial \tau^2} + \Delta_{\mathbb{R}^n} \tilde{f} = 0.$$  

Assume moreover that $\tilde{f}$ is homogeneous of degree $d$, $\tilde{f}(\rho x) = \rho^d \tilde{f}(x)$ for all $x \in \mathbb{R}^{n+1}$ and $\rho \in \mathbb{R}$. Then the restriction $f$ of $\tilde{f}$ to the hyperboloid $\mathbb{H}^n \hookrightarrow \mathbb{R}^{n,1}$ solves the equation

$$\Delta f = d(d + n - 1)f,$$  

where $\Delta$ is the Laplace-Beltrami operator of $\mathbb{H}^n$.

**Proof.** We observe first that the Minkowski metric $\eta$ of $\mathbb{R}^{n,1}$ can be decomposed (at least in the timelike region) as:

$$\eta = -d\rho^2 + \rho^2 b,$$

where $b$ is the metric of $\mathbb{H}^n$. Let $x$ a vector in the timelike region of $\mathbb{R}^{n,1}$. Then $x = \rho y$, where $(\rho, y) \in \mathbb{R}^* \times \mathbb{H}^n$. For $d > 0$ and $\tilde{f}$ as in the statement, we have at $x$:

$$0 = \Box_\eta \tilde{f}(\rho y) = \Box_\eta (\rho^d \tilde{f}(y))$$

$$= -\frac{1}{\rho^d} \partial_\rho (\rho^n d \rho^{d-1}) \tilde{f}(y) + \frac{1}{\rho^2} \rho^d \Delta \tilde{f}$$

$$= \rho^{d-2} \left[ -d(n-1 + d) \tilde{f}(y) + \Delta \tilde{f}(y) \right].$$

Hence for $n = 3$, the restrictions $\tilde{P}$ of the polynomials of $\mathcal{H}_{k-2}$ span the space

$$\left\{ \tilde{P} \in \mathbb{R}[x^1, x^2, x^3] \mid \deg \tilde{P} \leq k - 2, \Delta \tilde{P} = k(k-2)\tilde{P} \right\}.$$
Since we wish this space to take the form of \( \ker DF_{b}^{\ast} \) for a suitable \( F \), we set
\[
F_{k} := g \mapsto \Delta_{g} \text{Scal}_{g} - \alpha_{k} \text{Scal}_{g}
\]
and, in the light of the Lemma above, we choose
\[
\alpha_{k} = k(k - 2).
\]

Let us begin with the case \( k = 2 \) and the operator \( F_{2} \) defined over the space of Riemannian metrics on \( \mathbb{H}^{3} \) as
\[
F_{2}(g) = \Delta_{g} \text{Scal}_{g}.
\]
We use for short the notation \( P_{2} = DF_{2}\bigr|_{b} \) for the differential map evaluated at \( b \). As above, we aim at finding the space \( \ker P_{2}^{\ast} \), and the expression of the 1-form \( U_{2}(f,e) := U_{F_{2}}(f,e) \) such that
\[
\text{div} U_{2}(f,e) = fP_{2}\cdot e - \langle P_{2}^{\ast} f, e \rangle
\]
for all smooth functions \( f \) and \((0,2)\)-symmetric tensors \( e \).

We can first establish the following result:
\[
P_{2}^{\ast} = D \text{Scal}^{\ast}_{b} \circ \Delta_{b},
\]
which implies that \( \ker P_{2}^{\ast} = \{ f \in C^{\infty}(\mathbb{H}^{3}) \mid \text{Hess}(\Delta f) = (\Delta f) b \} \). This space is in fact exactly the direct sum of the space of lapse functions with the space of harmonic functions on \( \mathbb{H}^{3} \).

Note that the space of constant functions on \( \mathbb{H}^{n} \) is a one-dimensional subspace which is invariant under the action of the Lorentz group. Following [27, Remark 2.6], it is therefore associated to a number provided the suitable decay assumptions are made, and it is invariant under the action of \( O_{\uparrow}(3,1) \).

Furthermore, the number so obtained corresponds to the – trivial – intertwining map for the case \( k = 2 \) obtained in Theorem 4.6, that is, for \( g \in G_{2}^{T} \), and some constant \( C \neq 0 \),
\[
\lim_{R \to \infty} \int_{S_{R}} U_{2}(1,e)(\nu_{R}) dS_{R} = C \int_{\mathbb{S}^{2}} \text{tr}^{\tau} m \, d\mu^{\tau}.
\]

This follows from the computation of \( U_{2}(f,e) \), given by
\[
U_{2}(f,e) = f d(\text{div} \, \text{div} e) - (\text{div} \, \text{div} e) df - \iota_{\nabla(\Delta f)} e + (\Delta f) \text{div} e
\]
\[
- f d(\Delta(\text{tr} e)) + \Delta(\text{tr} e) df + (\text{tr} e) d(\Delta f) - (\Delta f) d(\text{tr} e)
\]
\[
+ 2(f d(\text{tr} e) - (\text{tr} e) df)
\]
for all \( f \) and \( e \), where we have used the fact that \( \text{Ric} = -2b \) for the last line. Using this for \( g \in G_{2}^{T} \), we indeed finally get
\[
U_{2}(1,e)(\nu_{R}) = 12 \, \text{tr}^{\tau} m \, \text{sh}^{-2} R + O(\text{sh}^{-3} R),
\]
proving the equality.

We now repeat the process for the operators \( F_{k} \) defined in (34) for \( k \geq 2 \). Let \( F_{k} := DF_{k}\bigr|_{b} \) and \( U_{k} := U_{F_{k}} \). Let \( \mathcal{N}_{k} := \ker P_{k}^{\ast} \). We have that
\[
\mathcal{N}_{k} = \{ f \in C^{\infty}(\mathbb{H}^{3}) \mid \text{Hess}(\Delta f - k(k - 2)f) = (\Delta f - k(k - 2)f) b \} .
\]

For \( k \neq 3 \), \( \mathcal{N}_{k} \) is easily seen as being the direct sum of the space of Lapse functions with the space of smooth functions \( f \) such that \( \Delta f = k(k - 2)f \). For \( k = 3 \), \( \mathcal{N}_{3} \) contains as a proper subspace the solutions to \( \Delta f = 3f \).
Therefore, we see from Lemma 5.3 that each space \( \{ \hat{P} \mid P \in \mathcal{H}_{k-2} \} \) where \( P \mapsto \hat{P} \) denotes the restriction to the hyperboloid \( \mathbb{H}^3 \hookrightarrow \mathbb{R}^{3,1} \) is a subspace of \( \mathcal{N}_k \) which is moreover invariant under the action of the Lorentz group.

We have in fact the result:

**Theorem 5.4.** Let \( n = 3 \) and \( k \geq 2 \). Then for all metrics \( g \in G_k^T \) with the asymptotic expansion \( g = b + \rho^{k-2}m + O(\rho^{k-1}) \), for all \( P \in \mathcal{H}_{k-2} \), we have

\[
\lim_{R \to +\infty} \int_{S_R} \mathbb{U}_k(\hat{P}, \nu_R) dS_R = \int_{S^2} P(x^0 = 1, x^1, x^2, x^3) \, \mathrm{tr}^\sigma m \, d\mu^\sigma.
\]

**Proof.** We first need to express \( \mathbb{U}_k(\hat{P}, \nu) \) for any \( P \in \mathcal{H}_{k-2} \), where \( e := g - b = \rho^{k-2}m + O(\rho^{k-1}) \). For this, we note that \( F_k = F_2 - k(k-2)\text{Scal} \). The linearity of the differentiation as well as of the map \( L \mapsto L^* \) yields that \( P_k^* = P_2^* - k(k-2)\text{Scal}_k^* \), and thus \( \mathbb{U}_k = \mathbb{U}_2 - k(k-2)\mathbb{U}_{\text{Scal}} \). Hence for all \( f, e \), the expression simplifies into

\[
\mathbb{U}_k(f, e) = f\rho (\text{div div } e - (\text{div div } e)df - f\rho(\Delta \text{tr } e) + (\Delta \text{tr } e)df + 2[f\rho \text{tr } e - (tr e)df].
\]

We now use the expression of this 1-form evaluated against \( \nu_R = \frac{\partial}{\partial r} \) with \( e = \sinh^{-(k-2)}r \, m + O(e^{-(k-1)r}) \) (valid up to second order derivatives), and we derive the following expansions:

- \( \text{div div } e = \sinh^{-(k+2)}r \, \text{div } \text{div } e \, m + O(e^{-(k+3)r}) \),
- \( \text{tr } e = \sinh^{-k}r \, \text{tr } \text{tr } e \, m + O(e^{-(k+1)r}) \),
- \( \Delta (\text{tr } e) = k(k-2) \sinh^{-k}r \, \text{tr } \text{tr } m + O(e^{-(k+1)r}) \).

These expansions can be differentiated in both sides, therefore we get

\[
\mathbb{U}_k(P, e)(\nu_R) = (k(k-2) - 2) \left( kP + \frac{\partial P}{\partial r} \right) \sinh^{-k}r \, \text{tr } \text{tr } m + O(e^{-3r})
\]

for all \( P \in \mathcal{H}_{k-2} \). For any given \( P \in \mathcal{H}_{k-2} \), its restriction \( \hat{P} \) on the unit hyperboloid \( \mathbb{H}^3 \) is homogeneous of degree \( k-2 \) in the variables \( \text{cosh } r, x^1 \sinh r, x^2 \sinh r, x^3 \sinh r \), where \( x^i \) denote the Cartesian coordinates of \( \mathbb{R}^3 \) restricted to \( S^2 \). We deduce easily from this fact that \( \partial_r \hat{P} = (k-2)\hat{P} + O(e^{(k-4)r}) \) and that

\[
\mathbb{U}_k(P, e)(\nu_R) = 2(k-1)(k(k-2) - 2)P(1, x^1, x^2, x^3) \, \text{tr } \text{tr } m \, \sinh^{-2}r + O(e^{-3r}),
\]

the integration of which on \( S_R \) returns the desired limit as \( R \to +\infty \).

In fact we can relax the decay assumptions and still get the convergence of the mass functional for a larger class of metrics:

**Theorem 5.5.** Let \( n = 3 \) and \( k \leq 2 \). Assume that \( |g - b|_b = O(e^{-\tau r}) \) with \( \tau > k/2 \) and that \( e^{(k-2)r}F_k(g) \) is integrable. Then the functional

\[
P \in \mathcal{H}_{k-2} \mapsto \lim_{R \to +\infty} \int_{S_R} \mathbb{U}_k(P, e)(\nu_R) dS_R
\]

is well defined.

**Proof.** Following [27], the only thing left to prove is that the remainder \( e^{kr}Q(e) \) is integrable.

**Remark 5.6.** The decay assumptions of the above theorem are in particular fulfilled by metrics \( g \) with \( |g - b|_b = O(e^{-kr}) \) along with at least two derivatives. Hence, there is no other linear real asymptotic invariant for the metrics of Theorem 5.5 than the ones given in Theorem 5.4.
Appendix A. Elements of representation theory

In this appendix, we collect basic facts about the Lie algebra \(\mathfrak{so}(n,1)\). In particular, we give an explicit description of certain representations that appear in the article.

A.1. Representations of \(\mathfrak{sl}(2)\). We rest on the conventions of [22]. The standard basis for the Lie algebra \(\mathfrak{sl}(2)\) of traceless \(2 \times 2\) matrices is made of

\[
\begin{align*}
    h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
    e &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
    f &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

It satisfies the following commutation relation:

\[
\begin{align*}
    [h, e] &= 2e, \\
    [h, f] &= -2f, \\
    [e, f] &= h.
\end{align*}
\]

The representations of \(\mathfrak{sl}(2)\) are best described as in terms of homogeneous polynomials in two variables, see [23, Chapter II]. For any integer \(m \geq 0\), there exists a unique (up to equivalence) irreducible representation \(V_m\) of \(\mathfrak{sl}(2)\) such that \(\dim V_m = m + 1\). This representation can be described as the vector space \(\mathbb{C}_m[X,Y]\) of homogeneous polynomials in two variables of degree \(m\). The action of the Lie group \(SL(2,\mathbb{C})\) is given as follows:

\[
\rho : SL(2,\mathbb{C}) \times \mathbb{C}_m[X,Y] \to \mathbb{C}_m[X,Y] \\
\left( M, P \left( \begin{array}{c} X \\ Y \end{array} \right) \right) \mapsto P \left( M^{-1} \begin{array}{c} X \\ Y \end{array} \right).
\]

The action of the Lie algebra can be explicitly computed:

\[
\begin{align*}
    h \cdot P &= -X \partial_X P + Y \partial_Y P, \\
    e \cdot P &= -Y \partial_X P, \\
    f \cdot P &= -X \partial_Y P.
\end{align*}
\]

The Casimir element \(C \in U(\mathfrak{sl}(2))\) is given by

\[
C = ef + \frac{h(h-2)}{4} = fe + \frac{h(h+2)}{4}.
\]

Its value on the representation \(\mathbb{C}_m[X,Y]\) is given by

\[
C \cdot P = \frac{m(m+2)}{4} P.
\]

A.2. The Lie algebra \(\mathfrak{so}(n,1)\). Following the notations introduced in Section 2.1, the canonical basis of the Lie algebra \(\mathfrak{so}(n,1)\) satisfies the following commutation relations:

\[
\begin{align*}
    [a_i, a_j] &= -r_{ij}, \\
    [r_{ij}, a_k] &= \delta_{ik} a_j - \delta_{jk} a_i, \\
    [r_{ij}, r_{kl}] &= \delta_{ik} r_{jl} - \delta_{jk} r_{il} - \delta_{il} r_{jk} + \delta_{jl} r_{ik}.
\end{align*}
\]

We now briefly describe the structure of these Lie algebras according to the value of \(n\).
A.2.1. The case $n = 3$. It is a standard fact that the complexified Lie algebra $\mathfrak{so}(3, 1) \otimes \mathbb{C}$ splits into two copies of $\mathfrak{sl}(2)$. Several splittings exist mainly because $\mathfrak{sl}(2)$ has many automorphisms. We chose the following convention:

\[
\begin{align*}
  h_A &= a_1 + ir_{23} \\
  e_A &= \frac{1}{2} (a_2 + ia_3 - r_{12} - ir_{13}) \\
  f_A &= \frac{1}{2} (a_2 - ia_3 + r_{12} - ir_{13}) \\
  h_B &= a_1 - ir_{23} \\
  e_B &= \frac{1}{2} (a_2 - ia_3 - r_{12} + ir_{13}) \\
  f_B &= \frac{1}{2} (a_2 + ia_3 + r_{12} + ir_{13})
\end{align*}
\]

It can be checked that the families $\{h_A, e_A, f_A\}$ and $\{h_B, e_B, f_B\}$ commute one with the other and each satisfies the commutation relations of $\mathfrak{sl}(2)$. These formulas can be inverted:

\[
\begin{align*}
  a_1 &= \frac{1}{2} (h_A + h_B), \\
  a_2 &= \frac{1}{2} (e_A + f_A + e_B + f_B), \\
  a_3 &= \frac{1}{2i} (e_A - f_A - e_B + f_B), \\
  r_{12} &= \frac{1}{2} (-e_A + f_A - e_B + f_B), \\
  r_{13} &= \frac{i}{2} (e_A + f_A - e_B - f_B), \\
  r_{23} &= \frac{1}{2i} (h_A - h_B)
\end{align*}
\]

Complex representations of the Lie algebra $\mathfrak{so}(n, 1)$ are then in bijection with tensor products $V^A \otimes V^B$ where $V^A$ (resp. $V^B$) is an irreducible representation of $\mathfrak{sl}_{A} = \text{vect}(h_A, e_A, f_A)$ (resp. $\mathfrak{sl}_{B} = \text{vect}(h_B, e_B, f_B)$), see [29]. Thus, we say that an irreducible representation $V = V^A \otimes V^B$ of $\mathfrak{so}(n, 1)$ is of type $(m_A, m_B)$ if $V^A$ (resp. $V^B$) is the irreducible representation of $\mathfrak{sl}_{A}$ (resp. $\mathfrak{sl}_{B}$) of dimension $m_A + 1$ (resp. $m_B + 1$).

From Section 4, we shall be interested in studying two classes of representations.

**Proposition A.1.** Irreducible representations of type $(m_A, m_B)$ with $m_A = m_B$ are identified with the space of homogeneous (wave) harmonic polynomials on $\mathbb{R}^{3,1}$.

**Proof.** Mimicking the action of $\mathfrak{so}_{2}$ on homogeneous polynomials, the action of the Lie algebra of the Lorentz group on polynomials is given by

\[
\mathfrak{so}(3, 1) \times \mathbb{C}[X^0, X^1, X^2, X^3] \rightarrow \mathbb{C}[X^0, X^1, X^2, X^3] \\
(M, P) \mapsto M \cdot P = -M^{i} x^{j} \partial_{i} P.
\]

The operator $\Box := -\partial_{0}^{2} + \partial_{1}^{2} + \partial_{2}^{2} + \partial_{3}^{2}$ commutes with this action of $\mathfrak{so}(3, 1)$. Thus for each $m$, the subspace $\mathcal{H}_{m} = \mathbb{C}[X^0, X^1, X^2, X^3] \cap \ker(\Box)$ is stable under the action of $\mathfrak{so}(3, 1)$. It is a standard result that the space $\mathbb{C}_{m}[X^0, X^1, X^2, X^3]$ decomposes as

\[
\mathbb{C}_{m}[X^0, X^1, X^2, X^3] = \mathcal{H}_{m} \oplus \left((-X^{0})^{2} + (X^{1})^{2} + (X^{2})^{2} + (X^{3})^{2}\right) \mathbb{C}_{m-2}[X^0, X^1, X^2, X^3].
\]
The dimension of the set of homogeneous polynomials of degree $m$ is \( \binom{m+3}{m} = \binom{m+3}{3} \). Hence, 
\[
\dim(H_m) = \binom{m+3}{3} - \binom{m+1}{3} = (m+1)^2.
\]
It is straightforward to check that \( P := (X^0 - X^1)^m \) belongs to \( H_m \) and that 
\[
e_A \cdot P = e_B \cdot P = 0, \quad h_A \cdot P = h_B \cdot P = mP,
\]
meaning that \( H_m \) contains a representation of type \( (m, m) \). The dimension of this representation being \( (m+1)^2 \), we conclude that \( H_m \) is an irreducible representation of type \( (m, m) \).

\[\square\]

Appendix B. Gauss-Bonnet-Chern masses

References


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