# Hopf Algebroids and Their Cyclic Theory

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# Contents

# Introduction

## Background of the Thesis

### Hopf Algebroids

The main objects of study in this thesis are *generalised symmetries* and their associated *(co)homologies* within the realm of noncommutative geometry. Some parts of the background picture for the notion of generalised symmetries in noncommutative geometry are summarised in the following table (see further down for a similar table for the respective (co)homology theories).

| | *Differential Geometry* | *Algebraic Geometry* | *Noncommutative Geometry* |
|---|---|---|---|
| *Spaces* | Manifolds | Commutative Algebras (, . . . , Schemes) | Noncommutative Algebras (, . . . , Spectral Triples) |
| *Symmetries* | Lie Groups | Algebraic Groups, Group Schemes | Hopf Algebras |
| *Generalised Symmetries* | Lie Groupoids and Pseudogroups | Groupoid Schemes | ? |

We now explain some of the entries of this table.

**Noncommutative Geometry**
The main idea of noncommutative is to study 'spaces' by means of their algebras of (continuous, smooth, etc.) functions. The novelty stems from the fact that these algebras are allowed to be noncommutative. In a certain sense, noncommutativity may be seen as a manifestation of the singular behaviour of the spaces involved. For instance, in many examples such as quotients by group actions or leaf spaces of foliations, the naive spaces may be highly pathological. Indeed, the noncommutative approach to such spaces starts by associating a noncommutative algebra to them, as the 'algebra of functions on the noncommutative space'.

**Hopf Algebras**
The concept of symmetry in noncommutative geometry, i.e. the noncommutative analogue of Lie groups from classical differential geometry, is given by the notion of Hopf algebras. More precisely, noncommutative symmetries are encoded in the action *or coaction* of some Hopf algebra on some algebra or coalgebra.

Roughly speaking, when passing from a Lie group $G$ to its algebra of (say) continuous functions $\mathcal{C}G$, the group multiplication transforms into a map $\mathcal{C}G \to \mathcal{C}(G \times G)$ or, using the appropriate tensor product, into a comultiplication $\Delta : \mathcal{C}G \to \mathcal{C}G \otimes \mathcal{C}G$. Moreover, the inversion in $G$ gives an involution $S : \mathcal{C}G \to \mathcal{C}G$. The algebra $\mathcal{C}G$ together with the comultiplication $\Delta$ and the involution (antipode) $S$ is the basic example of a Hopf algebra. Enveloping algebras of Lie algebras provide another (dual) basic example. Deforming a Lie group inside the larger world of noncommutative geometry refers to deforming the Hopf algebras associated to it. Hence typical examples of Hopf algebras arise as algebras of coordinates of a quantum group or, on some dual space, as the convolution algebra or the enveloping algebra of a quantum group.

It is important to note that the notion of Hopf algebra is self-dual: roughly speaking, under suitable circumstances the dual of a Hopf algebra is automatically a Hopf algebra again. From this point of view, the classical examples of enveloping algebras and function algebras are dual to each other.

Hopf algebras can be deployed to give a description of internal *quantum* symmetries of certain models in (low-dimensional) quantum field theory. More applications of Hopf algebras comprise e.g. the construction of invariants in topology and knot theory [OKoLeRoTu, Tu], and appear in connection with solutions of the quantum Yang-Baxter equation [Str]. As another example, (faithfully flat) Galois extensions by

Hopf algebras may be considered as the right generalisation of principal bundles towards the realm of noncommutative geometry [HPu, Kas4].

**Generalised (Noncommutative) Symmetries**

In classical differential geometry, generalised symmetries are encoded in the notions of Lie groupoids and pseudogroups—a fact that already emerges in the work of Lie [Lie] and Cartan [Car1, Car2]. Lie groupoids are a joint generalisation of manifolds and Lie groups and provide a symmetry concept that has found many applications, e.g. in the theory of foliations or for describing internal 'classical' symmetries (cf. e.g. [Mac, MoeMrč1, L2]. It is very natural to ask what the generalised symmetries in noncommutative geometry are (corresponding to the question mark in the table above). In other words, one is interested in the correct notion of:

$$\text{Noncommutative Groupoids} \Leftrightarrow \text{Quantum Groupoids} \Leftrightarrow \text{Hopf Algebroids.} \qquad (A)$$

An infinitesimal consideration of Lie groupoids leads to Lie algebroids (or, in an algebraic context, to Lie-Rinehart algebras). Hence, one could extend the picture by asking for the correct notion of

$$\text{Noncommutative Lie Algebroids/Lie-Rinehart Algebras} \Leftrightarrow \text{Hopf Algebroids.} \qquad (A')$$

The clear need for the generalisation of Hopf algebras was presumably stated for the first time in [Sw2] in the context of classification problems of algebras. A more recent motivation for such an extension of Hopf algebra concepts came from research on the index theory of transverse elliptic operators in [CoMos5], generalising the local approach in [CoMos2] towards *non-flat transversals*, globally described by an 'extended Hopf algebra' $H_{FM}$ associated to the frame bundle of a manifold (cf. also [CoMos6]).

Other examples that require extension of the Hopf algebraic framework are certain invariants [NiTuVai] in topology, or in Poisson geometry, where solutions of the dynamical Yang-Baxter equation that correspond to dynamical quantum groups elude a description by Hopf algebras, cf. [EtNi, NiVai, Lu, X3, DonMu, Kar]. In low-dimensional quantum field theories non-integral values of the *quantum* dimensions cannot be seen as a Hopf algebra symmetry [BSz1], emphasising the need for a noncommutative generalisation thereof.

**Quantum Groupoids**

In many of these approaches, problems have been handled by allowing for a not necessarily commutative ring $A$ replacing the commutative *ground* ring $k$ of a Hopf algebra. Considering a Hopf algebra as a $k$-bialgebra with an antipode, a Hopf algebroid should involve the notions of a generalised bialgebra over $A$ as well as an analogue of an antipode. Such a generalised bialgebra is commonly referred to as *bialgebroid*: it generalises a $k$-bialgebra towards an object (to which we will refer as the *total* ring) that is both a coalgebra and an algebra in (different) bimodule categories, determined by the ring $A$, to which we will refer as the *base* ring from now on. With the help of a new definition of tensor products over noncommutative rings (the so-called $\times_A$-products), bialgebroids were (presumably for the first time) introduced under the name $\times_A$-bialgebras in [Tak]. Ordinary $k$-bialgebras can be recovered if one uses the ground ring $k$ as base ring. Bialgebroids (under this name) were introduced in [Lu] (apparently independently of the work in [Tak]), and, motivated by problems in Poisson geometry, as *bialgebroids with anchor* in [X1, X3]. These notions were shown to be equivalent to that of a $\times_A$-bialgebra in [BrzMi].

Viewing bialgebroids as noncommutative analogues of groupoids, parallel to the relationship of bialgebras to groups as mentioned above, one also may justify the name *quantum groupoid* for (certain) bialgebroids. A precursor in this direction is [Mal1] for commutative base rings, and [Mal2] for an extension to the general, noncommutative case. From this viewpoint of quantum groupoids, one can also deduce what should be the basic ingredients of a bialgebroid. Recall first that a groupoid consists of a set of (invertible) arrows, a set of objects, two maps called source and target mapping arrows into objects, as well as a partially defined multiplication in the space of arrows, an inclusion of objects as zero arrows, and all these maps are subject to certain conditions which we conceal for the moment. A bialgebroid may then be considered to be a noncommutative analogue of the function algebra on a groupoid. More precisely, the total ring would play the rôle of the function algebra of the 'quantum space' of morphisms, whereas the base ring should be considered to be the function algebra on the 'quantum space' of objects. Since each arrow is provided with a source and a target, it is natural to assume corresponding source and target maps (in the opposite direction) to be part of the structure. The fact that composition of arrows in a groupoid is only partially defined is reflected in a bialgebroid by defining a comultiplication that takes values in a subspace of some tensor product of the total ring with itself, and only in this subspace a well-defined ring structure is given.

However, the precise definition of bialgebroids is quite technical, but evidence that it is the 'right' one is given in [Schau1]. Recall that a $k$-algebra $U$ is a $k$-bialgebra if and only if the category of left $U$-modules is a monoidal category such that the underlying (forgetful) functor to $k$-modules is monoidal: this means that the $k$-tensor product of two $U$-modules is again a $U$-module. This is a fundamental feature for Tannaka duality or reconstruction theory for quantum groups which make explicit use of their monoidal module categories [JoStr]. In an analogous fashion, a bialgebroid $U$ over some base ring $A$ is characterised by the fact that the category of its modules is again monoidal, with the (crucial) difference that only the forgetful functor from $U$-modules to $(A, A)$-bimodules (rather than $k$-modules) is monoidal. Hence the tensor product over $A$ of two $U$-modules is a $U$-module again.

**Concepts of Hopf Algebroids**

The next step in defining a Hopf algebroid consists in equipping a bialgebroid with some sort of antipode. In the preceding consideration of quantum groupoids, this would simply correspond to the inversion of arrows of a groupoid. The main difficulty here derives from the fact that the tensor category of $(A, A)$-bimodules is not symmetric, which impedes a straightforward generalisation of antipodes for Hopf algebras.

Motivated by topics in algebraic topology, Hopf algebroids were originally introduced as cogroupoid objects in the category of commutative algebras (see e.g. [Mor, Ra, Hov]), while they also arose in algebraic geometry in connection with stacks [FCha].

The underlying bialgebroids of the Hopf algebroids defined in [Ra] are special cases of the construction in [Tak] since the underlying algebra structure on both the total and base ring is commutative. Nevertheless, this is more general than a Hopf algebra since it is already equipped with characteristic features with respect to bimodule categories as mentioned above. In [Mrč1, Mrč2], non-commutative Hopf algebroids (but still over a commutative base ring) have been used for the study of principal fibre bundles with groupoid symmetry. The first general definition of a Hopf algebroid, in which both the total and base rings are not necessarily commutative, is presumably given in [Lu], although some auxiliary assumptions had to be made that in a sense lack a geometric or intuitive interpretation. More precisely, a section of a certain projection map is needed, so as to be able to impose axioms one would expect from a natural generalisation of the Hopf algebra axioms. Motivated by problems in cyclic cohomology (see below), the notion of *para*-Hopf algebroid was introduced in [KhR3]. Here, a *para*-antipode is introduced that avoids the section mentioned above, but as a price to be paid needs axioms that do not look like a conceptually straightforward generalisation of the Hopf algebra axioms anymore.

An alternative definition of Hopf algebroids from [B1, BSz2] steers clear of these problems by defining, roughly speaking, *two* distinct bialgebroid structures, assumed to exist on a given algebra: one considers *left* and *right* bialgebroids as introduced in [KSz] over an algebra $A$ and its opposite, and an antipode is then understood as a map intertwining them. In particular, this way one is able to circumvent another crucial problem when defining antipodes: in Hopf algebra theory, such a map is an anti-coalgebra morphism, a feature which is *a priori* not well-defined for bialgebroids. In the approach of [B1, BSz2] the antipode is still an anti-coalgebra morphism, but for different coalgebras, passing from the underlying left bialgebroid to the underlying right one. Not all information (left bialgebroid, right algebroid, antipode) is actually needed, but this way the axioms look most natural and symmetric. For example, one could equally well express (up to automorphisms) the right bialgebroid in terms of the left one and the antipode (provided it is invertible), but this does not quite reduce the amount of complexity. It is this definition which we consider the best suited for our purposes, and whenever no contrary mention is made, the term Hopf algebroid refers to this definition throughout the subsequent chapters. For example, we will see that étale groupoids and Lie-Rinehart algebras, and in particular their corresponding homology and cohomology operators, naturally ask for the existence of two bialgebroid structures of different kind. We also mention here that already [Mrč1, Mrč2] is tacitly dealing with both left and right bialgebroid structures for convolution algebras over étale groupoids, without, however, regarding these as being part of one global structure.

Furthermore, notice that for simplicity all 'competing' approaches [Lu, KhR3, BSz2] assume the antipode to be bijective (although this assumption was recently dropped in [B3], a slight reformulation of the definition in [B1, BSz2]). This is a class large enough for most interesting examples (if the antipode exists at all), such as quantum groups and certain quantum groupoids.

However, one should be aware of the fact that, in contrast to Hopf algebras, the notion of Hopf algebroid is not self-dual: the construction of a Hopf algebroid structure on (a suitable definition of) a dual of a Hopf algebroid is in general quite intricate [BSz2, KSz], and this also causes difficulties in the corresponding cyclic theory (see below).

Let us finally mention that a weaker approach of generalising Hopf algebras towards possibly noncommutative base algebras is given by the so-called $\times_A$-Hopf algebras from [Schau2]. We shall mostly refer to them as *left Hopf algebroids*, inasmuch as the Hopf algebroids from [B1, BSz2] are special cases of them. We are going to explain this later in more detail.

## Cyclic Theory of Hopf Algebras and Hopf Algebroids

Let us now depict the situation for the associated cohomologies:

| *Cohomology in* ☞ *for* ☟ | *Differential Geometry* | *Noncommutative Geometry* |
|---|---|---|
| *Spaces* | De Rham Cohomology | Cyclic Cohomology |
| *Symmetries* | Lie Algebra Cohomology | Hopf-Cyclic Cohomologies |
| *Generalised Symmetries* | Lie Algebroid Cohomology | ? |

A similar table can be formulated for the respective homology theories. Some of the entries of this table will be explained now.

**Cyclic (Co)Homology**
Among the first basic constructions in noncommutative geometry was the cyclic (co)homology of algebras, which may be seen as the correct noncommutative analogue of de Rham (co)homology. The building pieces for cyclic homology theories can be axiomatised so as to produce the more general notion of *cyclic objects*. There are two main avenues to cyclic cohomology: Connes [Co3] developed a cohomological theory in order to interpret index theorems of noncommutative Banach algebras, via a generalisation of the Chern character. The homological approach, introduced by Tsygan [Ts1] and Loday and Quillen [LoQ], shows that cyclic homology can be considered a Lie analogue of algebraic $K$-theory.

**Hopf-Cyclic Cohomology for Hopf Algebras**
Cyclic cohomology for Hopf algebras, or *Hopf-cyclic cohomology*, is the noncommutative analogue of Lie algebra homology (which is recovered in the case of universal enveloping algebras of Lie algebras). This was launched in the work of Connes and Moscovici [CoMos2] on the transversal index theorem for foliations and defined in general in [Cr3] (cf. also [CoMos3, CoMos4]).

In the transversal index theorem of Connes and Moscovici, the characteristic classes involved are *a priori* cyclic cocycles on the algebra $A$ modeling the (singular) leaf space of the foliation. Computing these cocycles turned out to be *tremendously* complicated, even in the 1-dimensional case. The key remark for understanding these cyclic cocycles is that they are quite special: their expression involves only some 'transversal differential operators' originating from the transversal geometry, and an 'integration map', determined by a trace on the algebra $A$. This translates into two conceptual pieces:

(*i*) The operators involved may be organised in a Hopf algebra $H$ acting on the algebra $A$ of functions on the leaf space (analogous to the description of universal enveloping algebras of Lie algebras as differential operators on the Lie group).

(*ii*) By means of the action of $H$ on $A$ and the trace, the cyclic theory of the algebra $A$ is reflected into a new cyclic theory, which is associated to $H$ (making use of the entire Hopf algebra structure).

With these conceptual pieces in mind, the special nature of the cyclic cocycles takes the following precise form: they arise from the cyclic cohomology of the Hopf algebra, via a canonical map (the characteristic map) associated to the action and the trace. In contrast to the cyclic cohomology of the algebra (which is pretty *wild*), the cyclic cohomology of $H$ is much easier to compute as Gel'fand-Fuchs cohomology.

Moreover, it is shown in [CoMos3, CoMos4, Cr3] that the cyclic theory makes sense for any Hopf algebra equipped with a so-called *modular pair in involution* (or twisted antipode). It is useful to keep in mind (as made clear in [Cr3]) that the resulting theory primarily makes use of the *coalgebra structure* of $H$ and of certain *coinvariants*.

**Dual Hopf-Cyclic Homology for Hopf Algebras**
While the notion of Hopf algebra is self-dual, Hopf-cyclic cohomology is not. For instance, while it gives interesting results for universal enveloping algebras of Lie algebras (recovering Lie algebra homology), it tends to be quite trivial for algebras of functions or group algebras (or whenever a *Haar measure* exists). The *dual Hopf-cyclic homology* appears as a companion to Hopf-cyclic cohomology that is better behaved for e.g. function algebras. In what sense these are dual to each other is best explained using the so-called cyclic

duality [Co2], see also below. While the Hopf-cyclic cohomology depends primarily on the *co*product, the unit and *coinvariants*, the dual theory makes use of the product, the *co*unit and certain *invariants* [Cr2, KhR2, KhR4, Tai]. It also shows that the passage from cyclic homology of algebras to the dual Hopf-cyclic cohomology has some similarities to the interpretation of Lie algebra cohomology (for a Lie algebra of a Lie group) as invariant de Rham cohomology of its Lie group manifold structure [CheE]. The need for such a dual theory is furthermore evident if one studies e.g. coactions of Hopf algebras (rather than the actions mentioned in the example of the transverse Hopf algebra above).

In general, Hopf-cyclic cohomology (and likewise dual Hopf-cyclic homology) cannot be seen as the cyclic cohomology of some coalgebra, but only makes sense as the cohomology of some specific cocyclic modules (which was known to describe the same theory right from the beginning [Co2], see e.g. [Lo1] for a full account). This observation will carry over to the cyclic theory of Hopf algebroids, see below.

**The Action and Coaction Picture**

As already mentioned, both theories of Hopf algebra cohomology and homology are 'parametrised' by a Hopf algebra character (to define coinvariants) and a grouplike element (to define invariants). In particular, this allows for cyclic cohomology (or dual homology) with coefficients, which is not possible for the 'standard' cosimplicial modules associated to coassociative coalgebras. General type (co)cyclic modules for Hopf-cyclic (co)homology with values in certain suitable modules were introduced in [HKhRSo2, HKhRSo1]. The need for this came from quantum groups and invariants of $K$-theory. Here, so-called *stable anti-Yetter-Drinfel'd* modules arise as generalisations of modular pairs in involution (more precisely, a modular pair in involution is equivalent to such a module structure on the ground ring $k$), and a generalisation of the characteristic map as a 'transfer' map allows to generally define para-(co)cyclic structures on (co)algebras on which a Hopf algebra acts or coacts, cf. [HKhRSo2, HKhRSo1, KhR2, Kay1] and also [JŞ] for a dual approach. Even more, a universal form suited to describe all examples of cyclic (co)homology arising from Hopf algebras (up to cyclic duality) was given in [Kay2], based on a construction of para-(co)cyclic objects in symmetric monoidal categories in terms of (co)monoids.

**The Cyclic Theory for Hopf Algebroids**

The generalisation of Hopf-cyclic cohomology to noncommutative base rings $A$, i.e. to Hopf algebroids, has been less explored. For instance, the general machinery from [Kay2] does not apply to this context (because the relevant category of modules is not symmetric, and in general is not even braided). Cyclic cohomology of Hopf algebroids appeared for the first time in the context of the transversal 'extended' Hopf algebra $H_{FM}$ mentioned above [CoMos5], i.e. in the case of a particular example rather than as a general theory. In this context, certain bialgebroids (in fact, *left* Hopf algebroids) carrying a cocyclic structure arise naturally. Extending this situation to general Hopf algebroids is not a totally straightforward issue. First of all, one encounters the problem what a Hopf algebroid is. For example, the notion of Hopf algebroid in [Lu] is apparently not well-suited to the problem. This led in [KhR3] to the definition of *para*-Hopf algebroids, in which the antipode of [Lu] is replaced by a *para*-antipode. Its axioms are principally designed for the cocyclic structure to be easily defined by just adapting the Hopf algebra case. However, the para-antipode axioms remain—as we think—too complicated to comprehend their intrinsic structure and purpose, beyond defining (co)cyclic structures; in particular, *guessing* an antipode (and hence the cyclic operator) in concrete examples remains intricate.

A general theory in [BŞ] that deals with cyclic (co)homology of bialgebroids (and $\times_A$-Hopf algebras) appeared while this thesis was written. There, a cyclic theory (in terms of so-called (co)monads) is developed that works in an arbitrary category and hence embraces the construction in [Kay2] for symmetric monoidal categories (in case the (co)monads in question are induced by (co)monoids). This approach is certainly related to our own method, but the precise relation is not completely clear to us.

# Principal Results of this Thesis

The main objective of this thesis is to clarify the notion and concepts of generalised symmetries in noncommutative geometry and their associated (co)homologies—that is, the question marks in the previous tables.

As for the notion of a Hopf algebroid itself, i.e. the question mark in the first table, we do not claim that we have developed this notion ourselves. Instead, we present our own point of view on the theory and in particular which of the 'competing' notions [B1, BSz2, KhR3, Lu, Schau2] appears to be best suited for our purposes (i.e. the question mark in the second table), with some contributions along the way.

**New Examples of Hopf Algebroids**

We reveal that the *universal enveloping algebra* of a Lie-Rinehart algebra (of a Lie algebroid) is always a *left Hopf algebroid* ($\times_A$-Hopf algebra) in a canonical way (Subsection 4.2.1). However, despite of what was originally believed, these enveloping algebras may fail to be Hopf algebroids—an aspect we completely clarify. In particular, we show that the *right connections* from [Hue2] are precisely the extra datum needed: we prove that a well-defined Hopf algebroid structure is only given in case of existence of such a connection, provided it is flat (Theorem 4.2.4, Proposition 4.2.9, Proposition 4.2.11). The next example deals with *jet spaces* associated to Lie-Rinehart algebras, which may be seen as a construction 'dual' to the previous example. This time, the Hopf algebroid structure only depends on the aforementioned canonical *left* Hopf algebroid structure on the universal enveloping algebra and hence always exists (Theorem 4.3.1), in contrast to the previous example.

Another class of natural examples for Hopf algebroids is given by *convolution algebras* over étale groupoids. As already mentioned, the existence of two (opposite) bialgebroid structures was already observed in [Mrč2], and we only need to connect these to give a Hopf algebroid in the sense of [B1, BSz2] (Proposition 4.4.1).

Further examples of Hopf algebroids and bialgebroids we give include *function algebras* over étale groupoids (Proposition 4.5.6) and (generalised) *Connes-Moscovici algebras* (or rather bialgebroids), i.e. the space of *transverse differential operators* on arbitrary étale groupoids, see below for further statements. These should be seen as a step towards the construction of Hopf algebroids associated to (Lie) pseudogroups. Because of these examples—together with the (co)homology computations, see below—we feel sufficiently encouraged to consider Hopf algebroids (in the sense of [B1, BSz2]) as the right noncommutative analogue of both Lie groupoids and Lie algebroids/Lie-Rinehart algebras, respectively (see the analogies (A) and (A') above).

**Left Hopf Algebroids versus Hopf Algebroids**

As a spin-off of the examples mentioned above, we give a first counterexample (see §4.2.13) that not each $\times_A$-Hopf algebra originates in a Hopf algebroid, answering a question in [B3]. This motivates us to refer to $\times_A$-Hopf algebras as *left Hopf algebroids* (which also solves a problem of pronunciation).

**Bicrossed Products; Connes-Moscovici Algebras**

As already outlined above, we use the bialgebroid examples arising from function algebras, Lie-Rinehart algebras and Connes-Moscovici algebras to describe the general 'background' procedure of the constructions in [CoMos5, CoMos6, MosR]. To this end, we introduce the concept of *matched pairs of bialgebroids* and develop the construction of a *bicrossed product bialgebroid* (Theorem 3.3.5), as a generalisation of similar considerations for bialgebras in [Maj]. This is a construction that establishes a (left or right) bialgebroid structure on a certain tensor product of (left or right) bialgebroids over commutative bases. The Connes-Moscovici algebras can then be shown to arise in such a way (Theorem 4.7.1, Proposition 4.7.3).

**Duality and (Co)Modules**

Another construction of how to produce new bialgebroids out of known ones is the construction of *left* and *right* (Hom-)duals for left bialgebroids from [KSz]. We add to this theory a theorem that proves a categorical equivalence between left bialgebroid comodules and modules over its duals (Theorem 3.1.11 and Proposition 3.1.9). Also, we prove an equivalence between grouplike elements of a left bialgebroid and *generalised right characters*, i.e. maps that behave like a right counit on the duals (Proposition 3.1.14). This generalises a similar statement for bialgebras and their duals (see e.g. [Sw1]).

**Hopf-Cyclic Cohomology for Hopf Algebroids**

Central to this thesis is our argument that Hopf-cyclic cohomology is naturally defined when using the Hopf algebroids from [B1, BSz2]. We are going to explain how Hopf-cyclic cohomology fits into the monoidal category of (Hopf algebroid) modules and show that it descends (more precisely: projects) in a canonical way from the cyclic cohomology of coalgebras, or rather corings, under the minimal condition $S^2 = \mathrm{id}$ for the antipode (Proposition 5.2.1, Theorem 5.2.5). This is a generalisation of the consideration of coinvariants for Hopf algebras in [Cr3].

Furthermore, we are able to introduce coefficients at the Hochschild level into the theory, and give an interpretation of the Hopf-Hochschild cohomology groups as a derived functor (Theorem 5.3.3). The main ingredient here is an appropriate resolution in the category of left bialgebroid comodules, the so-called cobar complex. We can show that the cobar complex in case of a commutative Hopf algebroid can be additionally equipped with a cocyclic structure (Proposition 5.4.2). As a consequence, we can express the cyclic cohomology of commutative Hopf algebroid by their Hochschild cohomology groups (Theorem 5.4.4). These statements generalise considerations in [KhR1] from Hopf algebras to Hopf algebroids.

**Dual Hopf-Cyclic Homology for Hopf Algebroids**
Besides cyclic cohomology of Hopf algebroids, we will also develop a dual cyclic homology theory for Hopf algebroids, by applying *cyclic duality* to the underlying cocyclic object (Theorem 6.1.1). This generalises the corresponding theory for Hopf algebras (see above), and produces—analogously as for Hopf algebras—interesting results even if the pertinent cyclic cohomology is trivial. This homology theory is related to a certain category of comodules over the Hopf algebroid: the main difficulty is here that the underlying $(A, A)$-bimodule category fails to be symmetric and on top of that differs from the one for cyclic cohomology. More precisely, the tensor product used for defining cochains in cohomology originates from the monoidal category of modules for the underlying left bialgebroid, whereas the tensor chains in homology make use of the monoidal structure of right bialgebroid comodules. We came to the conclusion that we need to generalise the Hopf-Galois map (see [Schau2]) and its inverse to 'higher degrees' (Lemma 6.1.2), to obtain the necessary tool to translate the two structures into each other such that cyclic duality can be applied. We remark here that this complex of problems does not appear for the symmetric category of $k$-modules in the Hopf algebra case.

However, since the notion of Hopf algebroid is not self-dual (see above), a statement—dual to the cohomology case—that dual Hopf-cyclic homology is obtained from the cyclic homology of algebras in a canonical way (by restriction on *invariants*) does not seem to hold in general (see Subsection 6.1.1, although we give such a construction in special cases, see Section 6.5 and Subsection 6.6.1).

Also in this dual theory, we are able to introduce coefficients at the Hochschild level, and give an interpretation of the Hopf-Hochschild homology groups as derived functors (Theorem 6.2.3), using a generalised bar complex. We can then analogously prove that the bar complex in case of a cocommmutative Hopf algebroid can be equipped with a cyclic structure (Proposition 6.3.1), and show in Theorem 6.3.3 that the dual cyclic homology of cocommutative Hopf algebroids can be expressed by Hopf-Hochschild homology groups, generalising again the corresponding statement in [KhR1] for Hopf algebras.

**Hopf-Cyclic (Co)Homology Computations**
We calculate Hopf-cyclic cohomology and dual Hopf-cyclic homology in concrete examples of Hopf algebroids, such as the universal enveloping algebra of a Lie-Rinehart algebra, jet spaces and convolution algebras over étale groupoids. The results of these computations establish a connection between Hopf-cyclic theory and Lie-Rinehart (co)homology and groupoid homology, respectively (Theorems 5.5.7, 5.6.2, 5.7.1, 6.4.1, 6.5.1, 6.6.4). This motivates to consider Hopf-cyclic (co)homology as the 'correct' noncommutative analogue of both Lie-Rinehart (co)homology and groupoid homology.

On top of that, we are able to construct a special method to obtain dual Hopf-cyclic homology for convolution algebras over étale groupoids, which shows how the theory fits into the monoidal category of (left and right bialgebroid) comodules. The dual Hopf-cyclic homology is then obtained by restricting the (generalised) algebra cyclic module structure to *invariants* (Proposition 6.6.8, Theorem 6.6.10), which is a procedure dual to the considerations of coinvariants in Sections 5.1 and 5.2, working (at least) in this particular example.

**Multiplicative Structures and Duality in (Co)Homology Theories**
Finally, we prove a theorem that suggests that left Hopf algebroids are a key concept for multiplicative structures (such as cup, cap and Yoneda products) and certain duality isomorphisms in algebraic (co)homology theories (Theorem 7.1.1). In particular, results on Hochschild (co)homology [VdB] and Lie-Rinehart (co)homology [Hue3] are included this way.

# Outline of the Thesis

**Chapter One**
In chapter one, we introduce preliminary concepts used throughout the thesis. We give a presentation of basic concepts in cyclic (co)homology in Section 1.1. We then discuss in Section 1.2 the fundamental notion of $A$-rings and $A$-corings for an arbitrary $k$-algebra $A$, which are the generalisations of $k$-algebras and $k$-coalgebras in bimodule categories, and explain how these generalised (co)algebras give rise to (co)cyclic modules. In Section 1.3, we proceed to define Hopf algebras and their cyclic cohomology, which are some

of the concepts that will be generalised in the following chapters. Finally, in Sections 1.4 and 1.5 we introduce Lie-Rinehart algebras and groupoids, as a generalisation for Lie algebras and groups. These will give fundamental examples in the theory of Hopf algebroids.

**Chapter Two**

Chapter two contains the notion of a Hopf algebroid as introduced in [B1, BSz2]. First, we will consider left bialgebroids in Section 2.1, and also the corresponding monoidal categories of bialgebroid modules and bialgebroids comodules in Section 2.3. We proceed in Section 2.4 with discussing how these comodules give rise to derived functors, which will be important for the computations of cyclic (co)homology of chapters five and six. Section 2.2 deals with a weaker version of Hopf algebroids, the so-called left Hopf algebroids ($\times_A$-Hopf algebras) from [Schau2]. These turn out to be a key concept for our considerations in chapter seven, and are also important for the construction of antipodes on jet spaces in chapter four. In the framework of right bialgebroids in Section 2.5, we also introduce in §2.5.1 the notion of (right) connections, as a generalisation to the Lie-Rinehart connections in [Hue2, Hue3], which will appear in examples in chapters three and four. Then, Hopf algebroids are discussed in the final Section 2.6, and we conclude the chapter by some comments on alternative notions of Hopf algebroids.

**Chapter Three**

In chapter three, we give several constructions of how to produce new bialgebroids out of known ones. Section 3.1 discusses the duals for left bialgebroids [KSz], and we prove categorical equivalences between modules and comodules and study the interplay between grouplike elements and (generalised) characters.

Section 3.2 gives a construction how to push forward bialgebroids in case a certain algebra morphism is given. We use the resulting construction to 'localise' certain Hopf algebroids, so as to give an associated Hopf algebra. The chapter continues with our construction of bicrossed product bialgebroids for *matched pairs* of bialgebroids in Section 3.3. The basic ingredients here are the generalised notions of *module rings* and *comodule corings*, as generalised notions of the action and coaction picture for bialgebras, i.e. module algebras and comodule coalgebras.

**Chapter Four**

Chapter four deals with examples of Hopf algebroids. We first indicate in Section 4.1 how enveloping algebras and Hopf algebras (with possibly twisted antipode) fit into the picture. We then devote our attention in Section 4.2 to construct the canonical left Hopf algebroid structure for the universal enveloping algebra $VL$ of a Lie-Rinehart algebra $(A, L)$, and to the relation of (certain) left bialgebroids to their primitive elements. To obtain an antipode on $VL$, we need to recall Lie-Rinehart connections [Hue2, Hue3], and can then describe the full Hopf algebroid structure on $VL$. In Section 4.3 we construct the Hopf algebroid structure on a certain dual of $VL$, the so-called jet spaces. Sections 4.4 and 4.5 indicate how étale groupoids give rise to Hopf algebroid structures in two different ways, where the one in Section 4.5 serves as a basic ingredient in Connes-Moscovici algebras (or rather bialgebroids), which we describe in Section 4.6. This very general construction is shown to be essentially a bicrossed product bialgebroid in Section 4.7.

**Chapter Five**

In chapter five we discuss Hopf-cyclic cohomology for Hopf algebroids. A fundamental step here is to define coinvariants in Section 5.1, which lead to necessary and sufficient conditions for a well-defined cocyclic module structure to exist on any Hopf algebroid, as we explain in Section 5.2. Also, we introduce coefficients into the Hochschild theory and then construct Hopf-Hochschild cohomology as a derived functor in Section 5.3. The following Section 5.4 specialises to the case of Hopf-cyclic cohomology for commutative Hopf algebroids. In the following three Example Sections 5.5–5.7, we discuss and compute Hopf-cyclic cohomology for Lie-Rinehart algebras, jet spaces, and convolution algebras.

**Chapter Six**

Chapter six deals with the dual Hopf-cyclic homology. In Section 6.1, we discuss and construct the corresponding chain complex as the cyclic dual of the cochain complex of chapter five, whereas in Subsection 6.1.1 we discuss a few problems attached to *invariants*. Parallel to chapter five, we introduce coefficients at the Hochschild level and consequently give an interpretation of dual Hopf-Hochschild homology as a derived functor in Section 6.2, whereas in Section 6.3 we discuss and compute dual Hopf-cyclic homology for cocommutative Hopf algebroids. Again, in the Example Section 6.4, we compute and discuss in detail the cases of Lie-Rinehart algebras, jet spaces and convolution algebras.

Parts of chapters five and six are versions of parts of our joint work with Hessel Posthuma [KowPo].

**Chapter Seven**
Finally, chapter seven is a version of our joint work with Ulrich Krähmer [KowKr]. It is mainly devoted to the proof of the central Theorem 7.1.1. Section 7.2 is concerned with the construction of cup and cap products, and with a certain functor that combines left and right modules over left Hopf algebroids. Section 7.3 explains the duality and concludes the proof of Theorem 7.1.1.

**Appendix**
In the Appendix we gather some standard algebraic facts used throughout the text, basically to fix some of our notation and terminology.

**Some conventions**
Throughout this work, 'ring' means 'unital associative ring', and we fix a commutative ground ring $k$. All other algebras, coalgebras, modules and comodules will have the underlying structure of an object of the symmetric monoidal category $k$-$\mathbf{Mod}$ of left $k$-modules. In general, for any ring $U$ the spaces $U$-$\mathbf{Mod}$ and $U^{\mathrm{op}}$-$\mathbf{Mod}$ (or $\mathbf{Mod}$-$U$) denote the category of left $U$-modules and right $U$-modules, respectively, in the standard sense. Also, we fix a (not necessarily commutative) $k$-algebra $A$, i.e. a ring with a ring homomorphism $\eta : k \to Z(A)$ into its centre. We denote by $A^{\mathrm{op}}$ the opposite and by $A^{\mathrm{e}} := A \otimes_k A^{\mathrm{op}}$ the enveloping algebra of $A$. Thus left $A^{\mathrm{e}}$-modules are $(A, A)$-bimodules with symmetric action of $k$. For $U, V$ any rings, we will write a $(U, V)$-bimodule $M$ as ${}_U M_V$ if need be. To indicate with respect to which structure a Hom-functor is defined, we shall write $\mathrm{Hom}_{(U,-)}(M, N)$ for $\mathrm{Hom}({}_U M, {}_U N)$ and analogously $\mathrm{Hom}_{(-,V)}(M, N)$ for $\mathrm{Hom}(M_V, N_V)$; also $\mathrm{Hom}_{(U,V)}(M, N)$ for bimodules appears, and the same kind of notation applies for the sake of uniformity when both $M$, $N$ or only one of them carries a one-sided module structure only.

# Chapter 1

# Preliminaries

## 1.1 Cyclic Theory

One of the basic constructions in noncommutative geometry is the cyclic (co)homology of algebras, which arises as the correct de Rham (co)homology in the noncommutative context. Cyclic homology theories can be axiomatised, giving rise to the more general notion of cyclic objects. In this chapter we recall some of the basic concepts and definitions regarding cyclic objects and their associated cyclic homologies. The main references for much of the material presented here are [FeTs, LoQ, Co3, Lo1, W]. We start by discussing simplicial objects, a notion which comes from algebraic topology and which determines the 'underlying' structure of a cyclic object.

**1.1.1 The Simplicial Category** Let $[k]$ be the ordered set of $k+1$ points $\{0 < 1 < \ldots < k\}$. A map is called *nondecreasing* if $f(i) \geq f(j)$ whenever $i > j$. The *simplicial category* $\Delta$ has as objects the sets $[k]$ for $k \geq 0$ and as morphisms the nondecreasing maps. Of particular interest are the *face* morphisms $\delta_i : [k-1] \to [k]$, the injection which misses $i$, and the *degeneracy* morphism $\sigma_j : [k+1] \to [k]$, the surjection that sends both $j$ and $j+1$ to $j$. We denote the set of morphisms between $[k]$ and $[m]$ by $\hom_\Delta([k],[m])$. In particular, one can show [Lo1, Thm. B.2] that any morphism $\phi : [n] \to [m]$ can be uniquely written as a composition of faces and degeneracies, i.e.,

$$\phi = \delta_{i_1} \ldots \delta_{i_r} \sigma_{j_1} \ldots \sigma_{j_s},$$

such that $i_1 \leq i_r$ and $j_1 < \ldots j_s$ with $m = n - s + r$, and $\phi = \mathrm{id}$ if the index set is empty. As a corollary one obtains a presentation of $\Delta$ with generators $\delta_i$, $\sigma_j$ for $0 \leq i, j \leq n$ (one for each $n$) subject to the relations

$$\begin{array}{rcll} \delta_j\, \delta_i & = & \delta_i\, \delta_{j-1} & \text{if } i < j, \\ \sigma_j\, \sigma_i & = & \sigma_i\, \sigma_{j+1} & \text{if } i \leq j, \\ \sigma_j\, \delta_i & = & \begin{cases} \delta_i\, \sigma_{j-1} & \text{if } i < j, \\ \mathrm{id}_{[n]} & \text{if } i = j,\ i = j+1, \\ \delta_{i-1}\, \sigma_j & \text{if } i > j+1. \end{cases} \end{array} \tag{1.1.1}$$

The opposite category of $\Delta$ is denoted by $\Delta^{\mathrm{op}}$. Observe that the isomorphisms in $\Delta$ are identities on $[k]$ since the identity is the only nondecreasing map that is bijective.

**1.1.2 (Co)Simplicial Objects** Let $M$ be an arbitrary category. A *simplicial object* $(X_\bullet, d_\bullet, s_\bullet)$ in $M$ is a functor $X_\bullet : \Delta^{\mathrm{op}} \to M$. Write $X_n := X([n])$ and $d_i = \delta_i^*$, $s_j = \sigma_j^*$ for the images of the morphisms $\delta_i$ and $\sigma_j$ under the functor $X$. By means of the presentation of $\Delta$ mentioned above, a simplicial object is therefore given by a set of objects $\{X_n\}_{n\geq 0}$ in $M$ as well as by two collections of morphisms $d_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$ and $s_j : X_n \to X_{n+1}$ for $0 \leq j \leq n$ for all $n \geq 0$, satisfying

$$\begin{array}{rcll} d_i\, d_j & = & d_{j-1}\, d_i & \text{if } i < j, \\ s_i\, s_j & = & s_{j+1}\, s_i & \text{if } i \leq j, \\ d_i\, s_j & = & \begin{cases} s_{j-1}\, d_i & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j,\ i = j+1, \\ s_j\, d_{i-1} & \text{if } i > j+1. \end{cases} \end{array} \tag{1.1.2}$$

A *cosimplicial object* $(Y^\bullet, \delta^\bullet, \sigma^\bullet)$ in $M$ is a functor $Y^\bullet : \Delta \to M$; this time write $Y^n := Y([n])$ for the images of the objects in the simplicial category under the functor $Y$ and $\delta^i := \delta_i$, $\sigma^j := \sigma_j$ for the images of the morphisms $\delta_i$, $\sigma_j$. They fulfill relations identical to those in (1.1.1). A *presimplicial* or *semisimplicial object* is similarly as above, if one ignores the degeneracies.

The main example for $M$ we will use is the category of modules (over some ring given by the context); correspondingly, we speak of simplicial *modules*. Define a morphism $f : X \to X'$ of simplicial modules to be a family of linear maps $f_n : X_n \to X'_n$ of modules commuting with both faces $f_{n-1}d_i = d_i f_n$ and degeneracies $f_{n+1}s_i = d_i s_n$ for all $i$ and $n$.

**1.1.3 The Cyclic Category** Next, we recall the definition of Connes' cyclic category $\Lambda$ from [Co2] and of its generalisations $\mathit{\Delta} C_r$ from [FeTs], defined for all integers $1 \leq r \leq \infty$; when $r = 1$, one has $\mathit{\Delta} C_1 = \Lambda$. Although these cyclic categories can be realised explicitly, for our purposes it suffices to recall their descriptions in terms of generators and relations. Roughly speaking, $\mathit{\Delta} C_r$ is a combination of the simplicial category $\mathit{\Delta}$ and the cyclic groups. More precisely, $\mathit{\Delta} C_r$ has the same objects as $\mathit{\Delta}$, but the morphisms are generated by the morphisms $\delta_i$, $\sigma_j$ of $\mathit{\Delta}$ and new morphisms $\tau_n : [n] \to [n]$, the *cyclic operators*, one for each integer $n \geq 0$. These operators serve to express elements in the automorphism groups $\mathrm{Aut}_{\mathit{\Delta} C_r}([n]) \simeq \mathbb{Z}/(n+1)r\mathbb{Z}$ for the case $r < \infty$, and $\mathrm{Aut}_{\mathit{\Delta} C_\infty}([n]) \simeq \mathbb{Z}$ in case of $r = \infty$. The relations they satisfy are the simplicial relations (1.1.1) together with new relations involving the cyclic operator:

$$
\begin{aligned}
\tau_n \delta_i &= \begin{cases} \delta_{i-1}\tau_{n-1} & \text{if } 1 \leq i \leq n \\ \delta_n & \text{if } i = 0, \end{cases} \\
\tau_n \sigma_i &= \begin{cases} \sigma_{i-1}\tau_{n+1} & \text{if } 1 \leq i \leq n \\ \sigma_n \tau_{n+1}^2 & \text{if } i = 0. \end{cases} \\
\tau_n^{(n+1)r} &= \mathrm{id}.
\end{aligned}
$$

In case $r = \infty$, the last equation is void.

**1.1.4 Cyclic Objects** Let $M$ be a category and $1 \leq r \leq \infty$. An *r-cyclic object* [FeTs] in $M$ is a functor $X : \mathit{\Delta} C_r^{\mathrm{op}} \to M$, that is, a simplicial object $(X_\bullet, d_\bullet, s_\bullet)$ together with morphisms $t_n : X_n \to X_n$ which are the images $\tau_n^*$ of $\tau_n$ under $X$ subject to

$$
d_i t_n = \begin{cases} t_{n-1} d_{i-1} & \text{if } 1 \leq i \leq n \\ d_n & \text{if } i = 0, \end{cases} \tag{1.1.3}
$$

$$
s_i t_n = \begin{cases} t_{n+1} s_{i-1} & \text{if } 1 \leq i \leq n \\ t_{n+1}^2 s_n & \text{if } i = 0, \end{cases} \tag{1.1.4}
$$

$$
t_n^{r(n+1)} = \mathrm{id}. \tag{1.1.5}
$$

Again, in case $r = \infty$ the last equation (1.1.5) is replaced by the empty relation. The resulting $\infty$-objects are also called *para-cyclic objects*. When $r = 1$, we recover Connes' cyclic category $\mathit{\Delta} C_1$, also denoted $\mathit{\Delta} C$ or $\Lambda$ and we speak of *cyclic objects*. Composition with the obvious functor $\mathit{\Delta}^{\mathrm{op}} \to \mathit{\Delta} C_r^{\mathrm{op}}$ reproduces the underlying simplicial object.

Throughout this thesis, we will be mainly interested in cyclic objects in the category of modules over a (not necessarily commutative) ring. In this case we speak of *cyclic modules* (the ring being clear from the context). A *morphism of cyclic modules* $f : X \to \tilde{X}$ is a morphism of simplicial modules that commutes with the cyclic structure, i.e., $f_n t_n = t_n f_n$ for all $n$. One can also define a *cyclic module with signs* [Lo1, Def. 2.5.1], with the same set of axioms but with the factor $\mathrm{sign}\, t_n = (-1)^n$ appearing in front of $t_n$ in (1.1.3) and (1.1.4).

**1.1.5 Examples** (*i*) The standard example (see e.g. [FeTs, Nis]) is the cyclic module associated to a (unital, associative) $k$-algebra $U$: set $U^\natural := \{U^{\otimes_k n+1}\}_{n \geq 0}$ with face, degeneracy and cyclic operators given by

$$
\begin{aligned}
d_i(u_0 \otimes_k \cdots \otimes_k u_n) &= \begin{cases} u_0 \otimes_k \cdots \otimes_k u_i u_{i+1} \otimes_k \cdots \otimes_k u_n & \text{if } 0 \leq i \leq n-1, \\ u_n u_0 \otimes_k u_1 \otimes_k \cdots \otimes_k u_{n-1} & \text{if } i = n, \end{cases} \\
s_i(u_0 \otimes_k \cdots \otimes_k u_n) &= u_0 \otimes_k \cdots \otimes_k u_i \otimes_k 1 \otimes_k u_{i+1} \otimes_k \cdots \otimes_k u_n \ \text{if } 0 \leq i \leq n, \\
t_n(u_0 \otimes_k \cdots \otimes_k u_n) &= u_n \otimes_k u_0 \otimes_k u_1 \otimes_k \cdots \otimes_k u_{n-1}.
\end{aligned}
$$

(*ii*) *Smooth Functions on a Compact Manifold.* Variations of the previous example arise when working in various categories of topological algebras and replacing the tensor product by topological (completed) versions of the algebraic one. The central example is that of smooth functions on a compact manifold $M$. In this case it is interesting to consider a completed tensor product $\hat{\otimes}_\pi$ (see e.g. [Gro]) such that $\mathcal{C}^\infty(M)\hat{\otimes}_\pi\mathcal{C}^\infty(M') \simeq \mathcal{C}^\infty((M \times M'))$ for any two compact manifolds $M, M'$. For any compact manifold one therefore has an associated cyclic module $\mathcal{C}^\infty(M)^\natural := \{\mathcal{C}^\infty(M^{\times(n+1)})\}_{n\geq 0}$, i.e. $\mathcal{C}^\infty(M^{\times(n+1)})$ in degree $n$. Considering that $\mathcal{C}^\infty(M)$ is commutative with the pointwise product, the above face, degeneracy and cyclic operators become, for any $f \in \mathcal{C}^\infty(M)$,

$$\begin{aligned} d_i f(x_0,\ldots,x_{n-1}) &= \begin{cases} f(x_0,\ldots,x_i,x_i,\ldots,x_{n-1}) & \text{if } 0 \leq i \leq n-2, \\ f(x_0,x_1,\ldots,x_n,x_0) & \text{if } i = n-1, \end{cases} \\ s_i f(x_0,\ldots,x_{n+1}) &= f(x_0,x_1,\ldots,\hat{x}_{i+1},\ldots,x_n,x_{n+1}) \quad \text{if } 0 \leq i \leq n, \\ t_n f(x_0,\ldots,x_n) &= f(x_1,\ldots,x_n,x_0). \end{aligned}$$

As long as they fulfill the mentioned property of $C^\infty(M)^{\hat{\otimes} n} \simeq C^\infty(M^{\times n})$, different tensor products $\hat{\otimes}$ (e.g. projective or inductive ones [Gro]) also lead to meaningful results in calculating cyclic homology, as will be seen in a moment in Example 1.1.10(*iii*). Further possibilities are given by defining tensor products $C^\infty(M)^{\otimes n} := \mathrm{germs}_\Delta C^\infty(M^{\times n})$ or $C^\infty(M)^{\otimes n} := \mathrm{jets}_\Delta C^\infty(M^{\times n})$ where $\Delta : M \to M^{\times n}, x \mapsto (x,\ldots,x)$ is the diagonal, confer [Ts2, Te] for details.

(*iii*) A generalisation of the first example is associated to an algebra $U$ endowed with an endomorphism $\phi \in \mathrm{End}_k U$: the resulting cyclic module $U^{\natural,\phi} := \{U^{\otimes_k n+1}\}_{n\geq 0}$ has as face, degeneracy and cyclic operators

$$\begin{aligned} d_i(u_0 \otimes_k \cdots \otimes_k u_n) &= \begin{cases} u_0 \otimes_k \cdots \otimes_k u_i u_{i+1} \otimes_k \cdots \otimes_k u_n & \text{if } 0 \leq i \leq n-1, \\ \phi(u_n)u_0 \otimes_k u_1 \otimes_k \cdots \otimes_k u_{n-1} & \text{if } i = n, \end{cases} \\ s_i(u_0 \otimes_k \cdots \otimes_k u_n) &= u_0 \otimes_k \cdots \otimes_k u_i \otimes_k 1 \otimes_k u_{i+1} \otimes_k \cdots \otimes_k u_n \ \text{ if } 0 \leq i \leq n, \\ t_n(u_0 \otimes_k \cdots \otimes_k u_n) &= \phi(u_n) \otimes_k u_0 \otimes_k u_1 \otimes_k \cdots \otimes_k u_{n-1}. \end{aligned}$$

Then $U^{\natural,\phi}$ is $r$-cyclic if the order of $\phi$ is less than infinity and cyclic if and only if $\phi = \mathrm{id}$. In this case we recover $U^\natural$ from (*i*).

(*iv*) In §1.2.4 we will discuss another generalisation of Example (*i*) (and simultaneously of (*iii*)) which arises when the commutative ground ring $k$ is replaced by a not necessarily commutative algebra.

**1.1.6 Hochschild and Cyclic Homology for Cyclic Objects** Next, we recall the definition of cyclic homologies associated to cyclic objects in an abelian category. Hence, let $M$ be an abelian category and let $X$ be an $r$-cyclic object in $M$. There are several equivalent ways to define the cyclic homology of $X$, all with their own advantages. We dedicate our attention first to Tsygan's double complex, which is one of the most complicated methods but has the best conceptual properties. Firstly, consider the operators

$$\begin{aligned} b'_n : X_n \to X_{n-1},\ b'_n &:= \sum_{j=0}^{n-1}(-1)^j d_j, \\ b_n : X_n \to X_{n-1},\ b_n &:= b'_n + (-1)^n d_n. \end{aligned} \tag{1.1.6}$$

Note that $b$ and $b'$ differ by the last face operator only. Secondly, set $\tilde{t}_n := (-1)^n t_{n+1}$ for $r \neq \infty$ and define the *norm operator*

$$N := \sum_{j=0}^{(n+1)r-1} \tilde{t}_n^j.$$

The cyclic homology groups $HC_\bullet(X)$ may be defined [FeTs] as the homology of the *associated cyclic bicomplex* $CC_{\bullet,\bullet}X$ (Tsygan's double complex, see the figure below). It has entries $CC_{p,q}X := X_q$ for $p, q \geq 0$, independently of $p$. In this complex, the columns are periodic of order 2; for $p$ even, the $p^{\mathrm{th}}$ column is the *Hochschild complex* $(C_\bullet X, b)$ (where $C_n X = X_n$); in case $p$ is odd, the respective column is the *acyclic complex* $C_\bullet^{\mathrm{acyc}} X := (C_\bullet X, b')$ (where $C_n X = X_n$ as before). The $q^{\mathrm{th}}$ row is the periodic complex associated to the action of the cyclic group $Z_{q+1}$ on $X_q$ in which the generator acts by multiplying with $\tilde{t}_q$; thus, the differential $X_q \to X_q$ is multiplication by $1 - \tilde{t}_q$ when $p$ is odd and by $N$ otherwise. Hence, in our sign convention, the bicomplex reads

$$CC_{\bullet,\bullet}X: \qquad \begin{tikzcd} {} \arrow[d, "b"'] & {} \arrow[d, "b'"'] & {} \arrow[d, "b"'] & {} \arrow[d, "b'"'] & \\ X_2 & X_2 \arrow[l, "1-t"'] & X_2 \arrow[l, "N"'] & X_2 \arrow[l, "1-t"'] & \cdots \arrow[l, "N"'] \\ X_1 \arrow[u, leftarrow, "b"] & X_1 \arrow[u, leftarrow, "b'"] \arrow[l, "1+t"'] & X_1 \arrow[u, leftarrow, "b"] \arrow[l, "N"'] & X_1 \arrow[u, leftarrow, "b'"] \arrow[l, "1+t"'] & \cdots \arrow[l, "N"'] \\ X_0 \arrow[u, leftarrow, "b"] & X_0 \arrow[u, leftarrow, "b'"] \arrow[l, "1-t"'] & X_0 \arrow[u, leftarrow, "b"] \arrow[l, "N"'] & X_0 \arrow[u, leftarrow, "b'"] \arrow[l, "1-t"'] & \cdots . \arrow[l, "N"'] \end{tikzcd}$$

Hochschild homology $HH_\bullet(X)$ of $X$ is now the homology of the zeroth column and its cyclic homology is defined as

$$HC_\bullet(X) := H_\bullet(\mathrm{Tot}\, CC_{\bullet,\bullet}X),$$

where we recall that the total complex is defined as

$$(\mathrm{Tot}\, CC_{\bullet,\bullet}X)_n := \bigoplus_{p+q=n} CC_{p,q}X.$$

Standard homological algebra leads to the fact that short exact sequences $0 \to X \to X' \to X'' \to 0$ of cyclic objects give rise to short exact sequences of both Hochschild complexes and Tsygan bicomplexes, which, in turn, give rise to long exact sequences in homology,

$$\begin{array}{l} \ldots \longrightarrow HH_n(X) \longrightarrow HH_n(X') \longrightarrow HH_n(X'') \longrightarrow HH_{n+1}(X) \longrightarrow \ldots \\ \ldots \longrightarrow HC_n(X) \longrightarrow HC_n(X') \longrightarrow HC_n(X'') \longrightarrow HC_{n-1}(X) \longrightarrow \ldots . \end{array}$$

**1.1.7 The *SBI*-sequence** Hochschild and cyclic homology are organised by three basic homomorphisms $I, S, B$ into a long exact sequence

$$\ldots \longrightarrow HC_{n+1}(X) \xrightarrow{S} HC_{n-1}(X) \xrightarrow{B} HH_n(X) \xrightarrow{I} HC_n(X) \xrightarrow{S} \ldots,$$

also called *Connes' exact sequence*, which is often (and often implicitly) used for concrete calculations. If $X$ is a cyclic object in an abelian category, inclusion of the Hochschild complex $C_\bullet X \hookrightarrow CC_{\bullet,\bullet}X$ into Tsygan's double complex as zeroth column yields a map $I : HH_n(X) \to HC_n(X)$. Considering only the zeroth and first column in $CC_{\bullet,\bullet}X$ leads to a double subcomplex denoted $CC^{\{2\}}_{\bullet,\bullet}X$; the inclusion $C_\bullet X \hookrightarrow CC^{\{2\}}_{\bullet,\bullet}X$ induces an isomorphism

$$HH_n(X) \simeq H_n(\mathrm{Tot}\,(CC^{\{2\}}_{\bullet,\bullet}X))$$

since the quotient is the first column that is acyclic. Also, there is an isomorphism

$$CC_{\bullet,\bullet}X[-2] := CC_{\bullet,\bullet}X/CC^{\{2\}}_{\bullet,\bullet}X \simeq CC_{\bullet,\bullet}X$$

of the quotient complex consisting of columns $p \geq 2$ with the original double complex itself, but shifted two columns to the right. The *shifting operator* $S : HC_n(X) \to HC_{n-2}(X)$ is therefore induced by the quotient map $\mathrm{Tot}\,(CC_{\bullet,\bullet}X) \to \mathrm{Tot}\,(CC_{\bullet,\bullet}X[-2])$. The resulting short exact sequence

$$0 \longrightarrow CC^{\{2\}}_{\bullet,\bullet}X \xrightarrow{I} CC_{\bullet,\bullet}X \xrightarrow{S} CC_{\bullet,\bullet}X[-2] \to 0$$

of double complexes yields a boundary map $B : HC_{n-1}(X) \to HH_n(X)$ which fits into Connes' long exact sequence above. The $SBI$-sequence is an efficient tool to compute cyclic homology once the Hochschild homology is known. In particular, it follows by induction as well as the 5-Lemma that every morphism of cyclic objects that induces an isomorphism on Hochschild homology induces an isomorphism on cyclic homology (note that $HH_0(X) = HC_0(X)$).

**1.1.8 Periodic Cyclic Homology** We now come to the most important version of cyclic homology, the periodic one. This the correct noncommutative analogue to the classical de Rham cohomology (see Examples 1.1.10(*ii*)–(*iii*) for illustrating results). The rôle of Hochschild and cyclic homology is merely that of intermediate steps towards the final, periodic theory; this philosophy also applies when doing computations.

As above, let $X$ be an $r$-cyclic object in an abelian category $M$, where we assume that $r \neq \infty$. Due to its obvious periodicity, Tsygan's double complex can be extended to the left to form the *upper half plane complex* $CP_{\bullet,\bullet}X$. The *periodic cyclic* homology, denoted $HP_\bullet(X)$, is the homology of the 'product' total complex

$$HP_\bullet(X) = H_\bullet(\mathrm{Tot}^{\,\Pi}(CP_{\bullet,\bullet}X)).$$

Here, by product total complex $\mathrm{Tot}^{\,\Pi}$ we mean the total complex formed by using products (thought of as 'infinite sums') rather than sums. Recall [Lo1, 5.1.2] that the homology of the standard 'sum' total complex does not lead to meaningful results (observe that in contrast to $CC_{\bullet,\bullet}X$ there is now an infinite number of non-zero modules $CC_{p,q}$ with $p+q=n$). It is visually evident from the periodicity of $CP_{\bullet,\bullet}X$ that each of the maps $S : HP_{n+2}(X) \to HP_n(X)$ is an isomorphism; hence its name: the modules $HP_n(X)$ are periodic of order 2.

**1.1.9 Mixed Complexes** There is another (simpler) double complex computing the cyclic homologies—*Connes' double complex*—which we now recall. This double complex arises as a simplification of Tsygan's double complex due to the fact that some of its columns are contractible: exploiting the fact that a cyclic object $X$ has degeneracies, one can eliminate the acyclic columns applying the 'killing contractible complexes lemma' [Lo1, Lem. 2.1.6] to obtain a double complex $BC_{\bullet,\bullet}X$, called Connes' double complex. To this end, introduce the 'extra' degeneracy

$$s_{-1} := t_{n+1}s_n : X_n \to X_{n+1}, \tag{1.1.7}$$

which can be shown to be a chain contraction of $C^{\mathrm{acyc}}_\bullet(X)$ (one may equally consider $s_{n+1} := t^{-1}_{n+1}s_0$). Also, define

$$B := (1-\tilde{t}_n)s_{-1}N : X_n \longrightarrow X_{n+1},$$

which is commonly called *Connes' coboundary map* or *Connes' cyclic operator* (notice, however, that it already appears in the early work of Rinehart [Rin]). One can easily see that $B^2 = 0$, $Bb + bB = 0$, besides $b^2 = 0$ for the Hochschild boundary. Define $BC_{\bullet,\bullet}X$ by $BC_{p,q}(X) := X_{q-p}$ for $0 \leq p \leq q$ and zero otherwise, and organise it into the following double complex:

$$BC_{\bullet,\bullet}X : \quad \begin{tikzcd} {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\ X_3 & X_2 \arrow[l, "B"'] & X_1 \arrow[l, "B"'] & X_0 \arrow[l, "B"'] \\ X_2 \arrow[u, leftarrow, "b"] & X_1 \arrow[l, "B"'] \arrow[u, leftarrow, "b"] & X_0 \arrow[l, "B"'] \arrow[u, leftarrow, "b"] & \\ X_1 \arrow[u, leftarrow, "b"] & X_0 \arrow[l, "B"'] \arrow[u, leftarrow, "b"] & & \\ X_0. \arrow[u, leftarrow, "b"] & & & \end{tikzcd}$$

Again, one obtains an exact sequence of complexes

$$0 \longrightarrow C_\bullet X \longrightarrow \mathrm{Tot}\, BC_{\bullet,\bullet}X \xrightarrow{S} \mathrm{Tot}\, BC_{\bullet,\bullet}X[2] \longrightarrow 0, \tag{1.1.8}$$

from which one may derive the $SBI$-sequence again, showing basically that the two operators which we both denoted by $B$ actually coincide.

The homology of the zeroth column in $BC_{\bullet,\bullet}X$ is still the Hochschild homology $HH_\bullet(X) = H_\bullet(C_\bullet X, b)$, whereas the morphism of complexes $\mathrm{Tot}\, CC_{\bullet,\bullet}X \leftarrow \mathrm{Tot}\, BC_{\bullet,\bullet}X$ is a quasi-isomorphism. Hence

$$HC_\bullet(X) = H_\bullet(\mathrm{Tot}\, CC_{\bullet,\bullet}X) \xleftarrow{\simeq} H_\bullet(\mathrm{Tot}\, BC_{\bullet,\bullet}X) \tag{1.1.9}$$

is an isomorphism if $X$ is a cyclic object, so that $BC_{\bullet,\bullet}X$ can be taken to compute cyclic homology. If $X$ happens to be a cyclic module for some ring $k$, each column can be replaced by its *normalised version*. That is, put $\bar{X}_i := X_i/k = \mathrm{Coker}\,(k \to X_i)$ in place of $X_i$ which leads to a new complex denoted $\bar{B}C_{\bullet,\bullet}X$ with horizontal differential of the form $B := s_{-1}N$. The main issue here is that one can replace $BC_{\bullet,\bullet}X$ in both (1.1.8) and (1.1.9) by $BC_{\bullet,\bullet}X$, hence it computes the same homology.

In a general framework, such an object that is both a chain and a cochain complex is called a *mixed complex* [Kas1]: this is a graded object $\{X_n\}_{n\geq 0}$ with two families of operators $b : X_n \to X_{n-1}$ and $B : X_n \to X_{n+1}$ subject to $b^2 = B^2 = Bb + bB = 0$. Hence each cyclic object gives rise to a mixed complex (but not necessarily the other way round). In any case, the isomorphism in (1.1.9) may serve as a definition of Hochschild and cyclic homologies of a mixed complex.

**1.1.10 Examples** Here we will give some typical 'classical' illustrations which will possibly be helpful later on.

(*i*) The cyclic homology of a $k$-algebra $U$ (as originally defined in [Co3]) is the cyclic homology of the cyclic object $U^\natural$ of Example 1.1.5(*i*). In particular, $HC_0(U) = U/[U, U]$.

(*ii*) *The Algebraic HKR-Theorem* [HoKoRos, LoQ] If $U$ is a unital commutative algebra over a commutative ring $k$ containing $\mathbb{Q}$, its $U$-module of *Kähler differentials* $\Omega^1_{U|k}$ is generated over $U$ by symbols $du$, $u \in U$ subject to the conditions

$$d(au) = a\,du,\ d(u + v) = du + dv,\ d(uv) = u\,dv + v\,du$$

for all $u, v \in U$, $a \in k$. Denote by $\Omega^\bullet_{U|k} := \wedge^\bullet_U \Omega^1_{U|k}$ the $U$-exterior algebra of Kähler differentials where $\Omega^0_{U|k} := U$ and write typical elements in $\Omega^n_{U|k}$ as $u_0\,du_1 \cdots du_n := u_0\,du_1 \wedge \cdots \wedge du_n$. The *de Rham differential* $d$ is the family of maps

$$d : \Omega^n_{U|k} \to \Omega^{n+1}_{U|k}, \quad d(u_0\,du_1 \cdots du_n) = du_0\,du_1 \cdots du_n,$$

for $u_i \in U$. It fulfills $d^2 = 0$, so that one may define the *de Rham cohomology* of $U$ as $H^\bullet_{\mathrm{dR}}(U) := H^\bullet(\Omega_{U|k})$.

One now has two natural maps, the antisymmetrisation map and its section, respectively,

$$\begin{aligned} &\pi_n : HH_n(U) \to \Omega^n_{U|k}, \quad (u_0, u_1, \ldots, u_n) \mapsto u_0\,du_1 \cdots du_n, \\ &\epsilon_n : \Omega^n_{U|k} \to HH_n(U), \quad u_0\,du_1 \cdots du_n \mapsto \sum_{\sigma \in S_n} \mathrm{sign}\,\sigma(u_0, u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(n)}), \end{aligned}$$

and $\pi_n\,\epsilon_n = n!\mathrm{id}$ holds. In particular, both maps commute with the map $B_* : HH_n(U) \to HH_{n+1}(U)$ induced by $B$, that is $B_*\,\epsilon_n = \epsilon_{n+1}\,d$ and $(n + 1)d\,\pi_n = \pi_{n+1}\,B_*$; hence $B$ is compatible with $d$. Under the assumption that $k$ contains $\mathbb{Q}$, the map $\frac{1}{n!}\pi_n$ induces a morphism of mixed complexes

$$(C_\bullet U, b, B) \to (\Omega^\bullet_{U|k}, 0, d),$$

and if $U$ is *smooth* (confer [Lo1, App. E] for the precise definition), one can prove

**1.1.11 Theorem** *If $U$ is a smooth algebra over $k$, the antisymmetrisation map $\epsilon : \Omega^\bullet_{U|k} \to HH_\bullet(U)$ is an isomorphism of graded algebras . As a consequence, if $k$ contains $\mathbb{Q}$, one additionally has a canonical isomorphism*

$$HC_n(U) \simeq \Omega^n_{U|k}/d\Omega^{n-1}_{U|k} \oplus H^{n-2}_{\mathrm{dR}}(U) \oplus H^{n-4}_{\mathrm{dR}}(U) \oplus \ldots,$$

*the last summand being $H^0_{\mathrm{dR}}(U)$ or $H^1_{\mathrm{dR}}(U)$ depending on $n$ even or odd, respectively. Finally,*

$$HP_n(U) \simeq \prod_{m \in \mathbb{Z}} H^{n+2m}_{\mathrm{dR}}(U).$$

Since $HC_\bullet(U)$ is defined even for noncommutative $U$, one may think of (periodic) cyclic homology as a generalisation of de Rham cohomology to a noncommutative setting.

(*iii*) *Connes' Theorem* [Co3] A differential geometric version of the preceding example is as follows. The algebraic theory $HC_\bullet(U)$ for a general algebra $U$ is usually hard to calculate; in applications, however, $U$ is often determined more specifically, for example it is given as a topological locally convex algebra. In such a situation the algebraic tensor product in Example 1.1.5(*i*) has to be replaced by one that does not ignore the topology since already in the case $C_c^\infty(M)$ of compactly supported smooth functions on a manifold $M$ the algebraic cyclic homology is not known. As already discussed in Example 1.1.5(*ii*), one has different possibilities for choosing such a tensor product $\hat{\otimes}$, as long as the property $C^\infty(M)^{\hat{\otimes} n} \simeq C^\infty(M^{\times n})$ is fulfilled. In either case, for $C_\bullet(\mathcal{C}^\infty(M)) = \mathcal{C}^\infty(M)^{\otimes \bullet+1}$ using any of the tensor products with the desired property, the map

$$\mu : f_0 \otimes f_1 \otimes \cdots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \cdots df_n$$

defines a quasi-isomorphism of complexes

$$(C_\bullet(\mathcal{C}^\infty(M)), b) \mapsto (\Omega^\bullet M, 0)$$

as well as a map of mixed complexes

$$(C_\bullet(\mathcal{C}^\infty(M)), b, B) \mapsto (\Omega^\bullet M, 0, d_{\mathrm{dR}}).$$

Note that the fact that $\mu$ is a map of mixed complexes indicates that, up to a factor, $B$ seems to be the correct replacement of $d_{\mathrm{dR}}$ to a noncommutative setting. In particular, if the manifold $M$ is compact one has the isomorphisms [Co3]

$$HH_\bullet(\mathcal{C}^\infty(M)) \simeq \Omega^\bullet M \quad \text{and hence} \quad HP_\bullet(\mathcal{C}^\infty(M)) \simeq H^{\mathrm{even}}_{\mathrm{dR}}(M) \oplus H^{\mathrm{odd}}_{\mathrm{dR}}(M).$$

(*iv*) *Almost Symmetric Algebras* Let $(U, \cdot)$ be a non-negatively filtered algebra over a commutative ring $k$, i.e., there is a sequence of $k$-modules $\{F_i U\}_{i\geq 0}$ such that

$$\begin{gathered} F_0U \subset F_1U \subset F_2U \subset \dots, \\ \cup_i F_iU = U, \quad \cap_i F_iU = 0, \quad (F_iU)\cdot(F_jU) \subset F_{i+j}U \quad \text{for } i,j \geq 0. \end{gathered}$$

Now $U$ is called an *almost symmetric algebra* if its associated graded algebra $\operatorname{gr} U = \oplus_i \operatorname{gr}_i U$, where $\operatorname{gr}_i U = F_iU/F_{i-1}U$, is isomorphic to the symmetric algebra $SV$ where $V := F_1U/F_0U$; this requires in particular $F_0U = k$ and $\operatorname{gr} U$ to be commutative. Moreover, $\operatorname{gr} U$ becomes a Poisson algebra, that is, a (possibly unital and commutative) associative algebra together with a Lie bracket that satisfies a Leibniz derivation rule in each of its arguments. Now consider the canonical map $p_i : F_iU \to \operatorname{gr} U$, take two elements $f \in \operatorname{gr}_i U$ and $g \in \operatorname{gr}_j U$, and choose $u \in F_iU$, $u' \in F_jU$ such that $f = p_i(u)$ and $g = p_j(u')$. Since $\operatorname{gr} U$ is commutative, the commutator $[u, u'] = uu' - u'u$ lies in $F_{i+j-1}U$ and one verifies that $\{f, g\} := p_{i+j-1}([u, u'])$ only depends on $f, g$ and indeed defines a Poisson structure on $\operatorname{gr} U$. Finally, the symmetric algebra $SV$ inherits a Poisson structure by pullback and the degree $-1$ differential

$$\begin{aligned} \delta(v_0\, dv_1 \cdots dv_n) := &\sum_{i=1}^{n} (-1)^i \{v_0, v_1\} dv_1 \cdots \hat{dv_i} \cdots dv_n \\ &+ \sum_{1\leq i<j<n} (-1)^{i+j} v_0 d\{v_i, v_j\} dv_1 \cdots \hat{dv_i} \cdots \hat{dv_j} \cdots dv_n \end{aligned}$$

on $\Omega^\bullet_{SV|k}$ anti-commutes with the degree $+1$ de Rham differential $d$. Hence this construction defines a mixed complex $(\Omega^\bullet_{SV|k}, \delta, d)$. Then one can prove [Kas2, Lo1]:

**1.1.12 Theorem** *If $k$ contains $\mathbb{Q}$ and $U$ is almost symmetric, one obtains isomorphisms*

$$\begin{aligned} HH_\bullet(U) &\xrightarrow{\simeq} H_\bullet(\Omega^\bullet_{SV|k}, \delta), \\ HC_\bullet(U) &\xrightarrow{\simeq} HC_\bullet(\Omega^\bullet_{SV|k}, \delta, d), \\ HP_\bullet(U) &\xrightarrow{\simeq} HP_\bullet(\Omega^\bullet_{SV|k}, \delta, d). \end{aligned}$$

In particular, one can apply this theorem to the universal enveloping algebra $UL$ of a (free as a $k$-module) Lie algebra $L$, reproducing e.g. the classical result $HH_\bullet(UL) \simeq H_\bullet(L, UL^{\mathrm{ad}})$ of [CarE, Chapt. XIII, Thm. 7.1], or to a similar construction in the framework of Lie-Rinehart algebras (provided one is interested in their *algebra* homology). We will, however, take a different route to the same result, see Theorem 6.4.1($i$).

**1.1.13 Cocyclic Objects** As before, let $N$ be an abelian category and pick $1 \leq r \leq \infty$. An *$r$-cocyclic object* in $N$ is a functor $Y : \Delta C_r \to N$, that is, a cosimplicial object $(Y^\bullet, \delta_\bullet, \sigma_\bullet)$ together with a morphism $\tau_n : Y^n \to Y^n$, subject to

$$
\begin{aligned}
\tau_n \delta_i &= \begin{cases} \delta_{i-1}\tau_{i-1} & \text{if } 1 \leq i \leq n, \\ \delta_n & \text{if } i = 0, \end{cases} \\
\tau_n \sigma_i &= \begin{cases} \sigma_{i-1}\tau_{n+1} & \text{if } 1 \leq i \leq n, \\ \sigma_n \tau_{n+1}^2 & \text{if } i = 0, \end{cases} \\
\tau_n^{r(n+1)} &= \mathrm{id}.
\end{aligned}
\tag{1.1.10}
$$

Again, in case $r = \infty$ the last equation is replaced by the empty relation. In such a case the resulting $\infty$-cocyclic objects are also called *para-cocyclic objects*; if $r = 1$, we simply speak of cocyclic objects. As before, we will be often dealing with the case in which $N$ is the category of modules over a ring, that is, we will be considering cocyclic modules. On the other hand, natural objects to deal with in this context are actually *comodules* with respect to some coalgebra and some ring (hence still some modules over some ring, see §1.2.2 for more details). In those cases we shall speak of cocyclic comodules.

**1.1.14 Example** Dually to Example 1.2.4($i$), one can assign to each $k$-coalgebra $C$ and $\psi \in \mathrm{End}_k C$ a cocyclic module $C_\natural^\psi$; cf. [FeTs]. We do not give the details here, but rather refer to the generalised version in §1.2.5, where $k$ is replaced by an arbitrary $k$-algebra.

**1.1.15 Hochschild and Cyclic Cohomology for Cocyclic Objects** For a cocyclic object $Y^\bullet$, define

$$
\begin{aligned}
\beta'_n : Y^n &\longrightarrow Y^{n+1}, \ \beta'_n := \sum_{j=0}^{n} (-1)^j \delta_j, \\
\beta_n : Y^n &\longrightarrow Y^{n+1}, \ \beta_n := \beta'_n + (-1)^{n+1}\delta_{n+1},
\end{aligned}
\tag{1.1.11}
$$

along with the extra codegeneracy $\sigma_{-1} := \sigma_n \tau_{n+1}$ which serves here as contracting homotopy (one could also take $\sigma_{n+1} = \sigma_0 \tau_{n+1}^{-1}$), and $\lambda_n := (-1)^n \tau_n$ for $r \neq \infty$. Moreover, set

$$
N := \sum_{j=0}^{(n+1)r-1} \lambda_n^j, \qquad B := N\sigma_{-1}(1 - \lambda_{n+1}).
$$

Then the resulting mixed complex $(Y^\bullet, \beta, B)$ defines the Hochschild and cyclic cohomologies $HH^\bullet(Y)$ and $HC^\bullet(Y)$, respectively; again, if $r = \infty$, set $HH^\bullet(Y) = H^\bullet(Y, \beta)$. As before, one could alternatively consider a 'cocyclic' bicomplex, but we refrain from spelling out the details here. See, however, (5.2.14) for an example of a reduced (normalised) bicomplex where $Y^\bullet$ is in the category of cocyclic modules.

**1.1.16 Tor and Ext Interpretation of Cyclic (Co)Homology** One can apply methods of homological algebra in the abelian category of (co)simplicial $k$-modules. If $X$ is a simplicial and $Y$ is a cosimplicial $k$-module, the groups $\mathrm{Tor}_n^{\Delta^{\mathrm{op}}}(Y, X)$ are well-defined and in fact are $k$-modules. The *trivial* simplicial or cosimplicial module $k$ is the functor $[n] \mapsto k$ for all $n \geq 0$ with (co)faces and (co)degeneracies given by the identity; one can show that $\mathrm{Tor}_n^{\Delta^{\mathrm{op}}}(k, X) \simeq HH_n(X)$ and $\mathrm{Ext}^n_{\Delta^{\mathrm{op}}}(Y, k) \simeq HH^n(Y)$, cf. e.g. [Lo1].

The interpretation of cyclic homology and cohomology as Tor and Ext functors, respectively, consists essentially in replacing the simplicial category $\Delta$ by $\Delta C$. A theorem in [Co2] (see also [Lo1, 6.2.8] for a proof) establishes canonical isomorphisms

$$
\mathrm{Tor}_n^{\Delta^{\mathrm{op}}C}(k, X) \simeq HC_n(X) \qquad \mathrm{Ext}^n_{\Delta^{\mathrm{op}}C}(Y, k) \simeq HC^n(Y)
$$

for $X, Y$ cyclic and cocyclic, respectively. The proof works essentially in an analogous manner as the one for Hochschild homology and simplicial modules; however, instead of constructing a particular resolution for the trivial cyclic module $k^\natural$ as in the case of simplicial modules, one rather constructs a certain *biresolution*;

see [Lo1] for details. In case of Example 1.2.4($i$), one obtains in a canonical way for a unital associative $k$-algebra $U$

$$\mathrm{Tor}_n^{\Delta^{\mathrm{op}}C}(k, U^\natural) \simeq HC_n(U), \qquad \mathrm{Ext}^n_{\Delta^{\mathrm{op}}C}(U^\natural, k) = HC^n(U).$$

**1.1.17 The Cyclic Dual** A remarkable property of the cyclic category $\Delta C_r$ is the existence of a natural isomorphism to its opposite category $\Delta^{\mathrm{op}}C_r$, so to say a self-duality. For the sake of simplicity we give the explicit construction for $r = 1$; confer [FeTs] for the more general case. Roughly speaking, the corresponding duality functor $\Delta C \to \Delta^{\mathrm{op}}C$ is the identity on objects, exchanges (co)faces and (co)degeneracies and sends the (co)cyclic operator to its inverse. More precisely, if $X = (X_\bullet, d_\bullet, s_\bullet, t_\bullet)$ is a para-cyclic object with $t_n$ assumed to be invertible for all $n \geq 0$, define its *cyclic dual* $\hat{X} := (\hat{X}^\bullet, \delta_\bullet, \sigma_\bullet, \tau_\bullet)$ where $\hat{X}^n := X_n$ in degree $n$ and

$$\begin{array}{llllll} \delta_i := s_{i-1} & : & \hat{X}^n & \to & \hat{X}^{n+1}, & 1 \leq i \leq n, \\ \delta_0 := t_n s_n & : & \hat{X}^n & \to & \hat{X}^{n+1}, & \\ \sigma_i := d_i & : & \hat{X}^n & \to & \hat{X}^{n-1}, & 0 \leq i \leq n-1, \\ \tau_n := t_n^{-1} & : & \hat{X}^n & \to & \hat{X}^n. & \end{array}$$

Also, if $Y = (Y^\bullet, \delta_\bullet, \sigma_\bullet, \tau_\bullet)$ is a para-cocyclic object with invertible operator $\tau$, its cyclic dual is defined as $\check{Y} := (\check{Y}_\bullet, d_\bullet, s_\bullet, t_\bullet)$, where $\check{Y}_n := Y^n$ in degree $n$ and

$$\begin{array}{llllll} d_i := \sigma_{i-1} & : & \check{Y}_n & \to & \check{Y}_{n-1}, & 1 \leq i \leq n, \\ d_0 := \sigma_{n-1}\tau_n & : & \check{Y}_n & \to & \check{Y}_{n-1}, & \\ s_i := \delta_i & : & \check{Y}_n & \to & \check{Y}_{n+1}, & 0 \leq i \leq n-1, \\ t_n := \tau_n^{-1} & : & \check{Y}_n & \to & \check{Y}_n. & \end{array}$$

Note, however, that there are other possible formulae for an isomorphism $\Delta C \to \Delta^{\mathrm{op}}C$ since $\Delta C$ has nontrivial automorphisms (see e.g. [Lo1, 6.1.14]). One easily proves [Co2]:

**1.1.18 Lemma** *Let $X$ be a para-cyclic object and $Y$ a para-cocyclic object, as above.*

($i$) *The quadruple $\hat{X} = (\hat{X}^\bullet, \delta_\bullet, \sigma_\bullet, \tau_\bullet)$ is a para-cocyclic object and is cocyclic if $X$ is cyclic.*

($ii$) *Analogously, the quadruple $\check{Y} = (\check{Y}_\bullet, d_\bullet, s_\bullet, t_\bullet)$ carries the structure of a para-cyclic object and is cyclic if $Y$ carries a cocyclic structure.*

The following proposition (see e.g. [KhR4]) reveals that the respective Hochschild (co)homology groups of the (co)cyclic duals of Examples 1.1.5($i$) and 1.1.14 turn out not to cause enormous excitement.

**1.1.19 Proposition** *Let $U$ be a unital algebra and let $C$ be a (counital) coalgebra, both over some field $k$.*

($i$) *The Hochschild cohomology of the cocyclic module $\hat{U}^\natural$ is trivial in positive dimensions.*

($ii$) *In the same fashion, the Hochschild homology groups of the cyclic module $\check{C}_\natural$ are trivial in positive dimensions.*

PROOF: The proof of both statements relies on giving a contracting homotopy. If $\phi : U \to k$ is a linear functional with the property $\phi(1_U) = 1_k$, defining

$$\sigma : U^{\otimes_k n+1} \to U^{\otimes_k n}, \quad u_0 \otimes_k \cdots \otimes_k u_n \mapsto \phi(u_0) u_1 \otimes_k \cdots \otimes_k u_n$$

and remembering that $\hat{U}_n := U^{\otimes n+1}$ in degree $n \geq 0$, one can verify that this defines a contracting homotopy for the Hochschild complex of $\hat{U}^\natural$. Likewise, for an element $\tau \in C$ with $\epsilon(\tau) = 1_k$, for example a *grouplike* element, the map

$$s : C^{\otimes_k n+1} \to C^{\otimes_k n+2}, \quad c_0 \otimes_k \cdots \otimes_k c_n \mapsto \tau \otimes_k c_0 \otimes_k \cdots \otimes_k c_n$$

defines a contracting homotopy for the Hochschild complex of $\check{C}_\natural$. □

This homological triviality generally fails; confer Chapter 6 for meaningful results for the homology of a cyclic dual.

## 1.2 $A$-rings and $A$-corings

This section presents the generalisations from $k$-algebras and $k$-coalgebras to algebras and coalgebras over not necessarily commutative ground rings; we also discuss their associated cyclic objects. The content of this section will be needed throughout all subsequent chapters.

**1.2.1 *A*-rings** An *A-ring* (cf. e.g. [BrzWi, B3, Str]) is a monoid in the monoidal category $(A^{\mathrm{e}}\text{-}\mathbf{Mod}, \otimes_A, A)$ of $(A, A)$-bimodules over a not necessarily commutative $k$-algebra $A$. We refer to $A$ as the *base* ring (or base algebra) whereas $k$ will still be called the *ground* ring. Let $A^{\mathrm{e}} := A \otimes_k A^{\mathrm{op}}$ be the enveloping algebra of $A$.

An $A$-ring $U$ is therefore a triple $(U, m_U, \eta)$ where $U \in A^{\mathrm{e}}\text{-}\mathbf{Mod}$ and $m_U : U \otimes_A U \to U, u \otimes v \mapsto uv$ as well as $\eta : A \to U$ are $(A, A)$-bimodule maps such that

$$\begin{aligned} m_U(m_U \otimes \mathrm{id}_U) &= m_U(\mathrm{id}_U \otimes m_U) && \text{(associativity)},\\ m_U(\eta \otimes \mathrm{id}_U) &= m_U(\mathrm{id}_U \otimes \eta) && \text{(unitality)}. \end{aligned}$$

We refer to $U$ as the *total* ring. Moreover, we will make frequent use of the well-known fact that $A$-rings $U$ correspond bijectively to $k$-algebra homomorphisms

$$\eta : A \to U, \tag{1.2.1}$$

see e.g. [B3] for a proof. With this characterisation, we may express the $(A, A)$-bimodule structure by $aub := \eta(a)u\eta(b)$ for $a, b \in A$ and $u \in U$, hence for $a \in A,\ u, v \in U$ one has

$$a(uv) = (au)v, \quad u(va) = (uv)a, \quad (ua)v = u(av), \tag{1.2.2}$$

A *morphism* of $A$-rings $f : U \to V$ is an $(A, A)$-bimodule morphism satisfying $\mu(f \otimes f) = f\mu$ as well as the property $f\eta = \eta$, which we shall baptise unitality again. Let $A$-$\mathbf{Ring}$ denote the category of $A$-rings and (unital) $A$-ring morphisms. If $A = k$ coincides with the commutative ground ring, $k$ being mapped into the centre of $U$ by means of $\eta$, one recovers the conventional notion of a $k$-algebra.

**1.2.2 *A*-corings** Dual to the notion of an $A$-ring is the concept of an $A$-coring: this is a comonoid in the monoidal category $(A^{\mathrm{e}}\text{-}\mathbf{Mod}, \otimes_A, A)$ of $(A, A)$-bimodules for a $k$-algebra $A$. Explicitly, an $A$-coring $C$ is a triple $(C, \Delta, \epsilon)$, where $C$ is an $(A, A)$-bimodule (with left and right actions $L_A$ and $R_A$) and $\Delta : C \to C \otimes_A C$, $\epsilon : C \to A$ are $(A, A)$-bimodule maps (called *coproduct* and *counit*) such that

$$\begin{aligned} (\Delta \otimes \mathrm{id}_C)\Delta &= (\mathrm{id}_C \otimes \Delta)\Delta && \text{(coassociativity)},\\ L_A(\epsilon \otimes_C \mathrm{id}_C)\Delta &= R_A(\mathrm{id}_C \otimes \epsilon)\Delta = \mathrm{id}_C && \text{(counitality)}. \end{aligned}$$

The notion of a *cocommutative* $A$-coring only makes sense if $A$ is commutative and $L_A$ and $R_A$ coincide. It is then defined by the condition $\sigma_{C,C}\Delta = \Delta$ where $\sigma_{C,C}(c \otimes_A c') = c' \otimes_A c$ is the tensor flip. An *A-coring morphism* is an $(A, A)$-bimodule morphism $f : C \to D$ with $\Delta f = (f \otimes_A f)\Delta$ and $\epsilon f = \epsilon$, which we call counitality again. Denote by $A$-$\mathbf{Coring}$ the category of (counital coassociative) $A$-corings and (counital) $A$-coring morphisms. If $A = k$ coincides with the commutative ground ring, one recovers the conventional notion of $k$-coalgebra. See [BrzWi] for more details on $A$-corings.

**1.2.3 Cyclic Tensor Products** In order to give a well-defined meaning to the cyclic theories for $A$-(co)rings, we first have to discuss the notion of cyclic tensor product of [Q2]. The tensor product $M_1 \otimes_A \cdots \otimes_A M_n$ of a sequence of $(A, A)$-bimodules $M_i,\ i = 1, \ldots, n$, is again an $(A, A)$-bimodule and can be therefore equipped with the structure of a (say, right) $A^{\mathrm{e}}$-module by means of

$$(m_1 \otimes_A \cdots \otimes_A m_n) \cdot (a \otimes_k b) := bm_1 \otimes_A \cdots \otimes_A m_n a, \qquad \forall\, a, b \in A.$$

On the other hand, $A$ itself carries a left $A^{\mathrm{e}}$-action by $(a \otimes_k b) \cdot c := acb$ where $a, b, c \in A$, and one forms the *cyclic tensor product*

$$M_1 \otimes_A \cdots \otimes_A M_n \otimes_A := M_1 \otimes_A \cdots \otimes_A M_n \otimes_{A^{\mathrm{e}}} A.$$

Analogously to usual tensor products, this $k$-module is universal for multilinear functions $f : M_1 \times \cdots \times M_n \to V$ into any space $V$ satisfying $f(\ldots, m_i a, m_{i+1}, \ldots) = f(\ldots, m_i, am_{i+1}, \ldots)$ as well as $f(m_1, \ldots, m_n a) = f(am_1, \ldots, m_n)$ for $a \in A,\ m_i \in M$. In particular, for the lowest degree one has

$$M \otimes_{A^{\mathrm{e}}} A = M/[A, M],$$

where $[A, M] = \{am - ma | a \in A,\ m \in M\}$. Of course, one can also consider $A \otimes_{A^e} M_1 \otimes_A \cdots \otimes_A M_n$, i.e., consider the left $A^e$-action on $M_1 \times \cdots \times M_n$ and the right $A^e$-action on $A$: this leads to the same universal $k$-module. For notational reasons we always put the abbreviation $- \otimes_A$ on the right, even if $A \otimes_{A^e} -$ stands on the left. If all $M_i = M$, $i = 1, \ldots, n$, are identical, its $n$-fold cyclic tensor product $M \otimes_A \cdots \otimes_A M \otimes_A$ carries two natural actions of the cyclic group $\mathbb{Z}/n\mathbb{Z}$, the generators of which are either given by

$$t(m_1 \otimes_A \cdots \otimes_A m_n \otimes_A) = m_n \otimes_A m_1 \otimes_A \cdots \otimes_A m_{n-1} \otimes_A,$$

or by the operators $\lambda = (-1)^{n-1} t$.

**1.2.4 $A$-rings as Cyclic Objects** There is a functor $\natural : A$-**Ring** $\to M^{\Delta C_r}$ from the category of $A$-rings to the category of cyclic modules, defined as follows. For an $A$-ring $U$, the right $A^e$-action on the $(n+1)$-th tensor power $U^{\otimes_A n+1}$ is, as before, given by

$$(u_0 \otimes_A \cdots \otimes_A u_n) \cdot (a \otimes_k b) := b u_0 \otimes_A \cdots \otimes_A u_n a.$$

Let

$$B^A_n U := U^{\otimes_A n+1} \otimes_{A^e} A = U^{\otimes_A n+1} \otimes_A .$$

For $\phi \in \mathrm{End}_{(A,A)}(U)$, we associate in degree $n$ the face, degeneracy and cyclic operators

$$\begin{aligned} d_i(u_0 \otimes_A \cdots \otimes_A u_n \otimes_A) &= \begin{cases} u_0 \otimes_A \cdots \otimes_A u_i u_{i+1} \otimes_A \cdots \otimes_A u_n \otimes_A & \text{if } 0 \le i \le n-1, \\ \phi(u_n) u_0 \otimes_A u_1 \otimes_A \cdots \otimes_A u_{n-1} \otimes_A & \text{if } i = n, \end{cases} \\ s_i(u_0 \otimes_A \cdots \otimes_A u_n \otimes_A) &= u_0 \otimes_A \cdots \otimes_A u_i \otimes_A 1_U \otimes_A u_{i+1} \otimes_A \cdots \otimes_A \ \text{if } 0 \le i \le n, \\ t_n(u_0 \otimes_A \cdots \otimes_A u_n \otimes_A) &= \phi(u_n) \otimes_A u_0 \otimes_A u_1 \otimes_A \cdots \otimes_A u_{n-1} \otimes_A \end{aligned} \tag{1.2.3}$$

to the space $U^{\natural,\phi}_A := \{B^A_n U\}_{n \ge 0}$. Then $U^{\natural,\phi}_A$ is $r$-cyclic if the order of $\phi$ is less than infinity, and cyclic if and only if $\phi = \mathrm{id}$, in which case we will write $U^{\natural}_A$. The notion of cyclic tensor products is required here to make these operators well-defined and allows one to drop the condition that $\eta(A) \subset Z(U)$ lies in the centre of $U$. The condition (1.2.2) is required to make the face operators well-defined. A unital $A$-ring morphism $f : U \to V$ induces a morphism of cyclic modules $f^\natural : U^{\natural,\phi}_A \to V^{\natural,f\circ\phi}_A$ by $f^\natural(u_0 \otimes_A \cdots \otimes_A u_n \otimes_A) := f(u_0) \otimes_A \cdots \otimes_A f(a_n) \otimes_A$. In case that $A = k$, the commutative ground ring (as a $k$-module over itself), the cyclic structure given here specialises to the conventional one for $k$-algebras from e.g. [FeTs] or [Nis] as presented in Example 1.1.5(*iii*).

**1.2.5 $A$-corings as Cocyclic Objects** Similarly, there is a functor $\natural : A$-**Coring** $\to N^{\Delta C_r}$ from the category of $A$-corings to the category of cocyclic modules, defined as follows: Let $(C, \Delta, \epsilon)$ be an $A$-coring, and use Sweedler's [Sw1] shorthand notation $\Delta c =: c_{(1)} \otimes_A c_{(2)}$ for the coproduct. Recall that the left $A^e$-action on the $(n+1)$-th tensor power $C^{\otimes_A n+1}$ is given by

$$(a \otimes_k b) \cdot (c^0 \otimes_A \cdots \otimes_A c^n) := a c^0 \otimes_A \cdots \otimes_A c^n b.$$

On the other hand, a right $A^e$-action on $A^{\mathrm{op}}$ is given by $a \cdot (b \otimes_k c) := cab$ and we consider the cyclic tensor products

$$C \otimes_A \cdots \otimes_A C \otimes_A := A^{\mathrm{op}} \otimes_{A^e} C \otimes_A \cdots \otimes_A C.$$

For $\psi \in \mathrm{End}_{(A,A)}(C)$, define the cocyclic module $C^{\psi}_{\natural} := \{B^n_A C\}_{n \ge 0}$ where $B^n_A C := A^{\mathrm{op}} \otimes_{A^e} C^{\otimes_A n+1} =: C^{\otimes_A n+1} \otimes_A$ in degree $n$ with coface, codegeneracies and cocyclic operators

$$\begin{aligned} \delta_i(c^0 \otimes_A \cdots \otimes_A c^n \otimes_A) &= \begin{cases} c^0 \otimes_A \cdots \otimes_A \Delta c^i \otimes_A \cdots \otimes_A c^n \otimes_A & \text{if } 0 \le i \le n \\ c^0_{(2)} \otimes_A c^1 \otimes_A \cdots \otimes_A \psi(c^0_{(1)}) \otimes_A & \text{if } i = n+1, \end{cases} \\ \sigma_i(c^0 \otimes_A \cdots \otimes_A c^n \otimes_A) &= c^0 \otimes_A \cdots \otimes_A c^i \epsilon(c^{i+1}) \otimes_A c^{i+2} \otimes_A \cdots \otimes_A \ \text{if } 0 \le i \le n-1, \\ \tau_n(c^0 \otimes_A \cdots \otimes_A c^n \otimes_A) &= c^1 \otimes_A c^2 \otimes_A \cdots \otimes_A \psi(c^0) \otimes_A . \end{aligned} \tag{1.2.4}$$

Again, $C^A_{\natural,\psi}$ is $r$-cocyclic if the order $r$ of $\psi$ is less than infinity and cocyclic if and only if $\psi = \mathrm{id}$ in which case we will only write $C^A_\natural$. Similarly as before, a counital coalgebra morphism $f : C \to D$ induces a morphism of cocyclic modules $f_\natural : C^A_{\natural,\psi} \to D^A_{\natural,f\circ\psi}$ by $f_\natural(c^0 \otimes_A \cdots \otimes_A c^n \otimes_A) = f(c^0) \otimes_A \cdots \otimes_A f(c^n) \otimes_A$ in degree $n$. In case $A = k$ coincides with the commutative ground ring (as $k$-module over itself), the cocyclic structure given here specialises with the conventional one for $k$-coalgebras as given e.g. in [FeTs].

## 1.3 Hopf Algebras and Their Cyclic Cohomology

Hopf algebras can be seen as a noncommutative analogue of Lie groups. More precisely, symmetries in noncommutative geometry are determined by the action or coaction of some Hopf algebra on an algebra or coalgebra.

A large part of this thesis is devoted to generalisations of Hopf algebra theories towards Hopf algebroids We therefore briefly give an overview on Hopf algebras and their Hopf-cyclic cohomology.

### 1.3.1 Bialgebras and Hopf Algebras

Let $U$ be a $k$-module equipped simultaneously with a $k$-algebra structure $(U, m_U, \eta)$ (cf. §1.2.1) and a $k$-coalgebra structure (cf. §1.2.2). Equip $U \otimes_k U$ with the induced structure of a tensor product of algebras (by factorwise multiplication and unit $1_U \otimes_k 1_U$) and also with the induced structure of a tensor product of coalgebras (with comultiplication $(\mathrm{id}_U \otimes_k \sigma_{U,U} \otimes_k \mathrm{id}_U)(\Delta \otimes \Delta)$ and counit $\epsilon \otimes \epsilon$). The proof of the following classical lemma can be e.g. found in [Kas3, Str]

**1.3.1 Lemma** *The following statements are equivalent*

(*i*) *The maps $m_U$ and $\eta$ are morphisms of $k$-coalgebras.*

(*ii*) *The maps $\Delta$ and $\epsilon$ are morphisms of $k$-algebras.*

This leads to the following definition.

**1.3.2 Definition** (*i*) A *$k$-bialgebra* is a quintuple $(U, m_U, \eta, \Delta, \epsilon)$, where $(U, m_U, \eta)$ is a $k$-algebra and $(U, \Delta, \epsilon)$ a $k$-coalgebra verifying the equivalent conditions of the preceding lemma. A *morphism of $k$-bialgebras* is a morphism for both the underlying $k$-algebra and $k$-coalgebra.

(*ii*) Let $(H, m_H, \eta, \Delta, \epsilon)$ be a $k$-bialgebra. An endomorphism $S : H \to H$ is called an *antipode* for $H$ if

$$m_H(S \otimes \mathrm{id}_H)\Delta = m_H(\mathrm{id}_H \otimes S)\Delta = \eta\epsilon.$$

A *Hopf algebra* is a $k$-bialgebra with an antipode. A *morphism of Hopf algebras* is a morphism between the underlying $k$-bialgebras commuting with the respective antipodes.

It can be shown that the antipode is unique (if it exists) and is both an anti-algebra morphism and anti-coalgebra morphism.

**1.3.3 Examples** Commutative or cocommutative Hopf algebras arise naturally from groups and Lie algebras.

(*i*) Let $\Gamma$ be a discrete (not necessarily finite) group with group algebra $\mathbb{C}\Gamma$. Extending the maps

$$\Delta g = g \otimes_{\mathbb{C}} g, \quad \epsilon g = 1_{\mathbb{C}}, \quad Sg = g^{-1}, \qquad \forall g \in \Gamma,$$

linearly to all of $\mathbb{C}\Gamma$, one obtains a cocommutative Hopf algebra structure on $\mathbb{C}\Gamma$. This Hopf algebra is commutative if and only if $\Gamma$ is commutative.

(*ii*) The universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a cocommutative Hopf algebra. The structure maps have the defining properties

$$\Delta X = X \otimes_{\mathbb{C}} 1 + 1 \otimes_{\mathbb{C}} X, \quad \epsilon X = 0, \quad SX = -X \qquad \forall X \in \mathfrak{g}.$$

$U\mathfrak{g}$ is a commutative Hopf algebra if and only if $\mathfrak{g}$ is abelian, in which case $U\mathfrak{g}$ coincides with the symmetric algebra $S\mathfrak{g}$ of $\mathfrak{g}$.

See e.g. [Sw1, Kas3, Str, ChPr] for extensive material on Hopf algebras.

### 1.3.2 Hopf-Cyclic Cohomology

In [CoMos2] a new cohomology theory of Hopf algebras was launched, nowadays referred to as Hopf-cyclic (co)homology. One may consider this as the correct noncommutative analogue of both group and Lie algebra homology, see below. The original motivation was to obtain a noncommutative characteristic map

$$\chi_{\mathrm{tr}} : HC^\bullet_{\delta,\sigma}(H) \to HC^\bullet(A) \tag{1.3.1}$$

from a certain cyclic cohomology group to the standard cyclic cohomology group of an algebra $A$ needed for the study of transverse elliptic operators, induced on the cochain level by a map

$$\chi_{\mathrm{tr}} : H^{\otimes_k n} \to C^n A, \; \chi_{\mathrm{tr}}(h_1, \ldots, h_n)(a_0, \ldots, a_n) = \mathrm{tr}\,(a_0 h_1(a_1), \ldots, h_n(a_n)) \tag{1.3.2}$$

in degree $n$. Here, $H$ is a Hopf algebra (with structure maps as before) acting on a unital $k$-algebra $A$ by the assignment $(h, a) \mapsto h(a)$, satisfying in particular the Leibniz rule

$$h(ab) = h_{(1)}(a)h_{(2)}(b), \qquad a, b \in A, \; h \in H.$$

Such a Hopf algebra action may be seen as a notion of 'quantum' symmetry on a noncommutative space, see e.g. [CoMos1]. The data $(\delta, \sigma)$ that appears in (1.3.1) form an algebraic analogue of the modular function of a locally compact group:

**1.3.4 Definition** A character $\delta \in H^*$ (i.e. a ring homomorphism $\delta : H \to k$) together with a grouplike element $\sigma \in H$ (i.e. an element that fulfills $\Delta\sigma = \sigma \otimes_k \sigma$, $\epsilon\sigma = 1_k$) related to each other by the condition $\delta\sigma = 1_k$ is called a *modular pair*.

Modular pairs turns out to be self-dual in some sense when passing to the dual Hopf algebra, see [CoMos4]. Finally, the linear map $\mathrm{tr} : A \to k$ that appears in (1.3.2) is a *$\sigma$-invariant $\delta$-trace*, i.e. a map fulfilling

$$\mathrm{tr}\,(h(a)) = \delta(h)\mathrm{tr}\,(a), \quad \mathrm{tr}\,(ab) = \mathrm{tr}\,(b\sigma(a)), \qquad a, b \in A.$$

Maps of the kind (1.3.2), originally introduced in [Co1], are typical ingredients in cyclic cohomology.

**1.3.5 Twisted Antipodes** The character $\delta$ gives rise to a *twisted antipode* $\tilde{S} : H \to H$, defined by [CoMos4]

$$\tilde{S}h := \eta(\delta h_{(1)})Sh_{(2)}, \qquad h \in H, \tag{1.3.3}$$

where $S$ is the antipode of the Hopf algebra $H$. The twisted antipode $\tilde{S}$ is an anti-algebra morphism and a twisted anti-coalgebra morphism, that is

$$\Delta\tilde{S}h = Sh_{(2)} \otimes_k \tilde{S}h_{(1)}, \qquad h \in H. \tag{1.3.4}$$

Furthermore, it satisfies

$$\epsilon\tilde{S} = \delta, \quad \delta\tilde{S} = \epsilon. \tag{1.3.5}$$

If $A$ is unital, one also proves that $\delta$-invariance of $\mathrm{tr}$ is equivalent to *partial integration*

$$\mathrm{tr}\,(h(a)b) = \mathrm{tr}\,(a(\tilde{S}h)(b)), \qquad a, b \in A, \; h \in H,$$

and $(\delta, \sigma)$ is called a *modular pair in involution* if

$$\tilde{S}^2 h = \sigma h \sigma^{-1}, \qquad h \in H, \tag{1.3.6}$$

where $\sigma^{-1} := S\sigma$.

**1.3.6 Example** For any commutative or cocommutative Hopf algebra we have $S^2 = \mathrm{id}_H$ (cf. e.g. [Kas3]). Hence $(\epsilon, 1_H)$ is a modular pair in involution. See, however, [CoMos2] for a less trivial example.

**1.3.7 The Hopf-Cocyclic Module** In [CoMos2] a cocyclic module is attached to the triple $(H, \delta, \sigma)$ as follows. Set $H^{\delta,\sigma}_\natural := \{H^{\otimes n}\}_{n \geq 0}$, i.e. $H^{\otimes n}$ in degree $n$ and $H^{\otimes 0} := k$ in degree zero. The cosimplicial operators are then given as

$$\begin{aligned}
\delta_i(h^1 \otimes_k \cdots \otimes_k h^n) &= \begin{cases} 1 \otimes_k h^1 \otimes_k \cdots \otimes_k h^n & \text{if } i = 0, \\ h^1 \otimes_k \cdots \otimes_k \Delta_\ell h^i \otimes_k \cdots \otimes_k h^n & \text{if } 1 \leq i \leq n, \\ h^1 \otimes_k \cdots \otimes_k h^n \otimes_k \sigma & \text{if } i = n+1, \end{cases} \\
\sigma_i(h^1 \otimes_k \cdots \otimes_A h^k) &= h^1 \otimes_k \cdots \otimes_k \epsilon h^{i+1} \otimes_k \cdots \otimes_k h^n \qquad \text{if } 0 \leq i \leq n-1,
\end{aligned}$$

and in degree zero

$$\delta_j 1_k \quad = \begin{cases} 1_H & \text{if } j = 0, \\ \sigma & \text{if } j = 1. \end{cases}$$

Finally, the cocyclic operator is defined as

$$\begin{aligned} \tau_n(h^1 \otimes_k \cdots \otimes_k h^n) &= (\Delta_\ell^{n-1} \tilde{S} h^1)(h^2 \otimes_k \cdots \otimes_k h^n \otimes_k \sigma) \\ &= S(h^1_{(n)})h^2 \otimes_k \cdots \otimes_k S(h^1_{(2)})h^n \otimes_k \tilde{S}(h^1_{(1)})\sigma. \end{aligned}$$

These operators were originally obtained by pulling back the cocyclic structure of $C^\bullet A$ and were dictated by requiring the characteristic map (1.3.2) to be a morphism of cocyclic modules. One can now show [CoMos4] that $H_\natural^{\delta,\sigma}$ is cocyclic if and only if (1.3.6) is fulfilled. In such a case we speak of the Hopf-cyclic cohomology of the triple $(H, \delta, \sigma)$, denoted $HC^\bullet_{\delta,\sigma}(H)$ and $HP^\bullet_{\delta,\sigma}(H)$, respectively.

**1.3.8 Examples** Let us state two results corresponding to the Examples 1.3.3; cf. [CoMos2, Cr3] for the proofs.

(*i*) For the group algebra $\mathbb{C}\Gamma$ of a discrete not necessarily finite group $\Gamma$ one obtains

$$HP^0_{\epsilon,1}(\mathbb{C}\Gamma) \simeq \mathbb{C}, \qquad HP^1_{\epsilon,1}(\mathbb{C}\Gamma) \simeq 0.$$

(*ii*) For a Lie algebra $\mathfrak{g}$, a character $\delta : \mathfrak{g} \to \mathbb{C}$ is a linear map with $\delta|_{[\mathfrak{g},\mathfrak{g}]} = 0$, and denote its unique extension to an algebra morphism $U\mathfrak{g} \to \mathbb{C}$ by the same symbol $\delta$. We write $\mathbb{C}_\delta$ for $\mathbb{C}$, seen as a $\mathfrak{g}$-module via $\delta$. For elements $X \in \mathfrak{g}$ one obtains $\tilde{S}X = -X + \delta X$, hence $\sigma = 1$: this is the *unimodular* case. One then computes [CoMos2, Cr3]

$$HP^\bullet_{\delta,1}(U\mathfrak{g}) \simeq H_{\mathrm{odd}}(\mathfrak{g}, \mathbb{C}_\delta) \oplus H_{\mathrm{even}}(\mathfrak{g}, \mathbb{C}_\delta),$$

where the right hand side denotes the Chevalley-Eilenberg homology of Lie algebras with values in the $\mathfrak{g}$-module $\mathbb{C}_\delta$.

Example 1.3.8(*ii*) gives a hint why one should consider Hopf-cyclic cohomology as a noncommutative analogue of Lie algebra homology. We will make a similar statement in the generalised context of Hopf algebroids and Lie algebroids (Lie-Rinehart algebras, respectively) as a consequence of Theorem 5.5.7(*ii*) below.

See e.g. [CoMos4, CoMos2, Cr3, HKhRSo1, HKhRSo2] for more material on Hopf-cyclic (co)homology of Hopf algebras. There is also a Hopf-cyclic homology theory dual to the one above in the sense of cyclic duality. We will not give the details here (cf. [KhR1, KhR4, KhR2, Tai]), but will immediately extend this theory to the realm of Hopf algebroids in Chapter 6.

## 1.4 Lie-Rinehart Algebras

In this section we collect some material on Lie-Rinehart algebras. Lie-Rinehart algebras can be thought of as algebraic versions of Lie algebroids. As for Lie algebroids (cf. [NisWeiX]), there is an associated universal enveloping object associated to it. In Section 4.2 we will discuss the possibilities how this object can be considered an example of a 'generalised Hopf algebra'.

As before, let $k$ be a commutative unital ring (containing $\mathbb{Q}$) and $A$ a *commutative* $k$-algebra. For the subsequent definition unitality of $A$ is not strictly required, but for convenience we will assume this as well.

**1.4.1 Definition** [Rin] Let $L$ be a $k$-Lie algebra $L$, $a \otimes_k X \mapsto aX$ for $a \in A$, $X \in L$ a left $A$-module structure on $L$, and $\omega : L \to \mathrm{Der}_k A$, $X \mapsto \{a \mapsto X(a)\}$ a morphism of $k$-Lie algebras. The pair $(A, L)$ is called a *Lie-Rinehart algebra* with *anchor* $\omega$, provided

$$(aX)(b) = a(X(b)) \qquad X \in L,\ a, b \in A, \tag{1.4.1}$$

$$[X, aY] = a[X, Y] + X(a)Y \quad X, Y \in L,\ a \in A. \tag{1.4.2}$$

A *morphism* $(A, L) \to (A', L')$ *of Lie-Rinehart algebras* is a pair of maps $(\phi : A \to A', \psi : L \to L')$ where $\phi$ is a morphism of $k$-algebras and $\psi$ a morphism of $k$-Lie algebras with the properties $\psi(aX) = \phi(a)\psi(X)$ and $\phi(X(a)) = \psi(X)(\phi(a))$.

**1.4.2 Examples** Two immediate examples are given by the following:

(*i*) The pair $(A, \mathrm{Der}_k A)$ of a *commutative* algebra $A$ and its $k$-derivations with obvious anchor yields a Lie-Rinehart algebra.

(*ii*) A *Lie algebroid* is a vector bundle $E \to M$ over a smooth manifold $M$, together with a map $\omega : E \to TM$ of vector bundles and a (real) Lie algebra structure $[.,.]$ on the vector space $\Gamma E$ of sections of $E$, such that the induced map $\Gamma(\omega) : \Gamma E \to \mathcal{X}(M)$ is a Lie algebra homomorphism, and for all $X, Y \in \Gamma E$ and any $f \in \mathcal{C}^\infty(M)$, one has $[X, fY] = f[X,Y] + \Gamma(\omega)(X)(f)Y$. Then the pair $(\mathcal{C}^\infty(M), \Gamma E)$ is obviously a Lie-Rinehart algebra; see e.g. [CanWei] and [Mac] for more details on Lie algebroids.

**1.4.3 Definition** [Rin, Hue1] The *universal object* $(VL, i_L, i_A)$ of a Lie-Rinehart algebra $(A, L)$ is a $k$-algebra $VL$ with two morphisms $i_A : A \to VL$ and $i_L : L \to VL$ of $k$-algebras and $k$-Lie algebras, respectively, subject to the conditions

$$i_A(a)i_L(X) = i_L(aX), \qquad i_L(X)i_A(a) - i_A(a)i_L(X) = i_A(X(a)), \quad a \in A,\ X \in L,$$

universal in the following sense: for any other triple $(W, \phi_L, \phi_A)$ of a $k$-algebra $W$ and two morphisms $\phi_A : A \to W$, $\phi_L : L \to W_{\mathrm{L}}$ of $k$-algebras and $k$-Lie algebras, respectively (where $W_{\mathrm{L}}$ is the commutator Lie algebra) that obey

$$\phi_A(a)\phi_L(X) = \phi_L(aX), \qquad \phi_L(X)\phi_A(a) - \phi_A(a)\phi_L(X) = \phi_A(X(a)), \tag{1.4.3}$$

there is a unique morphism

$$\Phi : VL \to W \tag{1.4.4}$$

of $k$-algebras such that $\Phi i_A = \phi_A$ and $\Phi i_L = \phi_L$.

Note that in case of a trivial anchor one obtains the universal enveloping algebra of $L$ as an $A$-Lie algebra.

**1.4.4 Remark** An alternative construction [Hue1] describes $VL$ as a *Massey-Peterson* algebra [MasPe]: this coincides with what we will call a 'smash ring' in Lemma 3.3.2, see below. Let $UL$ be the universal enveloping algebra of the $k$-Lie algebra $L$ with coproduct $\Delta_{UL} u = u_{(1)} \otimes_k u_{(2)}$. Then clearly $A$ is a left $UL$-module ring (under the canonical action (2.3.3) and considering (2.3.4)), and one can set $VL = A \rtimes_A UL$ with product

$$(a \rtimes_A u)(a' \rtimes_A v') = a u_{(1)}(a') \rtimes_A u_{(2)} u', \qquad a, a' \in A,\ u, u' \in UL.$$

We may now obviously set $i_L : L \to VL,\ X \mapsto 1_A \rtimes_A X$ and $i_A : A \to VL,\ a \mapsto a \rtimes_A 1_{UL}$; from this description it is obvious that $i_A$ is an algebra morphism and $i_L$ is a morphism of Lie algebras.

**1.4.5 The Poincaré-Birkhoff-Witt Theorem for *$VL$*** The algebra $VL$ carries a natural filtration

$$V_0 L \subset V_1 L \subset V_2 L \subset \dots,$$

where $V_{-1}L := 0$, $V_0 L := A$ and for $p \geq 0$, $V_p L$ is the left $A$-submodule of $VL$ generated by $i_L(L)^p$, i.e. products of the image of $L$ in $VL$ of length at most $p$. Since $au - ua \in V_{p-1}L$ for any $a \in A$ and $u \in V_p L$, left and right $A$-module structures coincide on $V_p L / V_{p-1} L$. It follows that the associated graded object $\mathrm{gr}\, VL$ inherits the structure of a graded commutative $A$-algebra. Denote the symmetric $A$-algebra by $S_A L$ and define $S_A^p L$ as the $p^{\mathrm{th}}$ symmetric power of $L$. If $L$ is projective over $A$, the canonical $A$-linear epimorphism $S_A L \xrightarrow{\simeq} \mathrm{gr}\, VL$ is an isomorphism of $A$-algebras (cf. [Rin, Thm. 3.1] see also [NisWeiX] for a more differential geometric version). Hence $i_L$ and $i_A$ are injective; we may therefore identify elements $a \in A$ and $X \in L$ with their images in $VL$. As in the classical Lie algebra case, the symmetrisation

$$\pi : S_A^p V \to V_p L, \quad v_1 \cdots v_p \mapsto \frac{1}{p!} \sum_{\sigma \in P(p)} v_{\sigma(1)} \otimes_A \cdots \otimes_A v_{\sigma(p)}, \tag{1.4.5}$$

(where $v_i \in L$ or $v_i \in A$) induces an isomorphism of (left) $A$-modules $S_A L \to VL$. As an algebra with multiplication $m_{VL}$, we may also describe $VL$ as generated by elements $X \in L$ and $a \in A$ respecting the relations $m_{VL}(a, X) = aX$ and $[X, a] := m_{VL}(X, a) - m_{VL}(a, X) = X(a)$.

## 1.5 Groupoids

This section contains some basic material on groupoids.

**1.5.1 (Etale) Groupoids** A *groupoid* $G$ is a small category in which each arrow is invertible. Somewhat more explicitly, a groupoid consists of a space of objects $G_0$, a space of arrows $G_1$ (often denoted $G$ as well) and five structure maps relating the two:

(*i*) source and target maps $s, t : G_1 \to G_0$, assigning to each arrow $g$ its source $s(g)$ and target $t(g)$; we say that $g$ 'goes from $s(g)$ to $t(g)$';

(*ii*) a partially defined composition of arrows, that is, only for those arrows $g, h$ for which source and target match, $s(g) = t(h)$, i.e. a map $m : G_2 := G_1{}^s\times^t_{G_0} G_1 \to G_1,\ (g,h) \mapsto gh$ that is associative whenever defined, producing the composite arrow going from $s(gh) = s(h)$ to $t(gh) = t(g)$;

(*iii*) a unit map $1 : G_0 \to G_1,\ x \mapsto 1_x$ that has the property $1_{t(g)}g = g1_{s(g)}$;

(*iv*) an inversion $\mathrm{inv} : G_1 \to G_1,\ g \mapsto g^{-1}$ that produces the inverse arrow going from $s(g^{-1}) = t(g)$ to $t(g^{-1}) = s(g)$, fulfilling $g^{-1}g = 1_{s(g)},\ gg^{-1} = 1_{t(g)}$.

These maps can be assembled into a diagram

$$G_2 \xrightarrow{m} G_1 \xrightarrow{\mathrm{inv}} G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0 \xrightarrow{1} G_1 \ .$$

An arrow may be denoted $x \xleftarrow{g} y$ to indicate that $y = s(g)$ and $x = t(g)$, but usually we abbreviate this to $\cdot \xleftarrow{g} \cdot$.

A *topological groupoid* is a groupoid in which $G_1$, $G_0$ are topological spaces and all the structure maps are continuous. Mutatis mutandis one defines *smooth* groupoids, where in addition $s$ and $t$ are required to be surjective submersions in order to guarantee that $G_2 = G_1{}^s\times^t_{G_0} G_1$ remains a manifold. A topological (or smooth) groupoid is called *étale* if the source map is a local homeomorphism (or local diffeomorphism). This implies that all structure maps are local homeomorphisms (or local diffeomorphisms, respectively). In the smooth case, this equivalently amounts to saying that $\dim G_1 = \dim G_0$. In particular, an étale groupoid has zero-dimensional source and target fibres, hence they are discrete. See e.g. [CanWei, L1, Mac, MoeMrč2] for more material on groupoids.

**1.5.2 Local Bisections and Germs** A *local bisection* of a Lie groupoid $G$ is a local section $\sigma : U \to G$ of $s : G \to G_0$ defined on an open subset $U \subset G_0$ such that $t\sigma$ is an open embedding. If $G$ is étale, any arrow $g$ induces a germ of a homeomorphism $\sigma_g : (U, s(g)) \to (V, t(g))$ from a neighbourhood $U$ of $s(g)$ to a neighbourhood $V$ of $t(g)$: choosing $U$ small enough such that a bisection $\sigma$ exists and $t|_{\sigma U}$ is a homeomorphism into $V := t(\sigma U)$, we simply set $\sigma_g := t\sigma$. We usually do not distinguish between $\sigma_g$ and the 'actual' germ of this map at the point $s(g)$.

**1.5.3 Fibre Sum. Notation** For a space $X$ we denote the set of sheaves over $X$ by $\mathrm{Sh}(X)$ (cf. [Br]). If $\mathcal{F} \in \mathrm{Sh}(X)$, $\mathcal{E} \in \mathrm{Sh}(Y)$ are ($c$-soft [Br]) sheaves over some spaces $X, Y$, respectively, $\phi : X \to Y$ an étale map (i.e. a local homeomorphism) and $\alpha : \mathcal{F} \to \phi^{-1}\mathcal{E}$ a sheaf morphism, we often consider maps of type

$$(\alpha, \phi)_* : \Gamma_c(X, \mathcal{F}) \to \Gamma_c(Y, \mathcal{E}), \quad ((\alpha, \phi)_* u)(y) = \sum_{y=\phi(x)} \alpha_x(u(x)) \in \mathcal{E}_y, \quad x \in X, y \in Y,$$

where $\Gamma_c(-,-)$ denote the groups of compactly supported sections (cf. e.g. [Br], and [Cr2] for an extension of the functor $\Gamma_c$ to non-Hausdorff spaces). This map is abbreviated to

$$(\alpha, \phi)_* : \Gamma_c(X, \mathcal{F}) \to \Gamma_c(Y, \mathcal{E}), \quad (u \mid x) \mapsto (\alpha(u) \mid \phi(x)), \quad x \in X, u \in \mathcal{F}_x.$$

In particular, if $X, Y$ are two manifolds and $\mathcal{C}^\infty_X$, $\mathcal{C}^\infty_Y$ the sheaves of smooth functions over $X$ and $Y$, respectively, a smooth map $\phi : X \to Y$ yields a homomorphism of commutative algebras $\phi^*_x : \mathcal{C}^\infty_{Y,\phi(x)} \to \mathcal{C}^\infty_{X,x}$ on each stalk for $x \in X$. If $\phi$ is étale, i.e. a local diffeomorphism, $\phi^*_x$ is an isomorphism with inverse $\phi_{*x}$, and $\phi$ induces the linear map

$$\phi_+ : \mathcal{C}^\infty_{\mathrm{c}}(X) \to \mathcal{C}^\infty_{\mathrm{c}}(Y), \quad \phi_+(u)_y = ((\phi_*, \phi)_* u)(y) = \sum_{y=\phi(x)} \phi_{*x}(u_x), \tag{1.5.1}$$

the *sum over the $\phi$-fibres*.

# Chapter 2

# Hopf Algebroids

Roughly speaking, a Hopf algebroid is an algebra carrying simultaneously

(1) a 'left' coalgebra structure,

(2) a 'right' coalgebra structure,

(3) an 'antipode' intertwining these two structures.

We emphasise here that although the information given in (1)–(3) is partly redundant, the Hopf algebroid axioms and resulting identities are much more natural and symmetric if one distinguishes the three different structures. For instance, the right coalgebra structure can be reconstructed from (1) and (3), but the Hopf algebroid axioms written just in terms of the left coalgebra structure and the antipode would then become unnatural and complicated.

More precisely, a Hopf algebroid should be a kind of generalised bialgebra—a so-called *bialgebroid*—with a certain notion of an antipode on it. A typical generalisation of a $k$-bialgebra consists in replacing the commutative ground ring $k$ by a noncommutative ring, involving the concepts of $A$-ring and coring from §1.2.1 and §1.2.2, respectively, and a certain interaction between them. However, as said, instead of only one even two such generalised bialgebra structures (*left* and *right* bialgebroids) of a different nature will be required, two concepts that we recall below.

## 2.1 Left Bialgebroid Structures

Like $k$-bialgebras, bialgebroids are both algebras and coalgebras, but over different base rings. In other words, they are monoids and comonoids in different monoidal categories and the interplay between these is far from obvious. The correct setup presumably appeared for the first time in [Tak] under the name $\times_A$-*bialgebras*. They were rediscovered several times, apparently independently, and baptised bialgebroid in [Lu], bialgebroid with anchor in [X3, X1], and all these notions were shown to be equivalent in [BrzMi].

Recall from §1.2.1 that an $A^{\mathrm{e}}$-ring $U$ can be described by a $k$-algebra map $\eta_U = \eta : A^{\mathrm{e}} \to U$. Equivalently, one can consider its restrictions

$$s := \eta(- \otimes_k 1_A) : A \to U \quad \text{and} \quad t := \eta(1_A \otimes_k -) : A^{\mathrm{op}} \to U,$$

and call these the *source* and *target* map of the $A^{\mathrm{e}}$-ring $U$; hence an $A^{\mathrm{e}}$-ring may be equally given by such a triple $(U, s, t)$, which is also called an $(s,t)$-*ring* [Tak]. Using the left $A^{\mathrm{e}}$-module structure $(a \otimes_k \tilde{a}, u) \mapsto \eta(a \otimes_k \tilde{a})u$ on $U$, one considers

$$\begin{aligned} U \otimes_A U :=& U \otimes_k U/\mathrm{span}\{\eta(1 \otimes_k a)u \otimes_k u' - u \otimes_k \eta(a \otimes_k 1)u' \mid a \in A,\ u, u' \in U\} \\ =& U \otimes_k U/\mathrm{span}\{t(a)u \otimes_k u' - u \otimes_k s(a)u' \mid a \in A,\ u, u' \in U\}. \end{aligned} \tag{2.1.1}$$

Note that by $(a \otimes_k \tilde{a}) \cdot (u \otimes_A u') := s(a)u \otimes_A t(\tilde{a})u'$, the tensor product $U \otimes_A U$ becomes a left $A^{\mathrm{e}}$-module or $(A, A)$-bimodule in a standard way (see (A.1.1)).

**2.1.1 Definition** The $k$-submodule $U \times_A U \subset U \otimes_A U$ defined by

$$U \times_A U := \{\sum_i u_i \otimes_A u'_i \in U \otimes_A U \mid \sum_i u_i t(a) \otimes_A u'_i = \sum_i u_i \otimes_A u'_i s(a),\ \forall a \in A\} \tag{2.1.2}$$

is called the (left) *Takeuchi product* of the $A^e$-ring $U$ with itself.

One easily verifies that $U \times_A U$ is an $A^e$-ring via factorwise multiplication, with unit element $1_U \otimes_A 1_U$ and $\eta_{U\times_A U}(a \otimes \tilde{a}) = s(a) \otimes_A t(\tilde{a})$. On the other hand, there is no well-defined algebra structure on $U \otimes_A U$, not even if $A$ were commutative, since we do not assume $\eta(A^e) \subset ZU$, the centre of $U$: it is precisely the defining property of $U \times_A U$ which makes factorwise multiplication well-defined on this subspace.

**2.1.2 Definition** A *left* $A$*-bialgebroid* or $\times_A$*-bialgebra* is a $k$-module $U$ that carries simultaneously the structure of an $A^e$-ring $(U, s^\ell, t^\ell)$ as above and an $A$-coring $(U, \Delta_\ell, \epsilon)$ (cf. §1.2.2), subject to the following compatibility axioms:

(*i*) The $(A, A)$-bimodule structure in the $A$-coring $(U, \Delta_\ell, \epsilon)$ is related to the $A^e$-ring $(U, s^\ell, t^\ell)$ by

$$a \triangleright u \triangleleft \tilde{a} := \eta^\ell(a \otimes \tilde{a})u = s^\ell(a)t^\ell(\tilde{a})u, \quad a, \tilde{a} \in A, \ u \in U, \tag{2.1.3}$$

and we refer to this structure by writing ${}_\triangleright U_\triangleleft$. In particular, we write $U_\triangleleft \otimes {}_\triangleright U := U \otimes_A U$.

(*ii*) Considering the bimodule ${}_\triangleright U_\triangleleft$, the (left) coproduct $\Delta_\ell$ is a (unital) $k$-algebra morphism taking values in $U \times_A U$.

(*iii*) For all $a, \tilde{a} \in A$, $u, u' \in U$, the (left) counit $\epsilon$ has the property

$$\epsilon(s^\ell(a)t^\ell(\tilde{a})u) = a\epsilon(u)\tilde{a} \quad \text{and} \quad \epsilon(uu') = \epsilon(us^\ell(\epsilon u')) = \epsilon(ut^\ell(\epsilon u')). \tag{2.1.4}$$

Observe that, being an $A^e$-ring, such a left bialgebroid in total carries *four* $A$-module structures: one also has

$$a \blacktriangleright u \blacktriangleleft \tilde{a} := u\eta^\ell(\tilde{a} \otimes a) = us^\ell\tilde{a}t^\ell a, \tag{2.1.5}$$

and whenever we refer to this situation, we denote it by ${}_\blacktriangleright U_\blacktriangleleft$. Also note that (*i*) combined with (*ii*) implies that

$$\Delta_\ell s^\ell a = s^\ell a \otimes_A 1, \qquad \Delta_\ell t^\ell a = 1 \otimes_A t^\ell a, \qquad \text{for } a \in A. \tag{2.1.6}$$

Hence $\Delta$ is also an $A^e$-module morphism for the action in (2.1.5), i.e., for both (2.1.3) and (2.1.5) one has

$$\begin{array}{rcl} \Delta_\ell(a \triangleright u \triangleleft \tilde{a}) & = & (a \triangleright u_{(1)}) \otimes_A (u_{(2)} \triangleleft \tilde{a}), \\ \Delta_\ell(a \blacktriangleright u \blacktriangleleft \tilde{a}) & = & (u_{(1)} \blacktriangleleft \tilde{a}) \otimes_A (a \blacktriangleright u_{(2)}). \end{array} \tag{2.1.7}$$

**2.1.3 Remarks** (*i*) Even if $A$ were commutative, source and target do not necessarily coincide, as we will see in examples.

(*ii*) Since this will be of frequent technical use in all that follows, let us also explicitly state the comonoid identities involved. If $m_U$ and $m_{U^{\mathrm{op}}}$ denote the multiplication in $U$ and $U^{\mathrm{op}}$, respectively, one has

$$m_U(s^\ell\epsilon \otimes \mathrm{id}_U)\Delta_\ell = m_{U^{\mathrm{op}}}(\mathrm{id}_U \otimes t^\ell\epsilon)\Delta_\ell = \mathrm{id}_U. \tag{2.1.8}$$

For the coproduct of a left bialgebroid, we will use the Sweedler notation $\Delta_\ell u = u_{(1)} \otimes_A u_{(2)}$ with *lower* indices: it will become clear in a moment why we stress this distinction. The identity (2.1.8) then reads

$$s^\ell(\epsilon u_{(1)})u_{(2)} = t^\ell(\epsilon u_{(2)})u_{(1)} = \mathrm{id}_U, \qquad u \in U.$$

Finally, let us recall the notion of morphisms of bialgebroids [Sz].

**2.1.4 Definition** A *left bialgebroid morphism* $(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon) \to (U', A', s^{\ell'}, t^{\ell'}, \Delta'_\ell, \epsilon')$ is a pair $(\phi : A \to A', \psi : U \to U')$ of ring homomorphisms that commute with the structure maps in the obvious fashion. It is called a *left bialgebroid isomorphism* if $\phi$ and $\psi$ are bijective ring homomorphisms.

**2.1.5 Primitive and Grouplike Elements** As for ordinary coalgebras, an element $X \in U$ is called *primitive* if $\Delta_\ell X = X \otimes_A 1 + 1 \otimes_A X$. Using (2.1.8), this means $\epsilon X = 0$ if $X$ is primitive. Likewise, an element $\sigma \in U$ is called *grouplike* if $\Delta_\ell\sigma = \sigma \otimes_A \sigma$ and $\epsilon\sigma = 1$. We denote the space of primitive elements and grouplike elements by $P^\ell U$ and $G^\ell U$, respectively.

## 2.2 Left Hopf Algebroids

In this section, we present a generalisation of the notion of a Hopf algebra, which is based upon the notion of left bialgebroids: namely, the so-called $\times_A$*-Hopf algebra* of [Schau2]. We propose the name *left Hopf algebroid* instead, the reason for which will be explained in §2.6.14 (apart from solving a pronunciation problem). We will need this concept at various points, e.g. in Section 6.1 and Section 4.3. In particular, it will be the main ingredient in Chapter 7.

Let $U$ be a left bialgebroid over $A$ and define the so-called *(Hopf-)Galois map* of $U$ by

$$\beta : {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} \to U_{\triangleleft} \otimes_A {}_{\triangleright}U, \quad u \otimes_{A^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_A u_{(2)}v, \tag{2.2.1}$$

where

$${}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} = U \otimes_k U/\mathrm{span}\{a \blacktriangleright u \otimes_k v - u \otimes_k v \triangleleft a \,|\, u, v \in U, a \in A\}. \tag{2.2.2}$$

One could flip the tensor components in order to avoid taking the tensor product over $A^{\mathrm{op}}$, but we find it more convenient to keep $\beta$ in a form which is standard for bialgebras over fields. For the latter it is easily seen that $\beta$ is bijective if and only if $U$ is a Hopf algebra with $\beta^{-1}(u \otimes_k v) := u_{(1)} \otimes S(u_{(2)})v$, where $S$ is the antipode of $U$. This motivates the following definition due to Schauenburg [Schau2]:

**2.2.1 Definition** A left $A$-bialgebroid $U$ is called a *left Hopf algebroid* (or $\times_A$*-Hopf algebra*) if $\beta$ is a bijection.

Following [Schau2], we adopt a Sweedler-type notation

$$u_+ \otimes_{A^{\mathrm{op}}} u_- := \beta^{-1}(u \otimes_A 1) \tag{2.2.3}$$

for the so-called *translation map*

$$\beta^{-1}(\cdot \otimes_A 1) : U \to {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}.$$

Since these are substantial for calculations, e.g. in Chapter 7, we list some properties of $\beta^{-1}$ as proven in [Schau2, Proposition 3.7]: one has for all $u, v \in U$, $a \in A$

$$\begin{aligned}
u_{+(1)} \otimes_A u_{+(2)}u_- &= u \otimes_A 1 \in U_{\triangleleft} \otimes_A {}_{\triangleright}U, &(2.2.4)\\
u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-}u_{(2)} &= u \otimes_{A^{\mathrm{op}}} 1 \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}, &(2.2.5)\\
u_+ \otimes_{A^{\mathrm{op}}} u_- &\in U \times_{A^{\mathrm{op}}} U, &(2.2.6)\\
u_{+(1)} \otimes_A u_{+(2)} \otimes_{A^{\mathrm{op}}} u_- &= u_{(1)} \otimes_A u_{(2)+} \otimes_{A^{\mathrm{op}}} u_{(2)-}, &(2.2.7)\\
u_+ \otimes_{A^{\mathrm{op}}} u_{-(1)} \otimes_A u_{-(2)} &= u_{++} \otimes_{A^{\mathrm{op}}} u_- \otimes_A u_{+-}, &(2.2.8)\\
(uv)_+ \otimes_{A^{\mathrm{op}}} (uv)_- &= u_+v_+ \otimes_{A^{\mathrm{op}}} v_-u_-, &(2.2.9)\\
u_+u_- &= \eta(\epsilon u \otimes 1), &(2.2.10)\\
\eta(a \otimes b)_+ \otimes_{A^{\mathrm{op}}} \eta(a \otimes b)_- &= \eta(a \otimes 1) \otimes_{A^{\mathrm{op}}} \eta(b \otimes 1), &(2.2.11)
\end{aligned}$$

where in (2.2.6) we used the Takeuchi product

$$U \times_{A^{\mathrm{op}}} U := \Big\{\sum_i u_i \otimes_{A^{\mathrm{op}}} v_i \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} \mid \sum_i u_i \triangleleft a \otimes_{A^{\mathrm{op}}} v_i = \sum_i u_i \otimes_{A^{\mathrm{op}}} a \blacktriangleright v_i\Big\} \tag{2.2.12}$$

which is an algebra by factorwise multiplication, but with opposite multiplication on the second factor. Note that in (2.2.8) the tensor product over $A^{\mathrm{op}}$ links the first and third tensor component (cf. [Schau2, Equation (3.7)]). By (2.2.4) and (2.2.6) one can write

$$\beta^{-1}(u \otimes_A v) = u_+ \otimes_{A^{\mathrm{op}}} u_-v, \tag{2.2.13}$$

which is easily checked to be well-defined over $A$ with (2.2.9) and (2.2.11).

**2.2.2 Examples** (*i*) The enveloping algebra $A^{\mathrm{e}}$ of an associative algebra $A$ that governs Hochschild (co)homology is an example of a left Hopf algebroid over $A$, as already pointed out in [Schau2]; see Subsection 4.1.1 for details.

(*ii*) Clearly, Hopf algebras over $k$—such as universal enveloping algebras of Lie algebras or group algebras—are also left Hopf algebroids over $k$. More precisely, it is well-known [Schau2, p. 9] that Hopf algebras are in bijective correspondence with left Hopf algebroids over $k$: the inverse of (2.2.1) for a Hopf algebra $(H, \eta, \Delta, \epsilon, S)$ is given by $\beta^{-1}(h \otimes_k h') = h_{(1)} \otimes_k Sh_{(2)}h'$. Conversely, if $H$ is a left Hopf algebroid over $k$, an antipode for $H$ is given by $Sh := \eta(\epsilon h_+)h_-$.

## 2.3 Left Bialgebroid Modules and Comodules

### 2.3.1 The Monoidal category $U$-Mod

As for rings, one can consider modules over a left bialgebroid. However, there are some peculiarities attached to it, which we discuss in this subsection.

A standard characterisation [P] of bialgebras (common in quantum group theory) is as follows. A $k$-algebra $U$ is a bialgebra if and only if the category $U$-$\mathbf{Mod}$ of left $U$-modules is a monoidal category such that the underlying forgetful functor $U$-$\mathbf{Mod} \to k$-$\mathbf{Mod}$ is monoidal. The following theorem indicates that the definition of bialgebroids, though somewhat complicated, appears to be the right notion in this more general context.

**2.3.1 Theorem** [Schau1, Thm. 5.1] *The left $A$-bialgebroid structures on an $A^e$-ring $\eta : A^e \to U$ correspond bijectively to monoidal structures on $U$-$\mathbf{Mod}$ for which the forgetful functor $U$-$\mathbf{Mod} \to A^e$-$\mathbf{Mod}$ induced by $\eta$ is strictly monoidal.*

In particular, given a left $A$-bialgebroid structure on $U$, for $M \in U$-$\mathbf{Mod}$ with action $(u, m) \mapsto um$ the induced $A^e$-module structure is given by

$$am\tilde{a} := t^\ell \tilde{a} s^\ell a m, \quad m \in M, \ a, \tilde{a} \in A. \tag{2.3.1}$$

The monoidal structure on $U$-$\mathbf{Mod}$ is defined analogously as for bialgebras: for $M, M' \in U$-$\mathbf{Mod}$, the tensor product $M \otimes_A M$ of $A$-bimodules carries a $U$-module structure given by the diagonal

$$u(m \otimes_A m') := u_{(1)}m \otimes_A u_{(2)}m', \quad m \in M, \ m' \in M', \ u \in U, \tag{2.3.2}$$

which is well-defined since $U$ is a left bialgebroid. The monoidal unit in $U$-$\mathbf{Mod}$ is $A$ and $U$ acts on $A$ from the left in a canonical way,

$$ua := \epsilon(us^\ell a) = \epsilon(ut^\ell a), \quad a \in A, \ u \in U, \tag{2.3.3}$$

where the $(A, A)$-bimodule structure ${}_\blacktriangleright U_\blacktriangleleft$ appears. This may be called a (left) anchor [X3] for the bialgebroid. One easily gets the following *Leibniz rule*,

$$u(aa') = \epsilon(us^\ell(aa')) = \epsilon(us^\ell a t^\ell a) = \epsilon(u_{(1)}s^\ell a)\epsilon(u_{(2)}t^\ell a') = (u_{(1)}a)(u_{(2)}b), \tag{2.3.4}$$

hence $PU \subset \mathrm{Der}_k A$ by means of the canonical action.

On the other hand, for a left bialgebroid $U$ there is in general no canonical monoidal structure on $U^{\mathrm{op}}$-$\mathbf{Mod}$, and in particular no right action of $U$ on $A$.

There is a straightforward generalisation of a (left) module structure to a *(left) connection*, a notion with respect to which module structures give the special cases of *flat* left connections. Since we will need explicit details of this concept only for the 'opposite' notion of *right* bialgebroids (see below), we refer to Section 2.5.1.

### 2.3.2 The Monoidal Categories $U$-Comod and Comod-$U$

Likewise, similarly as for coalgebras, one may define comodules over bialgebroids. This section contains some issues characteristic to the situation of bialgebroids.

Let $U$ be a left bialgebroid over $A$ with structure maps as above. For the following definition confer e.g. [Schau1, B2, BrzWi].

**2.3.2 Definition** (*i*) A *right $U$-comodule* for a left bialgebroid $U$ over $A$ is a right comodule of the underlying $A$-coring $(U, \Delta_\ell, \epsilon)$, i.e. a right $A$-module $N$ with action $R_A : (n, a) \mapsto na$ and a right $A$-module map

$${}_N\Delta : N \to N \otimes_A U, \quad n \mapsto n_{(0)} \otimes_A n_{(1)}, \tag{2.3.5}$$

where

$$N \otimes_A U := N \otimes_k U / \mathrm{span}\{na \otimes u - n \otimes s^\ell a u \mid a \in A\},$$

satisfying the usual coassociativity and counitality axioms, i.e.,

$$({}_N\Delta \otimes \mathrm{id})_N\Delta = (\mathrm{id} \otimes \Delta_\ell)_N\Delta \quad \text{and} \quad R_A(\mathrm{id} \otimes \epsilon)_N\Delta = \mathrm{id},$$

respectively. For two right $U$-comodules $N, N'$, the set of *right $U$-comodule morphisms* is given as

$$\mathrm{Com}_U(N, N') := \{\phi \in \mathrm{Hom}_{(-,A)}(N, N') | (\phi \otimes \mathrm{id})_N\Delta = {}_{N'}\Delta\phi\},$$

and the corresponding category of right $U$-comodules and right $U$-comodule morphisms will be denoted $\mathbf{Comod}$-$U$.

(*ii*) A *left $U$-comodule* for a left bialgebroid $U$ over $A$ is a left comodule of the underlying $A$-coring $(U, \Delta_\ell, \epsilon)$, i.e. a left $A$-module $M$ with action $L_A : (a, m) \mapsto am$ and a left $A$-module map

$$\Delta_M : M \to U \otimes_A M, \quad m \mapsto m_{(-1)} \otimes_A m_{(0)}, \tag{2.3.6}$$

where

$$U \otimes_A M := U \otimes_k M/\mathrm{span}\{t^\ell au \otimes m - u \otimes am \mid a \in A\},$$

satisfying the usual coassociativity and counitality axioms

$$(\Delta_\ell \otimes \mathrm{id})\Delta_M = (\mathrm{id} \otimes \Delta_M)\Delta_M \quad \text{and} \quad L_A(\mathrm{id} \otimes \epsilon)\Delta_M = \mathrm{id}.$$

For two left $U$-comodules $M, M'$, the set of *left $U$-comodule morphisms* is given as

$$\mathrm{Com}_U(M, M') := \{\psi \in \mathrm{Hom}_{(A,-)}(M, M') | (\mathrm{id} \otimes \psi)\Delta_M = \Delta_{M'}\psi\},$$

and the corresponding category of left $U$-comodules and left $U$-comodule morphisms will be denoted $U$-$\mathbf{Comod}$.

**2.3.3 Examples** (*i*) Obviously, the $k$-module $U$ underlying a left bialgebroid is both a left and right $U$-comodule through the left coproduct.

(*ii*) In particular (and in contrast to the situation of $U$-modules), the base algebra $A$ carries both right and left coaction. Let $\sigma \in G^\ell U$ be a grouplike element. Then

$${}_A\Delta a = t^\ell(a)\sigma \quad \text{and} \quad \Delta_A a = s^\ell(a)\sigma, \qquad a \in A, \sigma \in G^\ell U, \tag{2.3.7}$$

define a right and left $U$-comodule structure on $A$, which we shall refer to as induced by $\sigma$.

Furthermore, on any right $U$-comodule one can additionally define a left $A$-action

$$an := n_{(0)}\epsilon(n_{(1)}t^\ell a), \quad a \in A, n \in N, \tag{2.3.8}$$

being the unique action that turns $N$ into an $A^e$-module and with respect to which ${}_N\Delta$ becomes an $A^e$-module morphism

$${}_N\Delta : N \to N \times_A U := \{\sum_i n_i \otimes_A u_i \in N \otimes_A U \mid \sum_i an_i \otimes u_i = \sum_i n_i \otimes u_i s^\ell a, \ \forall a \in A\} \tag{2.3.9}$$

to the Takeuchi product of $N$ with $U$. That means, for $a, b \in A$ and $n \in N$ one obtains the identities

$$\begin{array}{rcl} {}_N\Delta(anb) & = & n_{(0)} \otimes_A t^\ell b n_{(1)} t^\ell a, \\ an_{(0)} \otimes_A n_{(1)} & = & n_{(0)} \otimes_A n_{(1)} s^\ell a. \end{array} \tag{2.3.10}$$

Analogous considerations hold for left $U$-comodules $M$: one has an additional right $A$-action

$$ma := \epsilon(m_{(-1)}s^\ell a)m_{(0)},$$

and as a result gets the coaction as an $A^e$-module morphism into yet another Takeuchi product

$$\Delta_N : M \to U \times_A M := \{\sum_i u_i \otimes_A m_i \in U \otimes_A M \mid \sum_i u_i t^\ell a \otimes m_i = \sum_i u_i \otimes m_i a, \ \forall a \in A\},$$

hence satisfying the identities

$$\begin{array}{rcl} \Delta_M(amb) & = & s^\ell a m_{(-1)} s^\ell b \otimes_A m_{(0)}, \\ m_{(-1)} \otimes_A m_{(0)} a & = & m_{(-1)} t^\ell a \otimes_A m_{(0)}. \end{array}$$

One can then prove (see [B3, Thm. 3.18] and [Schau1, Prop. 5.6]) that, say, the category $\mathbf{Comod}\text{-}U$ of right $U$-comodules is monoidal such that the forgetful functor $\mathbf{Comod}\text{-}U \to (A^{\mathrm{op}})^{\mathrm{e}}\text{-}\mathbf{Mod}$ is monoidal (and similar for $U\text{-}\mathbf{Comod}$): for any two comodules $N, N' \in \mathbf{Comod}\text{-}U$, their tensor product $N \otimes_{A^{\mathrm{op}}} N'$ is a right $U$-comodule by means of the coaction

$$\begin{array}{rcl} {}_{N\otimes_A N'}\Delta : N \otimes_{A^{\mathrm{op}}} N' & \to & (N \otimes_{A^{\mathrm{op}}} N') \otimes_A U, \\ n \otimes_{A^{\mathrm{op}}} n' & \mapsto & n_{(0)} \otimes_{A^{\mathrm{op}}} n'_{(0)} \otimes_A n_{(1)} n'_{(1)}, \end{array} \tag{2.3.11}$$

where

$$N \otimes_{A^{\mathrm{op}}} N' := N \otimes_k \mathbb{N}'/\mathrm{span}\{an \otimes_k n' - n \otimes_k n'a \mid a \in A\} \tag{2.3.12}$$

is a right $A$-module by $(n \otimes_{A^{\mathrm{op}}} n')a = na \otimes_{A^{\mathrm{op}}} n'$. The map ${}_{N\otimes_A N'}\Delta$ is easily checked to be well-defined. Turning the tensor product around to avoid $A^{\mathrm{op}}$, this appears to be equivalent to a right $U$-coaction

$${}_{N\otimes N'}\Delta : N \otimes_A N' \to (N \otimes_A N') \otimes_A U, \quad n \otimes_A n' \mapsto n_{(0)} \otimes_A n'_{(0)} \otimes_A n'_{(1)} n_{(1)},$$

but for technical reasons in later sections and also because it is the analogue to the form which is standard for bialgebras we will prefer (2.3.11). All statements can also be made *mutatis mutandis* for the category $U\text{-}\mathbf{Comod}$.

## 2.4 Homological Coalgebra for Left Bialgebroid Comodules

In this section we discuss the notion of cotensor product with its derived functor $\mathrm{Cotor}$ for $U$-comodules. This will be of importance for our cyclic (co)homology computations in Chapter 5 and 6. Here, the term 'homological coalgebra' [Do] refers to the corresponding standard notions for $k$-coalgebras (see also [EMo]), which we transfer to the categories $U\text{-}\mathbf{Comod}$ and $\mathbf{Comod}\text{-}U$ for a left bialgebroid $U$.

**2.4.1 Theorem** *Let $U$ be a left bialgebroid. If ${}_\triangleright U$ is flat over $A$, then the category $\mathbf{Comod}\text{-}U$ is abelian. The same statement can be made about $U\text{-}\mathbf{Comod}$ if $U_\triangleleft$ is flat.*

PROOF: The theorem appears in [Ra] in the special situation $U$ and $A$ are commutative, but can be generalised without major changes to the present situation: the proof relies entirely on standard arguments from homological algebra, so we omit the details. □

**2.4.2 Definition** Let $U$ be a left bialgebroid, and take $M \in U\text{-}\mathbf{Comod}$ and $N \in \mathbf{Comod}\text{-}U$.

(*i*) The *cotensor product* of $N$ and $M$ over $U$ is the $A$-module defined by the exact sequence

$$0 \longrightarrow N \Box_U M \longrightarrow N \otimes_A M \xrightarrow{{}_N\Delta \otimes \mathrm{id}_M - \mathrm{id}_N \otimes \Delta_M} N \otimes_A U \otimes_A M.$$

(*ii*) Consider the base algebra $A$ as a right $U$-comodule as in (2.3.7). The subspace

$$A \Box_U M = \{m \in M \mid |\Delta_M m = \sigma \otimes_A m\} \subset M, \tag{2.4.1}$$

where $\sigma \in G^\ell U$ is a grouplike element, will be called the subspace of *left bialgebroid (left) invariants* of $M$.

Form the terminology and (2.3.7) it is clear that one may analogously introduce left bialgebroid *right* invariants on a right $U$-comodule $N$.

**2.4.3 Lemma** *Let $M, M' \in U\text{-}\mathbf{Comod}$ be left $U$-comodules, with ${}_A M$ finitely generated projective over $A$. In such a case*

(*i*) $\mathrm{Hom}_{(A,-)}(M, A)$ *is a right $U$-comodule,*

(*ii*) $\mathrm{Com}_U(M, M') = \mathrm{Hom}_{(A,-)}(M, A) \Box_U M'$, *e.g.* $\mathrm{Com}_U(A, M') = A \Box_U M'$.

PROOF: Part (*i*): let $e_1, \ldots, e_n$ be a generating set of $M$. Hence, there are elements $e^1, \ldots, e^n$ in $M^* := \mathrm{Hom}_{(A,-)}(M, A)$ such that each element $m \in M$ can be decomposed as $m = \sum_{i=1}^n e^i(m)e_i$. Now consider the map

$$\zeta : \mathrm{Hom}_{(A,-)}(M, A) \to \mathrm{Hom}_{(A,-)}(M, {}_{\triangleright}U), \ f \mapsto m_{U^{\mathrm{op}}}(\mathrm{id}_U \otimes t^\ell f)\Delta_M.$$

Using the isomorphism

$$\xi : \mathrm{Hom}_{(A,-)}(M, {}_{\triangleright}U) \to (\mathrm{Hom}_{(A,-)}(M, A))_A \otimes {}_{\triangleright}U, \ \psi \mapsto \sum_{i=1}^n e^i \otimes_A \psi(e_i),$$

we obtain a map

$$\Delta_M^* : M^* = \mathrm{Hom}_{(A,-)}(M, A) \to (\mathrm{Hom}_{(A,-)}(M, A))_A \otimes {}_{\triangleright}U, \quad f \mapsto \sum_{i=1}^n e^i \otimes_A (\zeta f)(e_i),$$

and we claim that this defines a right $U$-coaction on $M^*$, where $\mathrm{Hom}_{(A,-)}(M, A)$ is seen as a right $A$-module with action $R_A : (f, a) \mapsto fa$ in the standard way. One verifies counitality as follows. For $f \in M^*$, $m \in M$, applying the inverse $\xi^{-1} : (g \otimes u)(m) \mapsto g(m)u$, we obtain

$$m_{R_A}(\mathrm{id} \otimes \epsilon)\Delta_M^* f = \sum_{i=1}^n \big(e^i \otimes_A \epsilon(e_{i(-1)})f(e_{i(0)})\big)(m) = \sum_{i=1}^n \big(e^i \otimes_A f(e_i)\big)(m) = f(m),$$

since $M \in U$-**Comod**. Also, coassociativity is straightforward:

$$\begin{aligned}
(\Delta_M^* \otimes \mathrm{id})\Delta_M^* f &= \sum_{i,j=1}^n e^j \otimes_A t^\ell e^i(e_{j\,(0)})e_{j\,(-1)} \otimes_A t^\ell f(e_{i(0)})e_{i(-1)} \\
&= \sum_{i,j=1}^n e^j \otimes_A e_{j\,(-1)} \otimes_A t^\ell f(e_{i(0)})s^\ell e^i(e_{j\,(0)})e_{i(-1)} \\
&= \sum_{i,j=1}^n e^j \otimes_A e_{j\,(-1)} \otimes_A t^\ell f((e^i(e_{j\,(0)})e_i)_{(0)})(e^i(e_{j\,(0)})e_i)_{(-1)} \\
&= \sum_{j=1}^n e^j \otimes_A e_{j\,(-2)} \otimes_A t^\ell f(e_{j\,(0)})e_{j\,(-1)} \\
&= \sum_{j=1}^n (\mathrm{id}_{M^*} \otimes \Delta_\ell)(\mathrm{id}_{M^*} \otimes m_{U^{\mathrm{op}}})(\mathrm{id}_{M^*} \otimes \mathrm{id}_U \otimes t^\ell f)(\mathrm{id}_{M^*} \otimes \Delta_M)(e^j \otimes_A e_j) \\
&= (\mathrm{id} \otimes \Delta_\ell)\Delta_M^* f.
\end{aligned}$$

It remains to show the $A$-module morphism property. For $a \in A$,

$$\begin{aligned}
\Delta_M^*(fa) &= \sum_{i=1}^n e^i \otimes_A t^\ell(f(e_{i(0)})a)e_{i(-1)} \\
&= \sum_{i=1}^n e^i \otimes_A t^\ell a t^\ell(f(e_{i(0)}))e_{i(-1)} = f_{(0)} \otimes t^\ell a f_{(1)}.
\end{aligned}$$

Part (*ii*) can be shown exactly as in [Ra, Lem. A.1.1.6]. □

As a consequence of part (*ii*), that is,

$$A \square_U M \simeq \mathrm{Com}_U(A, M) = \{f \in \mathrm{Hom}_{(A,-)}(A, M) | (\mathrm{id} \otimes f)\Delta_A = \Delta_M f\},$$

applying the isomorphism $\mathrm{Hom}_{(A,-)}(A, M) \simeq M, \ f \mapsto f(1_A) =: m$ and using $\Delta_A$ from (2.3.7) for a grouplike element $\sigma \in G^\ell U$, we obtain

$$\sigma \otimes_A m = (\mathrm{id} \otimes f)\Delta_A = \Delta_M f(1_A) = m_{(-1)} \otimes_A m_{(0)} = \Delta_M m,$$

as before.

**2.4.4 Lemma** *Let $I$ be an injective $A$-module. Then $U \otimes_A I$ is an injective (left) $U$-comodule; hence the category of $U$-$\mathbf{Comod}$ has enough injectives.*

PROOF: Also this lemma is a generalisation of a result in [Ra] to noncommutative $U$ and $A$, and the proof can be taken over with only minor modifications. For a left $A$-module $M$ with action $L_A : (a, m) \mapsto am$, define a left $U$-coaction on $U \otimes_A M$ by $\Delta_{U \otimes N} := \Delta_\ell \otimes \mathrm{id}_M$. With Lemma 2.4.3(*ii*) we get for any $N \in U$-$\mathbf{Comod}$ an isomorphism

$$\theta : \mathrm{Hom}_{(A,-)}(N, M) \to \mathrm{Com}_U(N, U \otimes_A M), \quad f \mapsto (\mathrm{id}_U \otimes f)\Delta_N$$

with inverse $\theta^{-1} : \psi \mapsto m_{L_A}(\epsilon \otimes \mathrm{id}_M)\psi$ for any $\psi \in \mathrm{Com}_U(N, U \otimes_A M)$. To show that $U \otimes_A I$ is injective we need to show now that if $P$ is a $U$-subcomodule of $N$ (i.e. both an $A$-submodule and a subcomodule in the conventional sense), then $\psi \in \mathrm{Com}_U(P, U \otimes_A I)$ extends to a map in $\mathrm{Com}_U(N, U \otimes_A I)$. We have

$$\mathrm{Com}_U(P, U \otimes_A I) \simeq \mathrm{Hom}_{(A,-)}(P, I) \subset \mathrm{Hom}_{(A,-)}(N, I) \simeq \mathrm{Com}_H(N, U \otimes_A I)$$

as subgroups by injectivity of $I$. Since the category of left $A$-modules already has enough injectives, one can therefore construct enough injectives in $U$-$\mathbf{Comod}$ as well. □

Now we are in a position to finally define the following derived functors, analogously as in [Do, EMo]:

**2.4.5 Definition** (*i*) For two left $U$-comodules $M, M' \in U$-$\mathbf{Comod}$, the group $\mathrm{Ext}^i_U(M, M')$ is the $i$-th right derived functor of $\mathrm{Com}_U(M, M')$, as functor in $M'$.

(*ii*) For $M \in U$-$\mathbf{Comod}$ and $N \in \mathbf{Comod}$-$U$, the group $\mathrm{Cotor}^i_U(N, M)$ is the $i$-th right derived functor of $N \Box_U M$, as functor in $M$.

As already mentioned, these notions will be used in Chapters 5 and 6.

## 2.5 Right Bialgebroids

In this section we proceed with the ingredients of a Hopf algebroid as mentioned in aspect (2) at the beginning of this chapter.

If one wants to turn a bialgebra into a Hopf algebra, one needs to hunt for an antipode, i.e. for a $k$-bialgebra morphism $U \to U^{\mathrm{op}}_{\mathrm{coop}}$ into the opposite and coopposite bialgebra. If one aims to naturally generalise this idea to the case of (left) bialgebroids, one observes that the 'opposite' bialgebroid (see below) does not fulfill the left bialgebroid axioms any more, but rather the ones of a 'mirrored', or opposite version of it, i.e. of a *right* bialgebroid. These objects were introduced in [KSz] for the first time, confer also e.g. [B3] for the subsequent definition.

Let $B$ be a $k$-algebra. Similarly as for left bialgebroids, we consider a $B^{\mathrm{e}}$-ring $V$ given by an $k$-algebra map $\eta_V = \eta : B^{\mathrm{e}} \to V$ with source and target maps $s := \eta(- \otimes_k 1_B)$ and $t := \eta(1_B \otimes_k -)$. In contrast to left bialgebroids, one now considers the right $B^{\mathrm{e}}$-module structure $(v, b \otimes_k \tilde{b}) \mapsto v\eta(b \otimes_k \tilde{b})$ on $V$, and forms correspondingly the tensor product

$$\begin{aligned} V \otimes^B V :=& V \otimes_k V / \mathrm{span}\{v\eta(b \otimes_k 1) \otimes v' - v \otimes v'\eta(1 \otimes_k b) \mid b \in B,\ v, v' \in V\} \\ =& V \otimes_k V / \mathrm{span}\{vs(b) \otimes v' - v \otimes v't(b) \mid b \in B,\ v, v' \in V\}. \end{aligned} \tag{2.5.1}$$

Again, $V \otimes^B V$ becomes a $(B, B)$-bimodule in a standard way (see (A.1.1)), that is $(b, v \otimes^B v', \tilde{b}) \mapsto vt(b) \otimes^B v's(\tilde{b})$, and as before, $V \otimes^B V$ does not carry a well-defined algebra structure. One correspondingly introduces the (right) Takeuchi product of $V$ with itself, i.e. the $k$ submodule $V \otimes^B V \subset V \times^B V$ given by

$$V \times^B V := \{\sum_i v_i \otimes^B v'_i \in V \otimes^B V \mid \sum_i s(b)v_i \otimes^B v'_i = \sum_i v_i \otimes^B t(b)v'_i,\ \forall b \in\},$$

and one easily verifies that now this is a $B^{\mathrm{e}}$-ring via factorwise multiplication, with unit element $1_V \otimes^B 1_V$ and $\eta_{V \times^B V}(b \otimes_k \tilde{b}) = t(\tilde{b}) \otimes^B s(b)$.

**2.5.1 Definition** A *right $B$-bialgebroid* or *$\times^B$-bialgebra* is a $k$-module $V$ which carries simultaneously the structure of a $B^{\mathrm{e}}$-ring $(V, s^r, t^r)$ and a $B$-coring $(V, \Delta_r, \partial)$, subject to the following compatibility axioms:

(*i*) The $(B, B)$-bimodule structure in the $B$-coring is related to the $B^e$-ring $(V, s^r, t^r)$ by

$$b \blacktriangleright v \blacktriangleleft \tilde{b} := vs^r(\tilde{b})t^r(b) = v\eta^r(\tilde{b} \otimes b), \quad b, \tilde{b} \in B, \ v \in V, \tag{2.5.2}$$

and we refer to this structure as $_\blacktriangleright V_\blacktriangleleft$. In particular, we write $V_\blacktriangleleft \otimes {}_\blacktriangleright V := V \otimes^B V$.

(*ii*) Considering the bimodule $_\blacktriangleright V_\blacktriangleleft$, the (right) coproduct $\Delta_r$ is a (unital) $k$-algebra morphism taking values in $V \times^B V$.

(*iii*) For all $b, \tilde{b} \in B$, $v, v' \in V$, the (right) counit has the property

$$\partial(v\eta^r(\tilde{b} \otimes b)) = b\partial(v)\tilde{b} \quad \text{and} \quad \partial(vv') = \partial(s^r(\partial v)v') = \partial(t^r(\partial v)v'). \tag{2.5.3}$$

For the right coproduct, we will use the Sweedler notation $\Delta_r v = v^{(1)} \otimes^B v^{(2)}$ with *upper* indices so as not to confuse it with the left coproduct for objects that carry both structures. Here $V$ clearly acts on its base algebra $B$ from the right in a canonical way, namely

$$bv := \partial(s^r(b)v) = \partial(t^r(b)v), \quad b \in B, \ v \in V, \tag{2.5.4}$$

where the remaining two $B$-module structures $_\triangleright V_\triangleleft$, given by

$$b \triangleright v \triangleleft \tilde{b} := \eta^r(b \otimes \tilde{b})v = s^r(b)t^r(\tilde{b})v, \tag{2.5.5}$$

appear. Similarly as for left bialgebroids, one has

$$\Delta_r s^r b = 1 \otimes^B s^r b, \qquad \Delta_r t^r b = t^r b \otimes^B 1, \qquad \text{for } b \in B,$$

as well as

$$\Delta_r(b \blacktriangleright v \blacktriangleleft \tilde{b}) = (b \blacktriangleright v^{(1)}) \otimes^B (v^{(2)} \blacktriangleleft \tilde{b}), \quad \Delta_r(b \triangleright v \triangleleft \tilde{b}) = (v^{(1)} \triangleleft \tilde{b}) \otimes^B (b \triangleright v^{(2)}),$$

and the comonoid identities in this case read

$$m_V(\mathrm{id}_V \otimes s^r\partial)\Delta_r = m_{V^{\mathrm{op}}}(t^r\partial \otimes \mathrm{id}_V)\Delta_r = \mathrm{id}_V, \tag{2.5.6}$$

that is, using the Sweedler notation for the right coproduct, we have

$$v^{(1)}s^r\partial v^{(2)} = v^{(2)}t^r\partial v^{(1)} = \mathrm{id}_V, \qquad v \in V.$$

**2.5.2 Remarks** (*i*) The 'opposite' of a left bialgebroid $U = (U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ is defined as $U^{\mathrm{op}} := (U^{\mathrm{op}}, A, t^\ell, s^\ell, \Delta_\ell, \epsilon)$. This can be shown to be a right bialgebroid, whereas its 'coopposite' given by $U_{\mathrm{coop}} := (U, A^{\mathrm{op}}, t^\ell, s^\ell, \Delta_\ell^{\mathrm{coop}}, \epsilon)$ with $\Delta_\ell^{\mathrm{coop}}$ seen as a map $U \to {}_\triangleright U \otimes_{A^{\mathrm{op}}} U_\triangleleft$, $h \mapsto u_{(2)} \otimes_{A^{\mathrm{op}}} u_{(1)}$ remains a left bialgebroid. Note that in both cases source and target map interchange their rôles. In total, the object $U^{\mathrm{op}}_{\mathrm{coop}}$ will be a right bialgebroid, as announced above.

(*ii*) The idea of a right bialgebroid for the first time seems to appear in [KSz]. The necessity of such an analogous bialgebroid structure with some sort of 'opposite' properties became clear to us while considering Lie-Rinehart homology and attempting to introducing antipodes for (the left bialgebroid associated to) a Lie-Rinehart algebra; see below.

(*iii*) One may be tempted to think that the concepts of left and right counits coincide whenever the base algebra is commutative. This is, however, not the case, not even if source and target map are equal or trivial (in a context-determined sense), as the following examples will reveal.

### 2.5.1 Right Bialgebroid Connections

**2.5.3 The Monoidal Category $\mathbf{Mod}$-$V$** Of course, all concepts from Sections 2.3 and 2.4 dealing with bialgebroid modules and comodules could be repeated applying all statements to the opposite $A^e$-ring. We will refrain from doing so in detail and rather refer to [B3]. However, for later use, let us explicitly mention that any $N$ in the category $\mathbf{Mod}$-$V$ of right $V$-modules for a right bialgebroid $V$ carries its induced right $B^e$-structure by

$$\tilde{b}nb := ns^r bt^r \tilde{b}, \qquad n \in N, b, \tilde{b} \in B.$$

The category $\mathbf{Mod}$-$V$ acquires a monoidal structure by deploying the right coproduct, i.e., for $N, N' \in \mathbf{Mod}$-$V$, their tensor product over $B$ is in $\mathbf{Mod}$-$V$, with right $V$-action

$$(n \otimes_B n')v := nv^{(1)} \otimes_B n'v^{(2)}, \qquad n \in N, n' \in N',\ v \in V.$$

Similarly as before, one also has $B \in \mathbf{Mod}$-$V$, with right $V$-action

$$bv := \partial(s^r bv) = \partial(t^r bv), \qquad b \in B,\ v \in V,$$

and the Leibniz rule this time reads

$$(b\tilde{b})v = (bv^{(1)})(\tilde{b}v^{(2)}), \qquad b, \tilde{b} \in B,\ v \in V.$$

One may also vary the notion of a right $V$-module: consider e.g. a $B^{\mathrm{e}}$-module $N$ and define a right $V$-action on $N$ such that the induced $B^{\mathrm{e}}$-module structure coincides with the *a priori* given one. For example, $B$ itself already carries a natural $B^{\mathrm{e}}$-module structure (by left and right multiplication) and one may try to find a right $V$-module structure on it which does not originate from the right counit $\partial$. For certain modules $N$, we will come back to a situation like that in Section 4.7. Starting at this point, one also may introduce *right $V$-connections*, which are called *flat* if they specialise with right $V$-module structures. We will encounter these constructions again e.g. in Proposition 3.1.14 and Section 4.2.

**2.5.4 Definition** Let $N$ be a $B^{\mathrm{e}}$-module. A *right $V$-connection* on $N$ is a map

$$\nabla^r : N \to \mathrm{Hom}_{(B,B)}(\,_{\blacktriangleright}V_{\blacktriangleleft}\,, N),$$

such that the *Leibniz rule*

$$\nabla^r_v(nb) = \nabla^r_{s^r bv} n, \quad \nabla^r_v(bn) = \nabla^r_{t^r bv} n, \qquad v \in V,\ n \in N, b \in B \tag{2.5.7}$$

holds. A right $V$-connection is called *flat* if $\nabla^r_v \nabla^r_{v'} = \nabla^r_{v'v}$ for all $v, v' \in U$.

One easily verifies that

$$\nabla^r_v(nb) = \big(\nabla^r_{v^{(1)}} n\big)\partial s^r bv^{(2)}, \qquad \nabla^r_v(bm) = \partial(t^r bv^{(1)})\big(\nabla^r_{v^{(2)}} m\big).$$

Hence for a primitive element $Y \in P^r U$ one obtains the more familiar formulae

$$\nabla^r_Y(nb) = n(bY) + (\nabla^r_Y n)b, \quad \nabla^r_Y(bn) = b\nabla^r_Y n + (bY)n, \tag{2.5.8}$$

where, as in (2.5.4), we denoted the canonical right $V$-action on $B$ by $bY = \partial(s^r(b)Y) = \partial(t^r(b)Y)$.

**2.5.5** In the particular case $N = B$, where $B$ carries the natural right $B^{\mathrm{e}}$-module structure by multiplication, evaluating a right connection on $1_B \in B$ defines a $k$-linear operator $D \in \mathrm{Hom}_{(B,B)}(\,_{\blacktriangleright}U_{\blacktriangleleft}\,, B)$ by

$$D^r : V \to B, \quad v \mapsto \nabla^r_v 1_B.$$

If the connection is flat, we have for all $v, v' \in V$

$$\begin{aligned} D^r(vv') &= \nabla^r_{vv'} 1_B = \nabla^r_{v'} \nabla^r_v 1_B = \nabla^r_{v'} D^r v \\ &= \nabla^r_{s^r(D^r v)v'} 1_B = \nabla^r_{t^r(D^r v)v'} 1_B \\ &= D^r(s^r(D^r v)v') = D^r(t^r(D^r v)v'), \end{aligned} \tag{2.5.9}$$

which is the property (2.5.3) of a right counit. In the terminology of [B3], a map $V \to B$ with such a property is called a *right character* for the $B$-rings $(V, s^r)$ and $(V, t^r)$, respectively.

## 2.6 Hopf Algebroids

As intimated at the beginning of this chapter, a Hopf algebroid is simultaneously both a left and a right bialgebroid, with an antipode intertwining these structures. The following definition is due to Böhm-Szlachányi [BSz2, B1], cf. in particular [B3].

**2.6.1 Definition** Let $A, B$ be two $k$-algebras and $H$ a $k$-module. A *(double-sided) Hopf algebroid* structure on $H$ consists of

(1) a left bialgebroid structure $(H, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ on $H$ over $A$,

(2) a right bialgebroid structure $(H, B, s^r, t^r, \Delta_r, \partial)$ on $H$ over $B$, such that the underlying $k$-algebra structure on $H$ is the same as in (1),

(3) a $k$-module map $S : H \to H$.

These structures are subject to the following compatibility axioms:

(*i*)
$$s^\ell \epsilon t^r = t^r, \quad t^\ell \epsilon s^r = s^r, \quad s^r \partial t^\ell = t^\ell, \quad t^r \partial s^\ell = s^\ell. \tag{2.6.1}$$

(*ii*) *Twisted coassociativity* holds, that is to say
$$(\Delta_\ell \otimes \mathrm{id}_H)\Delta_r = (\mathrm{id}_H \otimes \Delta_r)\Delta_\ell \quad \text{and} \quad (\Delta_r \otimes \mathrm{id}_H)\Delta_\ell = (\mathrm{id}_H \otimes \Delta_\ell)\Delta_r. \tag{2.6.2}$$

(*iii*)
$$S(t^\ell a h\, t^r b) = s^r b S h s^\ell a \quad a \in A,\ b \in B,\ h \in H. \tag{2.6.3}$$

(*iv*)
$$m_H(S \otimes_A \mathrm{id}_H)\Delta_\ell = s^r \partial \quad \text{and} \quad m_H(\mathrm{id}_H \otimes^B S)\Delta_r = s^\ell \epsilon. \tag{2.6.4}$$

We call $S$ the *antipode* of the Hopf algebroid.

If particular reference is needed, we will denote the underlying left and right bialgebroid structures of a Hopf algebroid $H$ by $H^\ell$ and $H^r$, respectively.

**2.6.2 Remarks** (*i*) Applying $\partial$ to the first two and $\epsilon$ to the second pair of identities in (2.6.1), one obtains that $A$ and $B$ are anti-isomorphic, i.e.,

$$\begin{array}{llll} \mu := \partial s^\ell : A^{\mathrm{op}} \xrightarrow{\simeq} B, & & \mu^{-1} := \epsilon t^r : B \xrightarrow{\simeq} A^{\mathrm{op}}, \\ \nu := \partial t^\ell : A \xrightarrow{\simeq} B^{\mathrm{op}}, & & \nu^{-1} := \epsilon s^r : B^{\mathrm{op}} \xrightarrow{\simeq} A. \end{array} \tag{2.6.5}$$

Hence the ranges of $s^\ell$ and $t^r$ as well as $s^r$ and $t^\ell$, respectively, are coinciding subalgebras in $H$.

(*ii*) In particular, *(i)* implies that $\Delta_\ell$ behaves as follows with respect to the $B^{\mathrm{e}}$-bimodule structure mentioned in (2.5.2) and (2.5.5). For $h \in H,\ b, \tilde{b} \in B$,

$$\begin{array}{rcl} \Delta_\ell(b \blacktriangleright h \blacktriangleleft \tilde{b}) & = & (b \blacktriangleright h_{(1)}) \otimes_A (h_{(2)} \blacktriangleleft \tilde{b}), \\ \Delta_\ell(b \triangleright h \triangleleft \tilde{b}) & = & (h_{(1)} \triangleleft \tilde{b}) \otimes_A (b \triangleright h_{(2)}). \end{array} \tag{2.6.6}$$

Likewise for $\Delta_r$ with respect to (2.1.3) and (2.1.5): for $h \in H,\ a, \tilde{a} \in A$ one has

$$\begin{array}{rcl} \Delta_r(a \triangleright h \triangleleft \tilde{a}) & = & (a \triangleright h^{(1)}) \otimes^B (h^{(2)} \triangleleft \tilde{a}), \\ \Delta_r(a \blacktriangleright h \blacktriangleleft \tilde{a}) & = & (h^{(1)} \blacktriangleleft \tilde{a}) \otimes^B (a \blacktriangleright h^{(2)}). \end{array} \tag{2.6.7}$$

Introducing a $(B, B)$-bimodule structure on $H \otimes_A H$ by $(b, h \otimes_A h', \tilde{b}) \mapsto (b \blacktriangleright h) \otimes_A (h' \blacktriangleleft \tilde{b})$ and an $(A, A)$-bimodule structure on $H \otimes^B H$ by $(a, h \otimes_A h', \tilde{a}) \mapsto (a \triangleright h) \otimes^B (h' \triangleleft \tilde{a})$, the respective first equations in (2.6.6) and (2.6.7) say that $\Delta_\ell$ is also a $(B, B)$-bimodule morphism, while $\Delta_r$ is also an $(A, A)$-bimodule morphism. This observation makes axiom (*ii*) meaningful, the first identity of which can be expressed as follows,

$$\begin{tikzcd} H \arrow[r, "\Delta_\ell"] \arrow[dr, "\Delta_r"'] & H \otimes_A H \arrow[r, "\mathrm{id}_H \otimes \Delta_r"] & H \otimes_A (H \otimes^B H) \arrow[d, "\simeq"] \\ & H \otimes^B H \arrow[r, "\Delta_\ell \otimes \mathrm{id}_H"] & (H \otimes_A H) \otimes^B H, \end{tikzcd}$$

and likewise for the second identity. As follows from Example 2.3.3(*i*), the underlying left bialgebroid $H^\ell$ of $H$ determines both left and right coactions via $\Delta_\ell$; the same is true for the underlying right bialgebroid $H^r$ of $H$ with respect to $\Delta_r$. Hence in total there are *four* bialgebroid coactions on $H$, and twisted coassociativity states that they all commute with each other. This can be expressed by saying that the $k$-module $H$ is both an $H^\ell$-$H^r$-bicomodule and an $H^r$-$H^\ell$-bicomodule. For future use, let us mention that twisted coassociativity immediately leads to its 'higher' version,

$$\begin{aligned} (\Delta_\ell^n \otimes \mathrm{id}_H^{\otimes m})\Delta_r^m &= (\mathrm{id}_H^{\otimes n} \otimes \Delta_r^m)\Delta_\ell^n \\ (\Delta_r^n \otimes \mathrm{id}_H^{\otimes m})\Delta_\ell^m &= (\mathrm{id}_H^{\otimes n} \otimes \Delta_\ell^m)\Delta_r^n \end{aligned} \tag{2.6.8}$$

for $n, m \in \mathbb{N}$.

(*iii*) The axiom (*iii*) may be expressed by saying that the map $S$ is a morphism of twisted bimodules. That is, it intertwines the left $A^{\mathrm{e}}$-module structure (2.1.3) on $H^\ell$ with the right one from (2.1.5), as a morphism $H^\ell_{\triangleleft} \to H^\ell_{\blacktriangleleft}$. This is worth mentioning, because (2.1.5) does not explicitly appear in the axioms of a left bialgebroid. Similarly, it links the right and left $B^{\mathrm{e}}$-module structures (2.5.2) and (2.5.5) of the right bialgebroid $H^r$, i.e., it is a morphism ${}_{\blacktriangleright}H^r \to {}_{\triangleright}H^r$.

(*iv*) The left hand side of the first equation in (2.6.4) is a composition of maps

$$H \xrightarrow{\Delta_\ell} H_{\triangleleft} \otimes {}_{\triangleright}H \xrightarrow{S\otimes \mathrm{id}_H} H_{\blacktriangleleft} \otimes {}_{\triangleright}H \xrightarrow{m_H} H, \tag{2.6.9}$$

where $H_{\triangleleft} \otimes {}_{\triangleright}H = H \otimes_A H$ is given as in (2.1.1), whereas

$$H_{\blacktriangleleft} \otimes {}_{\triangleright}H = H \otimes_k H/\mathrm{span}\{hs^\ell a \otimes h' - h \otimes s^\ell a h' \mid a \in A,\ h, h' \in H\}$$

is in a sense the tensor product naturally associated to the $A$-ring $(H, s^\ell)$, and $m_H$ is to be understood the multiplication in this ring. The composition (2.6.9) is well-defined due to (*iii*), and a similar consideration holds for the second equation in (2.6.4). Observe that in this second case $m_H$ refers to multiplication in the $B$-ring $(H, s^r)$, despite of the identical notation.

At the latest at this point one recognises the need for two kinds of Sweedler notations. Using lower indices for the left and upper ones for the right coproduct, (2.6.4) reads

$$Sh_{(1)}h_{(2)} = s^r\partial h \quad \text{and} \quad h^{(1)}Sh^{(2)} = s^\ell \epsilon h, \qquad h \in H.$$

(*v*) Although not explicitly required in the definition, we will usually assume $S$ to be invertible.

**2.6.3 Examples** For examples one may jump directly to Chapter 4.

See also [BSz2, B1] for further details on Hopf algebroids, discussion, and many examples, and [B3] for a comparison with alternative notions.

The fact that the antipode of a Hopf algebra is an anti-homomorphism for the algebra structure and an anti-cohomomorphism for the coalgebra structure has the following counterpart in the bialgebroid framework [BSz2, B3].

**2.6.4 Proposition** *Let $H$ be a Hopf algebroid with structure maps as before.*

(*i*) *The antipode $S$ is a homomorphism of both $A^{\mathrm{e}}$-rings $(H, s^\ell, t^\ell) \to (H^{\mathrm{op}}, s^r\mu, t^r\mu)$ and $B^{\mathrm{e}}$-rings $(H, s^r, t^r) \to (H^{\mathrm{op}}, s^\ell\nu^{-1}, t^\ell\nu^{-1})$, where the isomorphisms from (2.6.5) were used. In particular, $S$ is an $k$-algebra morphism $H \to H^{\mathrm{op}}$.*

(*ii*) *Likewise, $S$ is a cohomomorphism of $A$-corings $(H, \Delta_\ell, \epsilon) \to (H, \Delta_r^{\mathrm{coop}}, \nu^{-1}\partial)$ as well as of $B$-corings $(H, \Delta_r, \partial) \to (H, \Delta_\ell^{\mathrm{coop}}, \mu\epsilon)$, where $\Delta_r^{\mathrm{coop}}$ is a map $H \to H \otimes^{B^{\mathrm{op}}} H := {}_{\blacktriangleright}H \otimes H_{\blacktriangleleft} \simeq H \otimes_A H$ by means of $\nu$, and $\Delta_\ell^{\mathrm{coop}} : H \to H \otimes_{A^{\mathrm{op}}} H := {}_{\triangleright}H \otimes H_{\triangleleft} \simeq H \otimes^B H$ via $\mu$.*

*In particular, if $H^\ell$ and $H^r$ denote the underlying left and right bialgebroids of $H$, respectively, the pair $(S, \partial s^\ell)$ is a morphism $H^\ell \to (H^r)^{\mathrm{op}}_{\mathrm{coop}}$ of left bialgebroids and $(S, \epsilon s^r)$ is a morphism $H^r \to (H^\ell)^{\mathrm{op}}_{\mathrm{coop}}$ of right bialgebroids.*

**2.6.5 Remark** (*i*) If the antipode is invertible, one can make an analogous statement about $S^{-1}$, cf. [B1, BSz2]. Moreover, using (*i*) of the preceding proposition, one may now consider $S$ both as an $(A, A)$-bimodule morphism ${}_{\blacktriangleright}H^\ell{}_{\triangleleft} \to {}_{\triangleright}H^\ell{}_{\blacktriangleleft}$ and a $(B, B)$-bimodule morphism ${}_{\blacktriangleright}H^r{}_{\triangleleft} \to {}_{\triangleright}H^r{}_{\blacktriangleleft}$.

(*ii*) We want to stress here that $S$ being an algebra anti-homomorphism is a *consequence* of the antipode axioms in Definition 2.6.1 and also that the antipode is *unique* [B1] (if it exists), provided left and right bialgebroid structures were given.

(*iii*) If $S^2 = \mathrm{id}$, one obtains $\mu = \nu$ and hence a canonical identification of $A^{\mathrm{op}}$ with $B$.

Define the maps

$$\begin{array}{llllll} S_A^{\otimes 2}: H_{\triangleleft} \otimes_A {}_{\triangleright}H & \to & H_{\blacktriangleleft} \otimes^B {}_{\blacktriangleright}H, & h \otimes_A h' & \mapsto & Sh' \otimes^B Sh \\ S_{\otimes 2}^B: H_{\blacktriangleleft} \otimes^B {}_{\blacktriangleright}H & \to & H_{\triangleleft} \otimes_A {}_{\triangleright}H, & h \otimes^B h' & \mapsto & Sh' \otimes_A Sh, \end{array}$$

where the tensor products (2.1.1) and (2.5.1) are used. These maps also have 'higher' analogues for $n$ factors, consisting in totally reversing the order followed by applying the antipode. The preceding proposition can then be given as a table by (ignore the right hand side if the antipode is not invertible):

$$\begin{array}{cccc} s^r \partial s^\ell = S s^\ell & s^\ell \epsilon s^r = S s^r & s^r \partial t^\ell = S^{-1} s^\ell & s^\ell \epsilon t^r = S^{-1} s^r \\ t^r \partial s^\ell = S t^\ell & t^\ell \epsilon s^r = S t^r & t^r \partial t^\ell = S^{-1} t^\ell & t^\ell \epsilon t^r = S^{-1} t^r \\ \partial s^\ell \epsilon = \partial S & \epsilon s^r \partial = \epsilon S & \partial t^\ell \epsilon = \partial S^{-1} & \epsilon t^r \partial = \epsilon S^{-1} \\ S_A^{\otimes 2} \Delta_\ell = \Delta_r S & S_{\otimes 2}^B \Delta_r = \Delta_\ell S & (S_{\otimes 2}^B)^{-1} \Delta_\ell = \Delta_r S^{-1} & (S_A^{\otimes 2})^{-1} \Delta_r = \Delta_\ell S^{-1}. \end{array} \tag{2.6.10}$$

We now collect a list of basic technical identities involving the antipode (which can be ignored on a first reading; again, ignore the last three lines if the antipode is not invertible).

**2.6.6 Lemma** *For a Hopf algebroid $H$ with the above structure maps, the following identities hold.*

$$\begin{array}{rlrl} m_H(S \otimes s^\ell \epsilon)\Delta_\ell &= S, & m_H(s^r \partial \otimes S)\Delta_r &= S, \\ m_{H^{\mathrm{op}}}(S^2 \otimes t^\ell \epsilon S^2)\Delta_\ell &= S^2, & m_{H^{\mathrm{op}}}(t^r \partial S^2 \otimes S^2)\Delta_r &= S^2, \\ m_{H^{\mathrm{op}}}(S^2 \otimes S)\Delta_\ell &= t^r \partial S^2, & m_{H^{\mathrm{op}}}(S \otimes S^2)\Delta_r &= t^\ell \epsilon S^2, \\ & & & \\ m_{H^{\mathrm{op}}}(\mathrm{id}_H \otimes S^{-1})\Delta_\ell &= t^r \partial, & m_{H^{\mathrm{op}}}(S^{-1} \otimes \mathrm{id}_H)\Delta_r &= t^\ell \epsilon, \\ m_{H^{\mathrm{op}}}(t^\ell \epsilon \otimes S^{-1})\Delta_\ell &= S^{-1}, & m_{H^{\mathrm{op}}}(S^{-1} \otimes t^r \partial)\Delta_r &= S^{-1}, \\ m_H(S^{-1} \otimes S^{-2})\Delta_\ell &= s^r \partial S^{-2}, & m_H(S^{-2} \otimes S^{-1})\Delta_r &= s^\ell \epsilon S^{-2}. \end{array} \tag{2.6.11}$$

*Here $m_{H^{\mathrm{op}}}$ is the multiplication in the opposite ring of $H$.*

PROOF: All identities follow from a straightforward computation using (2.6.10), (2.6.4) as well as the comonoid identities (2.1.8) and (2.5.6). As an example, we prove

$$\begin{aligned} m_{H^{\mathrm{op}}}(S^{-1} \otimes t^r \partial)\Delta_r &= m_H(t^r \partial S \otimes \mathrm{id}_H)\Delta_\ell S^{-1} \\ &= m_H(t^r \partial s^\ell \epsilon \otimes \mathrm{id}_H)\Delta_\ell S^{-1} \\ &= m_H(s^\ell \epsilon \otimes \mathrm{id}_H)\Delta_\ell S^{-1} = S^{-1}. \end{aligned}$$

□

**2.6.7 Remarks** (*i*) For the identities given in the preceding lemma, the same comment applies we made in Remark 2.6.2(*iv*): strictly speaking, the operations $m_H$ and $m_{H^{\mathrm{op}}}$ refer to multiplication in one of the various underlying ring structures. For example, the left hand side of $m_{H^{\mathrm{op}}}(S^{-1} \otimes \mathrm{id}_H)\Delta_r = t^\ell \epsilon$ can be decomposed as

$$H \xrightarrow{\Delta_\ell} H_{\blacktriangleleft} \otimes {}_{\blacktriangleright}H \xrightarrow{S^{-1} \otimes \mathrm{id}_H} H_{\triangleleft} \otimes {}_{\blacktriangleright}H \xrightarrow{m_{H^{\mathrm{op}}}} H,$$

where $H_{\blacktriangleleft} \otimes {}_{\blacktriangleright}H = H \otimes^B H$ is given as in (2.5.1), whereas

$$H_{\triangleleft} \otimes {}_{\blacktriangleright}H = H \otimes_k H / \mathrm{span}\{t^r(b)h \otimes h' - h \otimes h' t^r(b) \mid b \in B,\ h, h' \in H\}$$

is the tensor product naturally associated to the $B^{\mathrm{op}}$-ring $(H, t^r)$, and $m_{H^{\mathrm{op}}}$ is to be understood as multiplication in this ring.

(*ii*) If $(S, m_S)$ is a $k$-algebra and $(C, \Delta_C)$ a $k$-coalgebra for some commutative ring $k$, the space $\mathrm{Hom}_k(C, S)$ can be given a $k$-algebra structure by means of the *convolution product* $(f * f') = m_S(f \otimes f')\Delta_C$ for $f, f' \in \mathrm{Hom}_k(C, S)$ (see e.g. [Str]. We do not address the question in detail how this can be transferred to the case of monoids and comonoids in bimodule categories (see, however, [B3, Section 4.5.2]), but the first line of equalities in the preceding Lemma reflects that $\epsilon$ and $\partial$ are counits, that is, units in some generalised convolution algebra. However, in the first equation $S$ is seen as an $(A, A)$-bimodule map on $_\blacktriangleright H^\ell{}_\triangleleft$ , whereas $\epsilon$ is an $(A, A)$-bimodule map with respect to $_\triangleright H^\ell{}_\triangleleft$ (cf. (2.1.3) and (2.1.5) for the notation). This gives a hint why $\epsilon$ is only a 'right unit' for $S$; analogously in the second case, $S$ is an $(B, B)$-bimodule map on $_\blacktriangleright H^r{}_\triangleleft$ and $\partial$ on $_\blacktriangleright H^r{}_\blacktriangleleft$ (cf. (2.5.2) and (2.5.5) for the notation). In these terms, also (2.6.4) can be reformulated saying that $S$ is a convolution inverse to $\mathrm{id}_H$, but from left and right in two different ways (involving different coalgebra structures and over two different base algebras). As a consequence and in contrast to Hopf algebras, in general there is no information for terms of the form, say, $h_{(1)}Sh_{(2)}$ or $Sh^{(1)}h^{(2)}$.

**2.6.8 Alternative Formulation** Evidently, constructing a Hopf algebroid by left and right bialgebroid structures plus an antipode leads to some redundancy. Alternatively, one may start with a left bialgebroid $(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ only, plus a bijective anti-algebra isomorphism $S : U \to U$ subject to

(*i*) $St^\ell = s^\ell$,

(*ii*) $m_U(S \otimes \mathrm{id}_U)\Delta_\ell = t^\ell \epsilon S$,

(*iii*) $S^{\otimes 2}_A \Delta_\ell S^{-1} = (S^{A^{\mathrm{op}}}_{\otimes 2})^{-1} \Delta_\ell S$

(*iv*) $(\Delta_\ell \otimes \mathrm{id}_H) S^{\otimes 2}_A \Delta_\ell S^{-1} = (S^{\otimes 2}_A \Delta_\ell S^{-1} \otimes \mathrm{id}_H)\Delta_\ell$.

It then follows from Proposition 2.6.4 that (up to a trivial bialgebroid isomorphism) the set

$$(H, B, Ss^\ell \nu^{-1}, s^\ell \nu^{-1}, S^{\otimes 2}_A \Delta_\ell S^{-1}, \nu \epsilon S^{-1})$$

constitutes a right bialgebroid (where $\nu : A^{\mathrm{op}} \to B$ is an arbitrary isomorphism). Together with the given data of a left bialgebroid and the map $S$ this yields a Hopf algebroid, as in Definition 2.6.1. However, for an arbitrary isomorphism $\mu : A^{\mathrm{op}} \to B$, also

$$(H, B, t^\ell \mu^{-1}, S^{-1} t^\ell \mu^{-1}, (S^B_{\otimes 2})^{-1} \Delta_\ell S, \mu \epsilon S),$$

fulfills the requirements; see [BSz2, Prop. 4.2] for yet another formulations of Hopf algebroids in this sense, and their mutual equivalence. However, in our opinion the version cited here serves best for maintaining a certain transparency in Hopf-cyclic (co)homology.

We already mentioned that if left and right bialgebroid structures are given, the antipode is unique if it exists. However, in case that only a left bialgebroid structure is given, there is a certain ambiguity in the choice of the antipode (which corresponds to the choice of certain connections in Section 4.2). This is different from what is known for (weak) Hopf algebras and is also reflected in the following definition, which allows Hopf algebroid isomorphisms that 'ignore' the antipode.

**2.6.9 Definition** A *Hopf algebroid (iso)morphism* $(H, S) \to (H', S')$ is an (iso)morphism $(\phi, \psi)$ of the underlying left bialgebroid structure. It is called *strict* if $\psi$ commutes with the respective antipodes, that is, $S'\psi = \psi S$.

**2.6.10 Primitive Elements** Formally, a Hopf algebroid has two kinds of primitive elements (cf. §2.1.5) with respect to $\Delta_\ell$ and $\Delta_r$, denoted $P^\ell H$ and $P^r H$. We have

$$\begin{aligned} SX &= -X + s^r \partial X \quad \forall X \in P^\ell H && (2.6.12) \\ SX' &= -X' + s^\ell \epsilon X' \quad \forall X' \in P^r H, && (2.6.13) \end{aligned}$$

since $P^r H$ is generally not contained in $\ker \epsilon$, and $P^\ell H$ is not in $\ker \partial$ either, again in contrast to the case of Hopf algebras.

**2.6.11 Grouplike Elements** Similarly, one has two kinds of grouplike elements (cf. §2.1.5) for $H$ with respect to the two underlying bialgebroids, denoted by $G^\ell H$ and $G^r H$. These may be called left and right grouplike, respectively. Proposition 2.6.4 then entails

$$\sigma \in G^\ell H \iff S\sigma \in G^r H.$$

However, only the set $GH := G^\ell H \cap G^r H$ forms a group.

**2.6.12 Proposition** *The sets $G^\ell U$ and $G^r V$ for left and right bialgebroids $U$ and $V$, respectively, are multiplicative monoids with unit $1_U$ and $1_V$, respectively. If $H$ is a Hopf algebroid, only an element $\sigma$ that is both left and right grouplike has (two-sided) inverse $S\sigma$, hence the set $GH = G^\ell H \cap G^r H$ is a group.*

Compare, however, the notions of *weakly grouplike* elements in [Mrč2] and *groupoidlike* elements in [Kap].

**2.6.13 Comparison with Alternative Definitions**

($i$) In [Lu], a Hopf algebroid is defined to be a certain 'bialgebroid' with a concept of an antipode plus some extra data. The definition of a bialgebroid in [Lu] appears almost identical to the one we use here, the only difference being that the axiom (2.1.4) is replaced by asking $\ker \epsilon$ to be a left ideal in $H$. However, this last statement can be shown [BrzMi] to be equivalent to (2.1.4). Next, such a bialgebroid $(H, A, s, t, \Delta, \epsilon)$ is equipped with an anti-algebra homomorphism $S : H \to H$, subject to the properties

(a) $St = s$,

(b) $m_H(S \otimes \mathrm{id}_H)\Delta = t\epsilon S$,

(c) $m_H(\mathrm{id}_H \otimes S)\gamma\Delta = s\epsilon$, where a section $\gamma : H \otimes_A H \to H \otimes H$ of the natural projection $H \otimes H \to H \otimes_A H$ is required to give a meaning to this identity,

is called a *Hopf algebroid* in [Lu].

($ii$) In [KhR3] a Hopf algebroid (baptised 'para-Hopf algebroid' there) consists of a sextuple $(H, A, s, t, \Delta, \epsilon)$ fulfilling the left bialgebroid axioms we use here, again without (2.1.4). Furthermore, one requires a map $T : H \to H$ obeying the conditions

(a) $T : H \to H$ is an anti-algebra homomorphism,

(b) $Tt = s$,

(c) $m_H(T \otimes \mathrm{id}_H)\Delta = t\epsilon T$,

(d) $T^2 = \mathrm{id}_H$, implying however [KhR3, Lem. 2.1] the 'missing' condition (2.1.4), i.e., $\epsilon(h'h) = \epsilon(h's(h)) = \epsilon(h't(h))$,

(e)
$$(Th_{(1)})_{(1)}h_{(2)} \otimes_A (Th_{(1)})_{(2)} = 1 \otimes_A Th. \tag{2.6.14}$$

Compared to the formulation in §2.6.8, the notion of Hopf algebroid as given in (*i*) above has the obvious handicap that the *additional* antipode axiom (*i*)(c) requires a section $\gamma$ of the natural projection $H \otimes_k H \to H_\triangleleft \otimes_A {}_\triangleright H$, which does not come into play quite naturally and seems to be deprived of any geometrical meaning or justification; see [BSz2, KSz, B1] for a discussion of this complex of problems for bialgebroids associated to a depth-2 Frobenius extension of rings. There, the authors also give an example of Hopf algebroid which is not a Hopf algebroid in the sense of point (*i*) above. Finally, as observed in [KhR3], this approach does not seem to be suitable for defining (Hopf-)cyclic cohomology. As for the second approach (point (*ii*) above), it was shown in [BSz2] to be contained in the concept we use here (for an invertible antipode), but, as we think, also has certain disadvantages in dealing with Hopf-cyclic homology; see §5.2.12 for a discussion.

**2.6.14 Hopf Algebroids versus Left Hopf Algebroids** For a Hopf algebroid $H$ with structure maps as before, one checks that [BSz2]

$$\beta^{-1}(h \otimes_A h') = h_+ \otimes_{A^{\mathrm{op}}} h_- h' = h^{(1)} \otimes_{A^{\mathrm{op}}} Sh^{(2)} h', \tag{2.6.15}$$

is an inverse of the Hopf-Galois map (2.2.1). Hence every (*Böhm-Szlachányi-*)Hopf algebroid is a left Hopf algebroid over $A$ (cf. Definition 2.2.1). However, the opposite implication is not true, as we will show in Example 4.2.13: this answers a question posed in [B3] whether every left Hopf algebroid over $A$ is the constituent left bialgebroid in a Hopf algebroid.

As the reader may expect, the Hopf-Galois map (2.2.1) is not the only possibility of such a kind, given the abundance of tensor products in this context. Indeed, in [BSz2, Prop. 4.2] a Hopf algebroid with invertible antipode is equivalently characterised as follows: a pair of a left bialgebroid over $A$ and a right bialgebroid over $B \simeq A^{\mathrm{op}}$, subject to (2.6.1) and (2.6.2), such that not only (2.2.1) is bijective, but also the map

$$\alpha : {}_{\triangleright}U \otimes_{A^{\mathrm{op}}} U_{\blacktriangleleft} \to U_{\triangleleft} \otimes_A {}_{\triangleright}U, \quad u \otimes_{A^{\mathrm{op}}} v \mapsto v_{(1)}u \otimes_A v_{(2)}. \tag{2.6.16}$$

For a Hopf algebroid with invertible antipode, the inverse $\alpha^{-1}$ is in this case given by $u \otimes_A v \mapsto S^{-1}(v^{(1)})u \otimes_{A^{\mathrm{op}}} v^{(2)}$.

However, both $\beta$ and $\alpha$ use the left coproduct of the underlying left bialgebroid. We expect that one can equivalently characterise a Hopf algebroid by $\beta$ and a Hopf-Galois map $\alpha'$ which rather uses some right coproduct. More precisely, a left Hopf algebroid and a *right Hopf algebroid* (subject to some compatibility conditions) should determine a (double-sided) Hopf algebroid: hence our new terminology for $\times_A$-Hopf algebras.

# Chapter 3

# Constructions

## 3.1 Left and Right Duals of Bialgebroids

The main result in this section is to show that the classical correspondence between modules and comodules for algebras and their $\mathrm{Hom}$-duals extends to more possibilities when considering bialgebroids, as we show in Theorem 3.1.11 and Proposition 3.1.9. This is a consequence of the fact that each left bialgebroid comes equipped with *two* natural duals. *Both* of them can be given—under certain projectivity assumptions—a right bialgebroid structure [KSz, Propositions 2.5 and 2.6], as we recall below, cf. §3.1.6.

Recall that the notion of Hopf algebra is self-dual [Sw1], so if one can define a dual of H (which is e.g. always possible if H is finite-dimensional, for $k$ a field), then it is automatically a Hopf algebra. However, this is not necessarily the case for Hopf algebroids, for which duality is in fact considerably more intricate (see [BSz2]), which is why we do not treat this complex of problems here. Nevertheless, we conclude this section by an analogue of the classical statement that a character on a Hopf algebra correspond to a grouplike element on its dual (see Proposition 3.1.14).

Since any bialgebroid carries *four* natural $A$-module structures, we often take the liberty of somewhat redundantly indicating the module structure in question in the hope of increasing clarity. For further notation see the conventions on page 9 at the end of the Introduction as well as §A.1.1 for standard constructions on bimodules (implicitly used in the following).

**3.1.1 Definition** Let $U$ be a left bialgebroid with structure maps as before.

(*i*) The *left* dual of $U$ is the space

$$U_* := \mathrm{Hom}_{(A,-)}({}_{\triangleright}U, {}_AA) = \{\phi : U \to A \mid \phi(s^\ell au) = a\phi(u), \forall\, a \in A, u \in U\}.$$

(*ii*) The *right* dual of $U$ is the space

$$U^* := \mathrm{Hom}_{(-,A)}(U_{\triangleleft}, A_A) = \{\psi : U \to A \mid \psi(t^\ell au) = \psi(u)a, \forall\, a \in A, u \in U\}.$$

**3.1.2 Notation** We write $\phi(u) =: \langle \phi, u\rangle$ for $\phi \in U_*$, $u \in U$, and also $\psi(u) =: \langle \psi, u\rangle$ for $\phi \in U_*$, $u \in U$, whenever we think that this may increase clarity.

Before we proceed, we will explain how these duals can be made into bialgebroids, following [KSz]. To this end, we give the details of their ring structures, their various $A$-modules structures and their coring structures.

**3.1.3 Ring Structures on $U_*$ and $U^*$** We recall from [KSz] that both duals can be equipped with a product structure:

(*i*) 'Target' transposing of the comonoid structure of $U$ yields the following monoid structure $m_{U_*}$ on $U_*$ with the (two-sided) unit $\epsilon$:

$$m_{U_*}(\phi \otimes \phi')(u) = (\phi\phi')(u) := \phi'\big(m_{U^{\mathrm{op}}}(\mathrm{id} \otimes t^\ell\phi)\Delta_\ell u\big), \tag{3.1.1}$$

where $\phi, \phi' \in U_*,\ u \in U$.

(*ii*) 'Source' transposing of the comonoid structure of $U$ yields the following monoid structure $m_{U^*}$ on $U^*$ with the (two-sided) unit $\epsilon$:

$$m_{U^*}(\psi \otimes \psi')(u) = (\psi\psi')(u) := \psi'\big(m_U(s^\ell\psi \otimes \mathrm{id})\Delta_\ell u\big), \tag{3.1.2}$$

where $\psi, \psi' \in U^*,\ u \in U$.

**3.1.4 Remark** There are some subtleties attached to the fact that $m_{U_*}$ and $m_{U_*}$ are well-defined as maps acting on a certain tensor product of the (left or right) dual with itself. We refer to the proof of Proposition 3.1.9 for the technical details, and in particular to (3.1.16)–(3.1.19) and (3.1.24)–(3.1.26) for further explanation why these ring structures make sense.

### 3.1.5 A-Module Structures on Left and Right Duals

Beyond the product structure on the duals, we also want them to become $A^{\mathrm{e}}$-rings, hence we equip them with source and target maps. As a straightforward consequence of how Hom-spaces of bimodules become (bi)modules again (cf. §A.1.1), left and right duals carry four $A$-module structures each:

(*i*) In case of the left dual $U_*$, one encounters the following four situations. Let $u \in U,\ \phi \in U_*$ and $a \in A$.

(a) The *left dual source map* $s^r_*$ is defined as

$$s^r_* : A \to U_*, \quad a \mapsto \epsilon(t^\ell(a)(\cdot)) = \epsilon(\cdot)a. \tag{3.1.3}$$

As in (2.5.2), write $\phi s^r_*(a) =: \phi \blacktriangleleft a$. Then $(\phi \blacktriangleleft a)(u) = (\phi s^r_*(a))(u) = \phi(u)a$, and this is the $A$-module structure that arises from the pair $({}_{\triangleright}U\,,\,{}_AA_A)$ of $A$-(bi)modules, cf. §A.1.1. Analogously to §A.1.1, we denote this situation by $({}_{\triangleright}U\,,\,{}_AA_A) \Longrightarrow U_{*\,\blacktriangleleft}$.

(b) The *left dual target map* $t^r_*$ is defined as

$$t^r_* : A \to U_*, \quad a \mapsto \epsilon((\cdot)t^\ell(a)). \tag{3.1.4}$$

As in (2.5.2), write $\phi t^r_*(a) =: a \blacktriangleright \phi$. Hence $(a \blacktriangleright \phi)(u) = (\phi t^r_*(a))(u) = \phi(us^\ell a)$, and this corresponds to the situation $({}_{\triangleright}U_{\blacktriangleleft}\,,\,{}_AA) \Longrightarrow {}_{\blacktriangleright}U_*$.

(c) $({}_{\blacktriangleright\,,\,\triangleright}U\,,\,{}_AA) \Longrightarrow U_{*\,\triangleleft}$, given by $(\phi \triangleleft a)(u) := \phi(ut^\ell a) = (t^r_*(a)\phi)(u)$.

(d) $({}_{\triangleright}U_{\triangleleft}\,,\,{}_AA) \Longrightarrow {}_{\triangleright}U_*$, given by $(a \triangleright \phi)(u) := \phi(t^\ell au) = (s^r_*(a)\phi)(u)$.

(*ii*) In case of the right dual $U^*$, things read as follows. Let $u \in U,\ \psi \in U^*$ and $a \in A$.

(a) Define the *right dual source map* $s^*_r$ by

$$s^*_r : A \to U^*, \quad a \mapsto \epsilon((\cdot)s^\ell(a)), \tag{3.1.5}$$

and write $\psi s^*_r(a) =: \psi \blacktriangleleft a$. Hence $(\psi \blacktriangleleft a)(u) = (\psi s^*_r(a))(u) = \psi(ut^\ell a)$, corresponding to the situation $({}_{\blacktriangleright}U_{\triangleleft}\,,\,A_A) \Longrightarrow U^{*}{}_{\blacktriangleleft}$.

(b) Define the *right dual target map* $t^*_r$ by

$$t^*_r : A \to U^*, \quad a \mapsto \epsilon(s^\ell(a)(\cdot)) = a\epsilon(\cdot), \tag{3.1.6}$$

and write $\psi t^*_r(a) =: a \blacktriangleright \psi$. Hence $(a \blacktriangleright \psi)(u) = (\psi t^*_r(a))(u) = a\psi(u)$, corresponding to $(U_{\triangleleft}\,,\,{}_AA_A) \Longrightarrow {}_{\blacktriangleright}U^*$.

(c) $(U_{\triangleleft\,,\,\blacktriangleleft}\,,\,A_A) \Longrightarrow {}_{\triangleright}U^*$, given by $(a \triangleright \psi)(u) := (s^*_r(a)\psi)(u) = \psi(us^\ell a)$.

(d) $({}_{\triangleright}U_{\triangleleft}\,,\,A_A) \Longrightarrow U^{*}{}_{\triangleleft}$, given by $(\psi \triangleleft a)(u) := (t^*_r(a)\psi)(u) = \psi(s^\ell au)$.

**3.1.6 Right Bialgebroid Structures on $U_*$ and $U^*$** We recall from [KSz] how the left and right dual can *both* be made into a *right* bialgebroid:

($i$) If ${}_{\triangleright}U$ is finitely generated projective over $A$, the left dual $U_*$ can be given the structure of a right bialgebroid over $A_* \equiv A$ with the following structure maps: the $A^{\mathrm{e}}$-ring structure is determined by the product (3.1.1) and source and target are as in (3.1.3), (3.1.4). The $A$-coring structure $(U_*, \Delta^r_*, \partial_*)$ is given by the following (right) coproduct and (right) counit:

$$\begin{array}{ll} \Delta^r_* \ : U_* \to \mathrm{Hom}_{(A,-)}({}_{\triangleright}(U_{\blacktriangleleft} \otimes {}_{\triangleright}U), {}_A A), & \phi \mapsto \{u \otimes_A u' \mapsto \phi(uu')\}, \\ \partial_* \ : U_* \to A, & \phi \mapsto \phi(1_U). \end{array} \tag{3.1.7}$$

To see that $\Delta^r_*$ is really a right coproduct, i.e. a map $U \to U_{*\blacktriangleleft} \otimes {}_{\blacktriangleright}U_*$, where $U_{*\blacktriangleleft} \otimes {}_{\blacktriangleright}U_*$ is defined with respect to (3.1.3) and (3.1.4), we will rewrite it in a different way that is also more convenient for our following considerations: by projectivity of ${}_{\triangleright}U$, elements in $U$ can be decomposed according to §A.1.2 as $u = \sum_i s^\ell(e^i(u))e_i$, where $\{e_i\}_{1\leq i\leq n} \in U$, $\{e^i\}_{1\leq i\leq n} \in U_*$ is a dual basis of ${}_{\triangleright}U$. Furthermore, introduce the following left $U$-module structures on $U_*$:

$$(u \rightharpoondown \phi)(u') := \phi(u'u) \qquad \text{for } u, u' \in U,\ \phi \in U_*. \tag{3.1.8}$$

Using the fact that $U_{*\blacktriangleleft} \otimes {}_{\blacktriangleright}U_* \to \mathrm{Hom}_{(A,-)}({}_{\triangleright}(U_{\blacktriangleleft} \otimes {}_{\triangleright}U), {}_A A)$, given by

$$(\phi \otimes^A \phi')(u \otimes_A u') := \phi'(us^\ell(\phi u')),$$

is an isomorphism because of the projectivity of $U$ (see [KSz] for a proof, or our similar considerations in the proof of Proposition 3.1.9), one may write

$$(\Delta^r_*\phi)(u, u') = \phi(uu') = \sum_i \phi(us^\ell e^i(u')e_i) = \sum_i (e_i \rightharpoondown \phi)(us^\ell e^i(u')).$$

Hence instead of (3.1.7), for the right coproduct and right counit one finds the more systematic form

$$\begin{array}{ll} \Delta^r_* \ : U_* \to U_{*\blacktriangleleft} \otimes^A {}_{\blacktriangleright}U_*, & \phi \mapsto \sum_i e^i \otimes (e_i \rightharpoondown \phi), \\ \partial_* \ : U_* \to A, & \phi \mapsto \phi(1_U). \end{array} \tag{3.1.9}$$

($ii$) Likewise, if $U_{\triangleleft}$ is finitely generated projective over $A$, the right dual $U^*$ is a right bialgebroid as well over the same base $A^* \equiv A$. The $A^{\mathrm{e}}$-ring structure is defined by the product (3.1.2) and source and target as in (3.1.5), (3.1.6). The $A$-coring structure $(U^*, \Delta^*_r, \partial^*)$ is given by the following (right) coproduct and (right) counit:

$$\begin{array}{ll} \Delta^*_r \ : U^* \to \mathrm{Hom}_{(-,A)}((U_{\triangleleft} \otimes {}_{\blacktriangleright}U)_{\triangleleft}, A_A), & \psi \mapsto \{u \otimes_A u' \mapsto \phi(uu')\}, \\ \partial^* \ : U^* \to A, & \psi \mapsto \psi(1_U). \end{array}$$

Again, by projectivity of $U_{\triangleleft}$ we may rewrite this, decomposing elements in $U_{\triangleleft}$ as $v = \sum_j t^\ell(f^j(v))f_j$, where $\{f_j\}_{1\leq j\leq m} \in U$, $\{f^j\}_{1\leq j\leq m} \in U^*$ is a dual basis for $U_{\triangleleft}$. With the left $U$-action on $U^*$ given by

$$(u \rightharpoonup \psi)(u') := \psi(u'u) \qquad \text{for } u, u' \in U,\ \psi \in U^*, \tag{3.1.10}$$

and the isomorphism $U^*{}_{\blacktriangleleft} \otimes {}_{\blacktriangleright}U^* \to \mathrm{Hom}_{(-,A)}((U_{\triangleleft} \otimes {}_{\blacktriangleright}U)_{\triangleleft}, A_A)$ given by

$$(\psi \otimes^A \psi')(u \otimes_A u') := \psi(ut^\ell(\psi' u')),$$

where $U^*{}_{\blacktriangleleft} \otimes {}_{\blacktriangleright}U^*$ is defined with respect to (3.1.5) and (3.1.6), we have the following expressions for the right coproduct and the right counit:

$$\begin{array}{ll} \Delta^*_r \ : U^* \to U^*{}_{\blacktriangleleft} \otimes {}_{\blacktriangleright}U^*, & \psi \mapsto \sum_j (f_j \rightharpoonup \psi) \otimes^A f^j, \\ \partial^* \ : U^* \to A, & \psi \mapsto \psi(1_U). \end{array} \tag{3.1.11}$$

**3.1.7 Remark** Under analogous assumptions, a right bialgebroid has two duals as well, which can be made into left bialgebroids.

**3.1.8 Module-Comodule Correspondence** Classically [AW, Cart], if $U$ happens to be a finite dimensional algebra over a field $k$ and $DU := \mathrm{Hom}_k(U, k)$ is its dual (carrying the structure of a coassociative coalgebra [Sw1, 1.1.2]), right $DU$-modules naturally correspond to left $U$-comodules, i.e., one has a categorical equivalence

$$\mathbf{Mod}\text{-}DU \simeq U\text{-}\mathbf{Comod}.$$

The situation in the bialgebroid context is richer, as summarised by the main result of this section, Theorem 3.1.11 below. The following proposition explains how module and comodule structures imply each other.

**3.1.9 Proposition** *Let $U$ be a left bialgebroid, as above.*

(*i*) *Given a right $U$-comodule $N \in$ **Comod**-$U$ with coaction ${}_N\Delta : N \to N \otimes_A U$, $n \mapsto n_{(0)} \otimes_A n_{(1)}$, the assignment*

$$\mu_*^N : N \otimes U_* \to N, \quad n \otimes_A \phi \mapsto n_{(0)}\phi(n_{(1)}) \tag{3.1.12}$$

*defines a right $U_*$-module structure on $N$. Conversely, for each right $U_*$-module $M$ with $U_*$-action $(m, \phi) \mapsto m\phi$, the assignment*

$${}_M\Delta(m)(\phi) := m\phi \quad \forall m \in M,\ \phi \in U_* \tag{3.1.13}$$

*defines a map ${}_M\Delta : M \to \mathrm{Hom}_{(-,A)}(U_*, M)$, and if ${}_\triangleright U$ is finitely generated projective over $A$, this yields* a right *$U$-comodule structure ${}_M\Delta : M \to M \otimes_A U$ on $M$. In particular, these processes of assigning modules and comodules are inverse to each other.*

(*ii*) *Similarly, given a left $U$-comodule $N$ with coaction $\Delta_N : N \to U \otimes_A N$, $n \mapsto n_{(-1)} \otimes_A n_{(0)}$, the assignment*

$$\mu_N^* : N \otimes U^* \to N, \quad n \otimes_A \psi \mapsto \psi(n_{(-1)})n_{(0)} \tag{3.1.14}$$

*defines a right $U^*$-module structure on $N$. Conversely, for each right $U^*$-module $M$ with $U^*$-action $(m, \psi) \mapsto m\psi$, the assignment*

$$\Delta_M(m)(\psi) := m\psi \quad \forall m \in M,\ \psi \in U^*, \tag{3.1.15}$$

*defines a map $\Delta_M : M \to \mathrm{Hom}_{(A,-)}(U^*, M)$, and if $U_\triangleleft$ is finitely generated projective over $A$, this yields* a left *$U$-comodule structure $\Delta_M : M \to U \otimes_A M$ on $M$. Again, these two processes of assigning modules and comodules are inverse to each other.*

**3.1.10 Remark** This result is not totally obvious since the category of $(A, A)$-bimodules is not symmetric and even if $A$ is commutative, parts (*i*) and (*ii*) will be distinguished by both the facts that source and target maps do not need to coincide, and in any case do not necessarily map into the centre of $U$.

PROOF: To check the respective comodule identities, one expresses the comonoid structure on $U$ in terms of the monoid structures on $U_*$ and $U^*$ to obtain (3.1.13) and (3.1.15), similar to [KSz], but in a sense dualised again (we give all details only in case of the left dual, inasmuch this case is less expected, and leave the rigorous elaboration of the second one to the reader).

(*i*) To prove the first statement, we need to show that $\mu_*^N(\mathrm{id}_N \otimes m_{U_*}) = \mu_*^N(\mu_*^N \otimes \mathrm{id}_{U_*})$, where $m_{U_*}$ is as in (3.1.1). One has

$$\begin{aligned}
\mu_*^N(\mathrm{id}_N \otimes m_{U_*})(n \otimes_A \phi \otimes^A \phi') &= n_{(0)}\langle \phi\phi', n_{(1)}\rangle \\
&= n_{(0)}\langle \phi', t^\ell(\langle \phi, n_{(2)}\rangle)n_{(1)}\rangle = \big(n_{(0)}\phi(n_{(1)})\big)_{(0)}\langle \phi', \big(n_{(0)}\langle \phi, n_{(1)}\rangle\big)_{(1)}\rangle \\
&= \mu_*^N(\mu_*^N \otimes \mathrm{id}_{U_*})(n \otimes_A \phi \otimes^A \phi') = n_{(0)}\langle \phi', t^\ell(\langle \phi, n_{(2)}\rangle)n_{(1)}\rangle,
\end{aligned}$$

since $N$ is assumed to be a right $U$-comodule. Hence the two expressions coincide.

For the second claim, consider the $(A, A)$-bimodule ${}_\triangleright U_{*\blacktriangleleft}$ ('source-source'). The corresponding tensor product $U_* \otimes^A U_* := U_{*\blacktriangleleft} \otimes {}_\triangleright U_*$ carries an $(A, A)$-bimodule structure in the standard way, given by ${}_\triangleright(U_* \otimes^A U_*)_\blacktriangleleft = {}_\triangleright U_* \otimes^A U_{*\blacktriangleleft}$. Assuming that ${}_\triangleright U$ is finitely generated $A$-projective with dual basis $\{e_i\}_{1\leq i\leq n} \in U$, $\{e^i\}_{1\leq i\leq n} \in U_*$ as before, the map

$$\begin{aligned}
&U \otimes_A U \to \mathrm{Hom}_{(-,A)}((U_* \otimes^A U_*)_\blacktriangleleft, A_A), \\
&u \otimes_A u' \mapsto \big(\phi \otimes^A \phi' \mapsto \phi'(t^\ell(\phi u')u) = \langle \phi', t^\ell(\langle \phi, u'\rangle)u\rangle\big)
\end{aligned} \tag{3.1.16}$$

is an isomorphism. Its inverse is determined by calculating

$$\begin{aligned}
u \otimes_A u' &= \sum_i u \otimes_A s^\ell(e^i(u'))e_i \\
&\equiv \sum_i t^\ell(e^i(u'))u \otimes_A e_i = \sum_{i,j} s^\ell\big(\langle e^j, t^\ell(\langle e^i, u'\rangle)u\rangle\big)e_j \otimes_A e_i.
\end{aligned}$$

Hence the inverse can be expressed as

$$\Phi \mapsto \sum_{i,j} s^{\ell}(\Phi(e^i \otimes^A e^j))e_j \otimes_A e_i.$$

Now for each $u \in U$ the map

$$\phi \otimes^A \phi' \mapsto \langle \phi\phi', u \rangle \tag{3.1.17}$$

lies in $\mathrm{Hom}_{(-,A)}((U_* \otimes^A U_*)_{\blacktriangleleft}, A_A)$, as follows from §3.1.5(*i*)(a) and the relation of the monoid structure on $U_*$. The identity

$$\langle \phi\phi', u \rangle = \langle \phi \otimes^A \phi', u_{(1)} \otimes_A u_{(2)} \rangle = \langle \phi', t^{\ell}(\langle \phi, u_{(2)} \rangle)u_{(1)} \rangle \tag{3.1.18}$$

can either be read from left to right, in which case it defines the ring structure $m_{U_*}$ on the dual in dependence of the coproduct on $U$, as was done in §3.1.1, i.e.,

$$m_{U_*} : U_* \otimes^A U_* \to U_*, \quad \phi \otimes^A \phi' \mapsto \phi\phi'. \tag{3.1.19}$$

Or, (3.1.18) can be read from right to left, so as to express the coproduct on $U$ using the product in $U_*$; this is the point of view we adopt here. The coproduct on $U$ then reads

$$\Delta_{\ell} u = \sum_{i,j} s^{\ell}(\langle e^i e^j, u \rangle)e_j \otimes_A e_i \quad \forall\, u \in U. \tag{3.1.20}$$

We need this formula to verify the comodule identities of ${}_M\Delta$: let $M$ be a right $U_*$-module with action $(m, \phi) \mapsto m\phi$, so that in particular $M$ is an $(A, A)$-bimodule. Define a map

$$M_A \otimes {}_{\triangleright}U \to \mathrm{Hom}_{(-,A)}({}_{\triangleright}U_{*\blacktriangleleft}, M_A), \quad m \otimes_A u \mapsto m\langle -, u \rangle. \tag{3.1.21}$$

Under the hypotheses that ${}_{\triangleright}U$ is finitely generated $A$-projective, this map is an isomorphism, with inverse

$$\mathrm{Hom}_{(-,A)}({}_{\triangleright}U_{*\blacktriangleleft}, M_A) \to M_A \otimes {}_{\triangleright}U, \quad f \mapsto \sum_i f(e^i) \otimes_A e_i. \tag{3.1.22}$$

Finally, define the *right* $U$-comodule structure on $M$

$${}_M\Delta : M \to \mathrm{Hom}_{(-,A)}({}_{\triangleright}U_{*\blacktriangleleft}, M_A), \quad {}_M\Delta(m)(\phi) := m\phi \quad \forall m \in M,\ \phi \in U_*,$$

that is, with the isomorphisms (3.1.22) this amounts to a map ${}_M\Delta : M \to M \otimes_A U$ that reads

$${}_M\Delta m = \sum_i m e^i \otimes_A e_i. \tag{3.1.23}$$

To check the comodule identities, calculate

$$({}_M\Delta \otimes \mathrm{id}_U){}_M\Delta m = \sum_{i,j} m(e^i e^j) \otimes_A e_j \otimes_A e_i,$$

and, by making use of (3.1.20),

$$\begin{aligned}
(\mathrm{id}_M \otimes \Delta_{\ell}){}_M\Delta m &= \sum_k m e^k \otimes_A \Delta_{\ell} e_k \\
&= \sum_{i,j,k} m e^k \otimes_A s^{\ell}(\langle e^i e^j, e_k \rangle)e_j \otimes_A e_i \\
&= \sum_{i,j,k} (m e^k)\langle e^i e^j, e_k \rangle \otimes_A e_j \otimes_A e_i \\
&= \sum_{i,j,k} m\big(e^k s^r_*(\langle e^i e^j, e_k \rangle)\big) \otimes_A e_j \otimes_A e_i \\
&= \sum_{i,j} m(e^i e^j) \otimes_A e_j \otimes_A e_i,
\end{aligned}$$

where we used the fact that $M$ is a right $U_*$-module and the projectivity of ${}_{\triangleright}U$, compare the construction of $s_*^r$ from §3.1.5(*i*)(a). Furthermore, if $R_A : (m, a) \mapsto ma$ denotes the right $A$-action $M$, we see that

$$R_A(\mathrm{id}_M \otimes \epsilon)_M\Delta m = \sum_i (me^i)\epsilon(e_i) = \sum_i m(e^i s_*^r(\epsilon e_i)) = m\epsilon = m,$$

since $\epsilon = 1_{U_*}$, and the right comodule identities are proven. It is now easy to see that ${}_M\Delta$ is a (right) $A$-module morphism (in fact an $(A, A)$-bimodule morphism under the left $A$-action (2.3.8)). With (3.1.21) one gets

$${}_M\Delta(ma)(\phi) = {}_M\Delta(m)(s_*^r(a)\phi) = m_{(0)}\langle \phi, t^\ell(a)m_{(1)}\rangle,$$

hence ${}_M\Delta(ma) = m_{(0)} \otimes_A t^\ell(a)m_{(1)}$.

To see that the two processes of defining $U_*$-modules and $U$-comodules are inverse to each other is straightforward. Assume that a right $U_*$-action on $M$ induces the right $U$-coaction ${}_M\Delta$, as in (3.1.13). Then, as in (3.1.12), ${}_M\Delta$ induces in turn a right $U_*$-action which is given with (3.1.23) as

$$\mu_*^N(m \otimes_A \phi) = \sum_i me^i(\phi(e_i)) = \sum_i me^i s_*^r(\phi(e_i)), \qquad m \in M.$$

With §3.1.5(*i*)(a), by projectivity we have for any $u \in U$

$$\sum_i \langle e^i s_*^r(\phi(e_i)), u\rangle = \langle \sum_i e^i, u\rangle\langle \phi, e_i\rangle = \langle \phi, \sum_i s^\ell(\langle e^i, u\rangle)e_i\rangle = \phi(u).$$

Hence $\sum_i m(e^i s_*^r(\phi e_i)) = m\phi$, and the two module structures $\mu_*^N$ and $(m, \phi) \mapsto m\phi$ coincide. *Vice versa*, if the right $U_*$-module structure $\mu_*^N$ on $N$ originates from a right $U$-comodule structure as in (3.1.12), it induces a right $U$-comodule structure ${}_N\Delta' : n \mapsto n_{(0)'} \otimes_A n_{(1)'}$ on $N$ by (3.1.13), hence for any $n \in N$ and all $\phi \in U_*$ one obtains

$$n_{(0)'}\phi(n_{(1)'}) = {}_N\Delta'(n)(\phi) = \mu_*^N(n \otimes_A \phi) = n_{(0)}\phi(n_{(1)}) = {}_N\Delta(n)(\phi).$$

This means ${}_N\Delta = {}_N\Delta'$.

(*ii*) The case for the right dual is proven analogously; as said, we just give the analogous formulae for the second statement for later use, but still in quite some detail due to the slightly confusing richness of choices in this context.

This time, instead of (3.1.16), consider the $(A, A)$-bimodule ${}_{\blacktriangleright}U^*{}_{\triangleleft}$ ('target-target'), and for the tensor product $U^* \otimes^A U^* := U^*{}_{\triangleleft} \otimes {}_{\blacktriangleright}U^*$, consider the $(A, A)$-bimodule ${}_{\blacktriangleright}(U^* \otimes^A U^*)_{\triangleleft} := {}_{\blacktriangleright}U^* \otimes^A U^*{}_{\triangleleft}$. Assuming that $U_{\triangleleft}$ is finitely generated $A$-projective, with $\{f_j\}_{1 \le j \le m} \in U, \{f^j\}_{1 \le j \le m} \in U^*$ a dual basis of $U_{\triangleleft}$, the map

$$\begin{aligned} U \otimes_A U &\to \mathrm{Hom}_{(A,-)}({}_{\blacktriangleright}(U^* \otimes^A U^*), {}_AA), \\ u \otimes_A u' &\mapsto \big(\psi \otimes^A \psi' \mapsto \langle \psi', s^\ell(\langle \psi, u\rangle)u'\rangle\big) \end{aligned}$$

is an isomorphism, with inverse

$$\Psi \mapsto \sum_{i,j} f_i \otimes_A t^\ell(\Psi(f^i \otimes^A f^j))f_j.$$

For each $u \in U$ the map

$$\psi \otimes^A \psi' \mapsto \langle \psi'\psi, u\rangle \tag{3.1.24}$$

lies in $\mathrm{Hom}_{(A,-)}({}_{\blacktriangleright}(U^* \otimes^A U^*), {}_AA)$. This map reverses the order, which makes it well-defined on the chosen quotient in the tensor product ('target-target'). Now use the pairing

$$\langle \psi'\psi, u\rangle = \langle \psi \otimes^A \psi', u_{(1)} \otimes_A u_{(2)}\rangle = \langle \psi, s^\ell(\langle \psi', u_{(1)}\rangle)u_{(2)}\rangle \tag{3.1.25}$$

to either define a ring structure on the right dual $U^*$ by

$$m_{U^*} : U^* \otimes^A U^* \to U^*, \quad \psi \otimes^A \psi' \mapsto \psi'\psi, \tag{3.1.26}$$

which is well-defined only in this order reversing way (cf. (3.1.2)). Or, deploy (3.1.25) to obtain the expression

$$\Delta_\ell u = \sum_{i,j} f_i \otimes_A t^\ell(\langle f^j f^i, u\rangle) f_j \quad \forall\, u \in U \tag{3.1.27}$$

for the coproduct on $U$. Furthermore, for a right $U^*$-module $M$, define

$$U_\triangleleft \otimes_A M \to \mathrm{Hom}_{(A,-)}(\blacktriangleright U^*{}_\triangleleft\, , {}_A M), \quad u \otimes_A m \mapsto \langle -, u\rangle m,$$

which, under the hypotheses that $U_\triangleleft$ is finitely generated $A$-projective, is an isomorphism, with inverse

$$\mathrm{Hom}_{(A,-)}(\blacktriangleright U^*{}_\triangleleft\, , {}_A M) \to U_\triangleleft \otimes_A M, \quad g \mapsto \sum_j f_j \otimes_A g(f^j).$$

Now, for the *left* $U$-comodule structure on $M$, set

$$\Delta_M : M \to \mathrm{Hom}_{(A,-)}(\blacktriangleright U^*{}_\triangleleft\, , {}_A M), \quad \Delta_M(m)(\psi) := m\psi \quad \forall m \in M,\ \psi \in U^*,$$

so that we finally obtain a map $\Delta_M : M \to U \otimes_A M$ given by

$$\Delta_M m = \sum_j f_j \otimes_A m f^j.$$

With these formulae at hand, the $U$-comodule identities can be verified as before.

□

We now have all necessary information to state the main result of this section.

**3.1.11 Theorem** *Let $U$ be a left bialgebroid with left and right duals $U_*$ and $U^*$, respectively.*

(*i*) *There is a canonical functor* $\mathbf{Comod}\text{-}U \to \mathbf{Mod}\text{-}U_*$ *from the category of* right *$U$-comodules to the category of* right *$U_*$-modules, induced by (3.1.12). If ${}_\triangleright U$ is finitely generated $A$-projective, this functor is an equivalence of categories:*

$$\mathbf{Comod}\text{-}U \simeq \mathbf{Mod}\text{-}U_*.$$

(*ii*) *There is a canonical functor* $U\text{-}\mathbf{Comod} \to \mathbf{Mod}\text{-}U^*$ *from the category of* left *$U$-comodules to the category of* right *$U_*$-modules, induced by (3.1.14). If $U_\triangleleft$ is finitely generated $A$-projective, this functor is an equivalence of categories:*

$$U\text{-}\mathbf{Comod} \simeq \mathbf{Mod}\text{-}U^*.$$

PROOF: It remains to show that module and comodule morphisms correspond to each other. This can be modelled after [Sw1, Thm. 2.1.3.(e)]. We only show the first part, part (*ii*) works mutatis mutandis. Recall the space of morphisms of right $U$-comodules in Definition 2.4.2(*ii*). Suppose $f \in \mathrm{Com}_U(M,N) \subset \mathrm{Hom}_{(-,A)}(M,N)$ is a comodule morphisms for two $M,N \in \mathbf{Comod}\text{-}U$. With the induced right $U$-module structure maps $\mu_*^M$ and $\mu_*^N$ as in (3.1.12), we have for $m \in M$ and $\phi \in U_*$,

$$\begin{aligned}\mu_*^N(f(m) \otimes_A \phi) &= f(m)_{(0)}\langle \phi, f(m)_{(1)}\rangle \\ &= f(m_{(0)})\langle \phi, m_{(1)}\rangle \\ &= f(m_{(0)}\langle \phi, m_{(1)}\rangle) \\ &= f(\mu_*^M(m \otimes_A \phi)).\end{aligned}$$

Hence $f$ is a morphism of $U_*$-modules. Conversely, if $g \in \mathrm{Hom}_{U_*}(M,N)$ for $M,N \in \mathbf{Mod}\text{-}U_*$, we have

$$g(m)_{(0)}\langle \phi, g(m)_{(1)}\rangle = \mu_*^N(g(m) \otimes_A \phi) = g(\mu_*^M(m \otimes_A \phi)) = g(m_{(0)})\langle \phi, m_{(1)}\rangle$$

for all $\phi \in U_*$, $m \in M$, and by the relation between (induced) module and comodule structures in the preceding Proposition, this implies ${}_N\Delta g(m) = (g \otimes \mathrm{id}_U)_M\Delta m$, i.e., $g \in \mathrm{Com}_U(M,N)$. Again, (*ii*) is proven analogously. □

**3.1.12 Remark** As the base algebra of right bialgebroids, $A$ carries a right $U_*$-action as well as a right $U^*$-action, respectively. One may be tempted to think that both of these lead to left $U$-coactions, but this is not the case, as shown. Considering $A$ as the base algebra of the left bialgebroid $U$, it carries *a priori* only one left $U$-action (cf. (2.3.3)), but two $U$-coactions from left and right (cf. (2.3.7)). At least conceptually, this is reflected by the preceding proposition. One has the following chain of structures for the base algebra:

Left $U$-action $\Rightarrow$ right $U_*$- as well as right $U^*$-action $\Rightarrow$ right as well as left $U$-coaction.

We will continue this discussion in the subsequent proposition.

**3.1.13 Grouplike Elements and Generalised Right Characters** Recall that a character on the dual $U^*$ of a Hopf *algebra* $U$ is equivalent to giving a grouplike element in $U$ and vice versa, compare the self-duality of a modular pair mentioned in Section 1.3.

A *generalised right character* [B3] on a right $B$-bialgebroid $V$ is a $(B, B)$-bimodule map $_\blacktriangleright V_\blacktriangleleft \to B$ with respect to the bimodule structure (2.5.2) which fulfills the property (2.5.3). Hence every right counit of a right bialgebroid is by definition a generalised right character.

Using the expressions (3.1.20) and (3.1.27) for the coproduct in $U$ depending respectively on the ring structures on $U_*$ and $U^*$, one proves the following result.

**3.1.14 Proposition** *Let $U$ be a left bialgebroid and assume that $_\triangleright U$ is finitely generated $A$-projective. Then there is a bijective correspondence between grouplike elements $G^\ell U$ and generalised right characters on $U_*$. Likewise, if $U_\triangleleft$ is finitely generated $A$-projective, there is a bijective correspondence between $G^\ell U$ and generalised right characters on $U^*$. In particular, each right $U_*$-action (resp. $U^*$-action) on $A$ corresponds to a grouplike element in $U$, which induces the canonical right (resp. left) $U$-coaction on $A$ as in (2.3.7). These are the only ways in which $U$-coactions on $A$ appear.*

PROOF: Denote a right $U_*$-action on $A$ by $\nabla^r_\phi : a \mapsto \nabla^r_\phi a$. From (3.1.21) follows that $u \mapsto \langle -, u\rangle$ gives the isomorphism $_\triangleright U \simeq \mathrm{Hom}_{(-,A)}({}_\triangleright U_{*\blacktriangleleft}, A_A)$. Hence set

$$\nabla^r 1_A =: \sigma \in U, \tag{3.1.28}$$

cf. §2.5.5, and for each $u \in U$ define correspondingly $u(-) := \langle -, u\rangle \in \mathrm{Hom}_{(-,A)}({}_\triangleright U_{*\blacktriangleleft}, A_A)$. Using (3.1.18) and §3.1.5(*i*), it is not difficult to see that $\sigma$ fulfills $\sigma(\phi\phi') = \sigma(s^r_*(\sigma(\phi))\phi') = \sigma(t^r_*(\sigma(\phi))\phi')$, i.e. (2.5.3) (cf. also (2.5.9)), and also that it is an $(A, A)$-bimodule map with respect to the structures $_\blacktriangleright U_{*\blacktriangleleft}$. With (3.1.20), it follows that

$$\begin{aligned}
\Delta_\ell \sigma &= \sum_{i,j} s^\ell(\langle e^i e^j, \sigma\rangle) e_j \otimes_A e_i \\
&= \sum_{i,j} s^\ell(\langle s^r_*(\sigma(e^i)) e^j, \sigma\rangle) e_j \otimes_A e_i \\
&= \sum_{i,j} s^\ell(\langle e^j, t^\ell(\sigma(e^i))\sigma\rangle) e_j \otimes_A e_i \\
&= \sum_{i} t^\ell(\sigma(e^i))\sigma \otimes_A e_i \\
&= \sigma \otimes \sigma,
\end{aligned} \tag{3.1.29}$$

hence $\sigma$ is a grouplike element in $U$. The opposite direction is proved by reading all statements backwards; that is, if $\sigma$ is grouplike, it can be concluded from (3.1.29) (since $s^\ell$ has the left inverse $\epsilon$) that the property (2.5.3) holds; defining an operator $\nabla^r$ by

$$\nabla^r 1_A := \sigma$$

yields a right $U_*$-action on $A$. In particular, we have $t^\ell a\sigma = \nabla^r a$, as seen by $\langle \phi, t^\ell a\sigma\rangle = \langle s^r_*(a)\phi, \sigma\rangle$ and

$$\begin{aligned}
{}_A\Delta\, a = \sum_i \nabla^r_{e^i} a \otimes_A e_i &= \sum_i \langle e^i, t^\ell(a)\sigma\rangle \otimes_A e_i \\
&= \sum_i 1_A \otimes_A s^\ell\big(\langle e^i, t^\ell(a)\sigma\rangle\big) e_i = 1_A \otimes_A t^\ell(a)\sigma.
\end{aligned}$$

To prove the statement about $A$ we need to make use of (3.1.28) as well as of (3.1.9), (3.1.3), (3.1.1) and (3.1.8). Since $U_*$ is a right bialgebroid, one notices that

$$\begin{aligned} {}_A\Delta(a)(\phi) = a\phi &= (1_A\phi^{(1)})\partial_*(s^r_*(a)\phi^{(2)}) \\ &= \sum_i \langle e^i, \sigma\rangle \, (s^r_*(a)(e_i \rightharpoondown \phi)(1_U)) \\ &= \sum_i \langle e^i, \sigma\rangle \, ((e_i \rightharpoondown \phi)(t^\ell a)) \\ &= \sum_i \langle e^i, \sigma\rangle \langle \phi, t^\ell(a) e_i\rangle, \end{aligned}$$

which reads $a_{(0)}\langle \phi, a_{(1)}\rangle = \langle \phi, t^\ell a\sigma\rangle$, and in view of (3.1.22) means that ${}_A\Delta a = 1_A \otimes_A t^\ell(a)\sigma$. Again, all assertions for $U^*$ work analogously. □

It is not by pure coincidence that in the preceding proposition we chose the symbol $\nabla^r$. In fact, $\nabla^r$ may be seen as a (flat) right connection as in Subsection 2.5.1. We will later see an example of the intimate relation between flat right connections and right counits (Propositions 4.2.9 and 4.2.11).

## 3.2 Push Forward Bialgebroids

In this section we shall construct a new bialgebroid from a known one, provided one has some extra data; namely, a ring extension of the base algebra.

Let $A, B$ be two $k$-algebras and $\sigma : A \to B$ a ring homomorphism. This defines an obvious $(A, A)$-bimodule structure on $B$ by $(a, b, \tilde a) \mapsto \sigma(a)b\sigma(\tilde a)$ for $a, \tilde a \in A$ and $b \in B$.

Now let $U$ be a left bialgebroid over $A$ with structure maps as before. We set

$$B^{\mathrm{op}} \otimes_A U \otimes_A B = B^{\mathrm{op}} \otimes_k U \otimes_k B/I, \tag{3.2.1}$$

where

$$I = \operatorname{span}\{b \otimes_k t^\ell \tilde a s^\ell a u \otimes_k \tilde b - b\sigma(a) \otimes_k u \otimes_k \sigma(\tilde a)\tilde b \mid a, \tilde a \in A,\ b, \tilde b \in B,\ u \in U\}.$$

We furthermore define the Takeuchi product $\sigma_* U := B^{\mathrm{op}} \times_A U \times_A B$, which, similarly as before, denotes the subspace in $B^{\mathrm{op}} \otimes_A U \otimes_A B$ given by

$$\begin{aligned} \sigma_* U = \{ & \textstyle\sum_i b_i \otimes_A u_i \otimes_A \tilde b_i \in B^{\mathrm{op}} \otimes_A U \otimes_A B \mid \\ & \mid \textstyle\sum_i b_i \otimes_A u_i s^\ell a t^\ell \tilde a \otimes_A \tilde b_i = \sum_i \sigma(a) b_i \otimes_A u_i \otimes_A \tilde b_i \sigma(\tilde a),\ a, \tilde a \in A\}. \end{aligned} \tag{3.2.2}$$

Then factorwise multiplication (with the opposite product on the first factor) gives a well-defined $k$-algebra structure on $\sigma_* U$ with unit $1_B \otimes_A 1_U \otimes_A 1_B$.

The following statement may appear surprising to some extent.

**3.2.1 Proposition** *Let $U$ be a left bialgebroid. Then the $k$-algebra $\sigma_* U$ carries the structure of a* right *bialgebroid over $B$.*

PROOF: The right source and target maps are given by

$$\begin{aligned} s^r_B &: B \to \sigma_* U, \qquad b \mapsto 1_B \otimes_A 1_U \otimes_A b, \\ t^r_B &: B \to \sigma_* U, \qquad b \mapsto b \otimes_A 1_U \otimes_A 1_B. \end{aligned}$$

In particular, this defines the structure of a $B^{\mathrm{e}}$-ring with the four $B$-module structures as in (2.5.2) and (2.5.5) for right source and target maps. We may also form the tensor product $\sigma_* U \otimes_B \sigma_* U$, which is defined as in (2.5.1). If now $\Delta_\ell u = u_{(1)} \otimes_A u_{(2)}$ describes the left coproduct on elements $u \in U$ and $\epsilon : U \to A$ is the left counit in $U$, we define the right coproduct and right counit on $\sigma_* U$ as

$$\begin{aligned} \Delta^B_r &: \sigma_* U \to \sigma_* U \otimes_B \sigma_* U, \quad b \otimes_A u \otimes_A \tilde b \mapsto (b \otimes_A u_{(1)} \otimes_A 1_B) \otimes_B (1_B \otimes_A u_{(2)} \otimes_A \tilde b), \\ \partial_B &: \sigma_* U \to B, \qquad\qquad\quad b \otimes_A u \otimes_A \tilde b \mapsto b\sigma(\epsilon u)\tilde b, \end{aligned}$$

which are easily seen to be well-defined, also with respect to the presentation of $\Delta_\ell$. While most of the properties in Definition 2.5.1 of right bialgebroids are obvious, let us just prove (2.5.6) and (2.5.3). One has

$$m_U(\mathrm{id}_U \otimes s^r \partial_B)\Delta^B_r(b \otimes_A u \otimes_A \tilde b) = b \otimes_A u_{(1)} \otimes_A \sigma(\epsilon(u_{(2)}))\tilde b = b \otimes_A u \otimes_A \tilde b,$$

using (2.1.8), and likewise one proves the second identity in (2.5.6). Furthermore, one has

$$\begin{aligned}
\partial_B\big((b\otimes_A u\otimes_A \tilde{b})(b'\otimes_A u'\otimes_A \tilde{b}')\big) &= b'b\sigma\big(\epsilon(uu')\big)\tilde{b}\tilde{b}' \\
&= b'b\sigma\big(\epsilon(us^\ell\epsilon u')\big)\tilde{b}\tilde{b}' \\
&= b'\sigma(\epsilon u')b\sigma(\epsilon u)\tilde{b}\tilde{b}' \\
&= b'\sigma(\epsilon u')\partial_B(b\otimes_A u\otimes_A \tilde{b})\tilde{b}' \\
&= \partial_B\big(b'\otimes u'\otimes_A \partial_B(b\otimes_A u\otimes_A \tilde{b})\tilde{b}'\big) \\
&= \partial_B\big((s^r_B\partial_B(b\otimes_A u\otimes_A \tilde{b}))(b'\otimes u'\otimes_A \otimes_A\tilde{b})\big),
\end{aligned}$$

where we used the fact that $\sigma_* U$ is given as a Takeuchi product. Putting $t^\ell$ instead of $s^\ell$ in the second equation then leads to the second equation in (2.5.3). □

**3.2.2 Remark** One may also alter the push forward construction by taking different tensor products, opposites, coopposites or even $\sigma$ as an anti-homomorphism; however, none of these possibilities seem to lead to a left bialgebroid, precisely due to the requirement (2.1.4).

**3.2.3 Examples** (Localisation of Hopf algebroids)

(*i*) Let $E \to M$ be a Lie algebroid over a smooth manifold $M$ with anchor $\omega$. Denote the corresponding Lie-Rinehart algebra by $(\mathcal{C}^\infty(M), \Gamma E)$, and let $VE := V\Gamma E$ be the associated left $\mathcal{C}^\infty(M)$-bialgebroid; cf. Subsection 4.2.2 for all details of this construction. Let $\mathrm{ev}_x : \mathcal{C}^\infty(M) \to \mathbb{C}, a \mapsto a(x)$ be the evaluation at a point $x \in M$ and write $\mathbb{C}_x$ for $\mathbb{C}$ seen as (left or right) $\mathcal{C}^\infty(M)$-module by this map. By the PBW Theorem one has a $\mathcal{C}^\infty(M)$-module isomorphism $VE \simeq \Gamma(M, SE)$ on sections of the symmetric algebra $SE$, hence in particular $VE \otimes_{\mathcal{C}^\infty(M)} \mathbb{C}_x \simeq SE_x$, given by $u \otimes_{\mathcal{C}^\infty(M)} \lambda \mapsto u(x)\lambda$. The condition (3.2.2) for the right tensor factor for $X \in \Gamma E$ yields $(Xa)(x) = a(x)X(x)$, which is true if $X(a)(x) = 0$ for all $a \in \mathcal{C}^\infty(M)$. Hence $X \in \ker \omega_x$, the fibre at $x$ of the isotropy of the Lie algebroid; this is a Lie algebra. An analogous consideration for the left tensor factor yields the same information (since the source and target maps are equal). By extension, one obtains $\mathrm{ev}_{x*}VE \simeq U(\ker \omega_x)$, i.e. a $\mathbb{C}$-bialgebra.

(*ii*) Let $s, t : G \rightrightarrows G_0$ be an étale groupoid over a smooth manifold $G_0$, and denote the compactly supported functions over $G$ by $\mathcal{C}^\infty_c(G)$; see Section 4.4 how this can be seen as a left bialgebroid over $\mathcal{C}^\infty(G_0)$. The left and right $\mathcal{C}^\infty(G_0)$-actions on $\mathcal{C}^\infty_c(G)$ used in (3.2.1) are given in (4.4.4), i.e., $(au)(g) = a(t(g))u(g)$ and $(ua)(g) = u(g)a(s(g))$, where $u \in \mathcal{C}^\infty_c(G)$, $a \in \mathcal{C}^\infty(G_0)$. Again, let $\mathrm{ev}_x : \mathcal{C}^\infty(G_0) \to \mathbb{C}, a \mapsto a(x)$ be the evaluation at $x \in G_0$. Then the right tensor product in (3.2.1) identifies elements $a(t(g))u(g) = a(x)u(g)$, i.e., $\mathcal{C}^\infty_c(G) \otimes_{\mathcal{C}^\infty(G_0)} \mathbb{C}_x \simeq \mathcal{C}^\infty_c(t^{-1}(x))$. The subspace in (3.2.2), however, consists of those elements in $\mathcal{C}^\infty_c(t^{-1}(x))$ for which $u(g)a(s(g)) = u(g)a(x)$. As in (*i*), analogous considerations on the first tensor factor do not lead to any further conditions. Hence if $G_x = \{g \mid s(g) = t(g) = x\}$ denotes the isotropy subgroup of $x$, which is a discrete group here, one obtains $\mathrm{ev}_{x*}\mathcal{C}^\infty_c(G) \simeq \mathcal{C}^\infty_c(G_x)$, i.e. a $\mathbb{C}$-bialgebra again.

## 3.3 Matched Pairs of Bialgebroids

In this section we present a method to construct a new bialgebroid out of *two* known ones. It will give a generalisation of a similar consideration for $k$-bialgebras (see e.g. [Maj]), which is needed in Section 4.7 to analyse the structure of (generalised) Connes-Moscovici algebras (cf. [CoMos5, MosR]). The main ingredients of the construction are:

(*i*) two (left) bialgebroids $U, F$,

(*ii*) an action of $U$ on $F$, satisfying certain properties,

(*iii*) a coaction of $F$ on $U$, satisfying certain properties.

If the structures from (*ii*) and (*iii*) 'match' in a sense to be specified, a particular tensor product $F \otimes U$ carries the structure of a left bialgebroid again.

Note that already for $k$-bialgebras there are various possibilities of how to produce new bialgebras in this spirit (cf. [Maj, Kas3]), corresponding to the various action-coaction pictures but we will only generalise the case of the 'left-right bicrossproduct bialgebra' from [Maj, Thm. 6.2.2] to bialgebroids.

### 3.3.1 Left Module Rings for Bialgebroids

Here, we give a more precise sense to the 'action picture' of point (*ii*) above.

The additional structure on a ring $U$ that makes it a left (or right) bialgebroid over $A$ is precisely a monoidal structure on the category $U$-**Mod** of left $U$-modules (right $U$-modules, respectively) together with a strictly monoidal forgetful functor $U\text{-}\mathbf{Mod} \to A^{\mathrm{e}}\text{-}\mathbf{Mod}$; see Theorem 2.3.1 from [Schau1]. Hence the following analogue of a 'module algebra' appears quite naturally. Let $(U, A, s^\ell_U, t^\ell_U, \Delta^U_\ell, \epsilon_U)$ be a left bialgebroid, as before.

**3.3.1 Definition** [KSz] A *left $U$-module ring $M$* is a monoid in $U$-**Mod**. That is, by strict monoidality of the forgetful functor $U\text{-}\mathbf{Mod} \to A^{\mathrm{e}}\text{-}\mathbf{Mod}$, the space $M$ carries a canonical $A$-ring structure with $A$-balanced multiplication $\mu_M(m \otimes_A m') = mm'$ for $m, m' \in M$, and unit map $A \to M,\ a \mapsto s^\ell_U(a)1_M = t^\ell_U(a)1_M$ such that for $u \in U,\ m, m' \in M$

$$u(mm') = (u_{(1)}m)(u_{(2)}m'), \quad \text{and} \quad u1_M = s^\ell_U \epsilon_U(u)1_M. \tag{3.3.1}$$

Here the $U$-action on $M$ is denoted by $(u, m) \mapsto um$.

For example, the base algebra $A$ is a left $U$-module ring, but $U$ itself usually is not.

Observe in particular that with the induced $A^{\mathrm{e}}$-module structure on $M$ given by

$$amb := t^\ell_U b s^\ell_U am, \tag{3.3.2}$$

one has

$$\begin{aligned} a(mm') &= s^\ell_U a(mm') = (s^\ell_U am)m' = (am)m' \\ (mm')a &= t^\ell_U a(mm') = m(t^\ell_U am') = m(m'a), \end{aligned} \tag{3.3.3}$$

and moreover

$$m(am') = m(s^\ell_U am') = (t^\ell_U am)m' = (ma)m'. \tag{3.3.4}$$

We can then prove the following fact similar as in [KSz].

**3.3.2 Lemma** *Let $U$ be a left bialgebroid as above and $M$ a left $U$-module ring. Then $M \otimes_A U$ carries an $A$-ring structure, called the* smash ring *or* crossed product ring, *denoted by $M \bowtie_A U$.*

PROOF: Here

$$M \otimes_A U := M \otimes_k U / \mathrm{span}\{ma \otimes u - m \otimes s^\ell_U au \mid a \in A\}, \tag{3.3.5}$$

where $ma = t^\ell_U am$ as above. The $A^{\mathrm{e}}$-module structure on $M \otimes_A U$ is given by

$$a(m \otimes_A u)b = am \otimes_A us^\ell_U b = s^\ell_U am \otimes_A us^\ell_U b \quad \text{for } a, b \in A. \tag{3.3.6}$$

The product structure $\mu$ is

$$(m \otimes_A u)(m' \otimes_A u') := \mu\big((m \otimes_A u) \otimes_A (m' \otimes_A u')\big) := m(u_{(1)}m') \otimes_A u_{(2)}u', \tag{3.3.7}$$

and the unit is

$$A \to M \otimes_A U, \quad a \mapsto a1_M \otimes_A 1_U = s^\ell_U a1_M \otimes_A 1_U = t^\ell_U a1_M \otimes_A 1_U = 1_M \otimes_A s^\ell_U a.$$

Both maps are clearly $A^{\mathrm{e}}$-module morphisms. While the associativity and unitality axioms are easily checked using coassociativity of $\Delta^U_\ell$, we restrict ourselves to showing that (3.3.7) is actually well-defined over $A$. One has, for $a, b \in A$,

$$\begin{aligned} (ma \otimes_A u)(m'b \otimes_A u') &= (ma)(u_{(1)}(m'b)) \otimes_A u_{(2)}u' = m((s^\ell_U au_{(1)}t^\ell_U b)m') \otimes_A u_{(2)}u' \\ &= m((s^\ell_U au)_{(1)}m') \otimes_A (s^\ell_U au)_{(2)}s^\ell_U bu' = (m \otimes_A s^\ell_U au)(m' \otimes_A s^\ell_U bu'). \end{aligned}$$

Also, if multiplication is thought of as a composition of single maps, one has to show well-definedness of the Sweedler components, i.e.,

$$\begin{aligned} m((t^\ell_U au_{(1)})m') \otimes_A u_{(2)}u' &= m((u_{(1)}m')a) \otimes_A u_{(2)}u' \\ &= (m(u_{(1)}m'))a \otimes_A u_{(2)}u' = m(u_{(1)}m') \otimes_A s^\ell_U au_{(2)}u'. \end{aligned}$$

With the mentioned $A^{\mathrm{e}}$-module structure on $M \otimes_A U$, the fact that $\mu$ is $A$-balanced is obvious. □

For example, the universal enveloping algebra $VL$ of a Lie-Rinehart algebra $(A, L)$ arises in such a way, i.e., $VL = A \bowtie_A UL$ where $UL$ is the universal enveloping algebra of the $k$-Lie algebra $L$; see also Remark 1.4.4 above.

We finally remark that $M \otimes_A U$ can even be seen as an $M$-ring [B3].

### 3.3.2 Right Comodule Corings for Bialgebroids

Now we specify what we mean by the 'coaction picture' mentioned in point (*iii*) at the beginning of Section 3.3.

Let $(F, A, s^\ell_F, t^\ell_F, \Delta^F_\ell, \epsilon_F)$ be an arbitrary left bialgebroid and denote the Sweedler components of its left coproduct by $\Delta^F_\ell f = f_{[1]} \otimes_A f_{[2]}$ for all $f \in F$.

**3.3.3 Definition** A *right $F$-comodule coring* N is a comonoid in $\mathbf{Comod}\text{-}F$. That is, by strict monoidality of the forgetful functor $\mathbf{Comod}\text{-}F \to (A^{\rm op})^{\rm e}\text{-}\mathbf{Mod}$, the space $N$ carries a canonical $A^{\rm op}$-coring structure $(N, \Delta^N, \epsilon_N)$ with an $(A^{\rm op})^{\rm e}$-linear coproduct $\Delta^N n =: n_{(1)} \otimes_{A^{\rm op}} n_{(2)}$ and a right $F$-comodule structure ${}_N\Delta n =: n_{[0]} \otimes_{A^{\rm op}} n_{[1]}$, such that

$$m_F(s^\ell_F\epsilon_N \otimes \mathrm{id}_F)_N\Delta = t^\ell_F\epsilon_N, \quad (\Delta^N \otimes \mathrm{id}_F)_N\Delta = {}_{N\otimes N}\Delta\,\Delta^N. \tag{3.3.8}$$

Here ${}_{N\otimes N}\Delta$ denotes the right $F$-coaction on $N \otimes_{A^{\rm op}} N$ from (2.3.11).

Again, the base algebra $A$ is a right $F$-comodule coring (see (2.3.7)), whereas this is generally not the case for $F$ itself.

Compare [BŞ] for the analogous definition of a left $F$-comodule coring. Observe also that it is this $A^{\rm op}$-construction that generalises the bialgebra case (cf. e.g. [Maj]), at least if one wants the same order of elements in the formulae.

For the reader's convenience, let us explicitly recall all maps and tensor products involved. For the coproduct on $N$ we have $\Delta^N : N \to N \otimes_{A^{\rm op}} N$, where

$$N \otimes_{A^{\rm op}} N := N \otimes_k N/\mathrm{span}\{an \otimes n' - n \otimes n'a \mid a \in A\}, \tag{3.3.9}$$

and $N \otimes_{A^{\rm op}} N$ is an $(A, A)$-bimodule in a standard way, i.e. by $(a, n \otimes_{A^{\rm op}} n', b) \mapsto nb \otimes_{A^{\rm op}} an'$; in particular

$$\Delta^N(anb) = n_{(1)}b \otimes_{A^{\rm op}} an_{(2)} \quad \text{for all } a, b \in A. \tag{3.3.10}$$

For the right $F$-coaction on $N$ we have ${}_N\Delta : N \to N \otimes^A F$, where

$$N \otimes_A F := N \otimes_k F/\mathrm{span}\{na \otimes_k f - n \otimes s^\ell_F af \mid a \in A\} \tag{3.3.11}$$

is the tensor product from (2.3.5). Written explicitly, the conditions (3.3.8) then read

$$\begin{array}{rcl} s^\ell_F\epsilon_N(n_{[0]})n_{[1]} &=& t^\ell_F\epsilon_N n, \\ n_{[0](1)} \otimes_{A^{\rm op}} n_{[0](2)} \otimes_A n_{[1]} &=& n_{(1)[0]} \otimes_{A^{\rm op}} n_{(2)[0]} \otimes_A n_{(1)[1]}n_{(2)[1]}. \end{array} \tag{3.3.12}$$

Note that these conditions are well-defined by (2.3.10), (3.3.10) and (3.3.11). For example, on the right hand side in the second equation of (3.3.12) one has

$$\begin{aligned} (an_{(1)})_{[0]} \otimes_{A^{\rm op}} n_{(2)[0]} \otimes_A n_{(1)[1]}n_{(2)[1]} &= n_{(1)[0]} \otimes_{A^{\rm op}} n_{(2)[0]} \otimes_A n_{(1)[1]}t^\ell an_{(2)[1]} \\ &= n_{(1)[0]} \otimes_{A^{\rm op}} (n_{(2)}a)_{[0]} \otimes_A n_{(1)[1]}n_{(2)[1]}, \end{aligned}$$

from which the well-definedness over the presentation of $\Delta^N$ follows.

**3.3.4 Lemma** *Let $(F, A)$ be a left bialgebroid and $N$ a right $F$-comodule coring, with all structures maps as above. Then the space $F \otimes_{A^{\rm op}} N$ carries the structure of an $A$-coring, called the* cocrossed product coring, *denoted by $F \blacktriangleright\!\!< _{A^{\rm op}} N$.*

PROOF: Firstly, we recall that the underlying $A$-linear space of $F \blacktriangleright\!\!<_{A^{\rm op}} N$ is

$$F \otimes_{A^{\rm op}} N := F \otimes_k N/\mathrm{span}\{ft^\ell_F a \otimes n - f \otimes na \mid a \in A\}, \tag{3.3.13}$$

which is an $(A, A)$-bimodule with left $A$-action $L_A$ and right $A$-action $R_A$ given by

$$a(f \otimes_{A^{\rm op}} n)b := s^\ell_F at^\ell_F bf \otimes_{A^{\rm op}} n, \tag{3.3.14}$$

with respect to which we define the tensor product $(F \otimes_{A^{\rm op}} N) \otimes_A (F \otimes_{A^{\rm op}} N)$. The coproduct and counit are maps $\Delta^{\rm ccr}_\ell : F \otimes_{A^{\rm op}} N \to (F \otimes_{A^{\rm op}} N) \otimes_A (F \otimes_{A^{\rm op}} N)$ and $\epsilon^{\rm ccr} : F \otimes_{A^{\rm op}} N \to A$, given by

$$\begin{aligned} \Delta^{\rm ccr}_\ell(f \blacktriangleright\!\!<_{A^{\rm op}} n) &:= (f_{[1]} \blacktriangleright\!\!<_{A^{\rm op}} n_{(1)[0]}) \otimes_A (f_{[2]}n_{(1)[1]} \blacktriangleright\!\!<_{A^{\rm op}} n_{(2)}), && (3.3.15)\\ \epsilon_{\rm ccr}(f \blacktriangleright\!\!<_{A^{\rm op}} n) &:= \epsilon_F(ft^\ell_F\epsilon_N n). && (3.3.16) \end{aligned}$$

While it is trivial to check that the counit is well-defined over $A$ (recall that $\epsilon_N$ is an $(A^{\mathrm{op}}, A^{\mathrm{op}})$-bimodule map), this is somewhat more tedious for $\Delta_\ell^{\mathrm{ccr}}$. With (2.3.10) and (3.3.10) one has

$$\begin{aligned}\Delta_\ell^{\mathrm{ccr}}(ft^\ell_F a \blacktriangleright\!\!<_{A^{\mathrm{op}}} n) &= (f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0]}) \otimes_A (f_{[2]} t^\ell_F a n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)}) \\ &= (f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} (na)_{(1)[0]}) \otimes_A (f_{[2]} (na)_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} (na)_{(2)}) \\ &= \Delta_\ell^{\mathrm{ccr}}(u \blacktriangleright\!\!<_{A^{\mathrm{op}}} na).\end{aligned}$$

The well-definedness over the presentation of $\Delta_\ell^F$ immediately follows from (3.3.14). Moreover, one has to check the well-definedness over the presentation of ${}_N\Delta$, i.e.,

$$\begin{aligned}&(f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0]} a) \otimes_A (f_{[2]} n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)}) = \\ &\qquad = (f_{[1]} t^\ell_F a \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0]}) \otimes_A (f_{[2]} n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)}) \\ &\qquad = (f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0]}) \otimes_A (f_{[2]} s^\ell_F a n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)}),\end{aligned}$$

by the very definition of all tensor products involved, as well as the property $\mathrm{im}(\Delta_\ell^F) \subset F \times_A F$. Now, by (2.3.10) again, the computation

$$\begin{aligned}&(f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} (an_{(1)})_{[0]}) \otimes_A (f_{[2]} n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)}) = \\ &\qquad = (f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0]}) \otimes_A (f_{[2]} n_{(1)[1]} t^\ell_F a \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)}) \\ &\qquad = (f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0]}) \otimes_A (f_{[2]} n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)} a),\end{aligned}$$

proves that the presentation of $\Delta^N$ is well-defined. Finally, we are left with checking the comonoid identities. With (3.3.8), (3.3.12) and (2.3.10) one obtains

$$\begin{aligned}L_A(\epsilon^{\mathrm{ccr}} \otimes \mathrm{id})\Delta_\ell^{\mathrm{ccr}}(f \blacktriangleright\!\!<_{A^{\mathrm{op}}} n) &= s^\ell_F \epsilon_F\big(f_{[1]} t^\ell_F \epsilon_N(n_{(1)[0]})\big) f_{[2]} n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)} \\ &= f s^\ell_F \epsilon_N(n_{(1)[0]}) n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)} \\ &= f t^\ell_F \epsilon_N(n_{(1)}) \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)} \\ &= f \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)} \epsilon_N(n_{(1)}) = f \blacktriangleright\!\!<_{A^{\mathrm{op}}} n,\end{aligned}$$

since $N$ is an $A^{\mathrm{op}}$-coring. Also, with the same kind of arguments,

$$\begin{aligned}R_A(\mathrm{id} \otimes \epsilon^{\mathrm{ccr}})\Delta_\ell^{\mathrm{ccr}}(f \blacktriangleright\!\!<_{A^{\mathrm{op}}} n) &= t^\ell_F \epsilon_F(f_{[2]} n_{(1)[1]} t^\ell_F \epsilon_N n_{(2)}) f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0]} \\ &= t^\ell_F \epsilon_F(f_{[2]} (\epsilon_N(n_{(2)}) n_{(1)})_{[1]}) f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} (\epsilon_N(n_{(2)}) n_{(1)})_{[0]} \\ &= t^\ell_F \epsilon_F(f_{[2]} n_{[1]}) f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{[0]} \\ &= t^\ell_F \epsilon_F(f_{[2]}) f_{[1]} t^\ell_F \epsilon_F n_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{[0]} = f \blacktriangleright\!\!<_{A^{\mathrm{op}}} n.\end{aligned}$$

Spelling out the coassociativity condition of $\Delta_\ell^{\mathrm{ccr}}$, one finds

$$\begin{aligned}&(\Delta_\ell^{\mathrm{ccr}} \otimes \mathrm{id})\Delta_\ell^{\mathrm{ccr}}(f \blacktriangleright\!\!<_{A^{\mathrm{op}}} n) = \\ &= (f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0](1)[0]}) \otimes_A (f_{[2]} n_{(1)[0](1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0](2)}) \otimes_A (f_{[3]} n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)}) \\ &= (f_{[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(1)[0]}) \otimes_A (f_{[2]} n_{(1)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(2)[0]}) \otimes_A (f_{[3]} n_{(1)[1]} n_{(2)[1]} \blacktriangleright\!\!<_{A^{\mathrm{op}}} n_{(3)}) \\ &= (\mathrm{id} \otimes \Delta_\ell^{\mathrm{ccr}})\Delta_\ell^{\mathrm{ccr}}(f \blacktriangleright\!\!<_{A^{\mathrm{op}}} n).\end{aligned}$$

Here we used coassociativity of both $\Delta_\ell^F$, $\Delta^N$, and in the third line the comodule coring property (3.3.12) was applied. □

### 3.3.3 Matched Pairs

In this subsection we merge the concepts of left $U$-module rings and right $F$-comodule corings to produce a new bialgebroid, which will be called a *bicrossed product bialgebroid.* This is possible if $F$ and $U$ and their mutual coaction and action meet certain conditions (so as to yield a *matched pair* of bialgebroids; see below). Assume that both $M = F$ in Definition 3.3.1 and $N = U$ in Definition 3.3.3 are (left) bialgebroids; we want to impose a bialgebroid structure on some tensor product $F \otimes U$. The obvious problem that the ring structure from Lemma 3.3.2 and coring structure from Lemma 3.3.4 are on different tensor products can be removed by assuming $F$ to be a bialgebroid over $A^{\mathrm{op}}$ and $U$ over $A$. On the other hand, further (seemingly unavoidable) compatibility conditions (see below) will force $A$ to be commutative, such that the tensor products underlying the crossed product ring and the cocrossed product coring become (automatically) the same.

**3.3.5 Theorem** *Let $(F, A, s^\ell_F, t^\ell_F, \Delta^F_\ell, \epsilon_F)$ and $(U, A, s^\ell_U, t^\ell_U, \Delta^U_\ell, \epsilon_U)$ be left bialgebroids over some commutative base algebra $A$, and let $F$ be a left $U$-module ring and $U$ a right $F$-comodule coring with maps*

$$U \times F \to F,\ (u, f) \mapsto u(f) \qquad \text{and} \qquad {}_U\Delta : U \to U \otimes_A F,\ u \mapsto u_{[0]} \otimes_A u_{[1]}.$$

*Furthermore, assume that for all $f, f' \in F,\ u, u' \in U,\ a \in A$ the compatibility conditions*

$$\begin{aligned} s^\ell_U &\equiv t^\ell_U, & (3.3.17)\\ t^\ell_F af &= ft^\ell_F a, & (3.3.18)\\ u_{[0]} \otimes_A s^\ell_F a u_{[1]} &= u_{[0]} \otimes_A u_{[1]} s^\ell_F a, & (3.3.19)\\ f_{[1]} \otimes_A s^\ell_F a f_{[2]} &= f_{[1]} \otimes_A f_{[2]} s^\ell_F a, & (3.3.20) \end{aligned}$$

*as well as*

$$\begin{aligned} \epsilon_F\big(u(f)\big) &= \epsilon_U\big(us^\ell_U \epsilon_F(f)\big), & (3.3.21)\\ {}_U\Delta 1_U &= 1_U \otimes_A 1_F, & (3.3.22)\\ \Delta^F_\ell\big(u(f)\big) &= u_{(1)[0]}(f_{[1]}) \otimes_A u_{(1)[1]} u_{(2)}(f_{[2]}), & (3.3.23)\\ u_{(2)[0]} \otimes_A u_{(1)}(f) u_{(2)[1]} &= u_{(1)[0]} \otimes_A u_{(1)[1]} u_{(2)}(f), & (3.3.24)\\ {}_U\Delta(uu') &= u_{(1)[0]} u'_{[0]} \otimes_A u_{(1)[1]} u_{(2)}(u'_{[1]}) & (3.3.25) \end{aligned}$$

*hold, where we used the Sweedler notation $\Delta^U_\ell u =: u_{(1)} \otimes_A u_{(2)}$ and $\Delta^F_\ell f =: f_{[1]} \otimes_A f_{[2]}$ for the left coproducts. Then the linear space*

$$F \otimes_A U := F \otimes_k U/\mathrm{span}\{t^\ell_F af \otimes u - f \otimes s^\ell_U au \mid a \in A\} \qquad (3.3.26)$$

*carries compatible structures of both an $A^{\mathrm{e}}$-coring with ring structure $F \rtimes_A U$ from Lemma 3.3.2 and an $A$-coring with coring structure $F \blacktriangleright\!\!<_A U$ from Lemma 3.3.4, so as to form a left $A$-bialgebroid, denoted by $F \blacktriangleright\!\!\lhd_A U$.*

PROOF: First, note that the tensor products in (3.3.20) and (3.3.23) refer to (2.1.1) for the left $A$-bialgebroid $F$, whereas the tensor product used in (3.3.19), (3.3.22) (3.3.24) and (3.3.25) is meant to be the one defined in (3.3.11); now (3.3.23)–(3.3.25) are well-defined precisely due to (3.3.17)–(3.3.20). We will first dedicate our attention to the conditions (3.3.17)–(3.3.20), which are sufficient for both the construction of the crossed product ring $F \rtimes_A U$ and the cocrossed product coring $F \blacktriangleright\!\!<_A U$ on the linear space (3.3.26). Clearly, once established, (3.3.17) then implies that $A$ needs to be commutative. Asking $U$ to be a right $F$-comodule requires compatibility in the sense of (2.3.10) with respect to one of the right $A$-actions $\lhd$ or $\blacktriangleleft$ on $U$ from (2.1.3) or (2.1.5) (provided one does not want to introduce extra data, i.e. even more $A$-actions on $U$). If one furthermore wants to impose $U$ to be a right $F$-comodule coring, one of the left and one of the right of the four natural $A$-actions on $U$ has to be compatible with (3.3.10). By (2.1.7), this implies that one needs to use *either* (2.1.3) *or* (2.1.5). For either choice, one has, by (2.3.10),

$${}_U\Delta(a \rhd 1_U) = {}_U\Delta(1_U \lhd a) = {}_U\Delta(a \blacktriangleright 1_U) = {}_U\Delta(1_U \blacktriangleleft a) = 1_U \otimes_A t^\ell_F a, \qquad a \in A,$$

and hence by the comodule properties

$$s^\ell_U a = a \rhd 1_U = R_A(\mathrm{id} \otimes \epsilon_F)_U\Delta(a \rhd 1_U) = R_A(\mathrm{id} \otimes \epsilon_F)_U\Delta(1_U \lhd a) = 1_U \lhd a = t^\ell_U a,$$

where $R_A$ is one of the two natural right $A$-actions $\lhd$ or $\blacktriangleleft$ on $U$ from either (2.1.3) or (2.1.5). Analogous considerations hold for the $(A, A)$-bimodule ${}_\blacktriangleright U_\blacktriangleleft$ , but since we aim to produce the same linear space in the tensor products (3.3.5) and (3.3.13), the canonical *right* $A$-module structure on the right $F$-comodule $U$ needs to be *left* multiplication with $s^\ell_U a\ (= t^\ell_U a)$, which coincides with the additional left $A$-action (2.3.8) on the right $F$-comodule $U$. Observe that *right* multiplication with $s^\ell_U a\ (= t^\ell_U a)$ does not come into the picture, i.e. is not induced by the $F$-coaction on $U$. In particular, the tensor product (3.3.9) is then the standard one from (2.1.1) for the left bialgebroid $U$ (things would not have changed if we assumed that $F$ was a left $A^{\mathrm{op}}$-bialgebroid).

On the other hand, $F$ is requested to be a left $U$-module ring with compatible $A$-action with respect to the induced $(A, A)$-bimodule structure (3.3.2). Since we want to match the tensor products (3.3.5) with (3.3.13), we obtain the condition

$$t^\ell_U af = ft^\ell_F a, \quad f \in F,\ a \in A,$$

and from (3.3.4) and (3.3.3) it follows, with $s^\ell_U = t^\ell_U$, that

$$ft^\ell_F af' = (ft^\ell_F a)f' = (s^\ell_U af)f' = s^\ell_U a(ff') = ff't^\ell_F a, \quad f, f' \in F, \ a \in A.$$

Hence $t^\ell_F af = ft^\ell_F a$, and $A$ needs to be central in $F$ by means of $t^\ell_F$. Observe that both the left and right $A$-actions induced by the $U$-action on $F$ (cf. (3.3.2)) coincide with the $A$-module structure defined by $t^\ell_F$ (i.e., the $A$-action by the source map $s^\ell_F$ does not arise from the fact that $F$ is a left $U$-module ring).

Now the underlying linear spaces of $F\rhd\!\!\lhd_A U$ and $F\blacktriangleright\!\!\lhd_A U$ are given by (3.3.26); moreover $F\rhd\!\!\lhd_A U$ can be seen as an $A^{\mathrm{e}}$-ring by defining source and target as in (3.3.14), i.e.,

$$s^\ell_{\mathrm{ccr}} : a \mapsto s^\ell_F a \otimes_A 1_U, \qquad t^\ell_{\mathrm{ccr}} : b \mapsto t^\ell_F b \otimes_A 1_U. \tag{3.3.27}$$

Then the canonical left $A^{\mathrm{e}}$-module structure on $F\rhd\!\!\lhd_A U$ from (2.1.3) is

$$a \rhd (f \otimes_A u) \lhd b = s^\ell_F at^\ell_F bf \otimes_A u = s^\ell_F af \otimes_A s^\ell_U bu,$$

whereas (2.1.5) reads

$$(f \otimes_A u) \blacktriangleleft a = fu_{(1)}(s^\ell_F a) \otimes_A u_{(2)} \qquad \text{and} \qquad b \blacktriangleright (f \otimes u) = f \otimes_A us^\ell_U b, \tag{3.3.28}$$

where in the second equation we used (3.3.7), (3.3.1), (3.3.26), and $s^\ell_U = t^\ell_U$. The bimodule structures on $F \otimes_A U$ corresponding to (3.3.14) and (3.3.6) are then $a \rhd (f \otimes_A u) \lhd b$ and $(f \otimes_A u) \blacktriangleleft b \lhd a$, respectively. So far, part (*i*) in Definition (2.1.2) of a left bialgebroid has been shown.

As for part (*ii*), we need to show that the coring structure $\Delta^{\mathrm{ccr}}_\ell$ on $F\blacktriangleright\!\!\lhd_A U$ from Lemma 3.3.4 is a $k$-algebra morphism with respect to the algebra structure on $F\rhd\!\!\lhd_A U$ from Lemma 3.3.2. We will do this in an analogous manner as in [Maj, Thm. 6.2.2]. For this to hold, the identities (3.3.23)–(3.3.25) will be sufficient. Note that these are well-defined due to (3.3.19), and that (3.3.20) and inserting (3.3.23)–(3.3.25) into the following calculation is permitted by (2.1.2) and (2.3.9):

$$\begin{aligned}
&\Delta^{\mathrm{ccr}}_\ell\big((f\blacktriangleright\!\!\lhd_A u)(f'\blacktriangleright\!\!\lhd_A u')\big) \underset{(3.3.7)}{=} \Delta^{\mathrm{ccr}}_\ell\big(fu_{(1)}(f')\blacktriangleright\!\!\lhd_A u_{(2)}u'\big) \\
&\underset{(3.3.15)}{=} \big(f_{[1]}(u_{(1)}(f'))_{[1]}\blacktriangleright\!\!\lhd_A (u_{(2)}u'_{(1)})_{[0]}\big) \otimes_A \big(f_{[2]}(u_{(1)}(f'))_{[2]}(u_{(2)}u'_{(1)})_{[1]}\blacktriangleright\!\!\lhd_A u_{(3)}u'_{(2)}\big) \\
&\underset{(3.3.23)}{=} \big(f_{[1]}u_{(1)[0]}(f'_{[1]})\blacktriangleright\!\!\lhd_A (u_{(3)}u'_{(1)})_{[0]}\big) \otimes_A \big(f_{[2]}u_{(1)[1]}u_{(2)}(f'_{[2]})(u_{(3)}u'_{(1)})_{[1]}\blacktriangleright\!\!\lhd_A u_{(4)}u'_{(2)}\big) \\
&\underset{(3.3.25)}{=} \big(f_{[1]}u_{(1)[0]}(f'_{[1]})\blacktriangleright\!\!\lhd_A u_{(3)[0]}u'_{(1)[0]}\big) \otimes_A \big(f_{[2]}u_{(1)[1]}u_{(2)}(f'_{[2]})u_{(3)[1]}u_{(4)}(u'_{(1)[1]})\blacktriangleright\!\!\lhd_A u_{(5)}u'_{(2)}\big) \\
&\underset{(3.3.24)}{=} \big(f_{[1]}u_{(2)[0]}(f'_{[1]})\blacktriangleright\!\!\lhd_A u_{(3)[0]}u'_{(1)[0]}\big) \otimes_A \big(f_{[2]}u_{(1)}(f'_{[2]})u_{(2)[1]}u_{(3)[1]}u_{(4)}(u'_{(1)[1]})\blacktriangleright\!\!\lhd_A u_{(5)}u'_{(2)}\big) \\
&\underset{(3.3.12)}{=} \big(f_{[1]}u_{(2)[0](1)}(f'_{[1]})\blacktriangleright\!\!\lhd_A u_{(2)[0](2)}u'_{(1)[0]}\big) \otimes_A \big(f_{[2]}u_{(1)}(f'_{[2]})u_{(2)[1]}u_{(3)}(u'_{(1)[1]})\blacktriangleright\!\!\lhd_A u_{(4)}u'_{(2)}\big) \\
&\underset{(3.3.24)}{=} \big(f_{[1]}u_{(1)[0](1)}(f'_{[1]})\blacktriangleright\!\!\lhd_A u_{(1)[0](2)}u'_{(1)[0]}\big) \otimes_A \big(f_{[2]}u_{(1)[1]}u_{(2)}(f'_{[2]})u_{(3)}(u'_{(1)[1]})\blacktriangleright\!\!\lhd_A u_{(4)}u'_{(2)}\big) \\
&\underset{(3.3.1)}{=} \big(f_{[1]}u_{(1)[0](1)}(f'_{[1]})\blacktriangleright\!\!\lhd_A u_{(1)[0](2)}u'_{(1)[0]}\big) \otimes_A \big(f_{[2]}u_{(1)[1]}u_{(2)}(f'_{[2]}u'_{(1)[1]})\blacktriangleright\!\!\lhd_A u_{(3)}u'_{(2)}\big) \\
&\underset{(3.3.7)}{=} \big((f_{[1]}\blacktriangleright\!\!\lhd_A u_{(1)[0]})(f'_{[1]}\blacktriangleright\!\!\lhd_A u'_{(1)[0]})\big) \otimes_A \big((f_{[2]}u_{(1)[1]}\blacktriangleright\!\!\lhd_A u_{(2)})(f'_{[2]}u'_{(1)[1]}\blacktriangleright\!\!\lhd_A u'_{(2)})\big) \\
&= \Delta^{\mathrm{ccr}}_\ell(f\blacktriangleright\!\!\lhd_A u)\Delta^{\mathrm{ccr}}_\ell(f'\blacktriangleright\!\!\lhd_A u').
\end{aligned}$$

Finally, as part (*iii*) in Definition 2.1.2, we need to prove the left counit property (2.1.4) for $\epsilon_{\mathrm{ccr}}$ with respect to the $A$-actions (3.3.28) on $F\blacktriangleright\!\!\lhd_A U$. Here (3.3.21), (3.3.18), (2.1.4) for $\epsilon_F, \ \epsilon_U$, and the commutativity of $A$ will be needed:

$$\begin{aligned}
\epsilon_{\mathrm{ccr}}\big((f\blacktriangleright\!\!\lhd_A u)(f'\blacktriangleright\!\!\lhd_A u')\big) &= \epsilon_{\mathrm{ccr}}\big(fu_{(1)}(f')\blacktriangleright\!\!\lhd_A u_{(2)}u'\big) \\
&= \epsilon_F\big(fu_{(1)}(f')t^\ell_F\epsilon_U(u_{(2)}u')\big) \\
&= \epsilon_F(u_{(1)}(f'))\epsilon_U(u_{(2)}u')\epsilon_F(f) \\
&= \epsilon_U(u_{(1)}s^\ell_F\epsilon_F f')\epsilon_U(u_{(2)}u')\epsilon_F(f) \qquad\qquad \text{with (3.3.21)} \\
&= \epsilon_U(s^\ell_U\epsilon_F(f)us^\ell_U\epsilon_F(f')u') \\
&= \epsilon_F\big(ft^\ell_F\epsilon_U(us^\ell_U\epsilon_F(f't^\ell_F\epsilon_U u'))\big) \\
&= \epsilon_{\mathrm{ccr}}\big(f\blacktriangleright\!\!\lhd_A(us^\ell_U\epsilon_F(f't^\ell_F\epsilon_U u'))\big) = \epsilon_{\mathrm{ccr}}\big((f\blacktriangleright\!\!\lhd_A u)t^\ell_{\mathrm{ccr}}\epsilon_F(f't^\ell_F\epsilon_U u')\big).
\end{aligned}$$

Likewise one can show that

$$\begin{aligned}\epsilon_{\mathrm{ccr}}\big((f \blacktriangleright\!\!\triangleleft_A u)s^{\ell}_{\mathrm{ccr}}\epsilon_F(f' t^{\ell}_F \epsilon_U u')\big) &= \epsilon_F\big(f u_{(1)}(s^{\ell}_F \epsilon_F(f' t^{\ell}_F \epsilon_U u'))t^{\ell}_F \epsilon_U u_{(2)}\big) \\ &= \epsilon_U(s^{\ell}_U \epsilon_F(f) u s^{\ell}_U \epsilon_F(f')u'),\end{aligned}$$

and then continue as above from the third line from below. □

**3.3.6 Definition** A pair $(F, U)$ of two left $A$-bialgebroids related to each other by the properties (3.3.17)–(3.3.25) is called a *matched pair* of left bialgebroids. The resulting left $A$-bialgebroid $F \blacktriangleright\!\!\triangleleft_A U$ is called the *bicrossed product bialgebroid* of the matched pair $(F, U)$.

In Sections 4.5-4.7 we will present a context in which such a construction appears quite naturally.

**3.3.7 Remark** Of course, there is an analogous construction for right bialgebroids. We hence conjecture that, analogously as for Hopf algebras [Maj], the bicrossed product bialgebroid $F \blacktriangleright\!\!\triangleleft_A M$ can be made into a Hopf algebroid with antipode

$$S(f \blacktriangleright\!\!\triangleleft_A u) = \big(1_F \blacktriangleright\!\!\triangleleft_A S_U u_{[0]}\big)\big(S_F(f u_{[1]}) \blacktriangleright\!\!\triangleleft_A 1_U\big) = \big(S_U u_{[0]}\big)_{(1)}\big(S_F(f u_{[1]})\big) \blacktriangleright\!\!\triangleleft_A (S_U u_{[0]})_{(2)},$$

where $u \in U,\ f \in F$, provided that $F, U$ are Hopf algebroids with antipodes $S_F$, $S_U$, respectively. This will probably require $U$ to be both a right $F^{\ell}$-comodule and a right $F^{r}$-comodule for the underlying left and right bialgebroid structures of $F$, and in particular additional compatibility conditions for these coactions will have to be added. We expect these conditions to correspond to those occurring in the definition of right *Hopf algebroid comodules*, a subtle notion which was completely clarified only recently in [B3, Def. 4.6].

# Chapter 4

# Examples of Hopf Algebroids

## 4.1 Immediate Examples

In this section we present a few examples that one expects or rather requires to fulfill the axioms of a Hopf algebroid; see e.g. [B3] for more straightforward examples.

### 4.1.1 The Enveloping Algebra $A^{\mathrm{e}}$

One of the most basic examples (see e.g. [Lu, Schau1, B3]) of a Hopf algebroid is the enveloping algebra $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$ of any $k$-algebra $A$. Now $A^{\mathrm{e}}$ is a left bialgebroid over $A$ by $\eta^{\ell}_{A^{\mathrm{e}}} := \mathrm{id}_{A^{\mathrm{e}}}$, i.e., $s^{\ell}a = a \otimes_k 1$, $t^{\ell}b = 1 \otimes_k b$, and left coproduct as well as left counit given by

$$\Delta_\ell : A^{\mathrm{e}} \to A^{\mathrm{e}} \otimes_A A^{\mathrm{e}}, \quad a \otimes_k b \mapsto (a \otimes_k 1) \otimes_A (1 \otimes_k b), \quad \epsilon : A^{\mathrm{e}} \to A, \quad a \otimes_k b \mapsto ab.$$

We recall that $A^{\mathrm{e}} \otimes_A A^{\mathrm{e}} := A^{\mathrm{e}} \otimes_k A^{\mathrm{e}}/\mathrm{span}_k\{(a \otimes_k bc) \otimes_k (a' \otimes_k b') - (a \otimes_k b) \otimes_k (ca' \otimes_k b'),\ c \in A\}$; here and in what follows we express the product structure on $A^{\mathrm{op}}$ by the one in $A$. Similarly, there is a right bialgebroid structure on $A^{\mathrm{e}}$ over $A^{\mathrm{op}}$ given by $\eta^{r}_{A^{\mathrm{e}}} := \mathrm{id}_{(A^{\mathrm{op}})^{\mathrm{e}}}$, i.e., $s^{r}a = 1 \otimes_k a$, $t^{r}b = b \otimes_k 1$, and right coproduct as well as right counit

$$\Delta_r : A^{\mathrm{e}} \to A^{\mathrm{e}} \otimes^{A^{\mathrm{op}}} A^{\mathrm{e}}, \quad a \otimes_k b \mapsto (1 \otimes_k a) \otimes_A (b \otimes_k 1), \quad \partial : A^{\mathrm{e}} \to A^{\mathrm{op}}, \quad a \otimes_k b \mapsto ba,$$

where $A^{\mathrm{e}} \otimes^{A^{\mathrm{op}}} A^{\mathrm{e}} = A^{\mathrm{e}} \otimes_k A^{\mathrm{e}}/\mathrm{span}_k\{(a \otimes_k cb) \otimes_k (a' \otimes_k b') - (a \otimes_k b) \otimes_k (a'c \otimes_k b'),\ c \in A\}$. Finally, the antipode is given by the tensor flip, i.e.,

$$S : A^{\mathrm{e}} \to A^{\mathrm{e}}, \quad a \otimes_k b \mapsto b \otimes_k a.$$

As for the left Hopf algebroid ($\times_A$-Hopf algebra) structure, the tensor product in question reads

$${}_{\blacktriangleright} A^{\mathrm{e}} \otimes_{A^{\mathrm{op}}} A^{\mathrm{e}}{}_{\triangleleft} = A^{\mathrm{e}} \otimes_k A^{\mathrm{e}}/\mathrm{span}_k\{(a \otimes_k cb) \otimes_k (a' \otimes_k b') - (a \otimes_k b) \otimes_k (a' \otimes_k b'c)\},$$

where $cb$ and $b'c$ is understood to be the product in $A$. One then easily verifies that

$$(a \otimes_k b)_+ \otimes_{A^{\mathrm{op}}} (a \otimes_k b)_- := (a \otimes_k 1) \otimes_{A^{\mathrm{op}}} (b \otimes_k 1)$$

yields an inverse of the Hopf-Galois map defined as in (2.2.13).

### 4.1.2 Hopf Algebras Twisted by a Character

In this subsection we indicate how Hopf algebras with a character and Hopf algebroids over a *commutative ground ring* $k$ correspond to each other. More precisely, we explain and prove

**4.1.1 Proposition** *There is a bijective correspondence between Hopf algebroids over a commutative ground ring $k$ and Hopf algebras over $k$ equipped with a character.*

A Hopf algebra $H = (H, m_H, \eta, \Delta, \epsilon, S)$ together with a character $\delta : H \to k$ and the twisted antipode$\tilde{S} = m_H(\delta \otimes S)\Delta$ from [CoMos4] (cf. (1.3.3) in Section 1.3) can be considered a Hopf algebroid over $k$ as follows: the underlying $k$-bialgebra $(H, m_H, \eta, \Delta, \epsilon)$ is clearly a left $k$-bialgebroid, source and target map both being given by $\eta : k \to ZH \subset H$, mapping into the centre of $H$. As for the right $k$-bialgebroid structure, source and target maps are again given by $\eta$, whereas the right coproduct and the right counit read

$$\begin{array}{rll} \Delta_r : & H \to H \otimes_k H, & h \mapsto h_{(1)} \otimes_k \eta(\delta S h_{(2)}) h_{(3)}, \\ \partial := \delta : & H \to k, & h \mapsto \delta h, \end{array}$$

where $\Delta h = h_{(1)} \otimes_k h_{(2)}$ are the Sweedler components of $\Delta$. Observe that $\delta\tilde{S} = \epsilon$ while generally $\delta S \neq \epsilon$. One easily checks that the comonoid identities (2.5.6) are fulfilled, for example for each $h \in H$,

$$\begin{aligned} m_H(id_H \otimes \partial)\Delta_r h &= h_{(1)} \eta\delta(\delta(S h_{(2)}) h_{(3)}) \\ &= h_{(1)} \eta\delta(S h_{(2)} h_{(3)}) \\ &= h_{(1)} \eta\delta\epsilon h_{(2)} = h_{(1)} \eta\epsilon h_{(2)} = h, \end{aligned}$$

since $(H, S)$ is a Hopf algebra. Likewise, one checks that the data $(H, m_H, \eta, \Delta, \Delta_r, \epsilon, \delta, \tilde{S})$ fulfill all requirements in Definition 2.6.1 of a Hopf algebroid. As an example, we prove (2.6.4)

$$m_H(\tilde{S} \otimes \mathrm{id}_H)\Delta h = \eta(\delta h_{(1)}) S h_{(2)} h_{(3)} = \eta(\delta h_{(1)}) \epsilon h_{(2)} = \eta\delta h,$$

and also

$$\begin{aligned} m_H(id_H \otimes \partial)\Delta_r h &= h_{(1)} \tilde{S}\big(\eta(\delta S h_{(2)}) h_{(3)}\big) \\ &= h_{(1)} \eta(\delta S h_{(2)}) \delta h_{(3)} S h_{(4)} \\ &= h_{(1)} \eta(\delta S h_{(2)} h_{(3)}) S h_{(4)} = h_{(1)} \eta(\epsilon h_{(2)}) S h_{(3)} = \eta\epsilon h. \end{aligned}$$

Observe that if $S$ is invertible, then the inverse of $\tilde{S}$ is given by $\tilde{S}^{-1} = m_{H^{\mathrm{op}}}(S^{-1} \otimes \delta)\Delta$.

Now consider the converse situation of a Hopf algebroid $(H, k, \eta, \Delta_\ell, \Delta_r, \epsilon, \partial, S)$ over a commutative ground ring $k$: since source and target (for both the left and right bialgebroid) are unital maps $k \to H$ for the unital $k$-algebra $H$, all of them coincide with the unit $\eta : k \to ZH \subset H$. The left and right counit are $k$-algebra characters $\epsilon, \partial : H \to k$, but the underlying left and right bialgebroid structures do not necessarily coincide. If $S$ denotes the antipode of the Hopf algebroid, define

$$S' := m_H(\eta\partial \otimes S)\Delta_\ell.$$

One then verifies $S'(hh') = S'h'S'h$ and

$$\begin{aligned} \Delta_r S' h &= \eta\partial h_{(1)} S h_{(3)} \otimes_k S h_{(2)} \\ &= S h_{(2)} \otimes_k S' h_{(1)} \\ &= S h_{(2)} \otimes_k \eta\partial h^{(2)}_{(1)} S' h^{(1)}_{(1)} = S' h_{(2)} 0 \otimes_k S' h_{(1)}. \end{aligned}$$

Hence $S'$ is an antipode for the $k$-bialgebra $(H, \Delta_r, \partial)$ and $\epsilon$ plays the role of a character for it: twisting again as in (1.3.3) returns the original Hopf algebroid antipode, i.e., $m_H(\eta\epsilon \otimes S')\Delta_r = S$. Of course, a similar construction can be made for the $k$-bialgebra $(H, \Delta_\ell, \epsilon)$: here the antipode reads $S' = m_H(S \otimes \eta\epsilon)\Delta_r$ and the character is $\partial$. These two Hopf algebras are not independent from each other, though: as for the underlying left and right bialgebroid in the Hopf algebroid $H$, they can be transferred into each other by means of $S$.

**4.1.2 Remarks** Let us conclude this section by pointing on some generalisations of the concepts used above:

(*i*) If on a Hopf algebra there is no character given (apart from its counit), the proposition above does not produce anything new. In such a case one simply has the bijective correspondence between Hopf algebras and left Hopf algebroids over $k$, as mentioned in Example 2.2.2(*ii*). Hence, as a summary of Proposition 4.1.1 and Example 2.2.2 one may state that left Hopf algebroids generalise Hopf algebras, whereas Hopf algebroids generalise Hopf algebras with a character (or twisted antipode).

(*ii*) As observed in [B1], for a cocommutative Hopf algebra one can always find nontrivial characters. Twisting the antipode with respect to these characters produces Hopf algebroids which do not fulfill the axioms of the different definition of Hopf algebroid in [Lu], cf. §2.6.13(*i*). This shows that the two definitions (i.e. the one from [Lu] and the one from [BSz2] used throughout this thesis) are not equivalent.

(*iii*) The procedure of twisting a Hopf algebra antipode by a character is only a special case of the notion of *twist of a Hopf algebroid* [B1], producing new Hopf algebroids out of known ones. On top of that, in [BŞ, Ex. 2.18] one can find the generalisation of a modular pair in involution (see Subsection 1.3.2) to the realm of Hopf algebroids.

# 4.2 Universal Enveloping Algebras of Lie-Rinehart Algebras

In this section we discuss the fact that a Lie-Rinehart algebra always gives rise to a left bialgebroid as well as a left Hopf algebroid. Adding some extra structure, one may even obtain a (double-sided) Hopf algebroid.

## 4.2.1 The Canonical Left Hopf Algebroid Structure on $VL$

Let $(A, L)$ be a Lie-Rinehart algebra. Several authors [X3, KhR2, MoeMrč3] have shown that the enveloping algebra $VL$ is a left $A$-bialgebroid, but it is in fact also a left Hopf algebroid over $A$.

Recall first from e.g. [X3, KhR2, MoeMrč3] its left bialgebroid structure: source and target are equal and are given by $\iota_A : A \to VL$. The $(A, A)$-bimodule structure ${}_\triangleright VL_\triangleleft$ is hence given by multiplication of elements in $VL$, i.e., $a \triangleright u \triangleleft \tilde{a} = au\tilde{a}$, which enables us to suggestively denote the tensor product (2.1.1) by

$$VL \otimes^{ll} VL := VL_\triangleleft \otimes_A {}_\triangleright VL, \tag{4.2.1}$$

and likewise $VL \times^{ll} VL := VL \times_A VL$ for the Takeuchi product (2.1.2). The prescriptions

$$\Delta_\ell X = 1 \otimes^{ll} X + X \otimes^{ll} 1, \quad \Delta_\ell a = a \otimes^{ll} 1, \tag{4.2.2}$$

which map $X \in L$ and $a \in A$ into $VL \times^{ll} VL$, can be extended by the universal property to a coproduct $\Delta : VL \to VL \times_A VL \subset VL \otimes^{ll} VL$. The counit is similarly given by extension of the anchor $\omega$ to $VL$, more precisely, by

$$\epsilon : VL \to A, \qquad u \mapsto \omega(u)(1_A).$$

As in (2.3.3), this defines a left $VL$-action on $A$, which we abbreviate as $u(a) := \epsilon(ua)$, and in particular, one has

$$\epsilon X = 0, \qquad \epsilon a = a, \qquad \forall X \in L,\ a \in A.$$

The defining property of a left Hopf algebroid, i.e. the bijectivity of the Galois map, is seen in the same way: denote the tensor product (2.2.2) by

$$VL \otimes^{rl} VL := {}_\blacktriangleright VL \otimes_{A^{\mathrm{op}}} VL_\triangleleft, \tag{4.2.3}$$

and write $VL \times^{rl} VL := VL \times_{A^{\mathrm{op}}} VL$ for the Takeuchi product (2.2.12). Then the translation map $\beta^{-1}$ is given on generators as

$$a_+ \otimes^{rl} a_- := a \otimes^{rl} 1, \quad X_+ \otimes^{rl} X_- := X \otimes^{rl} 1 - 1 \otimes^{rl} X. \tag{4.2.4}$$

These maps stay in $VL \times^{rl} VL$, which is an algebra through the product of $VL$ in the first and its opposite in the second tensor factor. By universality we obtain a map $VL \to VL \times^{rl} VL \subset VL \otimes^{rl} VL$, and then $\beta^{-1}$ is defined using (2.2.13).

Conversely, certain left bialgebroids give rise to Lie-Rinehart algebras:

**4.2.1 Proposition** *For a left bialgebroid $(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ with $A$ commutative and $s^\ell \equiv t^\ell$, the pair $(A, P^\ell H)$ of the base algebra and the left primitive elements forms a Lie-Rinehart algebra.*

PROOF: The proof is quite straightforward. Firstly, since $s^\ell \equiv t^\ell$ we usually refrain from mentioning these maps; the remaining two left and right $A$-module structures on $U$ read $au := s^\ell(a)u$ and $ua := us^\ell(a)$. Then the coproduct $\Delta_\ell$ is a map $U \to U \otimes^{ll} U$, where we use again the notation $U \otimes^{ll} U$, with its obvious meaning analogous to (4.2.1). The natural Lie algebra structure on $U$ is simply $[u, u'] := uu' - u'u$, which is closed in $P^\ell U$. We have $\Delta_\ell(au) = au \otimes^{ll} 1 + 1 \otimes^{ll} au$ for $u \in P^\ell U$, which is therefore a (left) $A$-submodule (since $s^\ell \equiv t^\ell$). The anchor is given by the Lie algebra action

$$P^\ell U \to \mathrm{Der}_k A,\ u \to \{a \mapsto \epsilon(ua) =: u(a)\}.$$

The required property (1.4.1) is obvious and for $u, u' \in P^\ell U$ and $a, b \in A$, we conclude that

$$\begin{aligned}([u, au'])(b) &= \epsilon(u(au'(b)) - au'(u(b))) \\ &= u(a)u'(b) - a([u, u'])(b) = (u(a)u')(b) - (a[u, u'])(b).\end{aligned}$$

Since $b \in A$ was arbitrary, this proves (1.4.2). □

**4.2.2 Remark** (*i*) Somewhat more generally, for any left bialgebroid $(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ for which $s^\ell|_{ZA} \equiv t^\ell|_{ZA}$ on the center $ZA$ of $A$, one can analogously show that the pair $(ZA, P^\ell U)$ forms a Lie-Rinehart algebra.

(*ii*) Observe that even if one omits the 'action axiom' $\epsilon(uu') = \epsilon(u\epsilon u')$ of a counit, one is still able to define a Lie algebra action: first, for $u \in P^\ell U$ we have $\Delta_\ell(ua) = ua \otimes^{ll} 1 + 1 \otimes^{ll} au$. Applying $m_U(\epsilon \otimes \mathrm{id})$ on both sides yields

$$ua = u(a) + au, \quad a \in A, u \in P^\ell U,$$

from which can be read off again that $\epsilon u = 0$ for $u \in P^\ell U$. Now we have for any two primitive $u, u'$,

$$\begin{aligned}([u, u'])(a) &= \epsilon(uu'a - u'ua) \\ &= \epsilon(u(u'(a)) + u(au') - u'(u(a)) + u'(au)) \\ &= u(u'(a)) - u'(u(a)) - \epsilon(u(a)u' + [au, u']) \\ &= [u, u'](a),\end{aligned}$$

since $P^\ell U \subset \ker \epsilon$ was both an $A$-submodule and a Lie subalgebra in $U$. Hence we again obtain the desired Lie algebra action.

**4.2.3 Proposition** *Let $(U, A, s^\ell, \Delta_\ell, \epsilon)$ and $(U', A', s^{\ell'}, \Delta'_\ell, \epsilon')$ be left bialgebroids over commutative bases and suppose $s^\ell \equiv t^\ell$ as well as $s^{\ell'} \equiv t^{\ell'}$. A left bialgebroid morphism $(\psi, \phi) : (U, A) \to (U', A')$ induces a morphism*

$$(P_\phi, P_\psi) : (A, P^\ell U) \to (A', P^\ell U')$$

*of the corresponding Lie-Rinehart algebras of primitive elements. In case $A = A'$, this leads to a functor $P : A\text{-}\mathbf{LBiAlgd} \to A\text{-}\mathbf{LieRine}$ from the category $A\text{-}\mathbf{LBiAlgd}$ of left $A$-bialgebroids to the category $A\text{-}\mathbf{LieRine}$ of Lie-Rinehart algebras over $A$. Conversely, a morphism $(A, L) \to (A', L')$ of Lie-Rinehart algebras induces a morphism $(VL, A) \to (VL', A')$ of left bialgebroids, which in case $A = A'$ in turn leads to a functor $V : A\text{-}\mathbf{LieRine} \to A\text{-}\mathbf{LBiAlgd}$.*

PROOF: Using $\psi(1_U) = 1_{U'}$, it is easy to check that $\psi(P^\ell U) \subset P^\ell U'$. Hence the induced map reads

$$(P_\phi, P_\psi) := (\phi, \psi|_{P^\ell U}) : (A, P^\ell U) \to (A', P^\ell U').$$

Moreover, we see that $P_\psi(au) = \psi(au) = \phi(a)\psi(u) = P_\phi(a)P_\psi(u)$, since the primitive elements were respective $A$-submodules. Secondly, $P_\psi$ is automatically a Lie algebra morphism since the Lie bracket is simply the commutator. Thirdly, $\phi u(a) = \phi\epsilon(ua) = \epsilon'(\psi(ua)) = \epsilon'(\psi(h)a) = (\psi(u))(\phi a)$. These three statements together prove $(P_\phi, P_\psi)$ to be a morphism of Lie-Rinehart algebras. By the universal property, the converse statement is similarly simple to see. □

See Corollary 5.5.8 and in particular [MoeMrč3] for further statements on the interplay between Lie-Rinehart algebras and primitive elements of bialgebroids.

### 4.2.2 Hopf Algebroid Structures on $VL$

We saw in the previous subsection that $VL$ is a left bialgebroid and even a left Hopf algebroid in a canonical way. Adding some further (non-canonical and not necessarily existing) datum, one could even establish the structure of a Hopf algebroid on it. This subsection will be dedicated to explain and prove the following result:

**4.2.4 Theorem** *Let $(A, L)$ be a Lie-Rinehart algebra. If a flat right $(A, L)$-connection exists on $A$, the universal enveloping algebra $VL$ can be equipped with an antipode, and in particular can be made into a Hopf algebroid.*

We start discussing the concepts needed in this theorem.

**4.2.5 Connections** A main ingredient in the following discussion is the notion of $(A, L)$-connections for a Lie-Rinehart algebra $(A, L)$ from [Hue1, Hue2], with the slight difference that we assume $A$ to be unital. Let $M \in A$-$\mathbf{Mod}$. A map

$$\nabla^\ell : M \to \mathrm{Hom}_A(L, M) \tag{4.2.5}$$

that fulfills

$$\nabla^\ell_X(am) = a\nabla^\ell_X(m) + X(a)m, \quad a \in A, m \in M, \tag{4.2.6}$$

is called a *left* $(A, L)$-connection on $M$. It is said to be *flat* if it establishes a (left) Lie algebra action $m \mapsto (X \mapsto [X, m])$ of $L$ on $M$, in which case $M$ is called a left $(A, L)$-module. Clearly, $A$ itself carries such a left $(A, L)$-connection (given by the anchor), and flat $(A, L)$-connections uniquely correspond to left $VL$-modules structures by the universal property of $VL$. A *right* $(A, L)$-connection on an $A$-module $N$ is a map $\nabla^r : N \to \mathrm{Hom}_k(L, N)$ that fulfills

$$\begin{aligned} \nabla^r_X(an) &= a\nabla^r_X n - X(a)n, & (4.2.7)\\ \nabla^r_{aX} n &= a\nabla^r_X n - X(a)n, \quad a \in A, n \in N. & (4.2.8) \end{aligned}$$

Again, the connection is called *flat* if it establishes a (right) Lie algebra module structure $n \mapsto (X \mapsto [n, X])$ on $N$, in which case $N$ is called a right $(A, L)$-module (which, in turn, uniquely correspond to a right $VL$-module). See §4.2.10 for a comment on the apparent asymmetry in the definitions of left and right connections.

**4.2.6 Lie Algebroid Connections** Assume now that $L$ is $A$-projective of finite constant rank $n$, so that $\Lambda^n_A L$ is the highest non-zero power of $L$ in the category of $A$-modules. A result in [Hue2, Thm. 3] says that *right* $(A, L)$-connections on $A$ are equivalent to *left* $(A, L)$-connections on $\Lambda^n_A L$: the latter were (if $(A, L) := (\mathcal{C}^\infty(M), \Gamma E)$ originates from a Lie algebroid $E \to M$) introduced in [X2] under the name *Lie algebroid connection* or $E$*-connection*. In this particular case, there is always an $E$-connection that is flat, implying existence of a flat right connection on $A = \mathcal{C}^\infty(M)$, although there is still no canonical choice for it; cf. [EvLuWei, Prop. 4.3] and [X2]. A flat $E$-connection on a vector bundle $F$ is also called a *representation* of the Lie algebroid [Mac, EvLuWei]. In general, right $(A, L)$-connections on $A$ need not exist: see Example 4.2.13.

Recall that a *Gerstenhaber algebra* [G, GSch] is a graded commutative $k$-algebra $V$ together with a Lie bracket $[.,.]_G : V \otimes_k V \to V$ of degree $-1$ (a graded Lie bracket in the usual sense when the degrees of the elements of $V$ are lowered by 1) that satisfies a graded Leibniz identity (cf. e.g. [Kos] for details). A $k$-linear operator $\partial$ of degree $-1$ is said to *generate* a Gerstenhaber algebra $V$ if for every homogeneous $v, w \in V$, one has

$$[v, w]_G = (-1)^{\deg v}(\partial(vw) - (\partial v)w - (-1)^{\deg v} v\partial w).$$

The operator $\partial$ is called *exact* if $\partial^2 = 0$ and a Gerstenhaber algebra with an exact generator is called a *Batalin-Vilkovisky algebra* [Kos, X2]. On the $A$-exterior algebra $\wedge^\bullet_A L$, one has the following Gerstenhaber bracket:

$$\begin{aligned} &[.,.]_G : \wedge^\bullet_A L \otimes_k \wedge^\bullet_A L \to \wedge^\bullet_A L, \\ &[u, v]_G = (-1)^{\deg u} \sum_{i \leq j < l} (-1)^{i+l}[X_i, X_l] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \hat{X}_l \cdots \wedge X_n, \end{aligned} \tag{4.2.9}$$

for $u = X_1 \wedge \cdots \wedge X_l \in \wedge^l_A L$ and $v = X_{l+1} \wedge \cdots \wedge X_n \in \wedge^{n-l}_A L$, where the bracket $[.,.]$ is the one from the Lie-Rinehart structure on $L$. Right connections can then be characterised by the following statement.

**4.2.7 Theorem** [Hue2, Thm. 1] *Right $(A, L)$-connections on $A$ bijectively correspond to $k$-linear operators $\partial$ of degree $-1$ generating the Gerstenhaber bracket on $\wedge^\bullet_A L$.* Exact *operators or differentials, that is, $k$-linear operators $\partial$ of degree $-1$ with $\partial^2 = 0$ generating the Gerstenhaber bracket on $\wedge^\bullet_A L$, correspond to* flat *right $(A, L)$-connections.*

See [Hue2, Thm. 1] for a detailed proof. For later use we repeat how the precise correspondence is given. A $k$-linear operator $\partial$ generating the Gerstenhaber bracket (4.2.9) defines a right $(A, L)$-connection on $A$ via

$$\nabla^r_X a := a\,\partial X - X(a) = a(\partial X) + [a, X]_G = \partial(aX).$$

In particular,

$$\nabla^r_X 1_A = \partial X, \tag{4.2.10}$$

see also our Proposition 3.1.14 for a dual construction in the general context of left bialgebroids. Conversely, if $(a, X) \mapsto \nabla^r_X a$ is such a connection, the operator $\partial$ on $\wedge^\bullet_A L$ defined by

$$\begin{aligned}\partial(X_1 \wedge \cdots \wedge X_n) = &\sum_{i=1}^{n}(-1)^{i-1}(\nabla^r_{X_i} 1_A) X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n \\ &+ \sum_{i<j}(-1)^{j+i}[X_i, X_j] \wedge X_1 \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n\end{aligned}$$

yields an $k$-linear operator $\partial$ generating $[.,.]_G$.

**4.2.8 Hopf Algebroid Structure on $VL$**

By now, we have gathered all necessary ingredients to establish a Hopf algebroid structure on $VL$. To start, a right bialgebroid structure on $VL$ is given as follows: since $A = A^{\mathrm{op}}$, set

$$s^r \equiv t^r \equiv s^\ell \equiv t^\ell \equiv i_A : A \hookrightarrow VL, \tag{4.2.11}$$

and the $(A, A)$-bimodule structure $_\blacktriangleright VL_\blacktriangleleft$ is given again by multiplication in $VL$, i.e., $a \blacktriangleright u \blacktriangleleft \tilde{a} = au\tilde{a}$. We therefore suggestively denote the tensor product (2.5.1) by

$$VL \otimes^{rr} VL := VL_\blacktriangleleft \otimes^A {}_\blacktriangleright VL.$$

**4.2.9 Proposition** *Flat left and right $(A, L)$-connections on $A$ correspond to left and right bialgebroid structures on $VL$ over $A$, respectively. In particular, a Lie-Rinehart algebra $(A, L)$ with a flat right $(A, L)$-connection on its base algebra carries both left and right bialgebroid structures.*

PROOF: Recall that, as morphisms of $k$-Lie algebras, *flat* left and right connections $\nabla^\ell : L \to \mathrm{End}_k A$ and $\nabla^r : L^{\mathrm{op}} \to \mathrm{End}_k A$ can be extended to $k$-algebra morphisms $\nabla^\ell : VL \to \mathrm{End}_k A$ and $\nabla^r : (VL)^{\mathrm{op}} \to \mathrm{End}_k A$, with respective properties $\nabla^\ell_{uu'} a = \nabla^\ell_u \nabla^\ell_{u'} a$ and $\nabla^r_{uu'} a = \nabla^r_{u'} \nabla^r_u a$ for all $u, u' \in VL,\ a \in A$. Given such flat connections as above, with associated operators

$$\tilde{\epsilon} u := \nabla^\ell_u 1_A, \qquad \partial u := \nabla^r_u 1_A, \qquad u \in VL,$$

both seen as maps $VL \to A$, one has

$$\partial(\partial(u)u') = \nabla^r_{u'} \partial u = \nabla^r_{u'} \nabla^r_u 1_A = \nabla^r_{uu'} 1_A = \partial(uu'), \tag{4.2.12}$$

and also

$$\tilde{\epsilon}(u\tilde{\epsilon}(u')) = \nabla^\ell_u \tilde{\epsilon}(u') = \nabla^\ell_u \nabla^\ell_{u'} 1_A = \nabla^\ell_{uu'} 1_A = \tilde{\epsilon}(uu'). \tag{4.2.13}$$

Define two coproducts by setting on generators

$$\begin{array}{rclcrcl}\Delta_\ell X &=& 1 \otimes^{ll} X + X \otimes^{ll} 1 - \tilde{\epsilon} X \otimes^{ll} 1, && \Delta_\ell a &=& a \otimes^{ll} 1, \\ \Delta_r X &=& 1 \otimes^{rr} X + X \otimes^{rr} 1 - \partial X \otimes^{rr} 1, && \Delta_r a &=& a \otimes^{rr} 1,\end{array} \tag{4.2.14}$$

and extend these maps to the whole of $VL$ by requiring them to corestrict to $k$-algebra morphisms $\Delta_\ell : VL \to VL \times_A VL$ and $\Delta_r : VL \to VL \times^A VL$. One then easily checks that $(VL, A, i_A, \Delta_\ell, \tilde{\epsilon})$ and $(VL, A, i_A, \Delta_r, \partial)$ are left and right bialgebroids, respectively. □

**4.2.10 Remarks** (*i*) Clearly, the anchor already defines such a left $(A, L)$-connection, which reproduces the canonical left bialgebroid structure on $VL$ from Subsection 4.2.1.

(*ii*) By (2.1.8) one has $X + \epsilon X = X$, hence $\epsilon(aX) = 0$, $\epsilon(a) = a$ and also $\epsilon(Xa) = X(a)$ which reveals that $\epsilon$ (and also $\partial$) is an algebra morphism if the (left respectively right) action of $L$ on $A$ is trivial. This is, for example, the case for the symmetric algebra $S_A L$, which expressed on generators has analogous structure maps as $VL$, but defines a trivial action on $A$ (since it is commutative).

(*iii*) The apparent asymmetry in the definition of left and right $(A, L)$-connections in (4.2.5)–(4.2.8) is essentially due to the fact that $A$ already carries a canonical $VL$-module structure from the left (namely by the anchor), but not from the right. Hence Lie-Rinehart algebras should actually be called *left* Lie-Rinehart algebras. In case there is a right $VL$-action $(a, u) \mapsto \partial(au)$ on $A$, one may reformulate the right connection identities (4.2.7) and (4.2.8) and thus obtain a more symmetric definition compared to left connections: a flat right $(A, L)$-connection on a right $A$-module $N$ is a map $\nabla^r : N \to \mathrm{Hom}_{(-,A)}(VL_{\blacktriangleleft}, N_A)$ of *right* $A$-modules, subject to

$$\nabla^r_X(na) = n\partial(aX) + (\nabla^r_X n)a - na\partial X, \qquad a \in A, X \in L, n \in N. \tag{4.2.15}$$

Hence using $[X, a] = X(a)$, one has $\nabla^r_{aX} n + X(a)n = \nabla^r_{Xa} n = (\nabla^r_X n)a$, which is (4.2.7) again, and inserting $\partial(aX) = a\partial X - X(a)$ into (4.2.15) reproduces (4.2.8). The fact that (4.2.15) contains three terms whereas there are only two in (4.2.6), is due to the fact that elements $X \in L$ are primitive with respect to $\Delta_\ell$ but not with respect to $\Delta_r$, hence $\epsilon X = 0$, whereas most generally $\partial X \neq 0$ (see also the next comment).

(*iv*) It is then evident that right $(A, L)$-connections are a special case of right bialgebroid connections from Section 2.5.1. Compare (2.5.8) to (4.2.15), again the additional summand derives from the fact that $X$ is not primitive with respect to $\Delta_r$.

Linking right and left bialgebroid structures on $VL$ by an antipode leads to the structure of a Hopf algebroid on $VL$: let $(A, L)$ be a Lie-Rinehart algebra and $\nabla^r$ a right $(A, L)$-connection on $A$ with associated operator $\partial X := \nabla^r_X 1_A$, seen as a map $\partial : L \to A$ (cf. Theorem 4.2.7). Define a pair of maps $S^L_\partial : L \to VL$, $S^A_\partial : A \to VL$ by

$$S^L_\partial(X) = -X + \partial X, \quad S^A_\partial(a) = a, \qquad \forall\, X \in L,\ a \in A. \tag{4.2.16}$$

Combining (4.2.7) with (4.2.8), this implies that

$$S^L_\partial(aX) = -aX + \nabla^r_X a \qquad\qquad X \in L,\ a \in A.$$

**4.2.11 Proposition** (Antipodes for Lie-Rinehart algebras) *Let $(A, L)$ be a Lie-Rinehart algebra and $\nabla^r$ a right $(A, L)$-connection on $A$, as above.*

(*i*) *The pair $(S^A_\partial, S^L_\partial)$ extends to a $k$-algebra anti-homomorphism $S : VL \to VL$ if and only if $\nabla^r$ is flat. In such a case, $S$ is an involutive antipode with respect to the canonical left bialgebroid structure (see Subsection 4.2.1) and the right bialgebroid structure from Proposition 4.2.9.*

(*ii*) *Conversely, given a unital map $S : VL \to VL$ that is an isomorphism of twisted bimodules, i.e., $S(aub) = bS(u)a$ and $S(1_{VL}) = 1_{VL}$, $a, b \in A$, $u \in VL$, the assignment*

$$\nabla : A \to \mathrm{Hom}_k(L, A), \quad a \mapsto \{X \mapsto \epsilon(S(X)a)\} \tag{4.2.17}$$

*defines a right $(A, L)$-connection on $A$ which is flat if and only if $S$ is a $k$-algebra anti-homomorphism.*

PROOF: Part (*i*): exploiting the universal property of $VL$, we show that $((VL)^{\mathrm{op}}, S^L_\partial, S^A_\partial)$ is a triple of the kind (1.4.3). First, $S^A_\partial : A \to VL,\ a \mapsto i_A(a) = a$ is clearly a morphism of $k$-algebras; considering $S^L_\partial : L \to VL,\ X \mapsto -i_L(X) + i_A(\partial X) = -X + \partial X$ (refraining from mentioning $i_L, i_A$ all the time), we have

$$\begin{aligned}
[S^L_\partial X, S^L_\partial Y] &= [X, Y] + [Y, \partial X] - [X, \partial Y] + \partial X \partial Y - \partial Y \partial X \\
&= [X, Y] + Y(\partial X) - X(\partial Y) \\
&= S^L_\partial([Y, X]) - \partial([Y, X]) + Y(\partial X) - X(\partial Y) + \partial X \partial Y - \partial Y \partial X \\
&= S^L_\partial([Y, X]) - \nabla^r_{[Y,X]} 1_A + \nabla^r_X \partial Y - \nabla^r_Y \partial X \\
&= S^L_\partial([Y, X]) - \nabla^r_{[Y,X]} 1_A + \nabla^r_X \nabla^r_Y 1_A - \nabla^r_Y \nabla^r_X 1_A \\
&= S^L_\partial([Y, X]) + \big([\nabla^r_X, \nabla^r_Y] - \nabla^r_{[Y,X]}\big)(1_A).
\end{aligned}$$

The last term is the curvature of $\nabla^r$, so $S^L_\partial : L \to (VL)^{\mathrm{op}}_{\mathrm{L}}$ is a homomorphism of $k$-Lie algebras if and only if the connection is flat. We now check

$$S^A_\partial(a) S^L_\partial(X) = -Xa + a\partial X = -aX + \nabla^r_X a = S^L_\partial(aX),$$

and also
$$S^L_\partial(X)S^A_\partial(a) - S^A_\partial(a)S^L_\partial(X) = -aX + a\partial X + Xa - a\partial X = S^A_\partial(X(a)),$$
hence the property (1.4.3) for the pair $(\phi_L, \phi_A) := (S^L_\partial, S^A_\partial)$. As in Definition 1.4.3, we infer the existence of a unique morphism $S_\partial : VL \to (VL)^{\rm op}$ of $k$-algebras such that $S_\partial i_A = S^A_\partial$, $S_\partial i_L = S^L_\partial$. If the connection is flat, the antipode axioms including $S^2_\partial = \mathrm{id}$ are straightforward to check by considering e.g. a PBW basis of $VL$, and making use of the anti-homomorphism property.

Part (*ii*): we need to check the properties (4.2.7) and (4.2.8) for a right connection. It is easy to see that

$$\begin{aligned}\nabla_X(ab) &= \epsilon(S(X)ab) = \epsilon((-Xa + a\partial X)b) = \epsilon((-aX - X(a) + a\partial X)b)\\ &= a\epsilon(S(X)b) - X(a)b = a\nabla_X b - X(a)b,\end{aligned}$$

which is (4.2.7), and similarly one proves the second identity. To show flatness if and only if $S$ is a $k$-algebra anti-homomorphism, compare (reintroducing $i_L$ here)

$$\begin{aligned}[\nabla_Y, \nabla_X](a) &= \epsilon(S(i_L(Y))\epsilon(S(i_L(X))a)) - \epsilon(S(i_L(X))\epsilon(S(i_L(Y))a))\\ &= \epsilon(S(i_L(Y))S(i_L(X))a)) - \epsilon(S(i_L(X))S(i_L(Y))a).\end{aligned}$$

with $\nabla_{[X,Y]}a = \epsilon\big(S^L([X,Y])a\big)$. The statement follows by the universal property. □

**4.2.12 Remarks** (*i*) We want to stress that flatness of the connection is needed in Proposition 4.2.11(*i*) to show that $S_\partial$ is a $k$-algebra anti-homomorphism.

(*ii*) There seems to be no way of introducing an antipode on $VL$ other than by flat right connections; furthermore, there does not seem to be a canonical choice for such a connection, or even a 'trivial' one. The analogue for the Lie algebra case, i.e., $SX = -X$ for $X \in L$ is not well-defined unless the anchor is trivial, which essentially leads back to Lie algebras. Related to this is the problem of how to define the 'opposite' of a Lie-Rinehart algebra.

(*iii*) There might be an obstruction for the existence of such a flat right connection on $A$ (as in the following Example 4.2.13); consequently, in such a case $VL$ cannot be given the structure of a Hopf algebroid. However, as we have seen, $VL$ is *always* a left Hopf algebroid. Hence the answer to the question posed in [B3] whether every left Hopf algebroid is the constituent left bialgebroid in a Hopf algebroid is *no*.

**4.2.13 A Counterexample** The simplest example of a Lie-Rinehart algebra that cannot be made into a Hopf algebroid might be $L = \Gamma(T^{1,0}S^2)$. Here $T^{1,0}S^2 \oplus T^{0,1}S^2 = TS^2 \otimes \mathbb{C}$ is the decomposition of the complexified tangent bundle of $S^2$ into the holomorphic and antiholomorphic part with respect to the standard complex structure. Together with $A = C^\infty(S^2, \mathbb{C})$ this defines a Lie-Rinehart algebra, where the action of $L$ on $A$ is the usual action of a vector field on a smooth function and the action of $A$ on $L$ is given by fibrewise multiplication. We know from [Hue2, Thm. 3] that the right $VL$-module structures on $A$ correspond bijectively to left $VL$-module structures on $L$ itself (as seen, in general on its top exterior power over $A$, but here this is $L$ because $T^{1,0}S^2$ is a line bundle). Such a left $VL$-action corresponds precisely to a flat connection $\nabla$ on the complex line bundle $T^{1,0}S^2$, with $X \in L$ acting on sections of $T^{1,0}S^2$ by the covariant derivative $\nabla_X$ (see [Hue2] for the details). But the curvature of any connection represents the first Chern class of the bundle, which is nonvanishing since $T^{1,0}S^2$ is not trivial. Therefore, there is no flat connection, i.e. left $VL$-action on $L$ and hence no right $VL$-action on $A$.

## 4.3 Jet Spaces of Lie-Rinehart Algebras

Now we describe another Hopf algebroid associated to a Lie-Rinehart algebra $(A, L)$, where, as an $A$-module, $L$ is finitely generated projective of constant rank (this means that it is the same for every prime ideal of $A$). We will prove the following theorem:

**4.3.1 Theorem** *The* space of $L$-jets $JL$ *is a Hopf algebroid with involutive antipode in the sense of Definition 2.6.1.*

Some of its structure maps have been used before in the literature, cf. [NeTs, CalVdB], but here we give a complete description: the Hopf algebroid of $L$-jets is in a certain sense dual to $VL$. As mentioned in Section 3.1, duality in the category of bialgebroids has been described in [KSz] under certain projectivity assumptions of the bialgebroid over their base algebra. These are clearly not satisfied for $VL$, but each successive quotient $VL_p/VL_{p-1}$ in the Poincaré-Birkhoff-Witt filtration of §1.4.5 is projective, provided $L$ is projective over $A$. With this, the bialgebroid structure for $JL$ can be seen to be given essentially by that of the dual of a left bialgebroid, as in Section 3.1. Observe that the left and right dual coincide (as $k$-modules) since source and target map are equal for $VL$; also note that the dual is a right bialgebroid, but since the jet spaces will be commutative (see below), we may equally consider it as a left bialgebroid.

Let $(A, L)$ be a Lie-Rinehart algebra and $L$ finitely generated $A$-projective of constant rank. The space of $p$-jets of $(A, L)$ is now defined as $J_pL := \mathrm{Hom}_A(VL_{\leq p}, A)$, where $VL_{\leq p}$ denotes the elements in $VL$ of degree $\leq p$. The infinite jet space is defined as the projective limit

$$JL = J_\infty L := \varprojlim J_pL.$$

We will now list the Hopf algebroid structure maps of $JL$ over $A$:

(*i*) (*Ring structure*) The monoid structure is given by a commutative product on $JL$ that can be described using the left coproduct on $VL$ by

$$\phi\phi'(u) = \phi(u_{(1)})\phi'(u_{(2)}), \qquad \phi, \phi' \in JL, u \in VL,$$

which is (3.1.18) adapted to this situation. The unit is given by the left counit on $VL$, for better distinction denoted by $\epsilon_{VL} : VL \to A$ in this section, since

$$\epsilon_{VL}\phi(u) = \epsilon_{VL}(u_{(1)})\phi(u_{(2)}) = \phi(\epsilon_{VL}(u_{(1)})u_{(2)}) = \phi(u).$$

(*ii*) (*Source and target*) As in (3.1.3) and (3.1.4), define source and target maps $s^\ell_{JL}, t^\ell_{JL} : A \to JL$ by

$$s^\ell_{JL}a(u) := \epsilon_{VL}(au) = a\epsilon_{VL}(u), \qquad t^\ell_{JL}a(u) := \epsilon_{VL}(ua) = u(a).$$

It is easy to see that their images commute, and hence that $(JL, s^\ell_{JL}, t^\ell_{JL})$ becomes an $A^{\mathrm{e}}$-ring. Observe that this gives an example of a commutative base algebra where source and target do not coincide.

(*iii*) (*Coring structure*) To define additionally the structure of an $A$-coring, we need:

**4.3.2 Lemma** *There is a canonical isomorphism*

$$JL_{\triangleleft} \otimes_A {}_{\triangleright}JL \simeq \varprojlim_p \mathrm{Hom}_A\big((VL \otimes^{rl} VL)_{\leq p}, A\big),$$

*where* $\otimes^{rl}$ *is defined as in (4.2.3).*

PROOF: By definition (cf. (2.1.1)),

$$JL_{\triangleleft} \otimes_A {}_{\triangleright}JL = JL \otimes_k JL/\mathrm{span}_k\{t^\ell_{JL}a\phi \otimes_k \phi' - \phi \otimes s^\ell_{JL}a\phi',\ a \in A\}.$$

The first term in the ideal, evaluated on $u \otimes_k u' \in VL \otimes_k VL$, reads

$$\begin{aligned}(t^\ell_{JL}a\phi \otimes_k \phi')(u \otimes_k u') &= t^\ell_{JL}a\phi(u) \otimes \phi'(u') \\ &= \epsilon_{VL}(u_{(1)}a)\phi(u_{(2)}) \otimes \phi'(u') = \phi(ua) \otimes \phi'(u'),\end{aligned}$$

whereas for the second

$$(\phi \otimes_k s^\ell_{JL}a\phi')(u \otimes_k u') = \phi(u) \otimes a\epsilon_{VL}(u'_{(1)})\phi'(u'_{(2)}) = \phi(u) \otimes \phi'(au').$$

Observe that these two expressions use exactly the $A$-bimodule structure on $VL$ used in the $\otimes^{rl}$-tensor product. It therefore follows that the map $\phi \otimes_k \phi' \mapsto \{u \otimes_k u' \mapsto \phi(u\phi'(u'))\}$ induces the desired isomorphism (by projectivity of $L$ this is an isomorphism in each degree). □

Clearly, the product on $VL$ descends to a map $VL \otimes^{rl} VL \to VL$ which allows one to dualise the product to obtain the coproduct $\Delta_\ell^{JL} : JL \to JL_{\triangleleft} \otimes_A {}_{\triangleright}JL$, i.e.,

$$\phi(uu') =: \Delta_\ell^{JL}(\phi)(u \otimes^{rl} u') = \phi_{(1)}\big(u\phi_{(2)}(u')\big), \tag{4.3.1}$$

similarly as in §3.1.6(*i*). Associativity of the multiplication in $VL$ implies that $\Delta_\ell^{JL}$ is coassociative. Finally, the left counit is given as in (3.1.9) by

$$\epsilon_{JL} : JL \to A, \quad \phi \mapsto \phi(1_{VL}),$$

and it is straightforward to see that $(JL, \Delta_\ell^{JL}, \epsilon_{JL})$ is an $A$-coring.

(*iv*) (*Antipodes for Jet Spaces*) It is now easy to verify that $(JL, A, s_{JL}^\ell, t_{JL}^\ell, \epsilon_{JL}, \Delta_\ell^{JL})$ is a left bialgebroid. Since $JL$ is commutative, it is also a right bialgebroid. To obtain a Hopf algebroid, all we need is an antipode.

As observed in [NeTs], there are two left $VL$-module structures on $JL$. First there is the 'obvious' module structure given by

$$(u \rightharpoondown \phi)(u') = \phi(u'u),$$

as in (3.1.8) or §A.1.1(*i*), induced by right multiplication of $VL$ on itself. Second, there is another left $VL$-action on $JL$, constructed as follows. Consider the $A$-module structure ${}_{\triangleright}JL$, i.e., $(a \triangleright \phi)(u) := (s_{JL}^\ell a\phi)(u) = \phi(au)$. On this $A$-module, there is a canonical left connection induced by the anchor (also called the *Grothendieck connection*), given by

$$\nabla_X^\ell(\phi)(u) := \epsilon_{VL}\big(X(\phi u)\big) - \phi(Xu), \qquad X \in L,\ \phi \in JL,\ u \in VL. \tag{4.3.2}$$

One easily checks that this connection is flat, and we can write the induced $VL$-module structure in terms of the canonical left Hopf algebroid structure on $VL$ from (4.2.4) as

$$(u\phi)(u') := u_+\big(\phi(u_-u')\big) = \epsilon_{VL}\big(u_+\phi(u_-u')\big). \tag{4.3.3}$$

With respect to the coproduct, these two module structure satisfy

$$\begin{aligned}\Delta_\ell^{JL}(u \rightharpoondown \phi) &= (u \rightharpoondown \phi_{(1)}) \otimes_A \phi_{(2)},\\ \Delta_\ell^{JL}(u\phi) &= \phi_{(1)} \otimes_A u\phi_{(2)}.\end{aligned} \tag{4.3.4}$$

We now define the antipode on $JL$ to be the following map $S_{JL} : JL \to JL$:

$$(S_{JL}\phi)(u) := \epsilon_{JL}(u\phi) = u_+(\phi(u_-)) = \epsilon_{VL}(u_+\phi(u_-)).$$

PROOF: (of Theorem 4.3.1) Since $L$ acts on $VL$ via (4.3.2) by derivations, $L \to \mathrm{Der}_k\, JL$ is a morphism of Lie algebras. It therefore follows from the PBW theorem that (4.3.3) satisfies

$$u(\phi\phi') = (u_{(1)}\phi)(u_{(2)}\phi').$$

Using this property, one finds that $S$ is a homomorphism of commutative algebras:

$$S_{JL}(\phi\phi')(u) = (u(\phi\phi'))(1) = ((u_{(1)}\phi)(u_{(2)}\phi'))(1) = ((S_{JL}\phi)(S_{JL}\phi'))(u).$$

To prove the theorem, we verify the axioms of Definition 2.6.1: since $s_{JL}^\ell = t_{JL}^r$, $t_{JL}^\ell = s_{JL}^r$, the first one is trivially satisfied, whereas the second is equivalent to the coassociativity of $\Delta_\ell^{JL}$, because $\Delta_\ell^{JL} = \Delta_r^{JL}$. For (2.6.3), we compute with (2.1.4), (2.3.4) and (2.2.4)

$$\begin{aligned}S_{JL}(s_{JL}^\ell a)(u) &= \epsilon_{VL}\big(u_+(s_{JL}^\ell a)(u_-)\big) = \epsilon_{VL}\big(u_+\epsilon_{VL}(au_-)\big)\\ &= \epsilon_{VL}(u_{+(1)}a)\epsilon_{VL}(u_{+(2)}u_-) = \epsilon_{VL}(ua) = (t_{JL}^\ell a)(u).\end{aligned}$$

and with (2.2.10)

$$\begin{aligned}S_{JL}(t_{JL}^\ell a)(u) &= \epsilon_{VL}\big(u_+(t_{JL}^\ell a)(u_-)\big)\\ &= \epsilon_{VL}\big(u_+\epsilon_{VL}(u_-a)\big)\\ &= \epsilon_{VL}(u_+u_-a)\\ &= a\epsilon_{VL}(u) = (s_{JL}^\ell a)(u).\end{aligned}$$

To prove that $S$ is an involution, one computes

$$\begin{aligned}(S_{JL}^2\phi)(u) &= \epsilon_{VL}\big(u_+(S_{JL}\phi)(u_-)\big)\\ &= \epsilon_{VL}\big(u_+\epsilon_{VL}(u_{-+}\phi(u_{--}))\big)\\ &= \epsilon_{VL}\big(u_+u_{-+}\phi(u_{--})\big).\end{aligned} \tag{4.3.5}$$

To find an identity for the term $u_+u_{-+}\otimes^{rl}u_{--}$ that appears in the last line, apply the Hopf-Galois map (2.2.1) to it:

$$\begin{aligned}\beta(u_+u_{-+}\otimes^{rl}u_{--}) &= u_{+(1)}u_{-+(1)}\otimes^{ll}u_{+(2)}u_{-+(2)}u_{--}\\ &= u_{+(1)}u_-\otimes^{ll}u_{+(2)}\\ &= 1\otimes^{ll}u,\end{aligned}$$

where (2.2.4) was used in the second line and the last line follows from the fact that $VL$ is cocommutative together with (2.2.4) again. Hence

$$u_+u_{-+}\otimes^{rl}u_{--} = \beta^{-1}(1\otimes^{ll}u) = 1_+\otimes^{rl}1_-u = 1\otimes^{rl}u,$$

and inserting this identity into (4.3.5) yields $S_{JL}^2 = \mathrm{id}_{JL}$. We are left with proving the axioms (2.6.4). Observe that since the antipode is involutive, $JL$ is commutative, and both left and right bialgebroid structures are given by $(JL, s_{JL}^\ell, t_{JL}^\ell, \Delta_\ell^{JL}, \epsilon_{VL})$, it suffices to verify one of the two identities in (2.6.4). For example,

$$\begin{aligned}\big(\phi_{(1)}S_{JL}\phi_{(2)}\big)(u) &= \phi_{(1)}(u_{(1)})S_{JL}\phi_{(2)}(u_{(2)})\\ &= \phi_{(1)}(u_{(1)})\epsilon_{VL}\big(u_{(2)+}\phi_{(2)}(u_{(2)-})\big)\\ &= \phi_{(1)}(u_{+(1)})\epsilon_{VL}\big(u_{+(2)}\phi_{(2)}(u_-)\big)\\ &= \phi_{(1)}\big(u_+\phi_{(2)}(u_-)\big)\\ &= \phi(u_+u_-) = \phi(1_{VL})\epsilon_{VL}(u) = (s_{JL}^\ell\epsilon_{JL}\phi)(u),\end{aligned}$$

where (2.2.7) and (2.2.10) were used. This proves the second identity and therefore concludes the proof that $JL$ carries the structure of a Hopf algebroid. □

**4.3.3 Remark** Theorem 4.3.1 is remarkable in the sense that whereas the universal enveloping algebra $VL$ of a Lie-Rinehart algebra carries no canonical Hopf algebroid structure, its dual $JL$ does. Close inspection of the preceding proof shows that the Hopf algebroid structure—more precisely the antipode—depends solely on the left Hopf algebroid structure on $VL$, which is canonical, i.e. does not depend on the choice of a flat right connection (cf. Subsection 4.2.13).

**4.3.4 Remark** In the previous construction of the jet space $J_\ell L := JL$ we regarded $VL$ as an $A$-module by left multiplication. *Right* multiplication leads to a space written $J_rL$ without much structure. Only after introducing a flat right $(A, L)$-connection on $A$, we can introduce a ring structure using the right comultiplication $\Delta_r$ on $VL$ and source and target maps using the right counit $\partial$. This does lead to a Hopf algebroid, but one easily proves that the map $\phi \mapsto \phi\circ S$ defines an isomorphism $J_\ell L \to J_rL$ of Hopf algebroids, where $S$ is the antipode on $VL$ constructed from the same flat right connection as in Proposition 4.2.11.

## 4.4 Convolution Algebras

If $G \rightrightarrows G_0$ is an étale groupoid over a compact Hausdorff manifold $G_0$, the space $\mathcal{C}_c^\infty(G)$ of smooth functions on $G = G_1$ with compact support carries a Hopf algebroid structure:

**4.4.1 Proposition** *The groupoid structure of an étale groupoid $G \rightrightarrows G_0$ over a compact manifold $G_0$ determines a Hopf algebroid structure on the convolution algebra $\mathcal{C}_c^\infty(G)$ over $\mathcal{C}^\infty(G_0)$.*

We will dedicate this section to explain the Hopf algebroid structure of $\mathcal{C}_c^\infty(G)$, i.e. prove this proposition.

**4.4.2 Overall Assumption** Although $G = G_1$ often happens to be non-Hausdorff in examples, we do assume this in the rest of this section as well as in Sections 5.7 and 6.6, to simplify the computations a bit. However, we presume that all results in the mentioned sections can be adapted to the non-Hausdorff case by combining the formalism of [CrMoe1] for the functor $\Gamma_c$ for non-Hausdorff spaces with the results in [Mrč2].

The required structure maps are induced by the *target sheaf* $t : G \to G_0$ (which is a sheaf since $G$ is étale) for the underlying left bialgebroid structure plus antipode, and are basically already mentioned in [Mrč1] (for G Hausdorff) or [Mrč2] (general case). We only need to add the corresponding right structure given by the *source sheaf* $s : G \to G_0$ to assemble all data into a Hopf algebroid. Compactness of $G_0$ is needed here to obtain unital algebras instead of merely algebras with local units.

Note that, corresponding to the identities $1_{t(g)}g = g1_{s(g)}$, the source and target sheaves induce two natural $\mathcal{C}^\infty(G_0)$-module structures on $\mathcal{C}^\infty_{\rm c}(G)$, seen as left and right module structures. Since $\mathcal{C}^\infty(G_0)$ with pointwise product is commutative, we can again define four different tensor products denoted $\otimes^{ll}_{\mathcal{C}^\infty(G_0)}$, $\otimes^{rr}_{\mathcal{C}^\infty(G_0)}$, $\otimes^{rl}_{\mathcal{C}^\infty(G_0)}$, $\otimes^{lr}_{\mathcal{C}^\infty(G_0)}$ with obvious meaning. We will frequently need the following isomorphisms

$$\begin{array}{rlcl} \Omega_{s,t}: & \mathcal{C}^\infty_{\rm c}(G)\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G) & \stackrel{\simeq}{\longrightarrow} & \mathcal{C}^\infty_{\rm c}(G^s{\times}^t_{G_0}G)=\mathcal{C}^\infty_{\rm c}(G_2), \\ \Omega_{t,t}: & \mathcal{C}^\infty_{\rm c}(G)\otimes^{ll}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G) & \stackrel{\simeq}{\longrightarrow} & \mathcal{C}^\infty_{\rm c}(G^t{\times}^t_{G_0}G)=\mathcal{C}^\infty_{\rm c}(G^2), \\ \Omega_{s,s}: & \mathcal{C}^\infty_{\rm c}(G)\otimes^{rr}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G) & \stackrel{\simeq}{\longrightarrow} & \mathcal{C}^\infty_{\rm c}(G^s{\times}^s_{G_0}G), \\ \Omega_{t,s}: & \mathcal{C}^\infty_{\rm c}(G)\otimes^{lr}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G) & \stackrel{\simeq}{\longrightarrow} & \mathcal{C}^\infty_{\rm c}(G^t{\times}^s_{G_0}G) \end{array} \tag{4.4.1}$$

all given by the formula

$$\Omega_{-,-}(u\otimes^{--}_{\mathcal{C}^\infty(G_0)}u')(g,g') = u(g)u'(g'), \tag{4.4.2}$$

for $u, u' \in \mathcal{C}^\infty_{\rm c}(G)$ and $(g,g')$ in the respective pull-back $G^-{\times}^-_{G_0}G$. The fact that these maps are isomorphisms was shown in [Mrč1] (for $G$ Hausdorff), and for non-Hausdorff spaces can be derived from a more general result on sheaves in [Mrč2, p. 271]. Moreover, one can combine the various isomorphisms to produce 'mixed' ones, e.g.,

$$\mathcal{C}^\infty_{\rm c}(G)\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G)\otimes^{lr}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G)\simeq\mathcal{C}^\infty_{\rm c}(G^s{\times}^t_{G_0}G^t{\times}^s_{G_0}G). \tag{4.4.3}$$

We now give a list of the Hopf algebroid structure maps of $\mathcal{C}^\infty_{\rm c}(G)$ over $\mathcal{C}^\infty(G_0)$:

(*i*) (*Ring structure*) On the base algebra $\mathcal{C}^\infty(G_0)$ one uses the commutative pointwise product, whereas the total algebra $\mathcal{C}^\infty_{\rm c}(G)$ will be equipped with a *convolution product*, defined as the composition

$$* : \mathcal{C}^\infty_{\rm c}(G)\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G) \stackrel{\Omega^2_{s,t}}{\to} \mathcal{C}^\infty_{\rm c}(G_2) \stackrel{m_+}{\to} \mathcal{C}^\infty_{\rm c}(G).$$

Explicitly,

$$(u*v)(g) := *(u\otimes v) = (m_+\Omega_{s,t}(u\otimes v))(g) = \sum_{g=g_1g_2} u(g_1)v(g_2),$$

which can be used to show associativity of the product $*$.

(*ii*) (*Source and target maps*) In particular, taking $f \in \mathcal{C}^\infty(G_0)$ and $u \in \mathcal{C}^\infty_{\rm c}(G)$ one has

$$(f*u)(g) = f(t(g))u(g) \qquad \text{and} \qquad (u*f)(g) = u(g)f(s(g)), \tag{4.4.4}$$

hence the left and right $\mathcal{C}^\infty(G_0)$-action by the (groupoid) source and target sheaf. One can now show that $\mathcal{C}^\infty(G_0)$, identified with those functions in $\mathcal{C}^\infty_{\rm c}(G)$ having support on $1_{G_0} \subset G$, is a commutative subalgebra of $\mathcal{C}^\infty_{\rm c}(G)$. Correspondingly, we put for the (left and right bialgebroid) source and target maps

$$s^\ell \equiv t^\ell \equiv s^r \equiv t^r \equiv 1_+ : \mathcal{C}^\infty(G_0) \to \mathcal{C}^\infty_{\rm c}(G),$$

i.e. the injection as subalgebra given by the fibre sum of the unit map $1 : G_0 \to G$ (which we usually refrain from mentioning at all). Explicitly,

$$s^\ell : f \mapsto \tilde f, \quad \text{where} \quad \tilde f(g) = \begin{cases} f(x) & \text{if } g = 1_x \text{ for some } x \in G_0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.4.5}$$

and the tensor products $\otimes^{ll}_{\mathcal{C}^\infty(G_0)}$, $\otimes^{rr}_{\mathcal{C}^\infty(G_0)}$, $\otimes^{rl}_{\mathcal{C}^\infty(G_0)}$, $\otimes^{lr}_{\mathcal{C}^\infty(G_0)}$ introduced before can now be interpreted with respect to this injection.

(*iii*) (*Left and right coproduct*) With the help of the isomorphisms $\Omega_{\cdot,\cdot}$, the left and right coproducts read as follows:

$$\begin{array}{ll} \Delta'_\ell := \Omega^2_{t,t}\,\Delta_\ell : \mathcal{C}^\infty_{\rm c}(G)\to\mathcal{C}^\infty_{\rm c}(G^t{\times}^t_{G_0}G), & (\Delta'_\ell u)(g,g') = \begin{cases} u(g) & \text{if } g=g', \\ 0 & \text{else,} \end{cases} \\ \Delta'_r := \Omega^2_{s,s}\,\Delta_r : \mathcal{C}^\infty_{\rm c}(G)\to\mathcal{C}^\infty_{\rm c}(G^s{\times}^s_{G_0}G), & (\Delta'_r u)(\tilde g,\tilde g') = \begin{cases} u(\tilde g) & \text{if } \tilde g=\tilde g', \\ 0 & \text{else.} \end{cases} \end{array} \tag{4.4.6}$$

If one introduces the diagonal maps $d^\ell : G \to G^t\times^t_{G_0} G,\ g \mapsto (g,g)$ as well as $d^r : G \to G^s\times^s_{G_0} G,\ g \mapsto (g,g)$, this can be obviously rewritten as $\Delta'_\ell = d^\ell_+$ and $\Delta'_r = d^r_+$ or even as $\Delta_\ell = \Omega^{-1}_{t,t}\, d^\ell_+$ and $\Delta_r = \Omega^{-1}_{s,s}\, d^r_+$. For later computations, let us also mention that higher coproducts are given by

$$(\Delta'^n_\ell u)(g_1,\ldots,g_n) = \begin{cases} u(g_1) & \text{if } g_1 = \ldots = g_n, \\ 0 & \text{else,} \end{cases}$$

for $(g_1,\ldots,g_n) \in G^n$, where $G^n := G^t\times^t_{G_0} \cdots {}^t\times^t_{G_0} G$, and likewise for $\Delta'^n_r$.

(*iv*) (*Left and right counit*) Both left and right counit are determined by the fibre sum of the germ bundle projection of the target and source sheaf, respectively. For any $x \in G_0$, set

$$\begin{aligned} \epsilon : \mathcal{C}^\infty_{\rm c}(G) \to \mathcal{C}^\infty(G_0), \quad \epsilon u(x) &= \sum_{t(g)=x} u(g) = t_+ u(x), \\ \partial : \mathcal{C}^\infty_{\rm c}(G) \to \mathcal{C}^\infty(G_0), \quad \partial u(x) &= \sum_{s(g)=x} u(g) = s_+ u(x). \end{aligned} \tag{4.4.7}$$

(*v*) (*Antipode*) Finally, the antipode is given by the groupoid inversion,

$$S : \mathcal{C}^\infty_{\rm c}(G) \to \mathcal{C}^\infty_{\rm c}(G), \quad (Su)(g) = u(g^{-1}) = (\mathrm{inv}_+ u)(g). \tag{4.4.8}$$

The proof of Proposition 4.4.1 is now a straightforward verification:
PROOF: (of Proposition 4.4.1) We remark once again that compactness of $G_0$ makes both algebras $(\mathcal{C}^\infty_{\rm c}(G), *)$ and $(\mathcal{C}^\infty(G_0), \cdot)$ unital. The fact that $(\mathcal{C}^\infty_{\rm c}(G), \mathcal{C}^\infty(G_0), \Delta_\ell, \epsilon)$ is a left bialgebroid having an antipode $S$ with certain properties was already shown in [Mrč2, Prop. 2.5]. This can be carried over, *mutatis mutandis*, by simply replacing the target sheaf by the source sheaf to prove that $(\mathcal{C}^\infty_{\rm c}(G), \mathcal{C}^\infty(G_0), \Delta_r, \partial)$ gives a right bialgebroid. As an example, for $u, v \in \mathcal{C}^\infty_{\rm c}(G)$ we have $(\partial u * v)(g) = \partial u(t(g)) v(g) = \sum_{s(g_1)=t(g)} u(g_1)v(g)$, hence for some $x \in G_0$,

$$\partial(u*v)(x) = \sum_{s(g)=x,\ g=g_1g_2} u(g_1)v(g_2) = \sum_{s(g_2)=x,\ s(g_1)=t(g_2)} u(g_1)v(g_2) = \partial(\partial u * v)(x),$$

and, using implicitly $\Delta_r = \Omega^{-1}_{s,s}\, d^r_+$,

$$\begin{aligned} (*(\mathrm{id}\otimes\partial)\Delta_r u)(g) &= (u^{(1)} * \partial u^{(2)})(g) \\ &= u^{(1)}(g)\partial u^{(2)}(s(g)) = \sum_{s(g')=s(g)} u^{(1)}(g)u^{(2)}(g') = u(g) \end{aligned}$$

and so forth. Hence $\partial$ is indeed a right counit for $\Delta_r$. In what follows, we can now restrict ourselves to verify the Hopf algebroid identities in which left and right bialgebroid structures are intertwined: for example, twisted coassociativity (2.6.2) is obvious, so we only prove the second identity in (2.6.4):

$$\begin{aligned} (*(\mathrm{id}\otimes S)\Delta_r u)(g) &= \sum_{g=g_1g_2} (\Omega^{-1}_{s,s}\, d^r_+ u)(g_1, g_2^{-1}) \\ &= \sum_{\{g_1\in G \mid t(g_1)=t(g),\, g_1=g^{-1}g_1\}} u(g_1) \\ &= \begin{cases} \sum_{t(g_1)=x} u(g_1) & \text{if } g = 1_x \text{ for some } x \in G_0, \\ 0 & \text{else} \end{cases} \\ &= (s^\ell \epsilon u)(g). \end{aligned}$$

We leave the verification of the remaining identities in Definition 2.6.1 to the reader, but as an illustration, we also state the third relation in (2.6.10), i.e.,

$$(\partial S u)(x) = \sum_{s(g)=x} Su(g) = \sum_{s(g)=x} u(g^{-1}) = \sum_{t(g)=x} u(g) = (\epsilon u)(x),$$

or $\partial S = s_+ S = t_+ = \epsilon$. □

A different way to obtain a (topological) Hopf algebroid from an étale groupoid is described in [KaTan].

## 4.5 Function Algebras

The convolution algebra in Section 4.4 is not the only Hopf algebroid which arises from an étale groupoid. In this section, we use pullbacks of the structure maps rather than push forwards. Let $s, t : G \rightrightarrows P$ be an étale groupoid and consider the algebra $\mathcal{C}^\infty(G)$ of smooth functions with its commutative pointwise multiplication. We are going to consider a subspace of $\mathcal{C}^\infty(G)$, invariant under an action of (lifted) differential operators on the base manifold $P$ (see Definition 4.5.1 for the precise construction), and prove that it is a Hopf algebroid (see Proposition 4.5.6).

The Lie-Rinehart algebra $(\mathcal{C}^\infty(P), \Gamma TP)$ of smooth sections of the tangent bundle $TP$ over $\mathcal{C}^\infty(P)$ acts in two ways from the left on $\mathcal{C}^\infty(G)$: firstly, the assignment

$$\begin{aligned} \mathcal{C}^\infty(P) \otimes \mathcal{C}^\infty(G) &\to \mathcal{C}^\infty(G), \qquad & (a, f) &\mapsto a^t(f) := t^*(a)f \\ \Gamma TP \otimes \mathcal{C}^\infty(G) &\to \mathcal{C}^\infty(G), \qquad & (X, f) &\mapsto X^t(f) := L_{t^*X} f \end{aligned} \tag{4.5.1}$$

can be extended by the universal property to a left action

$$VP \otimes \mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G), \quad (u, f) \mapsto u^t(f) \tag{4.5.2}$$

of the universal enveloping algebra $VP := V\Gamma TP$ on $\mathcal{C}^\infty(G)$. Secondly, the same considerations with respect to the source map $s$ lead analogously to the left action

$$VP \otimes \mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G), \quad (u, f) \mapsto u^s(f). \tag{4.5.3}$$

Furthermore, the assignment

$$(a, f, b) \mapsto t^*(a) f s^*(b), \qquad f \in \mathcal{C}^\infty(G),\ a, b \in \mathcal{C}^\infty(P), \tag{4.5.4}$$

equips $\mathcal{C}^\infty(G)$ with a $\mathcal{C}^\infty(P)$-bimodule or left $\mathcal{C}^\infty(P)^{\mathrm{e}}$-module structure and we denote the canonical injection by

$$\eta : \mathcal{C}^\infty(P)^{\mathrm{e}} \to \mathcal{C}^\infty(G), \quad a \otimes_{\mathbb{C}} b \mapsto t^*(a)s^*(b).$$

We can now make the following definition.

**4.5.1 Definition** The space $F = F_\infty$ is the smallest left $VP$-submodule with respect to the action (4.5.2) that contains $F_0 := \mathcal{C}^\infty(P)^{\mathrm{e}}$ and is closed under groupoid inversion.

As the notation suggests, $F$ carries a filtration, which can be seen to originate from the canonical filtration of $VP$. To see what it looks like, we need to calculate the $VP$-action on elements of the form $t^*(a)s^*(b)$. One obviously has

$$a^t(s^*b) = t^*(a)s^*(b) \in F_0, \qquad X^t(t^*a) = t^*(L_X a) \in F_0, \qquad a, b \in \mathcal{C}^\infty(P), X \in \mathcal{X}P.$$

To calculate elements of type $L_{t^*X} s^* a$ one most conveniently makes use of the following lemma (cf. Lemma A.1.4).

**4.5.2 Lemma** (Dual Basis Lemma for the tangent bundle) *Let $P$ be a smooth manifold. Then there exist vector fields $X_1, \ldots, X_n \in \mathcal{X}P$ and one-forms $\theta^1, \ldots, \theta^n \in \Omega^1 P$ such that each vector field $X \in \mathcal{X}P$ can be decomposed as $X = \sum_{i=1}^n \theta^i(X) X_i$.*

As we will explain now, there are functions $\eta_X^i \in \mathcal{C}^\infty(G)$ such that

$$t^* X = \sum_{i=1}^n \eta_X^i s^* X_i.$$

Equivalently, at a point $g \in G$, this means

$$(dt)_g^{-1}(X_{t(g)}) = \sum_{i=1}^n \eta_X^i(g)(ds)_g^{-1}(X_{i,s(g)}).$$

That is to say,

$$g^{-1} \cdot X_{t(g)} = \sum_{i=1}^n \eta_X^i(g) X_{i,s(g)},$$

where $g^{-1}\cdot : T_{t(g)}P \to T_{s(g)}P$ denotes the right $G$-action $ds \circ dt^{-1}$ on $TP$. On the other hand, from Lemma 4.5.2 one also obtains

$$g^{-1} \cdot X_{t(g)} = \sum_{i=1}^{n} \theta^i(g^{-1} \cdot X_{t(g)}) X_{i,s(g)}, \tag{4.5.5}$$

hence we choose

$$\eta_X^i(g) := \theta^i(g^{-1} \cdot X_{t(g)}).$$

An analogous consideration holds for the functions $\tilde{\eta}_X^j$ arising from the decomposition $s^*X = \sum_{i=1}^n \tilde{\eta}_X^i t^* X_i$, where one clearly obtains

$$\tilde{\eta}_X^i(g) := \theta^i(g \cdot X_{s(g)}),$$

that is

$$\tilde{\eta}_X^j = \eta_X^j \circ \mathrm{inv}, \tag{4.5.6}$$

where $\mathrm{inv}$ is the groupoid inversion. Observe that we chose here the notation such that $g^{-1}\cdot$ is a right and $g\cdot$ is a left action on $TP$. Applying Lemma 4.5.2 again yields the property

$$\eta_X^i = \sum_{j=1}^n t^*(\theta^j(X))\eta_j^i \quad \text{and} \quad \tilde{\eta}_X^i = \sum_{j=1}^n s^*(\theta^j(X))\tilde{\eta}_j^i, \tag{4.5.7}$$

with $\eta_i^j(g) := \theta^j(g^{-1} \cdot X_{i,t(g)})$ and $\tilde{\eta}_i^j(g) := \theta^j(g \cdot X_{i,s(g)})$. Since $g\cdot$ is a left action, one furthermore has the following useful properties:

$$\begin{aligned} \tilde{\eta}_X^i(gh) &= \theta^i((gh)\cdot X_{s(h)}) = \theta^i(g \cdot (h \cdot X_{s(h)})) \\ &= \sum_{j=1}^n \theta^i\big(g \cdot \big(\theta^j(h \cdot X_{s(h)}) X_{j,t(h)}\big)\big) \\ &= \sum_{j=1}^n \tilde{\eta}_X^j(h) \theta^i(g \cdot X_{j,s(g)}) \\ &= \sum_{j=1}^n \tilde{\eta}_X^j(h)\tilde{\eta}_j^i(g) = \sum_{j=1}^n \tilde{\eta}_j^i(g)\tilde{\eta}_X^j(h), \end{aligned} \tag{4.5.8}$$

and analogously

$$\eta_X^i(gh) = \sum_{j=1}^n \eta_X^j(g)\eta_j^i(h). \tag{4.5.9}$$

Correspondingly, we will decompose now $t^*X$ and $s^*X$ as follows:

$$\begin{aligned} t^*X &= \textstyle\sum_{i=1}^n t^*(\theta^i(X)) t^*X_i = \sum_{i,j=1}^n t^*(\theta^i(X))\eta_i^j s^*X_j = \sum_{j=1}^n \eta_X^j s^*X_j, \\ s^*X &= \textstyle\sum_{i=1}^n s^*(\theta^i(X)) s^*X_i = \sum_{i,j=1}^n s^*(\theta^i(X))\tilde{\eta}_i^j t^*X_j = \sum_{j=1}^n \tilde{\eta}_X^j t^*X_j. \end{aligned} \tag{4.5.10}$$

As a consequence, one has

$$L_{t^*X} s^*a = \sum_{i,j=1}^n t^*(\theta^i(X))\eta_i^j s^*(L_{X_j}a) = \sum_{j=1}^n \eta_X^j s^*(L_{X_j}a), \tag{4.5.11}$$

and this is the type of element lying in $F_1$. Likewise,

$$L_{s^*X} t^*a = \sum_{i,j=1}^n s^*(\theta^i(X))\tilde{\eta}_i^j t^*(L_{X_j}a) = \sum_{j=1}^n \tilde{\eta}_X^j t^*(L_{X_j}a). \tag{4.5.12}$$

**4.5.3 Lemma** *For each vector field $X \in \mathcal{X}P$, the functions $\eta_X^j$, $\tilde{\eta}_X^j$ and hence in particular elements of the form $L_{s^*X}(s^*at^*b)$ lie in the submodule $F \subset \mathcal{C}^\infty(G)$.*

PROOF: Let $\sigma = \sigma_g : (U, s(g)) \to (V, t(g))$ be the germ associated to $g$ as in §1.5.2. Assign to any $X \in \mathcal{X}P$ and $\theta \in \Omega^1 P$ functions $(X, \theta) \mapsto \eta_X^\theta \in \mathcal{C}^\infty(G)$, $(X, \theta) \mapsto \tilde{\eta}_X^\theta \in \mathcal{C}^\infty(G)$ by defining

$$\eta_X^\theta(g) := \theta_{s(g)}\big((d\sigma)^{-1}_{t(g)}(X_{t(g)})\big), \qquad \tilde{\eta}_X^\theta(g) := \theta_{t(g)}\big((d\sigma)_{s(g)}(X_{s(g)})\big).$$

Now decompose an arbitrary $\theta \in \Omega^1 P$ into $\theta = \sum_i a_i db_i$ where $a_i, b_i \in \mathcal{C}^\infty(P)$. We then have

$$\begin{aligned}\sum_i s^* a_i L_{t^*X}(s^* b_i)(g) &= \sum_i a_i(s(g))(db_i)_{s(g)}(d\sigma_g^{-1})_{t(g)}(X_{t(g)}) \\ &= \theta_{s(g)}\big((d\sigma_g^{-1})_{t(g)}(X_{t(g)})\big) = \eta_X^\theta(g),\end{aligned}$$

and the left hand side is by definition in $F$; hence the claim follows. In particular, this is true for $\eta_X^j \in F$; the same holds then for $\tilde{\eta}_X^j$ by (4.5.6) and the definition of $F$. □

Somewhat simplified, for functions $f_i, \tilde{f}_i \in F$ with $\tilde{f}_i = f_i \circ \mathrm{inv}$ one finds the identities

$$t^* X = \sum_{i=1}^n f_i s^* X_i \quad \text{and} \quad s^* X = \sum_{i=1}^n \tilde{f}_i t^* X_i. \tag{4.5.13}$$

Now it is clear how the higher degrees $F_k$ of the filtration of $F$ arise: taking $L_{t^*Y} L_{t^*X} s^* a$ for $X, Y \in \mathcal{X}P$ and $a \in \mathcal{C}^\infty(P)$, one obtains with (4.5.11) two summands that stay in $F_1$ and a term containing $L_{t^*Y}\eta_i^j$, which characterises the terms in $F_2$. It is also clear now that, seen this way, the filtration of $VP$ determines the one of $F$.

Next, we want to give $F$ the structure of a Hopf algebroid. To this end, set up the following.

**4.5.4 Definition** A function $f \in F$ is called *$F$-codecomposable* if

$$f(gh) = \sum_i f_i'(g) f_i''(h), \qquad g, h \in G, \tag{4.5.14}$$

for functions $f_i', f_i'' \in F$, where the sum is finite.

**4.5.5 Lemma** *Each element in $F$ is $F$-codecomposable.*

PROOF: If $f, f'$ are $F$-codecomposable, then so is their product $ff'$. Also, for $a, b \in \mathcal{C}^\infty(P)$, the function $s^* a t^* b$ is $F$-codecomposable. Hence it suffices to prove that for any $X \in \mathcal{X}P$ and a $F$-codecomposable function $f$, the expression $L_{t^*X} f$ is $F$-codecomposable. To this end, assume that $f \in F$ is $F$-codecomposable as in (4.5.14). Since $G$ is étale, its tangent bundle is a multiplicative distribution, i.e., for any $k, g, h \in G$ with $k = gh$ and $W_k \in T_k G$ there are paths $k(t), g(t), h(t)$ in $G$ with $k(t) = g(t)h(t)$, $\dot{k}(t) \in T_{k(t)}G$, $\dot{g}(t) \in T_{g(t)}G$ and $\dot{h}(t) \in T_{h(t)}G$ such that $k(0) = k$, $g(0) = g$ and $h(0) = h$ as well as $\dot{k}(0) = W_k$. Identifying spaces $T_k G$ with $T_{t(k)}P$ by means of $t_*$, we may in particular choose $W$ to correspond to the given vector field $X$, i.e., $\dot{k}(0) = W_k = (dt)_k^{-1}(X_{t(k)})$. Therefore,

$$(dt)_g(\dot{g}(0)) = (dt)_k(\dot{k}(0)) = X_{t(k)} = X_{t(g)},$$

and with (4.5.5)

$$\begin{aligned}(ds)_h(\dot{h}(0)) &= (ds)_k(\dot{k}(0)) = (ds)_k(dt)_k^{-1}(X_{t(k)}) \\ &= k^{-1} \cdot X_{t(k)} = \sum_{j=1}^n \eta_X^j(k) X_{j,s(k)} = \sum_{j=1}^n \eta_X^j(k) X_{j,s(h)}.\end{aligned}$$

With (4.5.9) one now calculates

$$\begin{aligned}
L_{t^*X} f(gh) &= \frac{d}{dt}\Big|_{t=0} \sum_i f_i'(g(t)) f_i''(h(t)) \\
&= \sum_i (L_{\dot g(0)} f_i')(g) f_i''(h) + \sum_i f_i'(g)(L_{\dot h(0)} f_i'')(h) \\
&= \sum_i (L_{t^*X} f_i')(g) f_i''(h) + \sum_i \sum_{j=1}^n f_i'(g) \eta_X^j(k)(L_{s^*X_j} f_i'')(h) \\
&= \sum_i (L_{t^*X} f_i')(g) f_i''(h) + \sum_i \sum_{j,l=1}^n f_i'(g) \eta_X^l(g) \eta_l^j(h)(L_{s^*X_j} f_i'')(h) \\
&= \sum_i (L_{t^*X} f_i')(g) f_i''(h) + \sum_i \sum_{l=1}^n \eta_X^l(g) f_i'(g)(L_{t^*X_l} f_i'')(h).
\end{aligned} \tag{4.5.15}$$

Now elements of type $L_{t^*X} f_i'$ and $\eta_X^i$ were already shown above to be functions in $F$, hence the claim follows. □

For later use we mention that one analogously obtains

$$L_{s^*X} f(gh) = \sum_i \sum_{l=1}^n (L_{s^*X_l} f_i')(g) \tilde\eta_X^l(h) f_i''(h) + \sum_i f_i'(g)(L_{s^*X} f_i'')(h). \tag{4.5.16}$$

With the left $\mathcal{C}^\infty(P)^{\mathrm{e}}$-module structure on $F$ from (4.5.4), define

$$F \otimes_{\mathcal{C}^\infty(P)} F := F \otimes_{\mathbb{C}} F / \mathrm{span}_{\mathbb{C}}\{s^*(a) f \otimes_{\mathbb{C}} f' - f \otimes_{\mathbb{C}} t^*(a) f',\ a \in \mathcal{C}^\infty(P)\},$$

and set $G_2 := G \times_P G$ and $G_3 := G \times_P G \times_P G$.

**4.5.6 Proposition** *Assume that both maps*

$$\begin{aligned}
\jmath : F^{\otimes_{\mathcal{C}^\infty(P)} 2} &\to \mathcal{C}^\infty(G_2), & f \otimes_{\mathcal{C}^\infty(P)} f' &\mapsto \{(g,h) \mapsto f(g) f'(h)\}, \\
\jmath : F^{\otimes_{\mathcal{C}^\infty(P)} 3} &\to \mathcal{C}^\infty(G_3), & f \otimes_{\mathcal{C}^\infty(P)} f' \otimes_{\mathcal{C}^\infty(P)} f'' &\mapsto \{(g,h,k) \mapsto f(g) f'(h) f''(k)\},
\end{aligned}$$

*are injective and let $m_G(g,h) = gh$ for $(g,h) \in G \times_P G$ be the groupoid multiplication. For a function $f \in F$ with $F$-codecomposition $f(gh) = \sum_i f_i'(g) f_i''(h)$, $f_i', f_i'' \in F$, the formula $\Delta_\ell^F f := m_G^* f$ gives a well-defined map*

$$\Delta_\ell^F : F \to F \otimes_{\mathcal{C}^\infty(P)} F, \quad f \mapsto \sum_i f_i' \otimes_{\mathcal{C}^\infty(P)} f_i''. \tag{4.5.17}$$

*Together with the map*

$$\epsilon_F : F \to \mathcal{C}^\infty(P), \quad f \mapsto 1_G^* f, \tag{4.5.18}$$

*where $1_G : P \to G$ is the groupoid embedding, the triple $(F, \Delta_\ell^F, \epsilon_F)$ becomes a $\mathcal{C}^\infty(P)$-coring with respect to the bimodule structure (4.5.4). In particular, with its pointwise product $m_F$ and defining source and target maps as maps $\mathcal{C}^\infty(P) \to F$ by*

$$s_F^\ell : a \mapsto t^* a \quad \text{and} \quad t_F^\ell : a \mapsto s^* a, \tag{4.5.19}$$

*the $\mathcal{C}^\infty(P)$-coring $F$ is a left bialgebroid over $\mathcal{C}^\infty(P)$. On the other hand, defining two maps $\mathcal{C}^\infty(P) \to F$ by*

$$s_F^r : a \mapsto s^* a \quad \text{and} \quad t_F^r : a \mapsto t^* a$$

*equips the (same) quintuple $(F, \mathcal{C}^\infty(P), m_F, \Delta_\ell^F, \epsilon_F)$ with the structure of a* right *bialgebroid. Finally, the map*

$$S_F : F \to F, \quad f \mapsto \mathrm{inv}^* f \tag{4.5.20}$$

*defines an antipode on $F$ and all data can be assembled into a Hopf algebroid.*

PROOF: Note firstly that $m_G^*(F) \subseteq \jmath(F \otimes_{\mathcal{C}^\infty(P)} F)$ and with the injectivity of $\jmath$ the well-definedness of $\Delta_\ell^F$ follows directly. Coassociativity then follows from the injectivity of the second map. The bialgebroid axioms are not difficult to verify. We have, for example,

$$(m_F(\mathrm{id} \otimes t_F^\ell \epsilon_F) \Delta_\ell^F f)(g) = \sum_i f_i'(g) f_i''(1_{s(g)}) = f(g 1_{s(g)}) = f(g),$$

and similar for all remaining identities; in particular it is easy to see that $\epsilon_F$ and $\Delta_\ell^F$ can be simultaneously considered to be left and right counit and coproduct of a left and right bialgebroid, respectively. The fact that these are the constituent bialgebroid structures of a Hopf algebroid with antipode $S$ is seen by computing

$$(\epsilon_F Sf)(x) = f\big((1_x)^{-1}\big) = f(1_x) = (\epsilon_F f)(x).$$

Hence right and left counits coincide. Also, using $S^{-1} = S$ and $Sf(gh) = \sum_i f_i'(h^{-1})f_i''(g^{-1})$, we have

$$(S_{\otimes 2}\Delta_\ell^F Sf)(g,h) = (\Delta_\ell^F Sf)(h^{-1}, g^{-1}) = \sum_i f_i'(g)f_i''(h).$$

Hence $S_{\otimes 2}\Delta_\ell^F S = \Delta_\ell^F$, and the right and left coproducts coincide as well. Finally, we check

$$\begin{aligned}(m_F(S\otimes \mathrm{id}_F)\Delta_\ell^F f)(g) &= \sum_i Sf_i'(g)f_i''(g) = \sum_i f_i'(g^{-1})f_i''(g)\\ &= f(g^{-1}g) = f(1_{s(g)}) = s^*1^*f(g)\\ &= (s_F^r\epsilon_F f)(g),\end{aligned}$$

and leave all remaining identities to the reader. □

For later use, we give some explicit coproduct expressions. One clearly has $m_G^*s^*a(g,h) = s^*a(gh) = s^*a(h)$ and $m_G^*t^*b(g,h) = t^*b(g)$ for $a, b \in \mathcal{C}^\infty(P)$, $g, h \in G$. Hence

$$\Delta_\ell^F s^*a = 1 \otimes_{\mathcal{C}^\infty(P)} s^*a \quad \text{and} \quad \Delta_\ell^F t^*b = t^*b \otimes_{\mathcal{C}^\infty(P)} 1, \quad a, b \in \mathcal{C}^\infty(P). \tag{4.5.21}$$

More interesting is the case of the elements $\tilde\eta_X^i$ and $\eta_X^i$. With (4.5.8) and (4.5.9) one obtains

$$\Delta_\ell^F \tilde\eta_X^i = \sum_{j=1}^n \tilde\eta_j^i \otimes_{\mathcal{C}^\infty(P)} \tilde\eta_X^j, \quad \text{and} \quad \Delta_\ell^F \eta_X^i = \sum_{j=1}^n \eta_X^j \otimes_{\mathcal{C}^\infty(P)} \eta_j^i. \tag{4.5.22}$$

## 4.6 Connes-Moscovici Algebras

'Extending' $F$ from the previous section by the (lifted) differential operators on $P$, one gets a subspace of the differential operators on $G$ which can be given the structure of a left bialgebroid again (see Proposition 4.6.4), and presumably even of a Hopf algebroid (see Remark 4.6.5). We see this as a general background picture from which the constructions in [CoMos5] and [MosR] can be understood.

Consider the convolution algebra $(\mathcal{C}_c^\infty(G), *)$ with multiplication $m_{\mathcal{C}_c^\infty(G)}(f, f') = f * f'$ of compactly supported functions on $G$ and denote the space of linear maps $\mathcal{C}_c^\infty(G) \to \mathcal{C}_c^\infty(G)$ by $C^1(\mathcal{C}_c^\infty(G))$. For example, one can consider operators $m_f$ of pointwise multiplication with an element $f \in F$, or, by locality, the restriction to compactly supported functions of the action $L_{t^*X}$ from (4.5.1) and (4.5.2).

**4.6.1 Definition** The space of *transverse differential operators* on an étale groupoid $s, t : G \rightrightarrows P$ is the subalgebra $H \subset VG := V\Gamma TG$ generated by elements in $F$ and $t^*X$ for $a, b \in \mathcal{C}^\infty(P)$, $X \in \mathcal{X}P$.

By construction, $H \cap \mathcal{C}^\infty(G) = F$. In particular, elements of the form $s^*X$ for $X \in \mathcal{X}P$ are contained in $H$ as well. Interpreting $VG$ as the space of differential operators on $\mathcal{C}^\infty(G)$ (with elements in $\mathcal{C}^\infty(G)$ as multiplication operators), elements in $VG$ act on the convolution algebra $(\mathcal{C}_c^\infty(G), *)$. One has the inclusions $H \subset VG \subset C^1(\mathcal{C}_c^\infty(G))$.

**4.6.2 Definition** An operator $D \in H$ is called *$H$-codecomposable* if one has a finite sum

$$D(f * f') = \sum_i D_i'(f) * D_i''(f'), \qquad f, f' \in \mathcal{C}_c^\infty(G),$$

for elements $D_i', D_i'' \in H$.

**4.6.3 Lemma** *Each element in $H$ is $H$-codecomposable.*

PROOF: For $a, b \in \mathcal{C}^\infty(P)$, the operators $m_{s^*a}$, $m_{t^*b}$ are evidently $H$-codecomposable. If $D, E$ are $H$-codecomposable, then so is their product $DE$. This follows from

$$ED(f * f') = E\big(\sum_i D_i'(f) * D_i''(f')\big) = \sum_{i,j} E_j' D_i'(f) * E_j'' D_i''(f').$$

Finally, for $X, Y \in \mathcal{X}P$, the operators $L_{t^*X}$, $L_{s^*Y}$ are $H$-codecomposable. This can be seen by repeating the argumentation in the proof of Lemma 4.5.5:

$$\begin{aligned}\big(L_{t^*X}(f * f')\big)(g) &= \sum_{g=hk} \big(L_{t^*X} f(h) f'(k) + \sum_{l=1}^n f(h)\eta_X^l(h) L_{t^*X_l} f'(k)\big) \\ &= \big(L_{t^*X} f * f'\big)(g) + \sum_{l=1}^n \big(f\eta_X^l * L_{t^*X_l} f'\big)(g),\end{aligned} \tag{4.6.1}$$

and similar for $L_{s^*Y}$ using (4.5.16). Hence $L_{t^*X}$ and $L_{s^*Y}$ are $H$-codecomposable. These statements are sufficient to prove the lemma. □

Observe that for an element $f \in F$ with $F$-codecomposition $f(gh) = \sum_i f_i'(g) f_i''(h)$, the $H$-codecomposition of the corresponding multiplication operator and $f_1, f_2 \in \mathcal{C}_c^\infty(G)$ reads

$$\begin{aligned}m_f(f_1 * f_2)(g) &= \sum_{g=hk} f(hk) f_1(h) f_2(k) \\ &= \sum_i \sum_{g=hk} m_{f_i'}(h) f_1(h) m_{f_i''}(k) f_2(k) = \sum_i m_{f_i'} f_1 * m_{f_i''} f_2.\end{aligned}$$

Hence $H$-codecomposition, restricted to elements in $F \subset H$, coincides with $F$-codecomposition.

Now note that the space $H$ carries an obvious left $\mathcal{C}^\infty(P)^{\mathrm{e}}$-module structure arising from (4.5.4), namely

$$(a, D, b) \mapsto m_{t^*(a)} m_{s^*(b)} D, \qquad D \in H,\ a, b \in \mathcal{C}^\infty(P), \tag{4.6.2}$$

with respect to which we define

$$H \otimes_{\mathcal{C}^\infty(P)} H = H \otimes_k H / \mathrm{span}\{m_{s^*(b)} D \otimes E - D \otimes m_{t^*(a)} E,\ a \in \mathcal{C}^\infty(P)\}.$$

Furthermore, we regard the space $\mathcal{C}_c^\infty(P)$ as subalgebra of $\mathcal{C}_c^\infty(G)$ by means of

$$\mathcal{C}_c^\infty(P) \hookrightarrow \mathcal{C}_c^\infty(G), \quad f \mapsto \tilde{f}, \quad \text{where} \quad \tilde{f}(g) = \begin{cases} f(x) & \text{if } g = 1_x \text{ for some } x \in G_0, \\ 0 & \text{otherwise.}\end{cases}$$

Observe that the $H$-action on $\mathcal{C}_c^\infty(G)$ leaves this subalgebra $\mathcal{C}_c^\infty(P)$ invariant. The restriction of $H$ to $\mathrm{End}\,\mathcal{C}_c^\infty(P)$ coincides with $VP$, and therefore admits a tautological extension to an action on $\mathcal{C}_c^\infty(P)$. With these preliminary remarks in mind (which will allow us to define a counit for $H$), we can prove

**4.6.4 Proposition** *Assume that the maps*

$$\begin{aligned} J : H \otimes_{\mathcal{C}^\infty(P)} H &\to C^2(\mathcal{C}_c^\infty(G)), \\ D \otimes_{\mathcal{C}^\infty(P)} D' &\mapsto \{f \otimes_{\mathbb{C}} f' \mapsto D(f) * D'(f')\}, \\ J : H \otimes_{\mathcal{C}^\infty(P)} H \otimes_{\mathcal{C}^\infty(P)} H &\to C^3(\mathcal{C}_c^\infty(G)), \\ D \otimes_{\mathcal{C}^\infty(P)} D' \otimes_{\mathcal{C}^\infty(P)} D'' &\mapsto \{f \otimes_{\mathbb{C}} f' \otimes_{\mathbb{C}} f'' \mapsto D(f) * D'(f') * D''(f'')\}, \end{aligned}$$

*are injective. Then the formula $\Delta_\ell^H D := m^*_{\mathcal{C}_c^\infty(G)} D$ gives a well-defined map*

$$\Delta_\ell^H : H \to H \otimes_{\mathcal{C}^\infty(P)} H, \quad D \mapsto \sum_i D_i' \otimes_{\mathcal{C}^\infty(P)} D_i'',$$

*where $D$ has $H$-codecomposition $D(f * f') = \sum_i D_i'(f) * D_i''(f')$. Together with the map*

$$\epsilon_H : H \to \mathcal{C}^\infty(P), \quad D \mapsto D(1_{\mathcal{C}^\infty(G)}),$$

*the triple $(H, \Delta^H_\ell, \epsilon_H)$ becomes a $\mathcal{C}^\infty(P)$-coring with respect to the $\mathcal{C}^\infty(P)$-bimodule structure (4.6.2). In particular, with the composition of operators and source and target maps*

$$s^\ell_H : a \to m_{t^* a} \quad \text{and} \quad t^\ell_H : a \to m_{s^* a}, \tag{4.6.3}$$

*as maps $\mathcal{C}^\infty(P) \to H$, the $\mathbb{C}$-module $H$ is a left bialgebroid over $\mathcal{C}^\infty(P)$.*

PROOF: Using $m^*_{\mathcal{C}^\infty_c(G)}(H) \subseteq J(H \otimes_{\mathcal{C}^\infty(P)} H)$ and the injectivity of $J$, the well-definedness for $\Delta^H_\ell$ follows directly. Also, coassociativity follows from the injectivity of the second map. The bialgebroid axioms are again easy to check. For example,

$$\begin{aligned}\big(m_H(s^\ell_H \otimes \mathrm{id})\Delta^H_\ell D(f)\big)(g) &= \sum_i \big(D'_i(1_{\mathcal{C}^\infty(G)})\big)(1_{t(g)})\big(D''_i(f)\big)(g) \\ &= \sum_i \big(D'_i(1_{\mathcal{C}^\infty(G)}) * D''_i(f)\big)(g) \\ &= D(1_{\mathcal{C}^\infty(G)} * f)(g) = D(f)(g).\end{aligned}$$

Also,

$$\begin{aligned}\epsilon_H(DD') &= DD'(1_{\mathcal{C}^\infty(G)}) = D(\epsilon_H D') \\ &= D\big(m_{s^*(\epsilon_H D')}(1_{\mathcal{C}^\infty(G)})\big) = Dm_{s^*(\epsilon_H D')}(1_{\mathcal{C}^\infty(G)}) = \epsilon_H\big(Dt^\ell_H(\epsilon_H D')\big).\end{aligned}$$

□

Let us also mention some explicit coproduct expressions for certain elements, e.g. generators of $H$. These are calculated as in the proof of Lemma 4.6.3. Let $m_f$ denote the multiplication operator associated to $f \in F$. One sees that for $a, b \in \mathcal{C}^\infty(P)$, one has

$$\begin{aligned}\big(m^*_{\mathcal{C}^\infty_c(G)}(m_{s^*a})(f \otimes f')\big)(g) &= \big(m_{s^*a}(f * f')\big)(g) \\ &= \sum_{g=hk} s^*a(hk)f(h)f'(k) = (f * m_{s^*a}f')(g),\end{aligned}$$

and analogously $\big(m^*_{\mathcal{C}^\infty_c(G)}(m_{t^*b})(f \otimes f')\big)(g) = (m_{t^*b}f * f')(g)$. Hence

$$\Delta^H_\ell m_{s^*a} = 1 \otimes_{\mathcal{C}^\infty(P)} m_{s^*a} \quad \text{and} \quad \Delta^H_\ell m_{t^*b} = m_{t^*b} \otimes_{\mathcal{C}^\infty(P)} 1, \qquad a, b \in \mathcal{C}^\infty(P).$$

It follows from (4.5.22) that

$$\big(m^*_{\mathcal{C}^\infty_c(G)}(m_{\eta^i_X})(f \otimes f')\big)(g) = \sum_{g=hk}\sum_{j=1}^n \eta^j_X(h)f(h)\eta^i_j(k)f'(k).$$

Hence, as expected,

$$\Delta^H_\ell m_{\eta^i_X} = \sum_{j=1}^n m_{\eta^j_X} \otimes_{\mathcal{C}^\infty(P)} m_{\eta^i_j} \quad \text{and} \quad \Delta^H_\ell m_{\tilde{\eta}^i_X} = \sum_{j=1}^n m_{\tilde{\eta}^i_j} \otimes_{\mathcal{C}^\infty(P)} m_{\tilde{\eta}^j_X}. \tag{4.6.4}$$

In the same fashion, one obtains from (4.5.15) and (4.5.16),

$$\Delta^H_\ell t^*X = t^*X \otimes_{\mathcal{C}^\infty(P)} 1 + \sum_{i=1}^n \eta^i_X(g) \otimes_{\mathcal{C}^\infty(P)} t^*X_i, \tag{4.6.5}$$

$$\Delta^H_\ell s^*X = \sum_{i=1}^n s^*X_i \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^i_X + 1 \otimes_{\mathcal{C}^\infty(P)} s^*X. \tag{4.6.6}$$

**4.6.5 Remark** Let $S_F$ be the antipode on $F$ given in (4.5.20) and $S_{VP}$ the antipode of the Lie algebroid $VP$ given in (4.2.16) (depending on a (Lie algebroid) connection as in §4.2.5). We conjecture that one can extend the assignment

$$\begin{aligned} f &\mapsto S_F f, & \forall\, f \in F, \\ t^*X &\mapsto s^*(S_{VP}X), & \forall\, X \in \mathcal{X}P \end{aligned}$$

to an antipode on $H$ turning $H$ into a Hopf algebroid; see also Remarks 3.3.7 and 4.7.4.

## 4.7 Bicrossed Product Realisation

In this section we describe how $H$ can be 'composed' from the bialgebroids $F$ and $VP$. For this, we will need the concepts of Section 3.3 of matched pairs and the bicrossproduct bialgebroid, i.e., we will discuss a $VP$-action on $F$ and an $F$-coaction on $VP$ to obtain the main result of this section:

**4.7.1 Theorem** *The pair $(F, VP)$ is a matched pair of left bialgebroids (see Definition 3.3.6), and the map (4.7.3) defines a left $\mathcal{C}^\infty(P)$-bialgebroid isomorphism*

$$F \blacktriangleright\!\!\triangleleft_{\mathcal{C}^\infty(P)} VP \xrightarrow{\simeq} H. \tag{4.7.1}$$

The procedure we give here generalises methods of [MosR] to bialgebroids.

The pullbacks of the groupoid source and target maps give two algebra morphisms $s^*, t^* : VP \to VG$, where we recall the notation $VP := V\Gamma TP$ and $VG := V\Gamma TG$. In particular, we have two vector space isomorphisms

$$VP \otimes_{\mathcal{C}^\infty(P)} \mathcal{C}^\infty(G) \xrightarrow{\simeq} VG, \qquad (u, f) \longmapsto f t^* u, \tag{4.7.2}$$

$$\mathcal{C}^\infty(G) \otimes_{\mathcal{C}^\infty(P)} VP \xrightarrow{\simeq} VG, \qquad (f, u) \longmapsto f s^* u, \tag{4.7.3}$$

where the left $\mathcal{C}^\infty(P)$-action on $VP$ is denoted by $(a, u) \mapsto au$, and where the $\mathcal{C}^\infty(P)$-bimodule structure on $\mathcal{C}^\infty(G)$ from (4.5.4) has been used to define the tensor products.

The fact that (4.7.2) (and likewise (4.7.3)) is a vector space isomorphism can be seen as follows: with the help of the PBW map (1.4.5), (4.7.2) can be seen to be induced degree-wise by the isomorphism $\Gamma S^p TP \otimes_{\mathcal{C}^\infty(P)} \mathcal{C}^\infty(G) \simeq \Gamma S^p TG$ on the respective symmetric algebras, considered to be vector bundles over $P$ and $G$, respectively (cf. also §1.4.5 for notation). Since $TG \simeq t^*TP$ in the étale case, the isomorphism $\Gamma S^p TP \otimes_{\mathcal{C}^\infty(P)} \mathcal{C}^\infty(G) \simeq \Gamma S^p TG$ in turn follows from a general result on pullback vector bundles.

The maps (4.7.2) and (4.7.3) serve to define a right $F$-comodule structure on $VP$:

**4.7.2 Lemma** *The assignment*

$$VP \ni u \mapsto s^* u \in VG \tag{4.7.4}$$

*defines a right $F$-coaction*

$$_{VP}\Delta : VP \to VP \otimes_{\mathcal{C}^\infty(P)} F, \quad u \mapsto u_{[0]} \otimes_{\mathcal{C}^\infty(P)} u_{[1]} \tag{4.7.5}$$

*on the universal enveloping algebra $VP$.*

PROOF: Firstly we state that the canonical right $\mathcal{C}^\infty(P)$-action $R_{\mathcal{C}^\infty(P)}$ on $VP$ from Definition 2.3.2(*i*) is given here by $R_{\mathcal{C}^\infty(P)}(u, a) = au = s^\ell_{VP} au$. Hence the tensor product in question is

$$VP \otimes_{\mathcal{C}^\infty(P)} F = VP \otimes F / \mathrm{span}\{au \otimes f - u \otimes t^*(a) f,\ a \in \mathcal{C}^\infty(P),\ u \in VP,\ f \in F\}.$$

Observe that $_{VP}\Delta$ maps into this space, indeed, as follows from the definition of (the filtration of) $F$. We use (4.7.2) and (4.5.10) to write on generators

$$_{VP}\Delta a = 1 \otimes_{\mathcal{C}^\infty(P)} s^* a \in VP \otimes_{\mathcal{C}^\infty(P)} F_0, \tag{4.7.6}$$

$$_{VP}\Delta X = \sum_{j=1}^n X_j \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^j_X \in VP \otimes_{\mathcal{C}^\infty(P)} F_1. \tag{4.7.7}$$

It is now enough to check the coaction axioms on a PBW basis of $VP$: let $I$ be a (finite) sequence $i_1 \leq \ldots \leq i_p$ and set $X_I := X_{i_1} \cdots X_{i_p}$ for $X_{i_j} \in \mathcal{X}TP \subset VP$ as elements in $VP$ (if $I$ is empty, set $X_I = 1$). Elements of the form $aX_I$ with $a \in \mathcal{C}^\infty(P)$ form a basis of $VP$; see [Rin] for details. Since $_{VP}\Delta$ and $\Delta^F_\ell$ are $\mathcal{C}^\infty(P)$-linear, we can restrict to the case $a = 1$.

Using (4.5.10), the action $(-)^t$ from (4.5.2) and abbreviating $\eta^j_{i_l} := \tilde{\eta}^j_{X_{i_l}}$ for $l = 1, \ldots, p$ one can write

$$s^* X_I = s^* X_{i_1} \cdots s^* X_{i_p} = \sum_{j_1, \ldots, j_p}^n \tilde{\eta}^{j_1}_{i_1} t^* X_{j_1} \cdots \tilde{\eta}^{j_p}_{i_p} t^* X_{j_p}$$

$$= \sum_{j_1, \ldots, j_p}^n \tilde{\eta}^{j_1}_{i_1} X_{j_1}{}^t_{(1)} (\tilde{\eta}^{j_2}_{i_2}) (X_{j_1 (2)} X_{j_2 (1)})^t (\tilde{\eta}^{j_3}_{i_3}) \cdots (X_{j_1 (p-1)} X_{j_2 (p-2)} \cdots X_{j_{p-1} (2)})^t (\tilde{\eta}^{j_p}_{i_p}) \cdot$$

$$\cdot\, t^* \big( X_{j_1 (p)} X_{j_2 (p-1)} \cdots X_{j_{p-1} (1)} X_{j_p} \big),$$

and hence with (4.7.2)

$$\begin{aligned}
{}_{VP}\Delta X_I &= X_{I\,[0]} \otimes_{\mathcal{C}^\infty(P)} X_{I\,[1]} \\
&= \sum_{j_1,\dots,j_p}^{n} X_{j_1\,(p)} X_{j_2\,(p-1)} \cdots X_{j_{p-1}\,(1)} X_{j_p} \\
&\quad \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1} X_{j_1\,(1)}^{\;t}(\tilde{\eta}^{j_2}_{i_2}) (X_{j_1\,(2)} X_{j_2\,(1)})^t(\tilde{\eta}^{j_3}_{i_3}) \cdots (X_{j_1\,(p-1)} \cdots X_{j_{p-1}\,(2)})^t(\tilde{\eta}^{j_p}_{i_p}).
\end{aligned} \tag{4.7.8}$$

We now show that the coaction axioms hold by induction over the length $p$ of a PBW basis: one has for $X \in \mathcal{X}P$ with (4.5.22)

$$({}_{VP}\Delta \otimes \mathrm{id}_F)\,{}_{VP}\Delta X = \sum_{i,j=1}^{n} X_j \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^j_i \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^i_X = (\mathrm{id}_{VP} \otimes \Delta^F_\ell)\,{}_{VP}\Delta X, \tag{4.7.9}$$

as the start of the induction, and we proceed to prove that if this coaction axiom is true for $X_{I'} := X_{i_2} \cdots X_{i_p}$, where $I'$ is the finite increasing series $i_2 \leq \ldots \leq i_p$, it is also true for the sequence $I$ introduced before. From (4.7.8) one checks

$$\begin{aligned}
{}_{VP}\Delta X_I &= {}_{VP}\Delta(X_{i_1} X_{I'}) \\
&= \sum_{j_1,\dots,j_p}^{n} X_{j_1\,(1)} X_{j_2\,(p-1)} \cdots X_{j_{p-1}\,(1)} X_{j_p} \\
&\qquad \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1} X_{j_1\,(2)}^{\;t}\big(\tilde{\eta}^{j_2}_{i_2} X_{j_2\,(1)}^{\;t}(\tilde{\eta}^{j_3}_{i_3}) \cdots (X_{j_2\,(p-2)} \cdots X_{j_{p-1}\,(2)})^t(\tilde{\eta}^{j_p}_{i_p})\big) \\
&= \sum_{j_1,\dots,j_p}^{n} X_{j_1} X_{j_2\,(p-1)} \cdots X_{j_{p-1}\,(1)} X_{j_p} \\
&\qquad \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1} \tilde{\eta}^{j_2}_{i_2} X_{j_2\,(1)}^{\;t}(\tilde{\eta}^{j_3}_{i_3}) \cdots (X_{j_2\,(p-2)} \cdots X_{j_{p-1}\,(2)})^t(\tilde{\eta}^{j_p}_{i_p}) \\
&\quad + \sum_{j_2,\dots,j_p}^{n} X_{j_2\,(p-1)} \cdots X_{j_{p-1}\,(1)} X_{j_p} \\
&\qquad\quad \otimes_{\mathcal{C}^\infty(P)} X^s_{i_1}\big(\tilde{\eta}^{j_2}_{i_2} X_{j_2\,(1)}^{\;t}(\tilde{\eta}^{j_3}_{i_3}) \cdots (X_{j_2\,(p-2)} \cdots X_{j_{p-1}\,(2)})^t(\tilde{\eta}^{j_p}_{i_p})\big),
\end{aligned}$$

since $VP$ is cocommutative and $X_{j_1}$ is primitive. Hence with (4.7.9) and the action $(-)^s$ from (4.5.3),

$${}_{VP}\Delta X_I = \sum_{j_1}^{n} X_{j_1} X_{I'\,[0]} \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1} X_{I'\,[1]} + X_{I'\,[0]} \otimes_{\mathcal{C}^\infty(P)} X^s_{i_1}(X_{I'\,[1]}), \tag{4.7.10}$$

and therefore with the same argument again, as well as using the induction assumption,

$$\begin{aligned}
({}_{VP}\Delta \otimes \mathrm{id})\,{}_{VP}\Delta X_I &= \sum_{j_1,k_1}^{n} X_{k_1} X_{I'\,[0]} \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{k_1}_{j_1} X_{I'\,[1]} \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1} X_{I'\,[2]} \\
&\quad + \sum_{j_1}^{n} X_{I'\,[0]} \otimes_{\mathcal{C}^\infty(P)} X^s_{j_1}(X_{I'\,[1]}) \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1} X_{I'\,[2]} \\
&\quad + X_{I'\,[0]} \otimes_{\mathcal{C}^\infty(P)} X_{I'\,[1]} \otimes_{\mathcal{C}^\infty(P)} X^s_{i_1}(X_{I'\,[2]}) \\
&= (\mathrm{id} \otimes \Delta^F_\ell)\,{}_{VP}\Delta X_I
\end{aligned}$$

by (4.5.22) and (4.5.16).

Counitality is proven by the same kind of argument: first calculate

$$\epsilon_F(\tilde{\eta}^i_X)(x) = \tilde{\eta}^i_X(1_x) = \theta^i((1_x)^{-1} \cdot X_{s(1_x)}) = \theta^i(X)(x) \tag{4.7.11}$$

at a point $x \in P$. Hence, the first step of the induction is

$$R_{\mathcal{C}^\infty(P)}(\mathrm{id}_{VP} \otimes \epsilon_F)\,{}_F\Delta X = \sum_{i=1}^{n} \epsilon_F(\tilde{\eta}^i_X) X_i = X.$$

Subsequently, with (4.7.10), (2.1.4), (4.5.18) and the fact that $F$ is commutative,

$$\begin{aligned}R_{\mathcal{C}^\infty(P)}(\mathrm{id}_{VP} \otimes \epsilon_F)_F\Delta X_I &= \sum_{j_1}^{n} \epsilon_F(X_{I'[1]})\epsilon_F(\tilde{\eta}^{j_1}_{i_1})X_{j_1}X_{I'[0]} + \epsilon_F\big(X^s_{i_1}(X_{I'[1]})\big)X_{I'[0]} \\ &= X_{i_1}\epsilon_F(X_{I'[1]})X_{I'[0]} - X^\omega_{i_1}\big(\epsilon_F(X_{I'[1]})\big)X_{I'[0]} + \epsilon_F\big(X^s_{i_1}(X_{I'[1]})\big)X_{I'[0]} \\ &= X_I,\end{aligned}$$

since the assumption holds for $X_{I'}$ (where $X^\omega$ denotes the action originating from the anchor of $VP$, i.e. the Lie derivative of $\Gamma TP$ on $\mathcal{C}^\infty(P)$). □

**4.7.3 Proposition** *The bialgebroid $F$ is a left $VP$-module ring (cf. Definition 3.3.1) with respect to the action (4.5.3), whereas $VP$ is a right $F$-comodule coring (cf. Definition 3.3.3) with respect to the coaction (4.7.5). In particular, the left $\mathcal{C}^\infty(P)$-bialgebroids $F$ and $U$ form a matched pair in the sense of Definition 3.3.6.*

PROOF: The first part is obvious: the action (4.5.3) clearly restricts to an action $(u, f) \mapsto u^s(f)$ on $F$ by the definition of $F$ and Lemma 4.5.3, and all properties in Definition 3.3.1 of left module rings are trivial to check (e.g., for a PBW basis of $VP$). In particular, the induced left and right $\mathcal{C}^\infty(P)$-actions from (3.3.2) coincide with $F_\triangleleft$, i.e., $(s^\ell_{VP}a)^s(f) = a^s(f) = s^*af = t^\ell_F af$.

Let $(VP, \Delta^{VP}_\ell, \epsilon_{VP})$ denote the coring structure of $VP$ as given in Proposition 4.2.9. To show that it is a right $F$-comodule coring we check the conditions (3.3.8): by (4.7.8), (2.1.4) and $\epsilon X = 0$ for all $X \in \mathcal{X}P$, the first condition is trivially fulfilled for any PBW basis $aX_I$ for $a \in \mathcal{C}^\infty(P)$ and $I$ the increasing sequence $i_1 \leq \ldots \leq i_p$; in case $I$ is empty, it follows by (4.7.6) and (4.5.19). The second equation in (3.3.8) is again proven by induction over the length $p$ of $X_I$ and by $\mathcal{C}^\infty(P)$-linearity it is again enough to consider the case $a = 1$. Now equation (3.3.8) is obviously fulfilled for any $X \in \mathcal{X}P$ using (4.7.7) and the primitivity of $X$; this gives the induction start. Hence assume (3.3.8) to be fulfilled for $X_{I'}$ where $I'$ is the sequence $i_2 \leq \ldots \leq i_p$. It follows from (4.7.10) that

$$\begin{aligned}(\Delta^{VP}_\ell \otimes \mathrm{id})_{VP}\Delta X_I &= (\Delta^{VP}_\ell \otimes \mathrm{id})_{VP}\Delta(X_{i_1}X_{I'}) \\ &= \sum_{j_1}^{n} X_{j_1}X_{I'[0](1)} \otimes_{\mathcal{C}^\infty(P)} X_{I'[0](2)} \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1}X_{I'[1]} \\ &\quad + \sum_{j_1}^{n} X_{I'[0](1)} \otimes_{\mathcal{C}^\infty(P)} X_{j_1}X_{I'[0](2)} \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1}X_{I'[1]} \\ &\quad + \sum_{j_1}^{n} X_{I'[0](1)} \otimes_{\mathcal{C}^\infty(P)} X_{I'[0](2)} \otimes_{\mathcal{C}^\infty(P)} X^s_{i_1}(X_{I'[1]}) \\ &= \sum_{j_1}^{n} X_{j_1(1)}X_{I'(1)[0]} \otimes_{\mathcal{C}^\infty(P)} X_{j_1(2)}X_{I'(2)[0]} \otimes_{\mathcal{C}^\infty(P)} \tilde{\eta}^{j_1}_{i_1}X_{I'(1)[1]}X_{I'(2)[1]} \\ &\quad + \sum_{j_1}^{n} X_{I'(1)[0]} \otimes_{\mathcal{C}^\infty(P)} X_{I'(2)[0]} \otimes_{\mathcal{C}^\infty(P)} X^s_{i_1}(X_{I'(1)[1]}X_{I'(2)[1]}) \\ &= {}_{VP\otimes VP}\Delta\,\Delta^{VP}_\ell(X_{i_1}X_{I'}) = {}_{VP\otimes VP}\Delta\,\Delta^{VP}_\ell X_I,\end{aligned}$$

since $F$ is commutative and a left $VP$-module ring. This concludes the induction and we have shown that $VP$ is a right $F$-comodule coring.

We proceed by considering the conditions in Theorem 3.3.5. The identities (3.3.17)–(3.3.20) are obviously fulfilled by (4.2.11) and the commutativity of $F$. For an $f \in F$ with $F$-codecomposition $f(gh) = \sum_i f'_i(g)f''_i(h)$, with (4.5.16) at a point $x \in P$ we have

$$\begin{aligned}\epsilon_F\big(X^s(f)\big) &= (L_{s^*X}f(1_x)) \\ &= \sum_i \sum_{l=1}^{n} (L_{s^*X_l}f'_i)(1_x)\tilde{\eta}^l_X(1_x)f''_i(1_x) + \sum_i f'_i(1_x)(L_{s^*X}f''_i)(1_x) \\ &= \sum_i L_X(1^*f'_i\,1^*f''_i)(x) = \sum_i \epsilon_{VP}\big(X(1^*(f'_if''_i))\big)(x) = \epsilon_{VP}(X\epsilon_F f)(x),\end{aligned}$$

which is (3.3.21) for $X \in \mathcal{X}P$ and analogously on a PBW basis $X_I$ of $VP$ using (2.1.4). Now (3.3.22) is obvious from (4.7.6). For (3.3.23) we argue by induction on the length $p$ of $X_I$ again: the induction start follows from (4.5.16), (4.5.17) and (4.7.6), and the induction step to pass from $X_{I'}$ to $X_I$ (see above) works as follows: considering $X^s_{I'}(f)$ as an element in $F$ again, the assumption is already true for $X^s_{i_1}\big(X^s_{I'}(f)\big)$. Hence with (4.7.10),

$$\begin{aligned}
\Delta^F_\ell\big(X^s_I(f)\big) &= \Delta^F_\ell\big(X^s_{i_1}(X^s_{I'}(f))\big)\\
&= X_{i_1\,(1)[0]}\big((X^s_{I'}(f))_{[1]}\big) \otimes_{\mathcal{C}^\infty(P)} X_{i_1\,(1)[1]}X_{i_1\,(2)}\big((X^s_{I'}(f))_{[2]}\big)\\
&= X_{i_1}{}^s_{(1)[0]}\big(X_{I'}{}^s_{(1)[0]}(f_{[1]})\big) \otimes_{\mathcal{C}^\infty(P)} X_{i_1\,(1)[1]}X_{i_1}{}^s_{(2)}\big(X_{I'\,(1)[1]}X_{I'}{}^s_{(2)}(f_{[2]})\big)\\
&= X_{i_1}{}^s_{(1)[0]}\big(X_{I'}{}^s_{(1)[0]}(f_{[1]})\big) \otimes_{\mathcal{C}^\infty(P)} X_{i_1\,(1)[1]}X_{i_1}{}^s_{(2)}(X_{I'\,(1)[1]})X_{i_1}{}^s_{(3)}\big(X_{I'}{}^s_{(2)}(f_{[2]})\big)\\
&= (X_{i_1}X_{I'})^s_{(1)[0]}(f_{[1]}) \otimes_{\mathcal{C}^\infty(P)} (X_{i_1}X_{I'})_{(1)[1]}(X_{i_1}X_{I'})^s_{(2)}(f_{[2]})\\
&= X_I{}^s_{(1)[0]}(f_{[1]}) \otimes_{\mathcal{C}^\infty(P)} X_{I\,(1)[1]}X_I{}^s_{(2)}(f_{[2]}),
\end{aligned}$$

which is (3.3.23) for $u = X_I$, as desired.

Next, (3.3.24) is automatically fulfilled since $F$ is commutative and $VP$ is cocommutative. Finally, for (3.3.25), let $J$ be another increasing sequence of indices of length $q$. Then it is enough to show this identity for $u = X_I$ and $u' = aX_J$ where $a \in \mathcal{C}^\infty(P)$. The statement is already implicit in (4.7.10): the induction start is (4.7.10), choosing $I' = J$ and observing the relation $X_{i_1}aX_J = aX_{i_1}X_J + X_{i_1}(a)X_J$ whereas the induction is concluded by the same argument replacing $X_{I'}$ in (4.7.10) by elements of the form $X_{I'}aX_J$. □

PROOF: (of Theorem 4.7.1) First observe that, by definition of $F$ and $H$, (4.7.3) restricts to a vector space isomorphism

$$\phi : F \otimes_{\mathcal{C}^\infty(P)} VP \to H, \quad f \otimes_{\mathcal{C}^\infty(P)} u \mapsto fs^*u. \tag{4.7.12}$$

As before we view elements in $F$ as multiplication operators. Hence it is enough to show that $\phi$ is both an isomorphism of $\mathcal{C}^\infty(P)^{\mathrm{e}}$-rings and of $\mathcal{C}^\infty(P)$-corings: with (4.6.3), (4.5.19), (3.3.27) and the commutativity of $F$, it is clear that $\phi$ is a $\mathcal{C}^\infty(P)^{\mathrm{e}}$-bimodule map; hence it remains to show that it is a morphism of algebras and coalgebras. As already seen, $F$ is a left $VP$-module ring and therefore we have for any $\xi \in \mathcal{C}^\infty(G)$, with the notation (4.5.3),

$$\begin{aligned}
\big(\phi(f \otimes_{\mathcal{C}^\infty(P)} u)\phi(f' \otimes_{\mathcal{C}^\infty(P)} u')\big)(\xi) &= fu^s\big(f'u'^s(\xi)\big)\\
&= fu^s_{(1)}(f')(u_{(2)}u')^s(\xi)\\
&= \phi\big((f \otimes_{\mathcal{C}^\infty(P)} u)(f' \otimes_{\mathcal{C}^\infty(P)} u')\big)(\xi),
\end{aligned}$$

where in the last term the product (3.3.7) on the space $F \rtimes_{\mathcal{C}^\infty(P)} VP$ is meant. To show that $\phi$ is also a coalgebra morphism, we proceed as in the proof of Proposition 4.7.3 by induction on the length $p$ of a PBW basis $aX_I$ with respect to a finite increasing sequence $I$ of indices $i_1 \leq \ldots \leq i_p$ and $a \in \mathcal{C}^\infty(P)$. Since $\phi$, $\Delta^H_\ell$ and $\Delta^{\mathrm{ccr}}_\ell$ are bimodule maps, we can set $a = 1$; since these are $\mathbb{C}$-algebra morphisms on the respective tensor products, we may even take $f = 1_F$. Now the induction start for $X \in \mathcal{X}P$ and (4.6.6) is easily checked, namely

$$(\phi \otimes \phi)\Delta^{\mathrm{ccr}}_\ell(1_F \otimes_{\mathcal{C}^\infty(P)} X) = \sum_{i=1}^n s^*X_i \otimes_{\mathcal{C}^\infty(P)} \tilde\eta^i_X + 1_H \otimes_{\mathcal{C}^\infty(P)} s^*X = \Delta^H_\ell\phi(1_F \otimes_{\mathcal{C}^\infty(P)} X).$$

Hence if the claim holds for the sequence $I'$ given by $i_2 \leq \ldots \leq i_p$, i.e., $\Delta^H_\ell s^*X_{I'} = s^*X_{I'\,(1)[0]} \otimes_{\mathcal{C}^\infty(P)} X_{I'\,(1)[1]}s^*X_{I'\,(2)}$, we show that it also holds for $I$: for $\xi, \xi' \in \mathcal{C}^\infty(G)$, we have with (3.3.25), $F$ commutative and Proposition 4.7.3,

$$\begin{aligned}
&(\phi \otimes \phi)\Delta^{\mathrm{ccr}}_\ell(1_F \blacktriangleright\!\!\lhd_{\mathcal{C}^\infty(P)} X_I)(\xi \otimes \xi')\\
&\quad= (\phi \otimes \phi)\Big(\big(1_F \blacktriangleright\!\!\lhd_{\mathcal{C}^\infty(P)} (X_{i_1}X_{I'\,(1)})_{[0]}\big)\\
&\qquad\qquad \otimes_{\mathcal{C}^\infty(P)} \big((X_{i_1\,(1)}X_{I'\,(1)})_{[1]} \blacktriangleright\!\!\lhd_{\mathcal{C}^\infty(P)} X_{i_1\,(2)}X_{I'\,(2)}\big)\Big)(\xi \otimes \xi')\\
&\quad= (\phi \otimes \phi)\Big(\big(1_F \blacktriangleright\!\!\lhd_{\mathcal{C}^\infty(P)} X_{i_1\,(1)[0]}X_{I'\,(1)[0]}\big)\\
&\qquad\qquad \otimes_{\mathcal{C}^\infty(P)} \big(X_{i_1\,(1)[1]}X_{i_1}{}^s_{(2)}(X_{I'\,(1)[1]}) \blacktriangleright\!\!\lhd_{\mathcal{C}^\infty(P)} X_{i_1\,(3)}X_{I'\,(2)}\big)\Big)(\xi \otimes \xi')\\
&\quad= X_{i_1}{}^s_{(1)[0]}X_{I'}{}^s_{(1)[0]}(\xi) \otimes_{\mathcal{C}^\infty(P)} X_{i_1\,(1)[1]}X_{i_1}{}^s_{(2)}\big(X_{I'\,(1)[1]}X_{I'}{}^s_{(2)}(\xi')\big)\\
&\quad= \Delta^H_\ell(s^*X_{i_1})\Delta^H_\ell(s^*X_{I'})(\xi \otimes \xi') = \Delta^H_\ell(\phi(1_F \blacktriangleright\!\!\lhd_{\mathcal{C}^\infty(P)} X_I))(\xi \otimes \xi').
\end{aligned}$$

□

**4.7.4 Remark** We conjecture that the isomorphism (4.7.1) is even an isomorphism of Hopf algebroids: it is not difficult to see (at least on generators) that the candidate for the antipode for matched pairs of bialgebroids from Remark 3.3.7 is mapped by means of (4.7.1) to the antipode candidate for $H$, mentioned in Remark 4.6.5.

# Chapter 5

# Hopf-Cyclic Cohomology

Hopf-cyclic cohomology (and also its dual homology in Chapter 6) cannot be defined as the cyclic theory of some algebra or coalgebra itself, but only as a theory deriving from certain cyclic and cocyclic modules. As suggested by the concept of the space of coinvariants introduced in the next section, there are some rough similarities to the procedure in group (co)homology. We shall indeed be able to associate a cyclic complex (more precisely, a cocyclic module) to any Hopf algebroid if and only if its antipode is an involution (possibly twisted by a grouplike element). In particular, we will show that this cocyclic structure 'descends' in a natural way from the canonical cocyclic structure of a Hopf algebroid, regarded as a coring.

The resulting cyclic cohomology could be considered (by Theorem 5.5.7 and (5.6.2)) to be a natural generalisation of Lie-Rinehart (Lie algebroid) (co)homology within the context of Hopf algebroids and hence of noncommutative geometry.

## 5.1 The Space of Coinvariants

In this section we introduce the notion of coinvariants as a first step towards Hopf-cyclic cohomology. The method introduced here is a generalisation of a similar procedure for Hopf algebras in [Cr3].

Let $H$ be a Hopf algebroid with structure maps as in Definition 2.6.1 and let $M \in H\text{-}\mathbf{Mod}$, with action denoted $(h, m) \mapsto hm$. In particular, $M$ carries an induced $(A, A)$-bimodule structure, which we use to define a $(B, B)$-bimodule structure with the help of the map $\nu^{-1} = \epsilon s^r : B^{\mathrm{op}} \xrightarrow{\simeq} A$ from (2.6.5), namely

$$b \triangleright m \triangleleft \tilde{b} := t^\ell(\nu^{-1}b)s^\ell(\nu^{-1}\tilde{b})m = s^r(b)Ss^r(\tilde{b})m, \qquad b, \tilde{b} \in B,\ m \in M.$$

**5.1.1 Definition** (*i*) The *space of coinvariants* $I_\partial$ of $M$ is the $k$-linear span of elements

$$\partial h \triangleright m - hm, \qquad \forall\, m \in M,\ h \in H.$$

(*ii*) The $\partial$-localised module $M_\partial$ is given as the quotient

$$M_\partial := B_\partial \otimes_H M, \tag{5.1.1}$$

where $B = B_\partial \in \mathbf{Mod}\text{-}H$ by (2.5.4).

In particular, this defines a functor $(-)_\partial : H\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$. We also introduce the *coinvariant localisation*

$$\pi_\partial : M \to M_\partial, \quad m \mapsto 1_B \otimes_H m.$$

One sees that

$$M_\partial = B \otimes_A M / \mathrm{span}_k\{\partial h \otimes_A m - 1_B \otimes_A hm\} = M/I_\partial,$$

where $A$ acts on $B$ by means of $\nu$. If in particular $M := A$ is the base algebra of the underlying left bialgebroid itself, one obtains the identification $\epsilon h \equiv \partial h$ in $A_\partial = B \otimes_H A$, i.e. up to coinvariants ($\nu^{-1}$ suppressed here).

**5.1.2 Lemma** (Partial Integration) *Let $H$ be a Hopf algebroid as before. For any two $M, N \in H\text{-}\mathbf{Mod}$, the identity*

$$hm \otimes_A n \equiv m \otimes_A (Sh)n \qquad \forall\, m \in M,\ n \in N,\ h \in H$$

*holds up to coinvariants.*

PROOF: The induced $H$-module structure (for the underlying left bialgebroid) on $M \otimes_A N$ is given by (2.3.2), i.e., $h(m \otimes_A n) = h_{(1)} m \otimes_A h_{(2)} n$. This induces not only an $(A, A)$-bimodule structure but also a $(B, B)$-bimodule structure on $M \otimes_A N$:

$$b \triangleright (m \otimes_A n) \triangleleft \tilde{b} := s^\ell(\nu^{-1}\tilde{b}) m \otimes_A t^\ell(\nu^{-1} b) n, \qquad b, \tilde{b} \in B,\ m \in M, n \in N.$$

Then one has

$$\begin{array}{rcll} m \otimes_A (Sh)n & = & m \otimes_A (s^r \partial h^{(1)} S h^{(2)}) n & \text{by Lemma 2.6.6,} \\ & = & \partial h^{(1)} \triangleright (m \otimes_A (Sh^{(2)})n) & \\ & \equiv & \Delta_\ell h^{(1)} \big(m \otimes_A (Sh^{(2)})n\big) & \text{modulo coinvariants} \\ & = & h_{(1)} m \otimes_A h^{(1)}_{(2)} (S h^{(2)}_{(2)} n) & \text{by twisted coassociativity (2.6.2),} \\ & = & h_{(1)} m \otimes_A (h^{(1)}_{(2)} S h^{(2)}_{(2)}) n & \text{by } N \in H\text{-}\mathbf{Mod}, \\ & = & h_{(1)} m \otimes_A (s^\ell \epsilon(h_{(2)})) n & \text{by (2.6.4),} \\ & = & h_{(1)} m \otimes_A \epsilon h_{(2)} \triangleright n & \\ & = & (h_{(1)} m) \triangleleft \epsilon h_{(2)} \otimes_A n & \text{in the tensor product } \otimes_A, \\ & = & (t^\ell \epsilon(h_{(2)}) h_{(1)}) m \otimes_A n & \\ & = & hm \otimes_A n, & \end{array}$$

where the last identity is simply one of the comonoid identities of a left bialgebroid. $\square$

Regarding $H$ as a module over itself with respect to multiplication, we obtain

**5.1.3 Corollary** *For a $M \in H$-$\mathbf{Mod}$, there is an isomorphism of $k$-modules*

$$\phi : (H \otimes_A M)_\partial \xrightarrow{\simeq} M, \tag{5.1.2}$$

*induced by the covariant localisation*

$$\pi_\partial : H \otimes_A M \to (H \otimes_A M)_\partial,\ h \otimes_A m \mapsto 1_B \otimes_H (h \otimes_A m).$$

*Hence the isomorphism (5.1.2) takes the form*

$$\phi : (H \otimes_A M)_\partial \to M, \quad h \otimes_A m \mapsto (Sh)m, \tag{5.1.3}$$

*with inverse*

$$M \to (H \otimes_A M)_\partial, \quad m \mapsto 1_H \otimes_A m. \tag{5.1.4}$$

## 5.2 Cocyclic Structures on Hopf Algebroids

The basic idea is to define the structure of a cocyclic module on the space of coinvariants obtained by projecting the structure of the canonical cyclic module of $H$ as an $A$-coring by the map $\pi_\partial$ as in §1.2.5. Next, the isomorphism $\phi$ from (5.1.3) maps the cocyclic and cosimplicial operators on the space we are interested in.

Consider the $A$-coring structure $(H^\ell, \Delta_\ell, \epsilon)$ of the Hopf algebroid $H$, originating from the left underlying bialgebroid. As in §1.2.5 for any coring, this defines a para-cocyclic module in a natural way: define

$$C^n_A H := H^{\otimes_A n},$$

and as in §1.2.5 set

$$H^A_{\natural, S^2} := \{B^n_A H\}_{n \geq 0}$$

for the choice $\psi := S^2$, where

$$B^n_A H = B \otimes_{A^e} C^{n+1}_A H = C^{n+1}_A H \otimes_A$$

in degree $n$, i.e. $B \otimes_{A^e} H$ in degree zero. Here, $A^e$ acts on $B$ by means of $\nu$ in an obvious way and the left $A^e$-action on $C^n_A H$ is given by

$$(a \otimes b) \cdot (h^1 \otimes_A \cdots \otimes_A h^n) := s^\ell(a) h^1 \otimes_A \cdots \otimes_A \tilde{t}^\ell(b) h^n,$$

where $\tilde{t}^\ell := t^\ell \nu^{-1}\mu$ with $\nu^{-1}\mu := \epsilon s^r \partial s^\ell \in \mathrm{End}_k A^{\mathrm{op}}$ from (2.6.5). This little modification of the left target map is necessary to make $S^2$ an $(A, A)$-bimodule map (hence the operators (1.2.4) for $\psi = S^2$ well-defined). As stated in §1.2.5, $H^A_{\natural,S^2}$ is cocyclic if and only if $S^2 = \mathrm{id}$. In such a case $\nu^{-1}\mu \equiv \mathrm{id}$ and $\tilde{t}^\ell \equiv t^\ell$. Of course, one may also define another para-cocyclic module on $H$ based on the $B$-coring structure $(H^r, \Delta_r, \partial)$ of $H$ originating from the underlying right bialgebroid $H^r$, but we are not going to pursue this here.

The step from coring (para-)cocyclic modules to 'Hopf-(para-)cocyclic' modules is now performed by projection on coinvariants. Hence, define

$$H_{\natural,\partial} := \{B \otimes_H C^{n+1}_A H\}_{n\geq 0},$$

the (degree-wise) coinvariant localisation of $H^A_{\natural,S^2}$. In a second move, this space is mapped (degree-wise) by the isomorphism $\phi$ from (5.1.3) onto $\{C^n_A H\}_{n\geq 0}$, which will again be denoted by $H_{\natural,\partial}$. Stated as a diagram, we have the situation

$$\begin{tikzcd} B^\bullet_A H \arrow[d, "\pi_\partial"'] \arrow[dr, dashed, "\bar{\phi}_\partial"] & \\ B \otimes_H C^{\bullet+1}_A H \arrow[r, "\phi"'] & C^\bullet_A H. \end{tikzcd}$$

In the next proposition, we will show that the cosimplicial and cocyclic operators on $B \otimes_H C^{n+1}_A H$ are essentially still given by the same formula expressions (1.2.4) as on $B \otimes_{A^e} C^{n+1}_A H$. However, the map $\bar{\phi}_\partial$ changes the form of these cosimplicial and cocyclic operators on $C^n_A H$ to quite some extent. We are going to come back to this point in a moment in more detail.

**5.2.1 Proposition** *The cosimplicial and cocyclic operators on $H^A_{\natural,S^2}$ descend to well-defined operators on $H_{\natural,\partial}$ if and only if $S^2 = \mathrm{id}_H$. In that case, $H_{\natural,\partial}$ is a cocyclic module.*

PROOF: First, note that in this case the covariant localisation $\pi_\partial : H^A_{\natural,S^2} \to H_{\natural,\partial}$ takes the form $(\pi_\partial)_n : B \otimes_{A^e} C^{n+1}_A H \to B \otimes_H C^{n+1}_A H$ in degree $n$. By the left $A^e$-action on the unit element $1_H \in C^1_A H = H$, we can consider $A^e$ as a subring of $H$ and the projection $\pi_\partial$ is induced by this inclusion. Correspondingly, consider the space of coinvariants as

$$B \otimes_H C^{n+1}_A H = (B \otimes_{A^e} C^{n+1}_A H)/I,$$

where

$$I = \mathrm{span}_k\{\partial h \otimes_{A^e} h^0 \otimes_A \cdots \otimes_A h^n - 1_B \otimes_{A^e} \Delta^n_\ell h(h^0 \otimes_A \cdots \otimes_A h^n)\}.$$

To show that the operators $\delta_i, \sigma_i, \tau$ in (1.2.4) for $\psi := S^2$ descend to maps $(B \otimes_{A^e} C^{n+1}_A H)/I \to (B \otimes_{A^e} C^{n+1}_A H)/I$, one needs to prove that $I$ is in the respective kernel of these maps if their image is again projected on the quotient with respect to $I$. We only prove some of the identities and leave the rest to the reader. With the notation $B \otimes_{A^e} C^{n+1}_A H =: C^{n+1}_A H \otimes_A$ as in §1.2.3, one obtains, for example

$$\begin{aligned}
&\delta_{n+1}(\partial h \otimes_{A^e} h^0 \otimes_A \cdots \otimes_A h^n - 1_B \otimes_{A^e} h_{(1)} h^0 \otimes_A \cdots \otimes_A h_{(n+1)} h^n) \\
&= h^0_{(2)} \otimes_A \cdots \otimes_A t^\ell \nu^{-1}(\partial h) h^n \otimes_A S^2 h^0_{(1)} \otimes_A \\
&\qquad - h_{(2)} h^0_{(2)} \otimes_A h_{(3)} h^1 \cdots \otimes_A h_{(n+2)} h^n \otimes_A S^2(h_{(1)} h^0_{(1)}) \otimes_A \\
&= h^0_{(2)} \otimes_A \cdots \otimes_A s^r \partial(h) h^n \otimes_A S^2 h^0_{(1)} \otimes_A \\
&\qquad - h_{(2)} h^0_{(2)} \otimes_A h_{(3)} h^1 \cdots \otimes_A t^\ell \epsilon(h_{(n+3)}) h_{(n+2)} h^n \otimes_A S^2(h_{(1)} h^0_{(1)}) \otimes_A \\
&= h^0_{(2)} \otimes_A \cdots \otimes_A s^r \partial(h) h^n \otimes_A S^2 h^0_{(1)} \otimes_A \\
&\qquad - h_{(2)} h^0_{(2)} \otimes_A h_{(3)} h^1 \cdots \otimes_A h_{(n+2)} h^n \otimes_A h^{(1)}_{(n+3)} S h^{(2)}_{(n+3)} S^2(h_{(1)} h^0_{(1)}) \otimes_A \\
&= h^0_{(2)} \otimes_A \cdots \otimes_A s^r \partial(h) h^n \otimes_A S^2 h^0_{(1)} \otimes_A \\
&\qquad - (\Delta^{n+1}_\ell h^{(1)}_{(2)})(h^0_{(2)} \otimes_A h^1 \cdots \otimes_A h^n \otimes_A S h^{(2)} S^2(h^{(1)}_{(1)} h^0_{(1)})) \otimes_A,
\end{aligned}$$

where higher twisted coassociativity (2.6.8) was repeatedly used. Projecting on coinvariants by the map

$\pi_\partial : B \otimes_{A^e} C_A^{n+1} H \to B \otimes_H C_A^{n+1} H$, this becomes

$$\begin{aligned}
&1_B \otimes_H h^0_{(2)} \otimes_A \cdots \otimes_A s^r \partial(h) h^n \otimes_A S^2 h^0_{(1)} \\
&\qquad - 1_B \otimes_H (\Delta_\ell^{n+1} h^{(1)}_{(2)})(h^0_{(2)} \otimes_A h^1 \cdots \otimes_A h^n \otimes_A S h^{(2)} S^2(h^{(1)}_{(1)} h^0_{(1)})) \\
&= 1_B \otimes_H h^0_{(2)} \otimes_A \cdots \otimes_A s^r \partial(h) h^n \otimes_A S^2 h^0_{(1)} \\
&\qquad - 1_B \otimes_H h^0_{(2)} \otimes_A h^1 \cdots \otimes_A h^n \otimes_A s^r \partial(h^{(1)}_{(2)}) S h^{(2)} S^2(h^{(1)}_{(1)} h^0_{(1)}) \\
&= 1_B \otimes_H h^0_{(2)} \otimes_A \cdots \otimes_A s^r \partial(h) h^n \otimes_A S^2 h^0_{(1)} \\
&\qquad - 1_B \otimes_H h^0_{(2)} \otimes_A h^1 \cdots \otimes_A h^n \otimes_A S h_{(2)} S^2 h_{(1)} S^2 h^0_{(1)} \\
&= 1_B \otimes_H h^0_{(2)} \otimes_A \cdots \otimes_A s^r \partial(h) h^n \otimes_A S^2 h^0_{(1)} \\
&\qquad - 1_B \otimes_H h^0_{(2)} \otimes_A h^1 \cdots \otimes_A t^\ell \epsilon(Sh) h^n \otimes_A S^2 h^0_{(1)} \\
&= 0.
\end{aligned}$$

Also, the cyclic relation works analogously,

$$\begin{aligned}
&\tau(\partial h \otimes_{A^e} h^0 \otimes_A \cdots \otimes_A h^n - 1_B \otimes_{A^e} h_{(1)} h^0 \otimes_A \cdots \otimes_A h_{(n+1)} h^n) \\
&= h^1 \otimes_A \cdots \otimes_A t^\ell \nu^{-1}(\partial h) h^n \otimes_A S^2 h^0 \otimes_A \\
&\qquad - h_{(2)} h^1 \otimes_A h_{(3)} h^2 \cdots \otimes_A h_{(n+1)} h^n \otimes_A S^2(h_{(1)} h^0) \otimes_A \\
&= h^1 \otimes_A \cdots \otimes_A s^r(\partial h) h^n \otimes_A S^2 h^0 \otimes_A \\
&\qquad - h_{(2)} h^1 \otimes_A h_{(3)} h^2 \cdots \otimes_A h_{(n+1)} h^n \otimes_A s^\ell \epsilon(h_{(n+2)}) S^2(h_{(1)} h^0) \otimes_A \\
&= h^1 \otimes_A \cdots \otimes_A s^r(\partial h) h^n \otimes_A S^2 h^0 \otimes_A \\
&\qquad - h^{(1)}_{(2)} h^1 \otimes_A h^{(1)}_{(3)} h^2 \cdots \otimes_A h^{(1)}_{(n+1)} h^n \otimes_A h^{(1)}_{(n+2)} S h^{(2)} S^2(h^{(1)}_{(1)} h^0) \otimes_A \\
&= h^1 \otimes_A \cdots \otimes_A s^r(\partial h) h^n \otimes_A S^2 h^0 \otimes_A \\
&\qquad - (\Delta_\ell^n h^{(1)}_{(2)})(h^1 \otimes_A h^2 \cdots \otimes_A h^n \otimes_A S h^{(2)} S^2(h^{(1)}_{(1)} h^0)) \otimes_A .
\end{aligned}$$

Again, projecting on coinvariants yields

$$\begin{aligned}
&1_B \otimes_H h^1 \otimes_A \cdots \otimes_A s^r(\partial h) h^n \otimes_A S^2 h^0 \\
&\qquad - 1_B \otimes_H (\Delta_\ell^n h^{(1)}_{(2)})(h^1 \otimes_A h^2 \cdots \otimes_A h^n \otimes_A S h^{(2)} S^2(h^{(1)}_{(1)} h^0)) \\
&= 1_B \otimes_H h^1 \otimes_A \cdots \otimes_A s^r(\partial h) h^n \otimes_A S^2 h^0 \\
&\qquad - 1_B \otimes_H h^1 \otimes_A h^2 \cdots \otimes_A h^n \otimes_A s^r \partial(h^{(1)}_{(2)}) S h^{(2)}_{(2)} S^2(h_{(1)} h^0)) \\
&= 1_B \otimes_H h^1 \otimes_A \cdots \otimes_A s^r(\partial h) h^n \otimes_A S^2 h^0 \\
&\qquad - 1_B \otimes_H h^1 \otimes_A \cdots \otimes_A h^n \otimes_A t^r \partial(S^2 h) S^2 h^0) \\
&= 0,
\end{aligned}$$

as above; similarly for the remaining relations. Hence all operators descend to well-defined maps $B \otimes_H C_A^{n+1} H \to B \otimes_H C_A^{n+1} H$. In particular,

$$\tau_n^{n+1}(b \otimes_H h^0 \otimes_A \cdots \otimes_A h^n) = b \otimes_H S^2 h^0 \otimes_A \cdots \otimes_A S^2 h^n, \qquad b \in B,$$

and $\tau_n^{n+1} = \mathrm{id}$ if and only if $S^2 = \mathrm{id}$, analogously to the consideration for $A$-corings. □

The next step is to apply the isomorphism (5.1.3). It follows from Corollary 5.1.3 follows that $\phi : (H \otimes_A C_A^n H)_\partial \xrightarrow{\simeq} C_A^n H$ is an isomorphism in all degrees $n$, and the map $\bar{\phi}_\partial := \phi \pi_\partial$ explicitly reads

$$\bar{\phi}_\partial : B_A^n H \to C_A^n H, \quad h^0 \otimes_A \cdots \otimes_A h^n \otimes_A \mapsto (\Delta_\ell^{n-1} S h^0)(h^1 \otimes_A \cdots \otimes_A h^n). \tag{5.2.1}$$

Correspondingly, define

$$H_{\natural,\partial} := \{C_A^n H\}_{n \geq 0},$$

(ab)using the same notation as before. In degree zero, set $C_A^0 H := A$. Note that $H_{\natural,\partial}$ is a cocyclic module over $k$ only (and not over $A$); we will refer to it as the *Hopf-cocyclic module* associated to a Hopf algebroid

$H$. The cosimplicial and cocyclic operators on $H_{\natural,\partial}$ can be described as follows (the fact that they define a cocyclic module will be proven in Theorem 5.2.5 below). The coface maps are

$$\delta_i(h^1 \otimes_A \cdots \otimes_A h^n) = \begin{cases} 1 \otimes_A h^1 \otimes_A \cdots \otimes_A h^n & \text{if } i = 0, \\ h^1 \otimes_A \cdots \otimes_A \Delta_\ell h^i \otimes_A \cdots \otimes_A h^n & \text{if } 1 \leq i \leq n, \\ h^1 \otimes_A \cdots \otimes_A h^n \otimes_A 1 & \text{if } i = n+1. \end{cases} \tag{5.2.2}$$

In degree zero, set

$$\delta_j a = \begin{cases} t^\ell a & \text{if } j = 0, \\ s^\ell a & \text{if } j = 1. \end{cases} \qquad \forall\, a \in A.$$

The codegeneracies read

$$\sigma_i(h^1 \otimes_A \cdots \otimes_A h^n) = h^1 \otimes_A \cdots \otimes_A \epsilon h^{i+1} \otimes_A \cdots \otimes_A h^n \qquad 0 \leq i \leq n-1. \tag{5.2.3}$$

For the cocyclic operation, we finally set in each degree $n$

$$\tau_n(h^1 \otimes_A \cdots \otimes_A h^n) = (\Delta_\ell^{n-1} S h^1)(h^2 \otimes_A \cdots \otimes_A h^n \otimes_A 1). \tag{5.2.4}$$

The property of $S$ being an isomorphism of twisted bimodules induces a similar property for the cocyclic operator: the map $\tau_n$ is a map of $(A, A)$-bimodules, from ${}_{\blacktriangleright}(C^n_A H)_{\triangleleft}$, that is '$t^\ell$ multiplied from the right on the first and from the left on the last factor' to ${}_{\triangleright}(C^n_A H)_{\blacktriangleleft}$, which means '$s^\ell$ multiplied from the left on the first and from the right on the last factor', as is the case for $\tau_1 = S$.

**5.2.2 Remark** The operators (5.2.2)–(5.2.4) first appeared in [CoMos5] in an explicit example, and were shown to make sense in general in [KhR3], defining a Hopf-cocyclic module for any Hopf algebroid. Our Theorem 5.2.5 below states that they can be obtained naturally from the standard $A$-coring cocyclic operators (1.2.4) associated to $H$ (by covariant localisation and the isomorphism (5.1.3)).

Since for the definition of the underlying cosimplicial module only the underlying left bialgebroid structure of $H$ is needed, one easily verifies:

**5.2.3 Proposition** *For an arbitrary left bialgebroid $U$, the space $C^\bullet_A U$ is a cosimplicial module by means of the above structure maps.*

**5.2.4 Remark** An analogous result holds for right bialgebroids.

Adding the cocyclic operator (5.2.4) to Proposition 5.2.3 gives the following theorem, generalising a similar result in [Cr3] from Hopf algebras to Hopf algebroids:

**5.2.5 Theorem** *For a Hopf algebroid $H$, the formulae (5.2.2)–(5.2.4) equip $H_{\natural,\partial}$ with the structure of a cocyclic module if and only if $S^2 = \mathrm{id}$. In particular,*

$$\tau_n^{n+1}(h^1 \otimes_A \cdots \otimes_A h^n) = S^2 h^1 \otimes_A \cdots \otimes_A S^2 h^n. \tag{5.2.5}$$

PROOF: We proceed in much the same way as in the proof of Proposition 5.2.1. Consider (1.2.4) for $C := H$ and $\psi := S^2$. Since $\bar\phi_\partial : H^A_{\natural,S^2} \to H_{\natural,\partial}$ is surjective with right inverse

$$h^1 \otimes_A \cdots \otimes_A h^n \mapsto 1_H \otimes_A h^1 \otimes_A \cdots \otimes_A h^n \otimes_A,$$

it suffices to show that $\bar\phi_\partial$ commutes with the structure maps, i.e.,

$$\begin{aligned} \delta_i \bar\phi_\partial &= \bar\phi_\partial \delta_i && \text{for} \quad 0 \leq i \leq n+1, \\ \sigma_i \bar\phi_\partial &= \bar\phi_\partial \sigma_i && \text{for} \quad 0 \leq i \leq n-1, \\ \tau \bar\phi_\partial &= \bar\phi_\partial \tau, \end{aligned} \tag{5.2.6}$$

where the left hand side refers to the maps (5.2.2)–(5.2.4) above, whereas the right hand side refers to (1.2.4) for $\psi = S^2$.

We will use the identities

$$\begin{aligned} \Delta_\ell^{n-1} S h &= S h^{(n)} \otimes_A \cdots \otimes_A S h^{(1)}, \\ \Delta_\ell^{n-1} S^2 h &= S^2 h_{(1)} \otimes_A \cdots \otimes_A S^2 h_{(n)}, \end{aligned} \tag{5.2.7}$$

which easily follow by induction from higher twisted coassociativity (2.6.8), as well as Proposition 2.6.4. Consider now left and right hand sides of the third equation in (5.2.6) for $\psi = S^2$. We have

$$\bar{\phi}_\partial \tau_{n+1}(h_0 \otimes_A h_1 \otimes_A \cdots \otimes_A h_n \otimes_A) = (\Delta_\ell^{n-1} S h_1)(h_2 \otimes_A \cdots \otimes_A h_n \otimes_A S^2 h_0).$$

On the other hand,

$$\begin{aligned}\tau_n \bar{\phi}_\partial(h_0 \otimes_A h_1 \otimes_A \cdots \otimes_A h_n \otimes_A) &= \tau_n((\Delta_\ell^{n-1} S h_0)(h_1 \otimes_A \cdots \otimes_A h_n)) \\ &= (\Delta_\ell^{n-1} S(S h_0^{(n)} h_1))(S h_0^{(n-1)} h_2 \otimes_A \cdots \otimes_A S h_0^{(1)} h_n \otimes_A 1)\end{aligned}$$

by (5.2.7). Since $S$ is an anti-algebra morphism on $H$, this reads

$$\begin{aligned}&\tau_n \bar{\phi}_\partial(h_0 \otimes_A h_1 \otimes_A \cdots \otimes_A h_n \otimes_A) = \\ &\quad = (\Delta_\ell^{n-1} S h_1 \Delta_\ell^{n-1} S^2 h_0^{(n)})(S h_0^{(n-1)} h_2 \otimes_A \cdots \otimes_A S h_0^{(1)} h_n \otimes_A 1).\end{aligned}$$

Hence, for $\bar{\phi}_\partial \tau = \tau \bar{\phi}_\partial$ to hold it suffices to show that

$$\begin{aligned}&h_1 \otimes_A \cdots \otimes_A h_{n-1} \otimes_A S^2 h_0 = \\ &\quad = (\Delta_\ell^{n-1}(S^2 h_0^{(n)}))(S h_0^{(n-1)} h_1 \otimes_A \cdots \otimes_A S h_0^{(1)} h_{n-1} \otimes_A 1),\end{aligned} \tag{5.2.8}$$

as an element in $H_\triangleleft \otimes_A {}_\triangleright H_\triangleleft \otimes_A \cdots \otimes_A {}_\triangleright H$. The case $n = 2$ is shown in the subsequent lemma, and higher degrees will follow by induction.

**5.2.6 Lemma** *For each $h \in H$, we have*

(*i*) $(S h_{(1)})_{(1)} h_{(2)} \otimes_A (S h_{(1)})_{(2)} = S h_{(1)}^{(2)} h_{(2)} \otimes_A S h_{(1)}^{(1)} = 1_H \otimes_A S h,$

(*ii*) $S^2 h_{(1)}^{(2)} S h^{(1)} \otimes_A S^2 h_{(2)}^{(2)} = 1_H \otimes_A S^2 h,$

*as elements in $H_\triangleleft \otimes_A {}_\triangleright H$.*

PROOF: With the right comonoid identities (2.5.6) as well as (2.6.10) and (2.6.4) we have

$$\begin{aligned}1 \otimes_A S h &= 1 \otimes_A S(h^{(1)} s^r(\partial h^{(2)})) \\ &= t^\ell \epsilon s^r \partial h^{(2)} \otimes_A S h^{(1)} \\ &= s^r \partial h^{(2)} \otimes_A S h^{(1)} \\ &= S h_{(1)}^{(2)} h_{(2)}^{(2)} \otimes_A S h^{(1)} = S h_{(1)}^{(2)} h_{(2)} \otimes_A S h_{(1)}^{(1)},\end{aligned}$$

which proves the first part; the second part can be shown by simply applying the first equation to an element of the form $h' = S h$. ☐

By the same calculation one proves $h_2 \otimes_A S^2 h = S^2 h_{(1)}^{(2)} S h^{(1)} h \otimes_A S^2 h_{(2)}^{(2)}$, i.e. (5.2.8) for $n = 2$. Henceforth, assume (5.2.8) to be already true for $n-1$ and show that it holds for $n$ as well. Again, with the help of (5.2.7), (2.6.11), (2.6.10) and (2.6.4), we obtain

$$\begin{aligned}&h_1 \otimes_A \cdots \otimes_A h_n \otimes_A S^2 h_0 = \\ &= h_1 \otimes_A (\Delta_\ell^{n-1} S^2 h_0^{(n)})(S h_0^{(n-1)} h_2 \otimes_A \cdots \otimes_A S h_0^{(1)} h_n \otimes_A 1) \\ &= h_1 \otimes_A S^2 h_{0(1)}^{(n)} S h_0^{(n-1)} h_2 \otimes_A \cdots \otimes_A S^2 h_{0(n-1)}^{(n)} S h_0^{(1)} h_n \otimes_A S^2 h_{0(n)}^{(n)} \\ &= h_1 \otimes_A s^\ell \epsilon(S^2 h_{0(1)}^{(n)}) S^2 h_{0(2)}^{(n)} S h_0^{(n-1)} h_2 \otimes_A \cdots \otimes_A S^2 h_{0(n)}^{(n)} S h_0^{(1)} h_n \otimes_A S^2 h_{0(n+1)}^{(n)} \\ &= s^r \partial(S h_{0(1)}^{(n)}) h_1 \otimes_A S^2 h_{0(2)}^{(n)} S h_0^{(n-1)} h_2 \otimes_A \cdots \otimes_A S^2 h_{0(n)}^{(n)} S h_0^{(1)} h_n \otimes_A S^2 h_{0(n+1)}^{(n)} \\ &= S^2 h_{0(1)}^{(n+1)} S h_0^{(n)} h_1 \otimes_A S^2 h_{0(2)}^{(n+1)} S h_0^{(n-1)} h_2 \otimes_A \cdots \\ &\qquad \cdots \otimes_A S^2 h_{0(n)}^{(n+1)} S h_0^{(1)} h_n \otimes_A S^2 h_{0(n+1)}^{(n+1)} \\ &= (\Delta_\ell^n S^2 h_0^{(n+1)})(S h_0^{(n)} h_1 \otimes_A \cdots \otimes_A S h_0^{(1)} h_n \otimes_A 1),\end{aligned}$$

i.e. the desired claim, where higher coassociativity was used again.

The remaining identities in (5.2.6) easily follow by deploying twisted coassociativity, Lemma 2.6.6 and Proposition 2.6.4. The proof of the identity for $\delta_{n+1}$ repeats a similar (and similarly tedious) induction as is done above. The opposite direction is obtained by considering (5.2.5) for the case $n = 1$. □

Consequently, in case $S^2 = \mathrm{id}$ the map $\bar{\phi}_\partial$ is a morphism of cocyclic modules and we are in a position to depict the situation by a commutative diagram (in each degree):

$$\begin{tikzcd} H_\natural \arrow[r, "\bar{\phi}_\partial"] \arrow[d, "(\beta,\sigma,\tau)"'] & H_{\natural,\partial} \arrow[d, "(\beta,\sigma,\tau)"] \\ H_\natural \arrow[r, "\bar{\phi}_\partial"'] & H_{\natural,\partial}. \end{tikzcd} \tag{5.2.9}$$

**5.2.7 Definition** In case $S^2 = \mathrm{id}$, we denote the associated Tsygan's cyclic bicomplex of the cocyclic module $H_{\natural,\partial}$ by $CC^{\bullet,\bullet}_\partial(H)$, and define $HH^\bullet_\partial(H)$, $HC^\bullet_\partial(H)$ and $HP^\bullet_\partial(H)$ to be its Hochschild and cyclic cohomology groups, respectively. We will refer to these as *Hopf-Hochschild* and *Hopf-cyclic* cohomology.

**5.2.8 Hopf-Cyclic Cohomology Twisted by a Grouplike Element** If $\sigma \in GH = G^\ell H \cap G^r H$ is a grouplike element for the Hopf algebroid $H$ (cf. §2.6.11), one may twist $H_{\natural,\partial}$ by a grouplike element, similarly as for Hopf algebras [CoMos4]. The motivation for such an extension (at least in the Hopf algebra case) came from examples of quantum groups or compact matrix pseudogroups [Wo], where the antipode is not involutive any more but rather fulfills $S^2h = \sigma h \sigma^{-1}$. In such a case, the cosimplicial and cocyclic operators read

$$\begin{aligned} \delta_i(h^1 \otimes_A \cdots \otimes_A h^n) &= \begin{cases} 1 \otimes_A h^1 \otimes_A \cdots \otimes_A h^n & \text{if } i = 0, \\ h^1 \otimes_A \cdots \otimes_A \Delta_\ell h^i \otimes_A \cdots \otimes_A h^n & \text{if } 1 \leq i \leq n, \\ h^1 \otimes_A \cdots \otimes_A h^n \otimes_A \sigma & \text{if } i = n+1. \end{cases} \\ \delta_j a &= \begin{cases} t^\ell a & \text{if } j = 0, \\ s^\ell a \sigma & \text{if } j = 1, \end{cases} \\ \sigma_i(h^1 \otimes_A \cdots \otimes_A h^n) &= h^1 \otimes_A \cdots \otimes_A \epsilon h^{i+1} \otimes_A \cdots \otimes_A h^n \quad 0 \leq i \leq n-1, \\ \tau_n(h^1 \otimes_A \cdots \otimes_A h^n) &= (\Delta_\ell^{n-1} S(\sigma h^1))(h^2 \otimes_A \cdots \otimes_A h^n \otimes_A 1), \end{aligned} \tag{5.2.10}$$

where $a \in A$, $h^i \in H$, $i = 1, \ldots, n$, which can be shown to determine a cocyclic module if $S^2h = \sigma h \sigma^{-1}$ (where $\sigma^{-1} = S\sigma$). We denote the corresponding cohomology groups by $HH^\bullet_{\partial,\sigma}(H)$, $HC^\bullet_{\partial,\sigma}(H)$ and $HP^\bullet_{\partial,\sigma}(H)$. Although we use the same symbol, one should not confuse the grouplike element with the codegeneracies.

## 5.2.1 Connes' Associated Bicomplex

Let us have a brief look at the associated Connes' bicomplex $(BC^{\bullet,\bullet}_\partial(H), \beta, B)$ for the cocyclic module $H_{\natural,\partial}$. It is defined as follows:

$$BC^{p,q}_\partial(H) = \begin{cases} C^{p-q}_A H & \text{if } q \geq p, \\ 0 & \text{if } q < p. \end{cases} \tag{5.2.11}$$

The Hochschild coboundary $\beta : C^n_A H \to C^{n+1}_A H$ is given, as generally in (1.1.11), by

$$\beta = \sum_{i=0}^{n+1} (-1)^i \delta_i, \tag{5.2.12}$$

using the operators (5.2.2) (or, if need be, using the twisted ones in (5.2.10)). In case $n = 0$ one has

$$\beta a = t^\ell a - s^\ell a, \quad a \in A.$$

The operator (not to be confused with the algebra $B$)

$$B : C^{n+1}_A H \to C^n_A H, \qquad B := N\sigma_{-1}(1 - \lambda_{n+1})$$

may be calculated with the explicit formula for the extra codegeneracy $\sigma_{-1} : C_A^{n+1}H \to C_A^n H$, $\sigma_{-1} := \sigma_n \tau_{n+1}$, which here reads

$$\begin{aligned}\sigma_{-1}(h \otimes_A h^1 \otimes_A \cdots \otimes_A h^n) &= \sigma_n((\Delta_\ell^n Sh)(h^1 \otimes_A \cdots \otimes_A h^n \otimes_A 1_H)) \\ &= (Sh)_{(1)}h^1 \otimes_A \cdots \otimes_A (Sh)_{(n)}h^n \otimes_A \epsilon((Sh)_{(n+1)}) \\ &\equiv (Sh)_{(1)}h^1 \otimes_A \cdots \otimes_A t^\ell \epsilon((Sh)_{(n+1)})(Sh)_{(n)}h^n \\ &= (\Delta_\ell^{n-1} Sh)(h^1 \otimes_A \cdots \otimes_A h^n),\end{aligned}$$

by (2.1.8) and (5.2.7). This is formally the same operation as the coinvariant localisation which, however, maps from a different space. Also, for $n = 0$ one finds $\sigma_{-1}h = \epsilon Sh = \nu^{-1}\partial$. In particular,

$$Bh = \nu^{-1}\partial h + \epsilon h \qquad h \in H.$$

As in the case of Hopf algebras [CoMos4], this expression can be simplified by passing to the quasi-isomorphic *normalised bicomplex* $(\bar{B}C_\partial^\bullet(H), \beta, \bar{B})$, which is defined in a similar manner:

$$\bar{B}C_\partial^{p,q}(H) \quad = \begin{cases} \bar{C}_A^{p-q}H & \text{if } q \geq p, \\ 0 & \text{if } q < p, \end{cases} \tag{5.2.13}$$

where

$$\bar{C}_A^n H \quad = \begin{cases} (\ker \epsilon)^{\otimes_A n} & \text{if } n \geq 1, \\ \ker(s^\ell - t^\ell) & \text{if } n = 0. \end{cases} \tag{5.2.14}$$

While the form of the Hochschild coboundary $\beta$ remains unchanged, the new horizontal operator becomes

$$\bar{B} = N\sigma_{-1} \quad \text{for } n \geq 0.$$

Specifically, for $n = 0$,

$$\bar{B}h = \nu^{-1}\partial h \qquad h \in H.$$

**5.2.9 Left Haar Measures on Left Bialgebroids** An important motivation to introduce the dual Hopf cyclic homology in Chapter Six is the subsequent Proposition 5.2.11 from [KhR3], which tells us that in some cases Hopf-cyclic cohomology is not sufficiently interesting. This hinges on the existence of a *Haar system*, a notion from [KhR3] which we can apply without major reformulations since it only relies on the left bialgebroid structure, and this is essentially the same [KhR3, Lem. 2.1] as used here.

**5.2.10 Definition** Let $U$ be a left bialgebroid with structure maps as before and let $T \in \mathrm{Hom}_{(-,A)}(U_\triangleleft, A_A)$, that is a map $T : U \to A$ with $T(t^\ell a u) = T(u)a$. The map $T$ is called a *left Haar system* for the left bialgebroid $U$ if

$$m_U(s^\ell T \otimes \mathrm{id})\Delta_\ell = t^\ell T.$$

It is called *normal* if $T(1_U) = 1_A$.

This is still sufficient but is slightly weaker than the version in [KhR3], which requires $m_U(s^\ell T \otimes \mathrm{id})\Delta_\ell = s^\ell T$ *and* $s^\ell T = t^\ell T$.

**5.2.11 Proposition** [KhR3] *Let $H$ be a Hopf algebroid that admits a normal left Haar system on its underlying left bialgebroid structure. Then we have*

$$HP_\partial^{\mathrm{even}}(H) = \ker(s^\ell - t^\ell) \quad \textit{and} \quad HP_\partial^{\mathrm{odd}}(H) = 0.$$

PROOF: Introduce the map $s : C_A^\bullet H \to C_A^{\bullet-1}H$ given by

$$s_n(h^1 \otimes_A \cdots \otimes_A h^n) = s^\ell(Th^1)h^2 \otimes_A \cdots \otimes_A h^n.$$

If one introduces the (co-augmented) complex $\ker(s^\ell - t^\ell) \xrightarrow{\iota} C_A^\bullet H$, where $\iota : \ker(s^\ell - t^\ell) \to A$ is the canonical embedding and $s_{-1}$ is defined as restriction, one sees that $s\beta + \beta s = \mathrm{id}$. Hence one obtains a contracting homotopy for the Hochschild complex of $H_{\natural,\partial}$ and consequently $HH^n(H) = 0$ for $n > 0$ and $HH^0(H) = \ker(s^\ell - t^\ell)$. Finally, apply an $SBI$-sequence kind of argument and the periodicity of $HP$. $\square$

See [Cr3] for this result in the context of Hopf algebras.

**5.2.12 Comparison to Earlier Approaches** In the definition of *para*-Hopf algebroids from [KhR3] as presented in §2.6.13(*ii*), the condition $T^2 = \mathrm{id}_H$ was already implemented in the definition of a *para*-antipode $T$, implying the missing left bialgebroid axiom (2.1.4) (see [KhR3, Lem. 2.1]). From the perspective of obtaining cocyclic structures we are mainly interested in the case $S^2 = \mathrm{id}_H$, too, see Theorem 5.2.5. Furthermore, the antipode axioms in Definition 2.6.1 from [BSz2] imply the ones (with the exception of the condition $T^2 = \mathrm{id}$) in §2.6.13(*ii*)(a)–(e), as is seen from Lemma 5.2.6 and Proposition 2.6.4(*i*). For the opposite direction cf. [BSz2, Prop. 4.2].

In particular, the approach via left and right bialgebroids used here avoids the somewhat technical condition (2.6.14), which in our context appears only in the auxiliary Lemma 5.2.6. Moreover, we gained the possibility to define coinvariants and to see the Hopf-cyclic structure 'descending' from a standard coalgebra (coring) cocyclic module (cf. §1.2.5) by means of the projection (5.2.1). This is particularly helpful in the example of Lie-Rinehart algebras when hunting for an antipode (cf. Subsection 4.2.2) and its corresponding cyclic structure (which cannot be so easily guessed from (2.6.14)).

## 5.3 Hopf-Hochschild Cohomology as a Derived Functor

In the next theorem we are going to show that Hopf-Hochschild cohomology given by the complex $(C^\bullet_A H, \beta_\sigma)$ for a grouplike element $\sigma$ from (5.2.10) can be seen as a derived functor of the cotensor product functor.

**5.3.1 Coefficients** Observe that the cosimplicial module given by (5.2.2) and (5.2.3) tacitly determines cohomology with values in the base algebra $A$. More generally, let $M \in \mathbf{Comod}\text{-}H$ with coaction ${}_M\Delta m =: m_{(0)} \otimes_A m_{(1)}$ and define $C^\bullet_M H := \{M \otimes_A H^{\otimes_A n}\}_{n \geq 0}$. For a grouplike element $\sigma \in GH$, the operators

$$\begin{aligned} \delta_i(m \otimes_A h^1 \otimes_A \cdots \otimes_A h^n) &= \begin{cases} {}_M\Delta m \otimes_A h^1 \otimes_A \cdots \otimes_A h^n & \text{if } i = 0, \\ m \otimes_A h^1 \otimes_A \cdots \otimes_A \Delta_\ell h^i \otimes_A \cdots \otimes_A h^n & \text{if } 1 \leq i \leq n, \\ m \otimes_A h^1 \otimes_A \cdots \otimes_A h^n \otimes_A \sigma & \text{if } i = n+1, \end{cases} \\ \delta_j(m) &= \begin{cases} {}_M\Delta m & \text{if } j = 0, \\ m \otimes_A \sigma & \text{if } j = 1, \end{cases} \\ \sigma_i(m \otimes_A h^1 \otimes_A \cdots \otimes_A h^n) &= m \otimes_A h^1 \otimes_A \cdots \otimes_A \epsilon h^{i+1} \otimes_A \cdots \otimes_A h^n \quad 0 \leq i \leq n-1 \end{aligned} \tag{5.3.1}$$

give $C^\bullet_M H$ the structure of a cosimplicial module (we do not address the question here how to extend this to a cocyclic module with coefficients). Denote by $H^\bullet_\sigma(H, M)$ the corresponding (Hopf-)Hochschild cohomology computed by $(C^\bullet_M H, \beta_\sigma)$. In particular, with the notation from Definition 5.2.7, one recovers $H^\bullet_\sigma(H, A) = HH^\bullet_{\partial,\sigma}(H)$. Observe at this point that the fact that one can define Hopf-Hochschild homology with 'trivial' coefficients (i.e. with values in $A$) may be interpreted as a consequence of the existence of both left and right $H$-coactions on $A$ as in (2.3.7) (whereas *a priori* there is only one left $U$-action on $A$). This, in turn, we saw to be strongly connected (under certain projectivity assumptions) to the appearance of two duals of $U$, cf. Proposition 3.1.9.

**5.3.2 The Cobar Complex** The cohomology groups of the complex associated to (5.3.1) are calculated by finding a suitable resolution provided by a generalisation of the classical *cobar* complex [Ad, Do]. It is in some sense the complex arising from the so-called *(co)path space* $PN^\bullet := \{N^{n+1}\}_{n \geq 0}$ associated to any cosimplicial object $N^\bullet$: one has $PN^n := N^{n+1}$ in degree $n$ and the cosimplicial operators are shifted correspondingly. More precisely, put $M := H$ in (5.3.1) with right $H$-coaction simply given by the left coproduct $\Delta_\ell$. Then the (co)path space

$$\mathrm{Cob}^\bullet_\sigma(H) := \{H \otimes_A H^{\otimes_A n}\}_{n \geq 0}$$

associated to the cosimplicial space $C^\bullet_H H$ has cosimplicial pieces given by

$$\begin{aligned} \tilde{\delta}_i(h^0 \otimes_A \cdots \otimes_A h^n) &= \begin{cases} h^0 \otimes_A \cdots \otimes_A \Delta_\ell h^i \otimes_A \cdots \otimes_A h^n & \text{if } 0 \leq i \leq n, \\ h^0 \otimes_A \cdots \otimes_A h^n \otimes_A \sigma & \text{if } i = n+1, \end{cases} \\ \tilde{\sigma}_i(h^0 \otimes_A \cdots \otimes_A h^n) &= h^0 \otimes_A \cdots \otimes_A \epsilon h^{i+1} \otimes_A \cdots \otimes_A h^n \qquad 0 \leq i \leq n-1. \end{aligned}$$

A coboundary $\beta'_\sigma : \mathrm{Cob}^n_\sigma(H) \to \mathrm{Cob}^{n+1}_\sigma(H)$ is defined by $\beta'_\sigma = \sum_{i=0}^{n+1} (-1)^i \tilde{\delta}_i$, and it is easy to see that that $\beta'_\sigma \beta'_\sigma = 0$. We refer to $(\mathrm{Cob}^\bullet_\sigma(H), \beta'_\sigma)$ as the *cobar complex* of $H$. One can then describe Hopf-Hochschild cohomology as follows:

**5.3.3 Theorem** *Let $H$ be a left bialgebroid and $M$ a right $H$-comodule. Then there is an isomorphism*

$$H^\bullet_\sigma(H,M) \simeq \mathrm{Cotor}^\bullet_H(M, A_\sigma).$$

*In particular, one has*

$$HH^\bullet_{\partial,\sigma}(H) \simeq \mathrm{Cotor}^\bullet_H(A, A_\sigma),$$

*where $A_\sigma$ is $A$ seen as left $H$-comodule induced by the grouplike element $\sigma$ and $A$ is seen as a right $H$-comodule with respect to the grouplike element $1_H$.*

PROOF: The proof follows standard homological algebra arguments, slightly adapted to the case at hand. Recall firstly from (2.3.7) that for any grouplike element $\sigma$ the maps $\Delta^A_\sigma a := s^\ell(a)\sigma$ and ${}^A\Delta_\sigma := t^\ell(a)\sigma$ induce left and right $H$-comodule structures on $A$, respectively. The left $H$-coaction (in each degree) on $\mathrm{Cob}^\bullet_\sigma(H)$ is simply $\Delta_\ell \otimes \mathrm{id}_H^{\otimes \bullet}$. Observe that $\beta'_\sigma$ is a morphism of left $H$-comodules. Moreover, the maps

$$s^{n-1} : \mathrm{Cob}^n_\sigma(H) \to \mathrm{Cob}^{n-1}_\sigma(H), \quad h^0 \otimes_A \cdots \otimes_A h^n \mapsto s^\ell(\epsilon h^0)h^1 \otimes_A \cdots \otimes_A h^n$$

fulfill $s\,\beta'_\sigma + \beta'_\sigma\, s = \mathrm{id}$, hence $s$ together with the maps $s^{-1} := \epsilon$ and the source map $s^\ell$ is a contracting homotopy for the complex $(\mathrm{Cob}^\bullet_\sigma(H), \beta'_\sigma)$ over $A_\sigma$. Correspondingly, $A_\sigma \xrightarrow{\Delta^A_\sigma} \mathrm{Cob}^\bullet_\sigma(H)$ is a resolution of $A_\sigma$ by (free hence injective) left $H$-comodules: exactness in degree $> 0$ was shown a moment ago and as for degree 0, observe that the space $\ker \beta'_\sigma = \{h \in H \mid \Delta_\ell h = h \otimes_A \sigma\}$ is precisely given by elements of the form $s^\ell(a)\sigma$ for all $a \in A$ and fixed $\sigma$, and hence is isomorphic to $A$.

Now let $M$ be a right $H$-comodule with coaction ${}_M\Delta : M \to M \otimes_A H$ and recall that the groups $\mathrm{Cotor}^\bullet_H(M, A_\sigma)$ are computed by $M \Box_H \mathrm{Cob}^\bullet_\sigma(H)$. To finish the proof it suffices to show that the isomorphism

$$\phi : M\Box_H \mathrm{Cob}^\bullet_\sigma(H) \xrightarrow{\simeq} C^\bullet_M H, \quad m \otimes_A h^0 \otimes_A \cdots \otimes_A h^n \mapsto m \otimes_A s^\ell(\epsilon h^0)h^1 \otimes_A \cdots \otimes_A h^n$$

is a morphism of complexes on the complex $(C^\bullet_M H, \beta_\sigma)$ that computes Hochschild cohomology, that is

$$\phi(\mathrm{id}_M \otimes \beta'_\sigma) = \beta_\sigma \phi. \tag{5.3.2}$$

Furthermore, for the left $H$-comodule $H \otimes_A N$ with coaction $\Delta_\ell \otimes \mathrm{id}_N$, where $N$ is any $A$-bimodule, one has

$$\begin{aligned} M\Box_H H \otimes_A N = \{ m \otimes_A h \otimes_A n \in M \otimes_A H \otimes_A N \mid \\ \mid m \otimes_A h_{(1)} \otimes_A h_{(2)} \otimes_A n = m_{(0)} \otimes_A m_{(1)} \otimes_A h \otimes_A n\}. \end{aligned}$$

Applying the operator $(\mathrm{id}_M \otimes m_{H^{\mathrm{op}}} \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes \mathrm{id}_H \otimes t^\ell\epsilon \otimes \mathrm{id}_N)$ yields the relation

$$m \otimes_A h \otimes_A n = m_{(0)} \otimes_A t^\ell\epsilon(h)m_{(1)} \otimes_A n = {}_M\Delta(m\epsilon(h)) \otimes_A n,$$

for all elements $m \otimes_A h \otimes_A n \in M\Box_H H \otimes_A N$. To prove (5.3.2) we now only consider equality of the respective first summands, the rest being evident. One has

$$\begin{aligned} \phi((\mathrm{id}_M \otimes \beta'_\sigma)(m \otimes_A h^0 \otimes_A h^1 \otimes_A \cdots \otimes_A h^n) &= m \otimes_A h^0 \otimes_A h^1 \otimes_A \cdots \otimes_A h^n + \ldots \\ &= {}_M\Delta(m\epsilon(h^0)) \otimes_A h^1 \otimes_A \cdots \otimes_A h^n + \ldots \\ &= {}_M\Delta m \otimes_A s^\ell(\epsilon h^0)h^1 \otimes_A \cdots \otimes_A h^n + \ldots \\ &= \beta_\sigma\phi(m \otimes_A h^0 \otimes_A h^1 \otimes_A \cdots \otimes_A h^n). \end{aligned}$$

□

## 5.4 Hopf-Cyclic Cohomology of Commutative Hopf Algebroids

In case of a *commutative* Hopf algebroid one can make more specific statements about the Hopf-cyclic cohomology groups: they are essentially determined by the Hopf-Hochschild groups, see Theorem 5.4.4. We thereby generalise an idea for (commutative) Hopf algebras in [KhR1, Thm. 4.2] to Hopf algebroids.

**5.4.1 Commutative Hopf Algebroids** A commutative Hopf algebroid $H$ necessarily has a commutative base algebra $A$. Furthermore, the left counit of the underlying left bialgebroid fulfills the required properties of a right counit and likewise the left coproduct can be used as right coproduct. Hence adopting the alternative perspective in §2.6.8, that is, constructing the right bialgebroid out of a left bialgebroid and an (invertible) anti-algebra isomorphism $S$ satisfying certain properties (see §2.6.8), a commutative Hopf algebroid may always be described (up to automorphism) by a left bialgebroid structure $(H, A, s^\ell, t^\ell, \Delta, \epsilon)$ and a right bialgebroid structure $(H, A, t^\ell, s^\ell, \Delta, \epsilon)$ plus an antipode fulfilling

$$St^\ell = s^\ell, \quad Ss^\ell = t^\ell, \quad m_H(S \otimes \mathrm{id}_H)\Delta_\ell = t^\ell \epsilon, \quad m_H(\mathrm{id}_H \otimes S)\Delta_\ell = s^\ell \epsilon. \tag{5.4.1}$$

One recovers this way the definition of commutative Hopf algebroids in [Ra]. In the rest of this section, we assume $H$ to be a commutative Hopf algebroids with this description.

For the sake of simplicity, we consider only the case where the grouplike element is $\sigma = 1$. Correspondingly, for the cobar complex denote $\mathrm{Cob}^\bullet(H) := \mathrm{Cob}^\bullet_1(H)$.

**5.4.2 Proposition** *Let $H$ be a commutative Hopf algebroid over commutative base algebra $A$. Then $\mathrm{Cob}^\bullet(H)$ is a para-cocyclic $H$-comodule with cocyclic operator*

$$\begin{aligned} \tilde{\tau}_n : \mathrm{Cob}^n(H) &\to \mathrm{Cob}^n(H), \\ h^0 \otimes_A \cdots \otimes_A h^n &\mapsto h^0_{(1)} \otimes_A h^0_{(2)} S h^1_{(n)} h^2 \otimes_A \cdots \otimes_A h^0_{(n)} S h^1_{(2)} h^n \otimes_A h^0_{(n+1)} S h^1_{(1)}, \end{aligned}$$

*which is cocyclic if and only if $S^2 = \mathrm{id}$. In particular,*

$$\tilde{\tau}_n^{n+1}(h^0 \otimes_A h^1 \otimes_A \cdots \otimes_A h^n) = h^0 \otimes_A S^2 h^1 \otimes_A \cdots \otimes_A S^2 h^n.$$

PROOF: We only prove the cocyclic identity $\tilde{\tau}_n^{n+1} = \mathrm{id}$ and leave the remaining identities for a cocyclic module to the reader. Using the commutativity of $H$, the identities (5.4.1) as well as (2.6.11) (for the structure maps of the commutative Hopf algebroid specified above), one obtains

$$\begin{aligned} \tilde{\tau}_n^2(h^0 \otimes_A \cdots \otimes_A h^n) &= h^0_{(1)} \otimes_A h^0_{(2)} S h^2_{(n)} h^3 S h^0_{(2n+1)} h^0_{(2n+2)} S^2 h^1_{(n)} S h^1_{(n-1)} \\ &\quad \otimes_A h^0_{(3)} S h^2_{(n-1)} h^4 S h^0_{(2n)} h^0_{(2n+3)} S^2 h^1_{(n+1)} S h^1_{(n-2)} \otimes_A \cdots \\ &\quad \cdots \otimes_A h^0_{(n-1)} S h^2_{(3)} h^n S h^0_{(n+4)} h^0_{(3n-1)} S^2 h^1_{(2n-3)} S h^1_{(2)} \\ &\quad \otimes_A h^0_{(n)} S h^2_{(2)} S h^0_{(n+3)} h^0_{(3n)} S^2 h^1_{(2n-2)} S h^1_{(1)} \otimes_A h^0_{(n+1)} S h^0_{(n+2)} S h^2_{(1)} S^2 h^1_{(2n-1)} \\ &= h^0_{(1)} \otimes_A h^0_{(2)} S h^2_{(n)} h^3 S h^0_{(2n)} h^0_{(2n+1)} t^\ell \epsilon h^1_{(n-1)} \\ &\quad \otimes_A h^0_{(3)} S h^2_{(n-1)} h^4 S h^0_{(2n-1)} h^0_{(2n+2)} S^2 h^1_{(n)} S h^1_{(n-2)} \otimes_A \cdots \\ &\quad \cdots \otimes_A h^0_{(n-1)} S h^2_{(3)} h^n S h^0_{(n+3)} h^0_{(3n-2)} S^2 h^1_{(2n-4)} S h^1_{(2)} \\ &\quad \otimes_A h^0_{(n)} S h^2_{(2)} S h^0_{(n+2)} h^0_{(3n-1)} S^2 h^1_{(2n-3)} S h^1_{(1)} \otimes_A s^\ell \epsilon h^0_{(n+1)} S h^2_{(1)} S^2 h^1_{(2n-2)} \\ &= h^0_{(1)} \otimes_A h^0_{(2)} S h^2_{(n)} h^3 S h^0_{(2n-1)} h^0_{(2n)} \\ &\quad \otimes_A h^0_{(3)} S h^2_{(n-1)} h^4 S h^0_{(2n-2)} h^0_{(2n+1)} S^2 h^1_{(n-1)} S h^1_{(n-2)} \otimes_A \cdots \\ &\quad \cdots \otimes_A h^0_{(n-1)} S h^2_{(3)} h^n S h^0_{(n+2)} h^0_{(3n-3)} S^2 h^1_{(2n-5)} S h^1_{(2)} \\ &\quad \otimes_A h^0_{(n)} S h^2_{(2)} S h^0_{(n+1)} h^0_{(3n-2)} S^2 h^1_{(2n-4)} S h^1_{(1)} \otimes_A S h^2_{(1)} S^2 h^1_{(2n-3)} \\ &= h^0_{(1)} \otimes_A h^0_{(2)} S h^2_{(n)} h^3 \otimes_A h^0_{(3)} S h^2_{(n-1)} h^4 S h^0_{(2n-2)} h^0_{(2n-1)} t^\ell \epsilon h^1_{(n-2)} \otimes_A \cdots \\ &\quad \cdots \otimes_A h^0_{(n-1)} S h^2_{(3)} h^n S h^0_{(n+2)} h^0_{(3n-5)} S^2 h^1_{(2n-6)} S h^1_{(2)} \\ &\quad \otimes_A h^0_{(n)} S h^2_{(2)} S h^0_{(n+1)} h^0_{(3n-4)} S^2 h^1_{(2n-5)} S h^1_{(1)} \otimes_A S h^2_{(1)} S^2 h^1_{(2n-4)} \\ &\qquad \vdots \\ &= h^0_{(1)} \otimes_A h^0_{(2)} S h^2_{(n)} h^3 \otimes_A \cdots \otimes_A h^0_{(n-1)} S h^2_{(3)} h^n \otimes_A h^0_{(n)} S h^2_{(2)} \otimes_A S^2 h^1 S^2 h^2_{(1)}, \end{aligned}$$

where the vertical dots mean another $n-3$ repetitions of the same three steps as before. Repeating the same procedure another $n-1$ times, one finds

$$\tilde{\tau}_n^n(h^0 \otimes_A \cdots \otimes_A h^n) = h^0_{(1)} \otimes_A h^0_{(2)} S h^n_{(n)} \otimes_A S^2 h_1 S h^n_{(n-1)} \otimes_A \cdots \otimes_A S^2 h_{n-1} S h^n_{(1)}.$$

Hence finally

$$
\begin{aligned}
&\tilde{\tau}_n^{n+1}(h^0 \otimes_A \cdots \otimes_A h^n) \\
&\quad = h^0_{(1)} \otimes_A h^0_{(2)} S h^0_{(2n+1)} S^2 h^1 S^2 h^n_{(n)} S h^n_{(n-1)} \\
&\qquad \otimes_A h^0_{(3)} S h^0_{(2n)} S^2 h^2 S^2 h^n_{(n+1)} S h^n_{(n-2)} \otimes_A \cdots \\
&\qquad\quad \cdots \otimes_A h^0_{(n)} S h^0_{(n+3)} S^2 h^n_{(2n-1)} S h^n_{(1)} S^2 h^{n-1} \otimes_A h^0_{(n+1)} S h^0_{(n+2)} S^2 h^n_{(2n)} \\
&\qquad\qquad \vdots \\
&\quad = h^0 \otimes_A S^2 h^1 \otimes_A \cdots \otimes_A S^2 h^n,
\end{aligned}
$$

with the same steps as before. □

**5.4.3 Lemma** *The injection*

$$C_A^\bullet H \hookrightarrow \mathrm{Cob}^\bullet(H), \quad h^1 \otimes_A \cdots \otimes_A h^n \mapsto 1_H \otimes_A h^1 \otimes_A \cdots \otimes_A h^n$$

*is a morphism of cosimplicial modules, and if $H$ is a commutative Hopf algebroid with $S^2 = \mathrm{id}$ it is even a morphism of cocyclic modules.*

PROOF: Straightforward computation. □

As a consequence, we can generalise a result in [KhR1] from commutative Hopf algebras to commutative Hopf algebroids:

**5.4.4 Theorem** *If $H$ is a commutative Hopf algebroid, its Hopf-cyclic cohomology is given as*

$$HC_\partial^\bullet(H) = \bigoplus_{i \geq 0} HH_\partial^{\bullet - 2i}(H).$$

PROOF: We do not give the proof here but rather refer to Theorem 6.3.3, the dual version of this theorem. Replacing there the bar resolution with the cobar resolution and the tensor product over $H$ with the cotensor product over $H$, one can easily dualise the proof given there. □

## 5.5 Example: Lie-Rinehart Algebras

We show in this section that the Hopf-cyclic cohomology of the universal enveloping algebra of a Lie-Rinehart algebra is given by its Lie-Rinehart homology, a generalised notion of Lie algebra homology. The corresponding Theorem 5.5.7 is not only a generalisation of the analogous statement for Lie algebras (Example 1.3.3(*ii*)), but may also justify why we regard Hopf-cyclic cohomology as a noncommutative analogue of Lie-Rinehart homology.

Before we dedicate our attention to Hopf-cyclic cohomology of $VL$, let us recall a few facts about the homology of Lie-Rinehart algebras [Rin, Hue2].

**5.5.1 Lie-Rinehart Homology** Let $L$ be projective as an $A$-module and consider the graded left $VL$-module $VL \otimes_A \wedge_A^\bullet L$ (both factors carry the obvious right and left $A$-module structures, respectively, and here $\otimes_A$ refers to them). Consider the $k$-linear operator $b'_{A,L} : VL \otimes_A \wedge^n L \to VL \otimes_A \wedge^{n-1} L$

$$
\begin{aligned}
&b'_{A,L}(u \otimes_A X_1 \wedge \cdots \wedge X_n) := \\
&\quad := \sum_{i=1}^n (-1)^{i-1} u X_i \otimes_A X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_n \\
&\qquad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} u \otimes_A [X_i, X_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_n,
\end{aligned}
\tag{5.5.1}
$$

where $uX_i$ is the right $(A, L)$-module structure that corresponds to right multiplication in $VL$. Now $b'_{A,L}$ is a $VL$-linear differential and if $s$ denotes suspension, we will call the resulting chain complex

$$K_\bullet^A L := (VL \otimes_A \wedge_A^\bullet sL, b'_{A,L}) \tag{5.5.2}$$

the *Koszul-Rinehart complex*. If $M$ is a right $(A, L)$-module (right $VL$-module), define *Lie-Rinehart homology* with coefficients in $M$ as

$$H^A_\bullet(L, M) := \mathrm{Tor}^{VL}_\bullet(M, A). \tag{5.5.3}$$

In case $L$ is projective as an $A$-module, the Koszul-Rinehart complex $K^A_\bullet L$ provides a projective resolution in the category of left $VL$-modules, and $(M \otimes_{VL} K^A_\bullet L,\ b_{A,L} := \mathrm{id}_M \otimes b'_{A,L})$ computes this homology with coefficients in $M$. The differential (to which we will refer as *Lie-Rinehart boundary*) is then explicitly given as

$$\begin{aligned} b_{A,L}&(m \otimes_A X_1 \wedge \cdots \wedge X_n) = \\ &= \sum_{i=1}^n (-1)^{i-1} [m, X_i] \otimes_A X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_n \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} m \otimes_A [X_i, X_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_n, \end{aligned} \tag{5.5.4}$$

for $m \in M$. If $L$ is finitely generated $A$-projective of constant rank $n$, the Koszul-Rinehart complex even yields a finite projective resolution of length $n$ with $VL \otimes_A \wedge^n_A L$ being the highest non-zero term; see [Rin, Hue2] for further details. In case $M = A$, Theorem 4.2.7 leads to

**5.5.2 Theorem** [Hue2, Thm. 2] *Let $\partial$ be an exact operator as in Theorem 4.2.7, making $A$ a right $(A, L)$-module, denoted $A_\partial$. The Batalin-Vilkovisky algebra $(\wedge^\bullet_A L, \partial)$ coincides with $(A_\partial \otimes_{VL} K^A_\bullet L, b_{A,L})$ as a chain complex. In particular, when $L$ is projective over $A$ the complex $(\wedge^\bullet_A L, \partial)$ computes $H^A_\bullet(L, A_\partial) = \mathrm{Tor}^{VL}_\bullet(A_\partial, A)$, i.e. the Lie-Rinehart homology of $L$ with values in its base algebra $A_\partial$.*

**5.5.3 Remark** Observe that the Lie-Rinehart boundary $b_{A,L}$ and coboundary $d_{A,L}$ (which will only be introduced in (5.6.2)) correspond precisely to left and right counits of $VL$: one has

$$b_{A,L}(a \otimes_{VL} X) = \partial(aX), \quad \text{whereas} \quad (d_{A,L}a)(X) = \epsilon(Xa), \quad a \in A,\ X \in L.$$

We stress here that in the absence of such a flat right connection on $A$, Lie-Rinehart homology with coefficients in $A$ cannot even be defined. The chains of the form $A_\partial \otimes_{VL} K^A_\bullet L$ in a sense correspond to what we called *coinvariant localisation* in Section 5.1, and will reappear in Section 6.3.

We turn to Hopf-cyclic cohomology. Let us state another corollary to Proposition 4.2.11.

**5.5.4 Corollary** *Let $(A, L)$ be a Lie-Rinehart algebra. Any flat right $(A, L)$-connection on $A$ defines a cocyclic module $VL_{\natural,\partial}$ associated to the universal enveloping algebra $VL$.*

**5.5.5 Remark** We want to underline again that flatness of the connection is needed in Proposition 4.2.11(*i*) to show that $S_\partial$ is an anti-algebra homomorphism; this, in turn, is crucial for the cyclic relations to hold in (the proof of) Theorem 5.2.5, in particular for $\tau_n^{n+1} = \mathrm{id}$.

The following lemma will serve as a tool to simplify the calculation of Hopf-Hochschild cohomology of $VL$. Strictly speaking, it is an immediate consequence of the generalised PBW theorem; we give a proof that fully relies on combinatorial arguments.

Like $VL$, the symmetric algebra $S_A L$ is generated by the elements both in $L$ and $A$, with the difference that $S_A L$ is commutative and hence acts trivially on $A$. Now any $A$-module $L'$ can be seen as a Lie-Rinehart algebra with trivial bracket and zero anchor; in such a case, $VL' = S_A L'$, and the $A$-coring structure is given again as $\Delta_{SL'} X = X \otimes^{ll} 1 + 1 \otimes^{ll} X$ for $X \in L'$ and $\Delta_{SL'} a = a \otimes^{ll} 1 = 1 \otimes^{ll} a$ for $a \in A$. Note that in this case the counit $\epsilon : S_A L' \to A$ becomes in this case a morphism of algebras. We then have

**5.5.6 Lemma** *For any Lie-Rinehart algebra $(A, L)$, the $A$-module isomorphism $\pi : S_A L \to VL$ from (1.4.5) is an isomorphism of $A$-corings.*

PROOF: Let $(\Delta_\ell, \epsilon)$ be the (left) comonoid structure on $VL$ and $(\Delta_{SL}, \epsilon)$ the one on $S_A L$. The assertions are $\epsilon\,\pi = \epsilon$, which is trivial, as well as $(\pi \otimes \pi)\Delta_{SL} = \Delta_\ell\,\pi$. Clearly, by the PBW theorem it suffices to prove this identity on elements $aX^p \in S_A L$ for $a \in A,\ X \in L$, and a natural number $p \geq 0$. For $k \geq 0$, denote the $k$th iterated action of the anchor on the algebra $A$ by $X^k(a)$. We have

$$\pi(aX^p) = \frac{1}{p+1} \sum_{k=0}^p \binom{p+1}{k+1} X^k(a) X^{p-k} = \frac{1}{p+1} \sum_{k=0}^p \binom{p+1}{k+1} X^{p-k}(a) X^k.$$

In particular, for $a = 1$ one gets $\pi(X^p) = X^p$. Furthermore, for the respective coproducts holds

$$\Delta_{SL}(aX^p) = \Delta_\ell(aX^p) = \sum_{j=0}^{p} \binom{p}{j} aX^j \otimes^{ll} X^{p-j} = \sum_{j=0}^{p} \binom{p}{j} aX^{p-j} \otimes^{ll} X^j,$$

since both sides are $A$-linear. Hence for $a = 1$ or a trivial $L$-action on $A$ the statement is immediate and reproduces the argument for Lie and Hopf algebras. Otherwise, one has to consider

$$\begin{aligned}
\Delta_\ell \pi(aX^p) &= \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k+1} X^k(a) \Delta_\ell X^{p-k} \\
&= \frac{1}{p+1} \sum_{k=0}^{p} \sum_{j=0}^{p-k} \binom{p+1}{k+1} \binom{p-k}{j} X^k(a) X^j \otimes^{ll} X^{p-k} \\
&= \frac{1}{p+1} \sum_{k=0}^{p} \sum_{j=0}^{p-k} \frac{(p+1)!}{(k+1)!\, j!\, (p-k-j)!} X^k(a) X^j \otimes^{ll} X^{p-k},
\end{aligned}$$

and compare this with

$$\begin{aligned}
(\pi \otimes \pi)\, \Delta_{SL}(aX^p) &= (\pi \otimes \pi)\, (\sum_{j=0}^{p} \binom{p}{j} aX^j \otimes^{ll} X^{p-j}) \\
&= \sum_{j=0}^{p} \sum_{k=0}^{j} \frac{1}{j+1} \binom{p}{j} \binom{j+1}{k+1} X^k(a) X^{j-k} \otimes^{ll} X^{p-j} \\
&= \sum_{j=0}^{p} \sum_{k=0}^{j} \frac{1}{p+1} \binom{p+1}{j+1} \binom{j+1}{k+1} X^k(a) X^{j-k} \otimes^{ll} X^{p-j} \\
&= \frac{1}{p+1} \sum_{j=0}^{p} \sum_{k=0}^{j} \frac{(p+1)!}{(k+1)!\, (p-j)!\, (j-k)!} X^k(a) X^{j-k} \otimes^{ll} X^{p-j}.
\end{aligned}$$

Observe here that the map $\pi \otimes \pi$ is only well defined on operations $aX_{(1)} \otimes^{ll} X_{(2)} = X_{(1)} \otimes^{ll} aX_{(2)}$ in the sum of the coproduct, rather than on single summands. To interchange the rôles of $j$ and $k$ in this last double sum, one lets $k$ run from 0 to $p$ and $j$ to $p-k$ only. Correspondingly, one has to raise the numbers $j \to j+k$ to obtain the same coefficients; one immediately sees that this operation produces the same result as above, hence $\Delta_\ell \pi(aX^p) = (\pi \otimes \pi)\, \Delta_{SL}(aX^p)$, as claimed. □

Recall that Proposition 4.2.11 states that it is essentially a right $(A, L)$-connection $\partial$ on $A$ determining a right counit and an antipode on a Lie-Rinehart algebra, hence a cocyclic structure. The following theorem may also be called a Hochschild-Kostant-Rosenberg-Theorem for Lie-Rinehart algebras or Lie algebroids. It is a generalisation of a similar result for universal enveloping algebras of Lie algebras [CoMos2, Cr3].

**5.5.7 Theorem** *Let $(A, L)$ be a Lie-Rinehart algebra, with $L$ projective over $A$, and $k$ containing $\mathbb{Q}$. Furthermore, let $\partial$ a flat right $(A, L)$-connection on its base algebra $A$ (or equivalently, an exact generator $\partial$ for the Gerstenhaber algebra $\wedge^\bullet_A L$).*

*(i) The Hochschild cohomology of the cocyclic module $VL_{\natural,\partial}$ is isomorphic to the exterior algebra of $L$ over $A$, i.e.,*

$$HH^\bullet_\partial(VL) \simeq \wedge^\bullet_A L.$$

*This map is induced by both the antisymmetrisation map*

$$\mathrm{Alt} : \wedge^n_A L \to S_A L^{\otimes^{ll} n}, \quad X_1 \wedge \cdots \wedge X_n \mapsto \frac{1}{n!} \sum_{\sigma \in P(n)} \mathrm{sign}\, \sigma X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)},$$

*as well as the map*

$$P : S_A L^{\otimes^{ll} n} \to \wedge^n_A L, \quad u_1 \otimes^{ll} \cdots \otimes^{ll} u_n \mapsto \mathrm{pr}\, u_1 \wedge \cdots \wedge \mathrm{pr}\, u_n,$$

*where $\mathrm{pr} : S_A L \to S^1_A L$ is the projection on the direct summand $S^1_A L = L = \wedge^1_A L$, and this isomorphism does not depend on $\partial$.*

(*ii*) *The periodic cyclic cohomology of* $VL_{\natural,\partial}$ *is isomorphic to the Lie-Rinehart algebra homology of* $L$ *(cf.* §5.5.1*), i.e.,*

$$HP^\bullet_\partial(VL) \xrightarrow{\simeq} H^A_{\mathrm{odd}}(L, A_\partial) \oplus H^A_{\mathrm{even}}(L, A_\partial),$$

*where* $A_\partial$ *is a right* $VL$*-module via* $\partial$.

PROOF: Part (*i*): since the Hochschild cohomology uses only the pertinent $A$-coring structure, the PBW map induces an isomorphism $HH^\bullet_\partial(SL) \xrightarrow{\simeq} HH^\bullet_\partial(VL)$ and we can restrict our investigation to $S_AL$. The proof follows standard homological algebra procedures.

Recall at first some facts about the symmetric algebra on modules, cf. e.g. [Eis, App. A2]. Assume first $L$ to be free of finite dimension $N$ over $A$, with basis $\{e_i\}_{1\leq i\leq N}$. Then $S_AL = \oplus_{p\geq 0}S^p_AL$ is the polynomial ring on the 'variables' $e_i$ and $S^p_AL$ is the free $A$-module of rank $\binom{N+p-1}{N-1}$, with basis the set of monomials of degree $p$ in $e_i$. The graded dual of $S_AL$ is defined to be $S_A(L)^* = \oplus_p S^p_A(L)^* = \oplus_p \mathrm{Hom}_{(A,-)}(S^p_AL, A)$. Observe that the distinction between left and right duals as in Section 3.1 disappears here since $s^\ell = t^\ell$. If $L$ is $A$-free and $A$ contains $\mathbb{Q}$, one has [Eis, Prop. A2.7] $S_A(L)^* \simeq S_A(L^*)$ as algebras, where $L^* := \mathrm{Hom}_A(L, A)$; as a consequence, we will just write $S_AL^*$. The module structures given in (3.1.14) or (3.1.12) transfer to this context as follows: one obtains an $S_AL^*$-module structure on $S_AL$ by

$$\nabla^r : S_AL \otimes S_AL^* \longrightarrow S_AL, \quad u \otimes v^* \longmapsto \langle v^*, u_{(1)}\rangle u_{(2)},$$

where the Sweedler components refer to the coproduct on $S_AL$. Each $\phi \in L^*$ acts as a derivation on $S_AL$: for two homogeneous elements $u, w \in S_AL$ one has $\nabla^r_\phi uw = \langle \phi, u_{(1)}w_{(1)}\rangle u_{(2)}w_{(2)}$ since $\Delta_{SL}$ is a homomorphism of $k$-algebras. Furthermore, since $\phi \in L^* = S^1_AL^*$, the only nonzero elements in $\langle .,.\rangle$ are those for which $u_{(1)}w_{(1)} \in S^1_AL$, i.e., one of the $u_{(1)}, w_{(1)}$ lies in $S^1_AL = L$ and the other one lies in $S^0_AL = A$. Now any element in $S_AL$ is a sum of products of elements of $L$, hence for any $u \in S_AL_+ := \sum_{j>0} S^j_AL$ one obtains $\Delta_{SL}u = 1 \otimes^{ll} u + u \otimes^{ll} 1 + x$ with $x \in S_AL_+ \otimes^{ll} S_AL_+$ (this is a general property of *connected* bialgebras, see e.g. [GrVaFi, Lemma 14.10]). Therefore,

$$\nabla^r_\phi uw = \langle \phi, u_{(1)}\rangle u_{(2)}w + \langle \phi, w_{(1)}\rangle uw_{(2)} = (\nabla^r_\phi u)w + u\nabla^r_\phi w.$$

To compute $HH^\bullet_\partial(S_AL)$ one may use a Koszul resolution of $A$ that is dual to the construction in [Kas3, XVIII.7]. We will, however, generalise a method in [Cr3] and proceed mainly as there, considering a coaugmented complex. Versions of this proof can also be found in [KhR2] and, in the framework of Lie algebroids, in [Cal].

Define the (dual) Koszul complex $\tilde{K}^\bullet_AL := S_AL \otimes_A \wedge^\bullet_AL$, where each $\tilde{K}^n_AL := S_AL \otimes_A \wedge^n_AL$ carries a left $S_AL$-coaction by $\Delta_{SL} \otimes \mathrm{id}_{\wedge^\bullet_AL}$. This yields a resolution of $A$ by left $S_AL$-comodules: consider the coaugmented complex

$$A \xrightarrow{1_{SL}} S_AL \otimes_A \wedge^0_AL \xrightarrow{d} S_AL \otimes_A \wedge^1_AL \xrightarrow{d} \ldots, \tag{5.5.5}$$

with coboundary $d$ defined as

$$d(u \otimes_A X_1 \wedge \cdots \wedge X_n) = \sum_{i=1}^N \nabla^r_{e^i} u \otimes_A e_i \wedge X_1 \wedge \cdots \wedge X_n,$$

which is easily seen not to depend on the chosen basis. One has

$$d\,d(u \otimes_A X_1 \wedge \cdots \wedge X_n) = \sum_{i=1}^N \sum_{j=1}^N \langle e^i, u_{(1)}\rangle\langle e^j, u_{(2)}\rangle u_{(3)} \otimes_A e_i \wedge e_j \wedge X_1 \wedge \cdots \wedge X_n,$$

and while $\langle e^i, u_{(1)}\rangle\langle e^j, u_{(2)}\rangle$ is only nonzero if $i = j$, the wedge powers in this case vanish. Hence $d\,1_{SL} = d\,d = 0$. The exactness of (5.5.5) is shown by the existence of a contracting homotopy: define $s : \tilde{K}^n_AL \to \tilde{K}^{n-1}_AL$ by

$$s(u \otimes_A X_1 \wedge \cdots \wedge X_n) = \sum_{j=1}^N (-1)^{j+1} uX_j \otimes_A X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n.$$

Then one obtains

$$
\begin{aligned}
&ds(u \otimes_A X_1 \wedge \cdots \wedge X_n) = \\
&\quad = \sum_{i=1}^{N} \sum_{j=1}^{N} (-1)^{j+1} (\nabla^r_{e^i} u) X_j \otimes_A e_i \wedge X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n \\
&\qquad + \sum_{i=1}^{N} \sum_{j=1}^{N} (-1)^{j+1} u \otimes_A X_1 \wedge \cdots \wedge X_{j-1} \wedge \langle e^i, X_j \rangle e_i \wedge X_{j+1} \cdots \wedge X_n,
\end{aligned}
$$

whereas

$$
\begin{aligned}
sd(u \otimes_A X_1 \wedge \cdots \wedge X_n) = &\sum_{i=1}^{N} \sum_{j=1}^{N} (-1)^{j} (\nabla^r_{e^i} u) X_j \otimes_A e_i \wedge X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n \\
&+ \sum_{i=1}^{N} \sum_{j=1}^{N} (\nabla^r_{e^i} u) e_i \otimes_A X_1 \wedge \cdots \wedge X_n.
\end{aligned}
$$

Hence $(d\,s + s\,d)(u \otimes_A \omega) = (p+n)(u \otimes_A \omega)$, where $p, n \in \mathbb{N}$ and $u \in S^p_A L$, $\omega \in \wedge^n_A L$. This shows exactness of (5.5.5) in degree $> 0$. As for degree zero, note that the kernel of $d : S_A L \to S_A L \otimes_A \wedge^1_A L,\ u \mapsto \sum_{i=1}^n \nabla^r_{e^i} u \otimes_A e_i$ is isomorphic to $A$. One concludes that the complex $A \xrightarrow{1_{SL}} \tilde{K}^\bullet_A L$ is acyclic and gives a resolution of $A$ by free (hence injective) left $S_A L$-comodules. Even simpler (cf. [Lo1, Thm. 3.2.2]), since $L$ is $A$-free, one can write $L = L_1 \oplus \ldots \oplus L_N$, where each $L_i$ is free of dimension one. By $S_A(L_1 \oplus L_2) \simeq S_A L_1 \otimes_A S_A L_2$ and $\wedge_A(L_1 \oplus L_2) \simeq \wedge_A L_1 \otimes_A \wedge_A L_2$, one reduces the consideration to the dimension one situation, and easily sees that tensoring resolutions $\tilde{K}^\bullet_A L = \otimes_i \tilde{K}^\bullet_A L_i$ leads to the same conclusion.

Now by Theorem 5.3.3 one has $HH^\bullet(S_A L) = \mathrm{Cotor}^\bullet_{S_A L}(A, A)$, hence it can be e.g. computed by $A \Box_{S_A L} \tilde{K}^\bullet_A L$. Under the isomorphism

$$
f : A \Box_{S_A L} \tilde{K}^\bullet_A L = A \Box_{S_A L} S_A L \otimes_A \wedge^\bullet_A L \xrightarrow{\simeq} \wedge^\bullet_A L, \quad u \otimes_A \omega \longmapsto \epsilon(u)\omega,
$$

one has $(\mathrm{id}_A \otimes_A f) du = 0$ for each homogeneous $u \in S^p_A L$ for $p > 0$; in case $p = 0$ this follows trivially from $A = \ker d$. Hence

$$
(A \Box_{S_A L} \tilde{K}^\bullet_A L, \mathrm{id}_A \otimes_A d) \simeq (\wedge^\bullet_A L, 0),
$$

as complexes.

To show that the isomorphism on cohomology is induced by $P$ and $\mathrm{Alt}$, we compare the Koszul resolution to the standard cobar resolution $(\mathrm{Cob}^\bullet(S_A L), \beta')$ from Theorem 5.3.3: the map

$$
\mathrm{id}_{S_A L} \otimes_A P : \mathrm{Cob}^\bullet(S_A L) \to \tilde{K}^\bullet_A L,\ u \otimes^{ll} v_1 \otimes^{ll} \cdots \otimes^{ll} v_n \mapsto u \otimes_A \mathrm{pr}\,(v_1) \wedge \cdots \wedge \mathrm{pr}\,(v_n)
$$

will be shown to be a chain map over the identity, that is

$$
(\mathrm{id}_{S_A L} \otimes_A P)\,\beta' = d\,(\mathrm{id}_{S_A L} \otimes_A P).
$$

Both sides, applied to an element $U = u \otimes^{ll} W \in \mathrm{Cob}^n(S_A L)$, where $W \in S^{p_1}_A L \otimes^{ll} \cdots \otimes^{ll} S^{p_n}_A L$, vanish under the projection $\mathrm{pr}$ if $(p_1, \ldots, p_n) \neq (1, \ldots, 1)$. Set $U = u \otimes^{ll} X_1 \otimes^{ll} \cdots \otimes^{ll} X_n$, where $X_i \in L$. Consequently, one has

$$
d\,(\mathrm{id}_{S_A L} \otimes_A P)(u \otimes^{ll} X_1 \otimes^{ll} \cdots \otimes^{ll} X_n) = \sum_{i=1}^{N} \nabla^r_{e^i} u \otimes_A e_i \wedge X_1 \wedge \cdots \wedge X_n,
$$

whereas with the same argument

$$
(\mathrm{id}_{S_A L} \otimes_A P)\,\beta'(u \otimes^{ll} X_1 \otimes^{ll} \cdots \otimes^{ll} X_n) = (\mathrm{id}_{S_A L} \otimes_A \mathrm{pr}\,)(\Delta_{SL}(u) \wedge X_1 \wedge \cdots \wedge X_n),
$$

by $\Delta_{SL} X_i = X_i \otimes^{ll} 1 + 1 \otimes^{ll} X_i$ and $\mathrm{pr}\,(1_{S_A L}) = \mathrm{pr}\,(a) = 0$. Comparing the two sides, one requires the identity

$$
(\mathrm{id}_{S_A L} \otimes_A \mathrm{pr}\,) \Delta_{SL} u = \sum_{i=1}^{N} \nabla^r_{e^i} u \otimes_A e_i
$$

to hold, which indeed follows from (3.1.23), observing again that all basis elements of degrees higher than one vanish under the projection. Hence $(\mathrm{id}_{S_A L} \otimes_A P)$ is a chain map between the two resolutions of $A$; applying the functor $A\Box_{S_A L}-$ produces the map $\mathrm{id}_A\Box_{S_A L}(\mathrm{id}_{S_A L} \otimes_A P) = P$, and by standard homological algebra this yields an isomorphism on cohomology. The property $P\,\mathrm{Alt} = \mathrm{id}_{\wedge^\bullet_A L}$ shows that this isomorphism is induced by Alt as well. Finally, it is easy to see that $P$ is a morphism of complexes, that is, annihilates elements of the form

$$1 \otimes^{ll} v_1 \otimes^{ll} \cdots \otimes^{ll} v_n + \sum_{i=1}^{n-1}(-1)^i v_1 \otimes^{ll} \cdots \otimes^{ll} \Delta_{SL} v_i \otimes^{ll} \cdots \otimes^{ll} v_n + (-1)^{n+1} v_1 \otimes^{ll} \cdots \otimes^{ll} v_n \otimes^{ll} 1.$$

It is equally easy to see that $\beta\,\mathrm{Alt} = 0$.

More generally, if $L$ is flat over $A$ (for example, if $L$ is $A$-projective; in fact, flatness suffices for part (*i*)), we continue as in [Lo1, Thm. 3.2.2]: there exists a filtered ordered set $J$ as well as an inductive system of free and finite dimensional $A$-modules $L_j$ such that

$$L \simeq \varinjlim_{j\in J} L_j,$$

cf. [Bou]. Since both $HH$ (which is the derived functor Cotor here) as well as $S$ commute with inductive limits over a filtered ordered set, the flat case follows from the finite dimensional (free) case.

Part (*ii*): this part of the proof is a generalisation of a method in [Cr3] for the universal enveloping of a Lie algebra.

Denote here by $(BC^\bullet_\partial(VL), \beta_\partial, B_\partial)$ Connes' bicomplex associated to the cocyclic module $VL_{\natural,\partial}$, cf. §5.2.1, and by $(BC^\bullet(VL), \beta, B)$ Connes' bicomplex associated to the standard $A$-coring cocyclic module $VL^A_\natural$, cf. §1.2.5. Recall that Theorem 4.2.7 tells us that Lie-Rinehart homology is computed by $(\wedge^\bullet_A L, \partial)$. Consider, therefore, the mixed complex

$$K : A \underset{b_{A,L}}{\overset{0}{\rightleftarrows}} \wedge^1_A L \underset{b_{A,L}}{\overset{0}{\rightleftarrows}} \wedge^2_A L \underset{b_{A,L}}{\overset{0}{\rightleftarrows}} \cdots,$$

where $b_{A,L}$ is the Lie-Rinehart boundary operator as in (5.5.4) with values in $A_\partial$, seen as a right $VL$-module via $\partial$. We will show that $(K, 0, b_{A,L})$ and $(BC^\bullet_\partial(VL), \beta_\partial, B_\partial)$ are quasi-isomorphic mixed complexes, which implies the claim. Similarly as in Theorem 5.2.5, it is easier to deduce the action of $B_\partial$ from the one of the operator $B$ (employing $\bar\phi_\partial$ from (5.2.1)) since it arises from a fairly simple cyclic operator.

Recall from Subsection 5.2 that $VL^A_\natural = \{B^n_A VL\}_{n\geq 0}$, that is, $B^n_A VL := VL^{\otimes^{ll} n+1}$ in degree $n$: the cyclic tensor product reduces tautologically to $\otimes^{ll}$ since source and target maps are equal. Generally, we have $B = N\sigma_{-1}(1-\tau)$, where for $VL^A_\natural$ the operators $N,\ \sigma_{-1},\ \tau$ are given as the following left $A$-module maps on $B^n_A VL$:

$$\begin{aligned}
\sigma_{-1}(u_0 \otimes^{ll} u_1 \otimes^{ll} \cdots \otimes^{ll} u_n) &= \epsilon(u_0) u_1 \otimes^{ll} \cdots \otimes^{ll} u_n,\\
\tau(u_0 \otimes^{ll} u_1 \otimes^{ll} \cdots \otimes^{ll} u_n) &= (-1)^n (u_1 \otimes^{ll} \cdots \otimes^{ll} u_n \otimes^{ll} u_0),\\
N &= 1 + \tau + \ldots + \tau^n.
\end{aligned}$$

The map $\bar\phi_\partial$ from (5.2.1) reads here, degree-wise

$$\bar\phi_\partial : VL^{\otimes^{ll} n+1} \to VL^{\otimes^{ll} n},\ u_0 \otimes^{ll} \cdots \otimes^{ll} u_n \mapsto (\Delta^{n-1}_\ell S_\partial u_0)(u_1 \otimes^{ll} \cdots \otimes^{ll} u_n), \tag{5.5.6}$$

and $B_\partial \bar\phi_\partial = \bar\phi_\partial B$ holds. On generators $a \in A,\ X \in L$ with the antipode (4.2.16), one calculates

$$\Delta^{n-1}_\ell S_\partial(aX) = -\sum_{i=1}^{n} 1^{\otimes^{ll} i-1} \otimes^{ll} \underset{i}{aX} \otimes^{ll} 1^{\otimes^{ll} n-i} + \partial(aX) \otimes^{ll} 1^{\otimes^{ll} n-1}. \tag{5.5.7}$$

The antisymmetrisation map

$$\mathrm{Alt} : \wedge^n_A L \to VL^{\otimes^{ll} n}, \quad X_1 \wedge \cdots \wedge X_n \mapsto \frac{1}{n!} \sum_{\sigma \in P(n)} \mathrm{sign}\,\sigma\, X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \tag{5.5.8}$$

can now be seen to be a quasi-isomorphism of mixed complexes $(K, 0, b_{A,L}) \to (BC^\bullet_\partial(VL), \beta_\partial, B_\partial)$ as follows (note that (5.5.8) is a well-defined map to the chosen tensor product): for the Hochschild boundaries this was precisely shown in part (*i*) and it only remains to prove that

$$B_\partial \circ \mathrm{Alt} = \mathrm{Alt} \circ b_{A,L}.$$

Using the right inverse (5.1.4) of $\bar\phi_\partial$, it is seen that

$$\mathrm{Alt}(aX_1 \wedge \cdots \wedge X_n) = \bar\phi_\partial\big(\frac{1}{n!}\sum_{\sigma\in P(n)} \operatorname{sign}\sigma\, a \otimes^{ll} X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)}\big).$$

Hence

$$\begin{aligned} B_\partial\big(\mathrm{Alt}(aX_1 \wedge \cdots \wedge X_n)\big) &= B_\partial(\bar\phi_\partial(\frac{1}{n!}\sum_{\sigma\in P(n)} \operatorname{sign}\sigma\, a \otimes^{ll} X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)})) \\ &= \bar\phi_\partial(B(\frac{1}{n!}\sum_{\sigma\in P(n)} \operatorname{sign}\sigma\, a \otimes^{ll} X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)})) \\ &= \bar\phi_\partial N\sigma_{-1}\big(\frac{1}{n!}\sum_{\sigma\in P(n)} \operatorname{sign}\sigma\, a \otimes^{ll} X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \\ &\quad - (-1)^n X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \otimes^{ll} a\big) \\ &= \bar\phi_\partial N(\frac{1}{n!}\sum_{\sigma\in P(n)} \operatorname{sign}\sigma\, aX_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)}) \\ &= \bar\phi_\partial\big(\frac{1}{(n-1)!}\sum_{\sigma\in P(n)} \operatorname{sign}\sigma\, aX_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)}\big), \end{aligned}$$

since $L \subset \ker\epsilon$ and $\epsilon$ is a left $A$-module map. Theorem 5.5.2 now explains how $\partial$ equips $A$ with a right $VL$-module structure induced by the right $(A, L)$-module structure $[a, X] := a\partial X - X(a)$ (note that $[1_A, X] = \partial X \neq 0$ in general). Using (5.5.6) and (5.5.7) gives

$$\begin{aligned} &B_\partial(\mathrm{Alt}(aX_1 \wedge \cdots \wedge X_n)) = \\ &= -\frac{1}{(n-1)!}\sum_{i=1}^{n}\sum_{\sigma\in P(n)} \operatorname{sign}\sigma\, aX_{\sigma(2)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(1)}X_{\sigma(i)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \\ &\quad + \frac{1}{(n-1)!}\sum_{\sigma\in P(n)} \operatorname{sign}\sigma\, \big(a\partial X_{\sigma(1)} - X_{\sigma(1)}(a)\big)X_{\sigma(2)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \\ &= \mathrm{Alt}\big(\sum_{i=1}^{n}(-1)^{i+1}[a, X_i]X_1 \wedge \cdots \wedge \hat X_i \wedge \cdots \wedge X_n \\ &\quad + \sum_{i<j}(-1)^{i+j}a[X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat X_i \wedge \cdots \wedge \hat X_j \wedge \cdots \wedge X_n\big) \\ &= \mathrm{Alt}\circ b_{A,L}(a \otimes_A X_1 \wedge \cdots \wedge X_n). \end{aligned}$$

This finishes the proof of the Theorem. □

Since $s^\ell \equiv t^\ell$ and $S^2 = \mathrm{id}$ (hence $\sigma = 1_H$ for the grouplike element in this example), the first Hopf-Hochschild cohomology group is formed precisely by the primitive elements of $VL$; by part (*i*) of the preceding Theorem, we have

$$\mathrm{im}\beta \oplus \wedge^\bullet_A L = \ker\beta.$$

Hence with Proposition 4.2.1 the following statement makes sense.

**5.5.8 Corollary** *The (left) primitive elements of the (left) bialgebroid given by the universal object $VL$ of a Lie-Rinehart algebra $(A, L)$ are isomorphic to $(A, L)$ as a Lie-Rinehart algebra, i.e.,*

$$(A, PVL) \xrightarrow{\simeq} (A, L).$$

In a certain sense, this is the 'easy half' of a Cartier-Milnor-Moore type theorem for Lie-Rinehart algebras; this essentially states that one has a left bialgebroid isomorphism $V(PU) \simeq U$, where $U$ is a cocommutative left bialgebroid (assumed to be filtered in a certain way and 'cocomplete') and where $P, V$ are the functors from Proposition 4.2.3. See [MoeMrč3] for the full theorem and details, [MiMo] for the original version for Lie algebras (cf. also [Q1, App. B.4] for a different approach), and [Lo2] for an extension to a vast choice of bialgebras of different type.

## 5.6 Example: Jet Spaces

In this section we calculate the Hopf-cyclic cohomology for the jet spaces $JL$ of a Lie-Rinehart algebra $(A, L)$: the outcome is in a certain sense dual to the result in the previous section. We therefore recall some facts about Lie-Rinehart *co*homology first.

**5.6.1 Lie-Rinehart Cohomology** Let $(A, L)$ be a Lie-Rinehart algebra and $M \in VL\text{-}\mathbf{Mod}$. Dually to §5.5.1, define the *Lie-Rinehart cohomology* groups with values in $M$ by

$$H^\bullet(L, M) := \mathrm{Ext}^\bullet_{VL}(A, M). \tag{5.6.1}$$

If $L$ is $A$-projective, $H^\bullet(L, M)$ is the homology of the complex $\mathrm{Hom}_{VL}(K^A_\bullet L, M) \simeq \mathrm{Hom}_{(A,-)}(\wedge^\bullet_A L, M)$, cf. (5.5.2), and the *Lie-Rinehart coboundary* is given by

$$\begin{aligned} d_{A,L}\phi(X_0 \wedge \cdots \wedge X_n) &:= \\ := \sum_{i=0}^{n} &(-1)^i \epsilon_{VL}\big(X_i \phi(X_0, \ldots, \hat{X}_i, \ldots, X_n)\big) \\ &+ \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n), \end{aligned} \tag{5.6.2}$$

where $\epsilon_{VL} : VL \to A$ denotes in this section the left counit of $VL$, induced by the anchor, as before.

**5.6.2 Theorem** *Let $(A, L)$ be a Lie-Rinehart algebra for which $L$ is finitely generated projective over $A$ of constant rank. There are canonical isomorphisms*

$$\begin{aligned} HH^\bullet(JL) &\simeq H^\bullet(L, A), \\ HC^\bullet(JL) &\simeq \bigoplus_{i \geq 0} H^{\bullet+2i}(L, A), \end{aligned}$$

*where the left hand side refers to the Hopf-cyclic cohomology groups.*

PROOF: Denote $L^* := \mathrm{Hom}_A(L, A)$. By the given conditions we have $\bigwedge^\bullet_A L^* \simeq \mathrm{Hom}_A(\bigwedge^\bullet_A L, A)$. To compute the Hochschild cohomology, instead of the cobar resolution one may use the dual of the Koszul-Rinehart resolution (5.5.2), given by the cochain complex (cf. [NeTs])

$$0 \longrightarrow A \xrightarrow{s^\ell_{JL}} JL \xrightarrow{\nabla} JL \otimes_A \wedge^1_A L^* \xrightarrow{\nabla} JL \otimes_A \wedge^2_A L^* \xrightarrow{\nabla} \ldots$$

where $\nabla$ is the continuation of the Grothendieck connection (4.3.2):

$$\begin{aligned} \nabla(\phi \otimes \omega)&(X_1, \ldots, X_{n+1}) = \\ = \sum_{i=1}^{n+1} &(-1)^{i-1} \nabla^\ell_{X_i}\phi \otimes \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \phi \otimes \omega([X_i, X_j], X_1 \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{n+1}), \end{aligned}$$

for $\phi \in JL$, $\omega \in \wedge^n_A L^*$ and $X_1, \ldots, X_{n+1} \in L$. It follows from (4.3.4) that this is indeed a resolution of $A$ in the category of free (hence injective) left $JL$-comodules (observe that $s^\ell_{JL} : A \to JL$ is a morphism of $JL$-comodules). To compute the $\mathrm{Cotor}$-groups, consider invariants as in (2.4.1): one has the isomorphism

$$\wedge^\bullet_A L^* \xrightarrow{\simeq} A \square_{JL} \big(JL \otimes_A \wedge^\bullet_A L^*\big),$$

given by $X_1 \wedge \cdots \wedge X_n \mapsto 1_A \otimes 1_{JL} \otimes X_1 \wedge \cdots \wedge X_n$. Since the unit $1_{JL} \in JL$ is given by the left counit $\epsilon_{VL} : VL \to A$, the induced differential is exactly the Lie-Rinehart coboundary $d_{A,L}$. This proves the isomorphism of the Hochschild cohomology groups.

The second isomorphism on cyclic cohomology now follows from Theorem 5.4.4. □

**5.6.3 Remark** Observe that our computations remain consistent: to be defined at all, both the Hopf algebroid structure for $VL$ and Lie-Rinehart homology with values in $A$ require an additional piece of information: the flat right connection $\partial$. In contrast to that, the Hopf algebroid structure on $JL$ (see Remark 4.3.3) as well as Lie-Rinehart cohomology already make sense without such a datum.

## 5.7 Example: Convolution Algebras

Let $G \rightrightarrows G_0$ be an étale groupoid over a compact base manifold $G_0$ and consider the Hopf algebroid given by its convolution algebra $\mathcal{C}^\infty_{\mathrm{c}}(G)$ over $\mathcal{C}^\infty(G_0)$, see Section 4.4.

Recall from our general considerations that the spaces of interest for cohomology were

$$\mathcal{C}^\infty_{\mathrm{c}}(G)_{\natural,\partial} := \{C^n_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\mathrm{c}}(G)\}_{n\geq 0}, \tag{5.7.1}$$

where

$$C^n_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\mathrm{c}}(G) = \mathcal{C}^\infty_{\mathrm{c}}(G) \otimes^{ll}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{ll}_{\mathcal{C}^\infty(G_0)} \mathcal{C}^\infty_{\mathrm{c}}(G) \simeq \mathcal{C}^\infty_{\mathrm{c}}(G^n), \tag{5.7.2}$$

$n$ times in degree $n$, where $G^n = G\,{}^t\!\times^t_{G_0} \cdots {}^t\!\times^t_{G_0} G$.

**5.7.1 Theorem** *For any étale groupoid $G \rightrightarrows G_0$ over a compact manifold $G_0$, the periodic cyclic cohomology of $\mathcal{C}^\infty_{\mathrm{c}}(G)_{\natural,\partial}$ is trivial, i.e.,*

$$HP^{\mathrm{even}}_\partial(\mathcal{C}^\infty_{\mathrm{c}}(G)) = \mathcal{C}^\infty_{\mathrm{c}}(G_0) \quad \textit{and} \quad HP^{\mathrm{odd}}_\partial(\mathcal{C}^\infty_{\mathrm{c}}(G)) = 0.$$

PROOF: The (restriction) map $T : \mathcal{C}^\infty_{\mathrm{c}}(G) \to \mathcal{C}^\infty(G_0),\ u \mapsto u(1_{(\cdot)})$ fulfills

$$(*(s^\ell T \otimes \mathrm{id})\Delta_\ell u)(g) = \begin{cases} u(g) & \text{if } g = 1_x \text{ for some } x \in G_0, \\ 0 & \text{else,} \end{cases}$$

hence $T$ is a left Haar system for $\mathcal{C}^\infty_{\mathrm{c}}(G)$. The statement follows now directly from Proposition 5.2.11 and the $SBI$-sequence. □

The triviality of this cohomology (and in general the cohomology for Hopf algebroids whose underlying bialgebroids carry a left Haar system) is one of the motivations to introduce a certain dual theory. This will be the subject of the next chapter.

# Chapter 6

# Dual Hopf-Cyclic Homology

Hopf-cyclic cohomology gives trivial results in some cases: as already seen in Theorem 5.7.1, in case of the existence of a (left) Haar measure, the Hopf-cyclic complex computing cohomology is shown to be acyclic [KhR3]. Clearly, taking any $\mathrm{Hom}$-duals also does not furnish any new kind of information: hence the need of a different, *dual* cyclic homology in the sense of cyclic duality [Co2].

## 6.1 The Cyclic Dual of Hopf-cyclic Homology

Recall from Section 3.1 that for a left bialgebroid $U$ there exist two natural $\mathrm{Hom}$-duals, corresponding to the bimodule structure ${}_{\triangleright}U_{\triangleleft}$ . Both of them are (under suitable conditions) right bialgebroids [KSz]. Hence it appears natural to start with a right bialgebroid to investigate the cyclic dual. Naively, a simplicial complex with faces dual to (5.2.2) should contain product and counit rather than coproduct and unit.

Let $V$ be a right bialgebroid over the base algebra $B$, with structure maps as before. To make the ring multiplication well-defined, to start with, one chooses a tensor product

$$V_B\!\otimes V := V_{\blacktriangleleft} \otimes {}_{\triangleright}V = V \otimes_k V/\mathrm{span}_k\{vs^r(b) \otimes v' - v \otimes s^r(b)v', v, v' \in V,\ b \in B\}; \qquad (6.1.1)$$

(taking e.g. the tensor product $V_{\triangleleft} \otimes {}_{\blacktriangleright}V$ with respect to the target maps and the monoid structure of $V^{\mathrm{op}}$ leads to a left bialgebroid again). At the second step, however, one runs into the problem that a bialgebroid is a monoid and comonoid in different monoidal categories: the algebra $B$ carries both left and right $V$-comodule structures, but has a single right $V$-module structure only. As follows from Subsection 5.3, these two comodule structures are those that appear in the coface operators $\delta_0$ and $\delta_{n+1}$ of (5.2.2), in form of the trivial coaction. For its dual version, already when defining Hochschild homology (with values in the base algebra) by means of the simplicial pieces, one therefore realises the necessity of a 'two-sided' bialgebroid equipped with respectively both left and right actions on the base algebra and its opposite; that is, one needs the full Hopf algebroid structure.

Consequently, let $H$ be a Hopf algebroid with structure maps as before and set

$$H^\natural_\partial := \{C^B_n H\}_{n\geq 0}, \qquad (6.1.2)$$

where $C^B_n H := H^{B\otimes n}$ in degree $n$ and $C^B_0 H := B$ in degree zero. To give $H^\natural_\partial$ the structure of a simplicial space, define face and degeneracy operators by

$$\begin{aligned} d_i(h^1{}_B\!\otimes \cdots {}_B\!\otimes h^n) &= \begin{cases} s^r(\partial h^1)h^2{}_B\!\otimes \cdots {}_B\!\otimes h^n & \text{if } i=0,\\ h^1{}_B\!\otimes \cdots {}_B\!\otimes h^ih^{i+1}{}_B\!\otimes \cdots {}_B\!\otimes h^n & \text{if } 1\leq i\leq n-1,\\ h^1{}_B\!\otimes \cdots {}_B\!\otimes h^{n-1}t^\ell(\epsilon h^n) & \text{if } i=n,\end{cases}\\ s_i(h^1{}_B\!\otimes \cdots {}_B\!\otimes h^n) &= \begin{cases} 1_B\!\otimes h^1{}_B\!\otimes \cdots {}_B\!\otimes h^n & \text{if } i=0,\\ h^1{}_B\!\otimes \cdots {}_B\!\otimes h^i{}_B\!\otimes 1_B\!\otimes h^{i+1}{}_B\!\otimes \cdots {}_B\!\otimes h^n & \text{if } 1\leq i\leq n-1,\\ h^1{}_B\!\otimes \cdots {}_B\!\otimes h^n{}_B\!\otimes 1 & \text{if } i=n.\end{cases}\end{aligned} \qquad (6.1.3)$$

Elements of degree zero (i.e. of $B$) are mapped to zero, $d_0(b) = 0$ for all $b \in B$. We write the degeneracies in this detail to stress that the 'extra' degeneracy $s_{-1} := t_{n+1}s_n$ used for cyclic homology (cf. (1.1.7)) is quite different from the classical one for $k$-algebras. To extend the simplicial structure to the structure of a cyclic module on $H^\natural_\partial$, we use the assumption that the antipode is invertible and set

$$t_n(h^1{}_B\!\otimes \cdots {}_B\!\otimes h^n) = S^{-1}(h^1_{(2)}\cdots h^{n-1}_{(2)}h^n)_B\!\otimes h^1_{(1)}{}_B\!\otimes h^2_{(1)}{}_B\!\otimes \cdots {}_B\!\otimes h^{n-1}_{(1)} \qquad (6.1.4)$$

as a cyclic operator on $C^B_n H$ for each degree $n \geq 2$, and $t_1(h) = S^{-1}h$ in degree one. One easily verifies that this operator is well-defined. Similarly as for Hopf-cyclic cohomology, the fact that $S^{-1}$ is a morphism ${}_{\triangleright}H_{\blacktriangleleft} \to {}_{\blacktriangleright}H_{\triangleleft}$ of twisted bimodules transfers to the cyclic operator: $t_n$ is a map ${}_{\triangleright}(C^B_n H)_{\blacktriangleleft} \to {}_{\blacktriangleright}(C^B_n H)_{\triangleleft}$ of (twisted) $(B, B)$-bimodules as well.

As in §5.2.8, one could also introduce grouplike elements in the dual theory, but we will avoid this here.

The main result of this section is:

**6.1.1 Theorem** *Let $H$ be a Hopf algebroid with invertible antipode. The para-cyclic module $H^\natural_\partial$ is the cyclic dual of the para-cocyclic module $H_{\natural,\partial}$ from Theorem 5.2.5 (and vice versa).*

PROOF: We will prove that the prescriptions in §1.1.17 turn the set of operators $(\delta_\bullet, \sigma_\bullet, \tau_\bullet)$ from (5.2.2), (5.2.3) and (5.2.4) into the set $(d_\bullet, s_\bullet, t_\bullet)$ from (6.1.3) and (6.1.4). The subtlety in this proof lies in the fact that this cannot simply done by replacing cofaces with degeneracies, codegeneracies with faces and so on, since $H_{\natural,\partial} = \{C^n_A H\}_{n \geq 0}$ from cohomology and $H^\natural_\partial := \{C^B_n H\}_{n \geq 0}$ from homology do not have the same underlying bimodule structures: the respective tensor products are different. Hence one first needs to find a $k$-module isomorphism $C^n_A H \to C^B_n H$, which amounts to a generalisation to higher degrees of the Hopf-Galois map from [Schau2, Thm. 3.5] (see also (2.2.1)) and its inverse from [BSz2] for Hopf algebroids, cf. (2.2.3) and (2.6.15).

**6.1.2 Lemma** *For each $n \geq 2$, the $k$-modules $C^n_A H$ and $C^B_n H$ are isomorphic as $k$-modules by means of the* (higher) Hopf-Galois map

$$\begin{aligned} \varphi_n : H {}_B\!\otimes \cdots {}_B\!\otimes H &\xrightarrow{\simeq} H \otimes_A \cdots \otimes_A H \\ h^1 {}_B\!\otimes \cdots {}_B\!\otimes h^n &\mapsto h^1_{(1)} \otimes_A h^1_{(2)} h^2_{(1)} \otimes_A \cdots \otimes_A h^1_{(n)} h^2_{(n-1)} \cdots h^{n-1}_{(2)} h^n \\ &= (\Delta^{n-1}_\ell h^1)(\Delta^{n-2}_\ell h^2) \cdots (\Delta_\ell h^{n-1})(1 \otimes_A \cdots \otimes_A 1 \otimes_A h^n), \end{aligned} \tag{6.1.5}$$

*with inverse given by*

$$\begin{aligned} \psi_n : H \otimes_A \cdots \otimes_A H &\xrightarrow{\simeq} H {}_B\!\otimes \cdots {}_B\!\otimes H \\ h_1 \otimes_A \cdots \otimes_A h_n &\mapsto h_1^{(1)} {}_B\!\otimes S(h_1^{(2)}) h_2^{(1)} {}_B\!\otimes S(h_2^{(2)}) h_3^{(1)} {}_B\!\otimes \cdots {}_B\!\otimes S(h_{n-1}^{(2)}) h_n, \\ &= h_1^+ {}_B\!\otimes h_1^- h_2^+ {}_B\!\otimes \cdots {}_B\!\otimes h^-_{n-2} h^+_{n-1} {}_B\!\otimes h^-_{n-1} h_n, \end{aligned} \tag{6.1.6}$$

*using the notation $h^+ {}_B\!\otimes h^- := h^{(1)} {}_B\!\otimes Sh^{(2)}$, see (2.2.3) and (2.2.13). Note that the first map employs the left coproduct only, whereas the inverse uses the right one plus the antipode. In case $n = 1$, set $\varphi_1 = \psi_1 = \mathrm{id}_H$; in case $n = 0$, one combines the isomorphism $\partial s^\ell : A^{\mathrm{op}} \to B$ with the canonical isomorphism of $k$-modules $A \to A^{\mathrm{op}}$.*

PROOF: This is proven by induction on $n \geq 2$. For $n = 2$, it can be directly checked. Hence assume that the statement is already true for $n$, i.e., $\varphi_n \psi_n = \mathrm{id}_{C^n_A H}$ and $\psi_n \varphi_n = \mathrm{id}_{C^B_n H}$. As for $n + 1$, note that one can decompose

$$\varphi_{n+1} = (\mathrm{id} \otimes \varphi_n)(\varphi_2 \otimes \mathrm{id}^{\otimes n-1}) \quad \text{and} \quad \psi_{n+1} = (\psi_2 \otimes \mathrm{id}^{\otimes n-1})(\mathrm{id} \otimes \psi_n)$$

and then one verifies directly that $\varphi_{n+1}$ and $\psi_{n+1}$ are mutually inverse. □

The cyclic dual of a cocyclic operator is given by its inverse, see §1.1.17. To continue the proof, we hence need the inverse of the cocyclic operator (5.2.4) from cohomology:

**6.1.3 Lemma** *Let $\sigma \in GH$ be a grouplike element. If the inverse of the antipode $S$ exists, the cocyclic operator given by the map (5.2.4) is an automorphism of the $k$-modules $C^n_A H$ for $n \geq 1$, with inverse*

$$\tau_n^{-1}(h^1 \otimes_A \cdots \otimes_A h^n) = (\Delta^{n-1}_\ell S^{-1}(h^n \sigma^{-1}))(1 \otimes_A h^1 \otimes_A \cdots \otimes_A h^{n-1}) \tag{6.1.7}$$

*for all $n \geq 1$ as an $(A, A)$-bimodule morphism ${}_{\triangleright}(C^n_A H)_{\blacktriangleleft} \to {}_{\blacktriangleright}(C^n_A H)_{\triangleleft}$. Likewise, the operator (6.1.4) is an automorphism of the $k$-modules $C^B_n H$ for all $n \geq 1$ with inverse*

$$t_n^{-1}(h_1 {}_B\!\otimes \cdots {}_B\!\otimes h_n) = h_2^{(1)} {}_B\!\otimes \cdots {}_B\!\otimes h_n^{(1)} {}_B\!\otimes S(h_1 h_2^{(2)} \cdots h_n^{(2)}) \tag{6.1.8}$$

*for all $n \geq 1$ as $(B, B)$-bimodule morphism ${}_{\blacktriangleright}(C^B_n H)_{\triangleleft} \to {}_{\triangleright}(C^B_n H)_{\blacktriangleleft}$.*

PROOF: Clearly, this can be verified directly; however, we pursue a strategy with more structural insight, that is, we express the operator $t_{n+1}$ in terms of $t_n$ and prove the lemma by complete induction. For $n = 1$, (6.1.7) reads $\tau_1^{-1} = S^{-1}$, hence the induction start. Now assume that the assertion already holds for degree $n$ and introduce the following bijective $k$-module morphism and its inverse,

$$\begin{array}{rclcl} \phi_\sigma : H \otimes_A H & \to & H \otimes_A H, & h {}_A\!\otimes \tilde{h} & \mapsto & \Delta_\ell h(\sigma^{-1}\tilde{h} \otimes 1) = h_{(1)}\sigma^{-1}\tilde{h} \otimes_A h_{(2)}, \\ \phi_\sigma^{-1} : H \otimes_A H & \to & H {}_A\!\otimes H, & h \otimes_A \tilde{h} & \mapsto & \tilde{h}^{(2)} {}_A\!\otimes \sigma S^{-1}(\tilde{h}^{(1)})h. \end{array}$$

Observe that the tensor product ${}_A\!\otimes$ is the left bialgebroid version of (6.1.1), i.e., $H {}_A\!\otimes H = H_\blacktriangleleft \otimes_A {}_\triangleright H$. Then $\phi_\sigma$, $\phi_\sigma^{-1}$ are morphisms with respect to the canonical left $A$-module structures ${}_\triangleright(H {}_A\!\otimes H) \to {}_\triangleright(H \otimes_A H)$. One finds now that

$$\begin{array}{rcl} \tau_{n+1} & = & (\mathrm{id}^{\otimes n-1} \otimes \phi_\sigma)\,(\tau_n \otimes \mathrm{id}), \\ \tau_{n+1}^{-1} & = & (\tau_n^{-1} \otimes \mathrm{id})\,(\mathrm{id}^{\otimes n-1} \otimes \phi_\sigma^{-1}). \end{array}$$

Note that $\tau_n \otimes \mathrm{id}$ is only well-defined as a map $H \otimes_A \cdots \otimes_A H \otimes_A H \to H \otimes_A \cdots \otimes_A H {}_A\!\otimes H$, which is why the map $\phi$ is required. It can be directly seen that $\tau_{n+1}$ and $\tau_{n+1}^{-1}$ are mutually inverse; hence the induction is completed. As for the second part, introduce the maps

$$\begin{array}{rclcl} \tilde{\phi} : H {}_B\!\otimes H & \to & H \otimes_B H, & h {}_B\!\otimes h' & \mapsto & \Delta_\ell^{\mathrm{coop}} h(h' \otimes 1) = h_{(2)}h' \otimes_B h_{(1)}, \\ \tilde{\phi}^{-1} : H \otimes_B H & \to & H {}_B\!\otimes H, & h \otimes_B h' & \mapsto & h'^{(1)} {}_B\!\otimes \sigma S(h'^{(2)})h, \end{array}$$

which again can be directly checked to be mutually inverse. Here $\otimes_B$ is the tensor product given as

$$H \otimes_B H := H \otimes_k H/\mathrm{span}_k\{t^r bh \otimes_k h' - h \otimes_k s^r(\mu\nu^{-1}b)h',\ b \in B\},$$

where $\mu, \nu$ are given in (2.6.5) and $\mu\nu^{-1} = \mathrm{id}$ in case $S^2 = \mathrm{id}$. Then one has

$$\begin{array}{rcl} t_{n+1} & = & (t_n \otimes \mathrm{id})\,(\mathrm{id}^{\otimes n-1} \otimes \tilde{\phi}), \\ t_{n+1}^{-1} & = & (\mathrm{id}^{\otimes n-1} \otimes \tilde{\phi}^{-1})\,(t_n^{-1} \otimes \mathrm{id}), \end{array}$$

continuing to argue in the same fashion as in the first part of the proof. □

One can now immediately write down

$$\begin{aligned} &\tau_n^{-1}\varphi_n(h^1 {}_B\!\otimes \cdots {}_B\!\otimes h^n) = \\ &= \tau_n^{-1}(h^1_{(1)} \otimes_A h^1_{(2)}h^2_{(1)} \otimes_A h^1_{(3)}h^2_{(2)}h^3_{(1)} \otimes_A \cdots \otimes_A h^1_{(n)}h^2_{(n-1)} \cdots h^{n-1}_{(2)}h^n) \\ &= S^{-1}(h^{1}{}^{(n)}_{(n)} \cdots h^{n-1}{}^{(n)}_{(2)}h^{n(n)}) \otimes_A S^{-1}(h^{1}{}^{(n-1)}_{(n)} \cdots h^{n-1}{}^{(n-1)}_{(2)}h^{n(n-1)})h^1_{(1)} \otimes_A \cdots \\ &\qquad \cdots \otimes_A S^{-1}(h^{1}{}^{(1)}_{(n)} \cdots h^{n-1}{}^{(1)}_{(2)}h^{n(1)})h^1_{(n-1)}h^2_{(n-2)} \cdots h^{n-1}_{(1)}, \end{aligned}$$

and by higher twisted coassociativity one obtains

$$\begin{aligned} &\varphi_n t_n(h^1 {}_B\!\otimes \cdots {}_B\!\otimes h^n) = \varphi_n(S^{-1}(h^1_{(2)} \cdots h^{n-1}_{(2)}h^n) {}_B\!\otimes h^1_{(1)} {}_B\!\otimes h^2_{(1)} {}_B\!\otimes \cdots {}_B\!\otimes h^{n-1}_{(1)}) \\ &= S^{-1}(h^{1}{}^{(n)}_{(n)} \cdots h^{n-1}{}^{(n)}_{(2)}h^{n(n)}) \otimes_A S^{-1}(h^{1}{}^{(n-1)}_{(n)} \cdots h^{n-1}{}^{(n-1)}_{(2)}h^{n(n-1)})h^1_{(1)} \otimes_A \cdots \\ &\qquad \cdots \otimes_A S^{-1}(h^{1}{}^{(1)}_{(n)} \cdots h^{n-1}{}^{(1)}_{(2)}h^{n(1)})h^1_{(n-1)}h^2_{(n-2)} \cdots h^{n-1}_{(1)}. \end{aligned}$$

Hence $\varphi_n\, t_n = \tau_n^{-1}\, \varphi_n$ or $t_n = \psi_n\, \tau_n^{-1}\, \varphi_n$. In the same fashion,

$$\begin{aligned}
&\sigma_{n-1}\tau_n\varphi_n(h^1\,{}_B\!\otimes \cdots {}_B\!\otimes h^n) \\
&= \sigma_{n-1}(S(h^{1}{}^{(n)}_{(1)})h^1_{(2)}h^2_{(1)} \otimes_A \cdots \otimes_A S(h^{1}{}^{(2)}_{(1)})h^1_{(n)}h^2_{(n-1)} \cdots h^{n-1}_{(2)}h^n \otimes_A S(h^{1}{}^{(1)}_{(1)})) \\
&= S(h^{1}{}^{(n-1)}_{(1)})h^1_{(2)}h^2_{(1)} \otimes_A \cdots \otimes_A S(h^{1}{}^{(1)}_{(1)})h^1_{(n)}h^2_{(n-1)} \cdots h^{n-1}_{(2)}h^n \\
&= s^r\partial(h^{1}{}^{(n-1)}_{(1)})h^2_{(1)} \otimes_A S(h^{1(n-2)})h^{1}{}^{(n-1)}_{(2)}h^2_{(2)}h^3_{(1)} \otimes_A \cdots \\
&\qquad \cdots \otimes_A S(h^{1(1)})h^{1}{}^{(n-1)}_{(n-1)}h^2_{(n-1)} \cdots h^{n-1}_{(2)}h^n \\
&= h^2_{(1)} \otimes_A S(h^{1(n-2)}s^r\partial(h^{1}{}^{(n-1)}_{(1)}))h^{1}{}^{(n-1)}_{(2)}h^2_{(2)}h^3_{(1)} \otimes_A \cdots \\
&\qquad \cdots \otimes_A S(h^{1(1)})h^{1}{}^{(n-1)}_{(n-1)}h^2_{(n-1)} \cdots h^{n-1}_{(2)}h^n \\
&= h^2_{(1)} \otimes_A S(h^{1}{}^{(n-2)}_{(1)})h^1{}_{(2)}h^2_{(2)}h^3_{(1)} \otimes_A \cdots \otimes_A S(h^{1}{}^{(1)}_{(1)})h^1{}_{(n-1)}h^2_{(n-1)} \cdots h^{n-1}_{(2)}h^n \\
&\quad\vdots \\
&= h^2_{(1)} \otimes_A h^2_{(2)}h^3_{(1)} \otimes_A \cdots \otimes_A s^r\partial(h^1)h^2_{(n-1)} \cdots h^{n-1}_{(2)}h^n,
\end{aligned}$$

where the dots denote another $n-2$ repetitions of the same procedure of the lines before: one observes that the third compared to the sixth line from bottom has analogous left and right coproduct Sweedler components of $h^1$ with respectively one degree less and moved one factor to the right. On the other hand,

$$\varphi_{n-1}d_0(h^1\,{}_B\!\otimes \cdots {}_B\!\otimes h^n) = h^2_{(1)} \otimes_A h^2_{(2)}h^3_{(1)} \otimes_A \cdots \otimes_A s^r\partial(h^1)h^2_{(n-1)} \cdots h^{n-1}_{(2)}h^n,$$

hence $d_0 = \psi_{n-1}\,\sigma_{n-1}\tau_n\varphi_n$ as claimed. The proof of the remaining identities is left to the reader. □

**6.1.4 Corollary** *$H^\natural_\partial$ is a cyclic module if $H_{\natural,\partial}$ is cocyclic, i.e. if and only if $S^2 = \mathrm{id}$.*

**6.1.5 Definition** In case $S^2 = \mathrm{id}$, denote the associated Tsygan's cyclic bicomplex of the cyclic module $H^\natural_\partial$ by $CC^\partial_{\bullet,\bullet}(H)$, and the associated Connes' bicomplex by $(BC^\partial_{\bullet,\bullet}(H), b, B)$, analogously as in §5.2.1. Define $HH^\partial_\bullet(H)$, $HC^\partial_\bullet(H)$ and $HP^\partial_\bullet(H)$ to be the associated Hochschild and cyclic homology groups. We will refer to these as *dual Hopf-Hochschild* and *dual Hopf-cyclic* homology, respectively.

### 6.1.1 The Space of Invariants

In this subsection we make a few comments on a notion dual to coinvariants from Subsection 5.1, and its significance for dual Hopf-cyclic homology.

**6.1.6 Invariants** Let $H$ be a Hopf algebroid with structure maps as before, and denote the underlying right bialgebroid over $B$ by $H^r$.

Furthermore, let $M \in \mathbf{Comod}\text{-}H^r$ be a right $H^r$-comodule (hence in particular a $(B,B)$-bimodule) with coaction ${}_M\Delta : M \to M \otimes^B {}_\blacktriangleright H^r,\ m \mapsto m^{(0)} \otimes^B m^{(1)}$. We do not treat the details of right bialgebroid comodules here: these can be formulated *mutatis mutandis* as in Sections 2.3 and 2.4 for left bialgebroid comodules (see e.g. [B3]). Analogously to (2.4.1), for $M \in \mathbf{Comod}\text{-}H^r$ define the *right bialgebroid (right) invariants* of $M$ by

$$M_\epsilon := M \Box_{H^r} B = \{m \in M \mid {}_M\Delta m = m \otimes^B 1\} \subset M.$$

Considering $H^r$ as a right $H^r$-comodule over itself by the right coproduct $\Delta_r$, we can analogously to (2.3.11) equip the tensor product $M\,{}_B\!\otimes_\triangleright H^r$ with the following right $H^r$-comodule structure

$${}_{M\otimes H}\Delta : M\,{}_B\!\otimes_\triangleright H^r \to M \otimes^B {}_\triangleright H^r{}_\blacktriangleleft \otimes^B {}_\blacktriangleright H^r, \quad m\,{}_B\!\otimes h \mapsto m^{(0)}\,{}_B\!\otimes h^{(1)} \otimes^B m^{(1)}h^{(2)}.$$

One then finds, dually to Corollary 5.1.3:

**6.1.7 Lemma** *For any $M \in \mathbf{Comod}\text{-}H^r$, there is a canonical isomorphism*

$$M \xrightarrow{\simeq} (M\,{}_B\!\otimes_\triangleright H^r)_\epsilon$$

*of $k$-modules, given by*

$$m \longmapsto m^{(0)}\,{}_B\!\otimes Sm^{(1)}.$$

PROOF: We have

$$\begin{aligned} {}_{M\otimes H}\Delta(m^{(0)}{}_B\otimes Sm^{(1)}) &= m^{(0)}{}_B\otimes Sm^{(2)}_{(2)} \otimes^B m^{(1)}Sm^{(2)}_{(1)} \\ &= m^{(0)}{}_B\otimes Sm^{(1)}_{(2)} \otimes^B s^\ell\epsilon m^{(1)}_{(1)} \\ &= m^{(0)}{}_B\otimes Sm^{(1)} \otimes^B 1_H, \end{aligned}$$

where (2.1.8) and (2.6.10) have been used. Hence $m^{(0)}{}_B\otimes Sm^{(1)} \in (M_B\otimes_{\triangleright}H^r)_\epsilon$, indeed. It is easy to see that this is a bijective map, with inverse

$$(M_B\otimes_{\triangleright}H^r)_\epsilon \to M, \quad m_B\otimes h \mapsto m\nu(\epsilon h).$$

$\square$

**6.1.8 Relation to Dual Hopf-Cyclic Homology** Let $n \geq 0$ and compare the chain spaces $C^B_n H$ of the Hopf-cyclic module (6.1.2) with the chain spaces $B^B_n H = C^B_{n+1}H\otimes_{B^e}B$ associated to the canonical cyclic module of $H$ as $B$-ring (see §1.2.4). Since $C^B_{n+1}H \in \mathbf{Comod}\text{-}H^r$, the preceding lemma yields the isomorphism $C^B_n H \simeq (C^B_{n+1}H)_\epsilon = C^B_{n+1}H\Box_{H^r}B$ via the embedding

$$C^B_n H \to C^B_{n+1}H, \quad h_{1B}\otimes\cdots{}_B\otimes h_n \mapsto h^{(1)}_1{}_B\otimes\cdots{}_B\otimes h^{(1)}_n \otimes^B S\big(h^{(2)}_1\cdots h^{(2)}_n\big).$$

Combining this embedding with the canonical projection

$$C^B_{n+1}H = C^B_n H_B\otimes H \to C^B_n H \otimes^{B^e} H \simeq B^B_n H$$

yields a map

$$\Psi_\epsilon : C^B_n H \to B^B_n H.$$

Related to this map, one would expect a commutative diagram that is in some sense dual to the diagram (5.2.9) in cohomology, where the vertical arrows should rather be injections than surjections. Such a statement at least holds for Hopf algebras [KhR1, KhR2]. Unfortunately, for Hopf algebroids this does not appear to be that simple, and only works in special cases (see Section 6.5 and Subsection 6.6.1). First of all, the precise nature of the map $\Psi_\epsilon$ is in general not clear to us. A related problem appears to be that $H$ usually cannot be given the structure of a coring over $B^e \simeq B\otimes_k A$, which is possibly required for the existence of a well-defined injection $C^B_\bullet H \simeq C^B_{\bullet+1}H\Box_{H^r}B \hookrightarrow B^B_\bullet H \simeq C^B_{\bullet+1}H\otimes_{B^e}B$ that would do the job.

Apparently, Hopf-cyclic cohomology and dual Hopf-cyclic homology are 'not dual enough' for such a symmetric picture. Possible ways to investigate include *Hopf algebroid comodules* ([B3, Def. 4.6], cf. Remark 3.3.7) or, as was suggested to us by G. Böhm, (co)tensor products over so-called *bicoalgebroids* [BrzMi].

## 6.2 Dual Hopf-Hochschild Homology as a Derived Functor

In the next theorem we are going to show that the dual Hopf-Hochschild homology given by the complex $\big(C^B_\bullet H, b = \sum^n_{j=0}(-1)^j d_j\big)$ from (6.1.3) can be seen as a derived functor of the tensor product functor.

**6.2.1 Coefficients** As in the cohomological case, one may consider coefficients in the Hopf-Hochschild complex. Let $M$ be a right $H$-module with action $(m,h)\mapsto mh$ and define $C^M_\bullet H := \{M\otimes_A H^{B\otimes n}\}_{n\geq 0}$. Face and degeneracy operators are now given by

$$\begin{aligned} d_i(m_B\otimes h^1{}_B\otimes\cdots{}_B\otimes h^n) &= \begin{cases} mh^1{}_B\otimes h^2{}_B\otimes\cdots{}_B\otimes h^n & \text{if } i=0 \\ m_B\otimes h^1{}_B\otimes\cdots{}_B\otimes h^i h^{i+1}{}_B\otimes\cdots{}_B\otimes h^n & \text{if } 1\leq i\leq n-1 \\ m_B\otimes h^1{}_B\otimes\cdots{}_B\otimes h^{n-1}t^\ell(\epsilon h^n) & \text{if } i=n, \end{cases} \\ s_i(m_B\otimes h^1{}_B\otimes\cdots{}_B\otimes h^n) &= \begin{cases} m_B\otimes 1_{H\,B}\otimes h^1{}_B\otimes\cdots{}_B\otimes h^n & \text{if } i=0 \\ m_B\otimes\cdots{}_B\otimes h^i{}_B\otimes 1_B\otimes h^{i+1}{}_B\otimes\cdots{}_B\otimes h^n & \text{if } 1\leq i\leq n-1 \\ m_B\otimes h^1{}_B\otimes\cdots{}_B\otimes h^n{}_B\otimes 1 & \text{if } i=n. \end{cases} \end{aligned} \tag{6.2.1}$$

Elements of degree zero (i.e. of $M$) are mapped to zero, $d_0(m) = 0$ for all $m\in M$. The corresponding (dual Hopf-)Hochschild homology computed by $\big(C^M_\bullet H, b = \sum^n_{j=0}(-1)^j d_j\big)$ will be denoted $H_\bullet(H,M)$; in particular, if $M = B_\partial$, one has, with the notation of Definition 6.1.5, $H_\bullet(H,B_\partial) = HH^\partial_\bullet(H)$.

**6.2.2 The Bar Complex** As in the cohomology case, we will calculate the homology groups of the complex associated to (6.2.1) by finding an appropriate resolution, dual to the cobar complex in Section 5.3. Such a resolution is provided by an analogue of the classical bar complex. This is again in some sense the complex originating from the so-called *path space* $PM_\bullet := \{M_{n+1}\}_{n\geq 0}$ associated to any simplicial object $M_\bullet$; hence $PM_n = M_{n+1}$ in degree $n$, and face and degeneracy operators of the underlying simplicial object are shifted correspondingly. For the simplicial space $C^H_\bullet H$ using $M := H$ in (6.2.1), the simplicial pieces of its path space

$$\mathrm{Bar}_\bullet H := \{H_{\scriptscriptstyle B}\!\otimes H^{{\scriptscriptstyle B}\otimes n}\}_{n\geq 0}$$

therefore read

$$\tilde d_i(h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n) = \begin{cases} h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^i h^{i+1}{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n & \text{if } 0\leq i\leq n-1 \\ h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^{n-1}t^\ell(\epsilon h^n) & \text{if } i=n, \end{cases}$$
$$\tilde s_i(h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n) = \begin{cases} h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^i{}_{\scriptscriptstyle B}\!\otimes 1{}_{\scriptscriptstyle B}\!\otimes h^{i+1}{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n & \text{if } 0\leq i\leq n-1 \\ h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n{}_{\scriptscriptstyle B}\!\otimes 1 & \text{if } i=n. \end{cases}$$

Defining $b' = \sum_{i=0}^n (-1)^i \tilde d_i$, it is easily checked that $b'\, b' = 0$, and $(\mathrm{Bar}_\bullet H, b')$ is called the *bar complex* of $H$. A simple characterisation of the dual Hopf-Hochschild homology is then the following.

**6.2.3 Theorem** *Let $H$ be a Hopf algebroid with structure maps as before, and let $M$ be a right $H$-module. If $H$ is projective as a left $B$-module ${}_\triangleright H$, there is an isomorphism*

$$H_\bullet(H,M) \xrightarrow{\simeq} \mathrm{Tor}^H_\bullet(M, A_\epsilon).$$

*In particular,*

$$HH^\partial_\bullet(H) \xrightarrow{\simeq} \mathrm{Tor}^H_\bullet(B_\partial, A_\epsilon).$$

*Here $B_\partial$ and $A_\epsilon$ carry the canonical right and left $H$-module structures, originating from the right and left counit, respectively.*

PROOF: The proof works in a conceptually analogous manner to the classical statement for Hochschild homology of algebras. Recall (cf. e.g. [M, p. 146]) that for a projective left $B$-module $P$ the space $H_\blacktriangleleft \otimes_B P$ is projective as a left $H$-module, where the multiplication is simply $h'(h_{\scriptscriptstyle B}\!\otimes p) := h'h_{\scriptscriptstyle B}\!\otimes p$. Evidently, $\mathrm{Bar}_n H$ is $H$-projective if $H$ is $B$-projective (with respect to the module structure $b \triangleright h := s^r b\, h$ for $b \in B,\ h \in H$). Now $\epsilon : H \to A$ defines an augmentation of the bar complex and we need to show that $A_\epsilon \xleftarrow{\epsilon} \mathrm{Bar}_\bullet H$ is a projective resolution of $A_\epsilon$ in the category of (left) $H$-modules. The map $b' : \mathrm{Bar}_n H \to \mathrm{Bar}_{n-1} H$ in degree $n > 0$ is a morphisms of left $H$-modules if the $H$-module structure mentioned above is used; as for degree zero, this property of the map $\epsilon : \mathrm{Bar}_0 H \to A_\epsilon$ follows from (2.1.4). Then introduce the 'extra degeneracy'

$$s_{n+1} : \mathrm{Bar}_n H \to \mathrm{Bar}_{n+1} H, \quad h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n \mapsto 1_{H\,{\scriptscriptstyle B}}\!\otimes h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n$$

for $n \geq 0$. It is not particularly problematic to realise that $s_{n-1}\, b' + b'\, s_n = \mathrm{id}$ for $n > 0$; furthermore, set $s_{-1} := t^\ell = s^r \partial \nu$ to have $\epsilon\, s_{-1} = \mathrm{id}_A$ and $s_{n-1}\, b' + b'\, s_n = \mathrm{id}$ for $n \geq 0$. Hence, $s$ is a contracting homotopy and $(\mathrm{Bar}_n H, b')$ is acyclic. In particular, the latter yields a resolution of $A_\epsilon$ by projective left $H$-modules and can therefore be used to compute the groups $\mathrm{Tor}^H_\bullet(M, A_\epsilon)$ for a right $H$-module $M$. Note that in case $M = B$ the functor $(-)_\partial := B_\partial \otimes_H -$ of coinvariants reappears here. For each $n \geq 0$ one then obtains an isomorphism

$$\psi : M \otimes_H \mathrm{Bar}_n H \xrightarrow{\simeq} C^M_n H,\ m \otimes_H h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n \longmapsto m_{\scriptscriptstyle B}\!\otimes s^r \partial(h^0) h^1{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n,$$

and we are left to show that

$$\psi\,(\mathrm{id}_{\scriptscriptstyle B} \otimes b') = b\,\psi,$$

i.e. that $(\mathrm{id}_{\scriptscriptstyle B} \otimes b')$ becomes the Hochschild differential $b$ formed by the simplicial pieces from (6.2.1) under the isomorphism $\psi$. We write this down only for the first summand, the rest being clear. One has

$$\begin{aligned}\psi(\mathrm{id}_M \otimes b')(m \otimes_H h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n) &= m\partial(h^0h^1){}_{\scriptscriptstyle B}\!\otimes h^2{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n + \ldots \\ &= b\, f'(m \otimes_H h^0{}_{\scriptscriptstyle B}\!\otimes \cdots {}_{\scriptscriptstyle B}\!\otimes h^n),\end{aligned}$$

using the properties of a right counit. □

**6.2.4 Remark** Hence the dual Hopf-Hochschild homology groups are the left derived functors of the functor (5.1.1) of coinvariants. In the context of Hopf algebras in [KhR1] these are baptised *Hopf algebra homology groups*.

## 6.3 Dual Hopf-Cyclic Homology of Cocommutative Hopf Algebroids

The aim of the following consideration is to generally calculate the dual Hopf-cyclic homology of a *cocommutative* Hopf algebroid, as a generalisation to [KhR1, Thm. 4.1], where this is done for Hopf algebras.

Clearly, cocommutativity only makes sense for a special kind of $A$-corings: namely those for which both left and right $A$-module structures coincide, and as a consequence $A$ needs to be commutative. Observe that in case of a Hopf algebroid over commutative $A$ with, say, cocommutative right coproduct, the left coproduct is automatically cocommutative as well, as follows from (2.6.10). In particular, $A = B$ and both $s^\ell = t^\ell$ as well as $s^r = t^r$.

**6.3.1 Proposition** *Let $H$ be a cocommutative Hopf algebroid over commutative base algebra $A$ with invertible antipode $S$. Then $\mathrm{Bar}_\bullet\, H$ is a para-cyclic $H$-module with cyclic operator*

$$\begin{aligned}&\tilde{t}_n : \mathrm{Bar}_n\, H \to \mathrm{Bar}_n\, H,\\ &\quad h_0 {}_B\!\otimes \cdots {}_B\!\otimes h_n \mapsto\\ &\quad h_0 h_1{}^{(2)} \cdots h_n{}^{(2)} {}_B\!\otimes S^{-1}(h_1{}^{(1)}_{(2)} \cdots h_{n-1}{}^{(1)}_{(2)} h_n{}^{(1)}) {}_B\!\otimes h_1{}^{(1)}_{(1)} {}_B\!\otimes \cdots {}_B\!\otimes h_{n-1}{}^{(1)}_{(1)},\end{aligned}$$

*and is cyclic if and only if $S^{-2} = \mathrm{id}$. In particular,*

$$\tilde{t}_n^{n+1}(h_0 {}_B\!\otimes \cdots {}_B\!\otimes h_n) = h_0 {}_B\!\otimes S^{-2} h_1 {}_B\!\otimes \cdots {}_B\!\otimes S^{-2} h_n.$$

PROOF: We only prove the cyclicity condition $\tilde{t}_n^{n+1} = \mathrm{id}$ and leave the remaining identities defining a cyclic module to the reader. Since otherwise the Sweedler notation in this proof may be ambiguous, we indicate the order in which the respective coproducts are taken by inserting, at times, little gaps. By higher twisted coassociativity and the various antipode properties in (2.6.4) and (2.6.11), one finds

$$\begin{aligned}&\tilde{t}_n^2(h_0 {}_B\!\otimes \cdots {}_B\!\otimes h_n) =\\ &\quad = \tilde{t}_n\big(h_0 h_1{}^{(2)} \cdots h_n{}^{(2)} {}_B\!\otimes S^{-1}(h_1{}^{(1)}_{(2)} \cdots h_{n-1}{}^{(1)}_{(2)} h_n{}^{(1)}) {}_B\!\otimes h_1{}^{(1)}_{(1)} {}_B\!\otimes \cdots {}_B\!\otimes h_{n-1}{}^{(1)}_{(1)}\big)\\ &\quad = h_0 h_1{}^{(4)} \cdots h_{n-1}{}^{(4)} h_n{}^{(3)} S^{-1}(h_1{}^{(2)}{}_{(2)} \cdots h_{n-1}{}^{(2)}{}_{(2)} h_n{}^{(1)}_{(1)}) h_1{}^{(2)}{}_{(1)} \cdots h_{n-1}{}^{(2)}{}_{(1)}\\ &\qquad {}_B\!\otimes S^{-1}\big(S^{-1}(h_n{}^{(1)}_{(2)}) S^{-1}(h_1{}^{(2)}{}_{(3)} \cdots h_{n-1}{}^{(2)}{}_{(3)}) h_1{}^{(1)}{}_{(2)} \cdots h_{n-2}{}^{(1)}{}_{(2)} h_{n-1}{}^{(1)}\big)\\ &\qquad {}_B\!\otimes S^{-1}(h_1{}^{(3)} \cdots h_{n-1}{}^{(3)} h_n{}^{(2)}) {}_B\!\otimes h_1{}^{(1)}{}_{(1)} {}_B\!\otimes \cdots {}_B\!\otimes h_{n-2}{}^{(1)}{}_{(1)}\\ &\quad = h_0 h_1{}^{(3)} \cdots h_{n-1}{}^{(3)} h_n{}^{(3)} S^{-1}(h_n{}^{(1)}_{(1)})\\ &\qquad {}_B\!\otimes S^{-1}\big(S^{-1}(h_n{}^{(1)}_{(2)}) S^{-1}(h_1{}^{(1)}_{(3)} \cdots h_{n-2}{}^{(1)}_{(3)} h_{n-1}{}^{(1)}_{(2)}) h_1{}^{(1)}_{(2)} \cdots h_{n-2}{}^{(1)}_{(2)} h_{n-1}{}^{(1)}_{(1)}\big)\\ &\qquad {}_B\!\otimes S^{-1}(h_1{}^{(2)} \cdots h_{n-1}{}^{(2)} h_n{}^{(2)}) {}_B\!\otimes h_1{}^{(1)}_{(1)} {}_B\!\otimes \cdots {}_B\!\otimes h_{n-2}{}^{(1)}_{(1)}\\ &\quad = h_0 h_1{}^{(3)} \cdots h_{n-1}{}^{(3)} t^\ell \epsilon h_n{}^{(1)}_{(2)} {}_B\!\otimes s^r \partial(h_1{}^{(1)}_{(2)} \cdots h_{n-2}{}^{(1)}_{(2)} h_{n-1}{}^{(1)}) S^{-2}(h_n{}^{(1)}_{(1)})\\ &\qquad {}_B\!\otimes S^{-1}(h_1{}^{(2)} \cdots h_{n-1}{}^{(2)} h_n{}^{(2)}) {}_B\!\otimes h_1{}^{(1)}_{(1)} {}_B\!\otimes \cdots {}_B\!\otimes h_{n-2}{}^{(1)}_{(1)}\\ &\quad = h_0 h_1{}^{(2)} \cdots h_{n-1}{}^{(2)} {}_B\!\otimes S^{-2}(h_n{}^{(1)})\\ &\qquad {}_B\!\otimes S^{-1}(h_1{}^{(1)}_{(2)} \cdots h_{n-2}{}^{(1)}_{(2)} h_{n-1}{}^{(2)} h_n{}^{(2)}) {}_B\!\otimes h_1{}^{(1)}_{(1)} {}_B\!\otimes \cdots {}_B\!\otimes h_{n-2}{}^{(1)}_{(1)},\end{aligned}$$

where to obtain e.g. the second line the identity

$$\begin{aligned}&(\Delta_\ell \otimes \mathrm{id}^{\otimes 5})(\Delta_r \otimes \mathrm{id}^{\otimes 4})(\mathrm{id}^{\otimes 2} \otimes \Delta_r \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_\ell \otimes \mathrm{id})(\Delta_\ell \otimes \mathrm{id})\Delta_r =\\ &\quad = (\Delta_\ell \otimes \mathrm{id}^{\otimes 5})(\mathrm{id} \otimes \Delta_\ell^2 \otimes \mathrm{id}^{\otimes 2})\Delta_r^3\end{aligned}$$

was used. By repeating the above computation another $n-2$ times, one obtains

$$\begin{aligned}&\tilde{t}_n^n(h_0 {}_B\!\otimes \cdots {}_B\!\otimes h_n) =\\ &\quad = h_0 h_1{}^{(2)} {}_B\!\otimes S^{-2}(h_2{}^{(1)}) {}_B\!\otimes \cdots {}_B\!\otimes S^{-2}(h_n{}^{(1)}) {}_B\!\otimes S^{-1}(h_1{}^{(1)} h_2{}^{(2)} \cdots h_n{}^{(2)}).\end{aligned}$$

To end the proof we can now calculate

$$
\begin{aligned}
&\tilde{t}_n^{n+1}(h_0 {}_B\!\otimes \cdots {}_B\!\otimes h_n) = \\
&= h_0 h_1{}^{(2)} S^{-2}(h_2{}^{(2)}_{(2)} \cdots h_n{}^{(2)}_{(2)}) S^{-1}(h_2{}^{(3)}_{(2)} \cdots h_n{}^{(3)}_{(2)}) S^{-1}(h_1{}^{(1)}_{(1)}) \\
&\qquad {}_B\!\otimes S^{-1}\big(S^{-2}(h_2{}^{(1)}_{(2)} \cdots h_n{}^{(1)}_{(2)}) S^{-1}(h_{2(3)} \cdots h_{n(3)}) S^{-1}(h_1{}^{(1)}_{(2)})\big) \\
&\qquad\qquad {}_B\!\otimes S^{-2}(h_{2(1)}) {}_B\!\otimes \cdots {}_B\!\otimes S^{-2}(h_{n(1)}) \\
&= h_0 h_1{}^{(2)} S^{-1}(h_1{}^{(1)}_{(1)} t^\ell \epsilon S^{-2}(h_2{}^{(2)}_{(2)} \cdots h_n{}^{(2)}_{(2)})) \\
&\quad {}_B\!\otimes S^{-2}(h_1{}^{(1)}_{(2)} h_{2(3)} \cdots h_{n(3)}) S^{-3}(h_2{}^{(1)}_{(2)} \cdots h_n{}^{(1)}_{(2)}) {}_B\!\otimes S^{-2}(h_{2(1)}) {}_B\!\otimes \cdots {}_B\!\otimes S^{-2}(h_{n(1)}) \\
&= h_0 {}_B\!\otimes t^\ell \epsilon(h_{1(2)} S^{-2}(h_2{}^{(2)}_{(2)} \cdots h_n{}^{(2)}_{(2)})) S^{-2}(h_{1(1)} h_{2(3)} \cdots h_{n(3)}) S^{-3}(h_2{}^{(1)}_{(2)} \cdots h_n{}^{(1)}_{(2)}) \\
&\qquad {}_B\!\otimes S^{-2}(h_{2(1)}) {}_B\!\otimes \cdots {}_B\!\otimes S^{-2}(h_{n(1)}) \\
&= h_0 {}_B\!\otimes S^{-2}(h_1 h_2{}^{(2)}_{(2)} \cdots h_n{}^{(2)}_{(2)}) S^{-3}(h_2{}^{(1)}_{(2)} \cdots h_n{}^{(1)}_{(2)}) \\
&\qquad {}_B\!\otimes S^{-2}(h_{2(1)}) {}_B\!\otimes \cdots {}_B\!\otimes S^{-2}(h_{n(1)}) \\
&= h_0 {}_B\!\otimes S^{-2}(h_1) {}_B\!\otimes t^\ell \epsilon S^{-2}(h_{2(2)} \cdots h_{n(2)}) S^{-2}(h_{2(1)}) \\
&\qquad {}_B\!\otimes S^{-2}(h_{3(1)}) {}_B\!\otimes \cdots {}_B\!\otimes S^{-2}(h_{n(1)}) \\
&= h_0 {}_B\!\otimes S^{-2}(h_1) {}_B\!\otimes \cdots {}_B\!\otimes S^{-2}(h_n).
\end{aligned}
$$

□

**6.3.2 Lemma** *The projection*

$$\pi_\partial : \mathrm{Bar}_\bullet H \to C_\bullet^B H, \quad h_0 {}_B\!\otimes \cdots {}_B\!\otimes h_n \mapsto s^r \partial h_0 h_1 {}_B\!\otimes \cdots {}_B\!\otimes h_n$$

*is a morphism of simplicial modules. If $H$ is a cocommutative Hopf algebroid with $A = B$ and $S^{-2} = \mathrm{id}$, the map $\pi_\partial$ is even a morphism of cyclic modules.*

PROOF: Straightforward computation. □

The following theorem generalises [KhR1, Thm. 4.1] from cocommutative Hopf algebras (which in turn for $H = kG$ generalised Karoubi's theorem [Karou]) to cocommutative Hopf algebroids.

**6.3.3 Theorem** *If $H$ is a cocommutative Hopf algebroid over commutative base $A$, one has*

$$HC_\bullet^\partial(H) = \bigoplus_{i \geq 0} HH_{2\bullet - i}^\partial(H).$$

PROOF: We follow the pattern in [KhR1, Thm. 4.1]. As already mentioned in the proof of Theorem 6.2.3, the complex $\mathrm{Bar}_\bullet H$ becomes a left $H$-module by multiplication on the first tensor factor. One then has the relation (since $A = B$) $C_\bullet^A H = A_\partial \otimes_H \mathrm{Bar}_\bullet H$ which transfers to the level of (cyclic) bicomplexes $CC_\bullet^\partial(C_\bullet^A H) = A_\partial \otimes_H CC_\bullet(\mathrm{Bar}_\bullet H)$. With the fact that the augmented complex $\epsilon : \mathrm{Bar}_\bullet H \to A_\epsilon$ is a resolution for $A$, the bicomplex $CC_\bullet(\mathrm{Bar}_\bullet H)$ is in turn a resolution of the complex $A_\bullet : A \leftarrow 0 \leftarrow A \leftarrow 0 \leftarrow \ldots$. One then computes

$$HC_\bullet^\partial(H) = H_\bullet\big(\mathrm{Tot}\, CC_\bullet^\partial(C_\bullet^A H)\big) = H_\bullet\big(A_\partial \otimes_H \mathrm{Tot}\, CC_\bullet(\mathrm{Bar}_\bullet H)\big) = \mathbf{Tor}(A_\partial, A_\bullet),$$

where the last term denotes the hyper-derived Tor groups [W, 5.7.8]. As [W, Lem. 5.7.2] reveals, we may find a Cartan-Eilenberg resolution for $A_\bullet$ with only zeros in every second column. One therefore concludes $HC_\bullet^\partial(H) = \mathbf{Tor}(A_\partial, A_\bullet) = \bigoplus_{i \geq 0} HH_{\bullet - 2i}^\partial(H)$. □

## 6.4 Example: Lie-Rinehart Algebras

As we have seen, already at the Hochschild level the dual theory requires the full Hopf algebroid structure; as we learned from Proposition 4.2.9, we can equip $VL$ with such a structure by choosing a flat right $(A, L)$-connection $\partial$ on the base algebra $A$. Recall also that in contrast to the cohomology theory, we will need to consider the tensor product $VL \otimes^{rl} VL$ from (4.2.3).

**6.4.1 Theorem** *Let $(A, L)$ be a Lie-Rinehart algebra and $n \geq 1$. Under the same assumptions as in Theorem 5.5.7, the following holds.*

(*i*) *The composition of the antisymmetrisation map $\mathrm{Alt}_n : \wedge^n_A L \to VL^{\otimes^{ll} n}$ with the inverse Hopf-Galois map $\psi_n : VL^{\otimes^{ll} n} \to VL^{\otimes^{rl} n}$ from (6.1.6), more precisely, the map $\overline{\mathrm{Alt}}_n := \psi_n n! \,\mathrm{Alt}_n : \wedge^n_A L \to VL^{\otimes^{rl} n}$ given by*

$$X_1 \wedge \cdots \wedge X_n \mapsto \sum_{\sigma \in P(n)} \mathrm{sign}\,\sigma\, X^+_{\sigma(1)} \otimes^{rl} X^-_{\sigma(1)} X^+_{\sigma(2)} \otimes^{rl} \cdots \otimes^{rl} X^-_{\sigma(n-1)} X_{\sigma(n)},$$

*induces an isomorphism*

$$HH^\partial_\bullet(VL) \xleftarrow{\simeq} H^A_\bullet(L, A_\partial). \tag{6.4.1}$$

*Hence the dual (Hopf-)Hochschild homology of the Hopf algebroid $VL$ is isomorphic to the Lie-Rinehart homology of $L$ with values in its base algebra $A$.*

(*ii*) *Furthermore, for the dual Hopf-cyclic homology we have an isomorphism*

$$HC^\partial_\bullet(VL) \xleftarrow{\simeq} \bigoplus_{i \geq 0} H^A_{\bullet - 2i}(L, A_\partial).$$

PROOF: Part (*i*): to prove at first that the homology groups in (6.4.1) are isomorphic, it suffices to apply Theorem 6.2.3 to the case $H = VL$ and compare this to (5.5.3). Secondly, we show that the isomorphism is induced by $\overline{\mathrm{Alt}}$ by comparing the Koszul-Rinehart resolution $K^A_\bullet L = (VL \otimes_A \wedge^\bullet_A L, b'_{A,L})$ from (5.5.2) to the bar resolution $\mathrm{Bar}_\bullet VL = (VL^{\otimes^{rl} n+1}, b')$ from §6.2.2. The map $\mathrm{id}_{VL} \otimes n\overline{\mathrm{Alt}}_n : K^A_n L \to \mathrm{Bar}_n VL$ given by

$$\begin{aligned} &u \otimes_A X_1 \wedge \cdots \wedge X_n \mapsto \\ &\sum_{\sigma \in P(n)} \mathrm{sign}\,\sigma\, u \otimes^{rl} X^+_{\sigma(1)} \otimes^{rl} X^-_{\sigma(1)} X^+_{\sigma(2)} \otimes^{rl} \cdots \otimes^{rl} X^-_{\sigma(n-1)} X_{\sigma(n)} \end{aligned} \tag{6.4.2}$$

is obviously a map of left $VL$-modules (where the left $VL$-module structure on both sides is just multiplication on the first tensor factor $u$) and does not depend on $\partial$ any more:

$$X^+ \otimes^{rl} X^- = X \otimes^{rl} 1 - 1 \otimes^{rl} X,$$

see (4.2.4). Note that the 'conventional' antisymmetrisation Alt cannot be seen directly as a map from $\wedge^\bullet_A L$ into $VL^{\otimes^{rl} \bullet}$ since it would not be well-defined; whereas $\overline{\mathrm{Alt}}$ is. As an illustration, in degree $n = 2$, this reads

$$\overline{\mathrm{Alt}}(X \wedge Y) = X \otimes^{rl} Y - Y \otimes^{rl} X - 1 \otimes^{rl} [X, Y], \qquad X, Y \in L,$$

where $[X, Y] = XY - YX$ (as elements in $VL$). We now show that (6.4.2) is a chain map, i.e.,

$$b'\,(\mathrm{id}_{VL} \otimes \overline{\mathrm{Alt}}_n) = (\mathrm{id}_{VL} \otimes \overline{\mathrm{Alt}}_{n-1})\, b'_{A,L},$$

or, equivalently, and slightly simpler to see,

$$n(\mathrm{id}_{VL} \otimes \varphi_{n-1})\, b'\,(\mathrm{id}_{VL} \otimes \psi_n \,\mathrm{Alt}_n) = (\mathrm{id}_{VL} \otimes \mathrm{Alt}_{n-1})\, b'_{A,L},$$

where $\varphi = \psi^{-1}$ from (6.1.5). To start with, observe first that the right hand side $(\mathrm{id}_{VL} \otimes \mathrm{Alt}_{n-1})\, b'_{A,L} : K^A_n L \to VL^{\otimes^{ll} n}$ can be written as

$$\begin{aligned} &(\mathrm{id}_{VL} \otimes \mathrm{Alt}_{n-1})\, b'_{A,L}(u \otimes_A X_1 \wedge \cdots \wedge X_n) = \\ &\quad = \frac{1}{(n-1)!} \sum_{\sigma \in P(n)} \mathrm{sign}\,\sigma \big( u X_{\sigma(1)} \otimes^{rl} X_{\sigma(2)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \\ &\qquad - \sum_{i=1}^{n-1} (-1)^i u \otimes^{rl} X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(i)} X_{\sigma(i+1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \big), \end{aligned}$$

as a little thought reveals (or as proven easily by induction). As for the left hand side, with (4.2.16), (4.2.14) and $S(X^{(1)})X^{(2)} = 0$ for any $X \in L$, we calculate at first

$$\begin{aligned}
& nb' \, (\mathrm{id}_{VL} \otimes \psi_n \, \mathrm{Alt}_n)(u \otimes_A X_1 \wedge \cdots \wedge X_n) = \\
&= \frac{1}{(n-1)!} \sum_{\sigma \in P(n)} \operatorname{sign} \sigma \big( u X_{\sigma(1)} \otimes^{rl} X^+_{\sigma(2)} \otimes^{rl} X^-_{\sigma(2)} X^+_{\sigma(3)} \otimes^{rl} \cdots \otimes^{rl} X^-_{\sigma(n-1)} X_{\sigma(n)} \\
& \qquad - u \otimes^{rl} X_{\sigma(1)} X^+_{\sigma(2)} \otimes^{rl} X^-_{\sigma(2)} X^+_{\sigma(3)} \otimes^{rl} \cdots \otimes^{rl} X^-_{\sigma(n-1)} X_{\sigma(n)} \big) \\
&= \frac{1}{(n-1)!} \sum_{\sigma \in P(n)} \operatorname{sign} \sigma \big( u X_{\sigma(1)} \otimes^{rl} \psi_{n-1} (X_{\sigma(2)} \otimes^{ll} X_{\sigma(3)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)}) \\
& \qquad - u \otimes^{rl} X_{\sigma(1)} X^+_{\sigma(2)} \otimes^{rl} X^-_{\sigma(2)} X^+_{\sigma(3)} \otimes^{rl} \cdots \otimes^{rl} X^-_{\sigma(n-1)} X_{\sigma(n)} \big).
\end{aligned}$$

Using $\Delta^n_\ell(uv) = \Delta^n_\ell u \, \Delta^n_\ell v$ for $u, v \in VL$ and $n \geq 0$, one obtains for an element $X \in L$, as a primitive element,

$$\Delta^n_\ell X = \sum_{i=1}^{n+1} 1 \otimes^{ll} \cdots \otimes^{ll} 1 \otimes^{ll} \underset{i}{X} \otimes^{ll} 1 \otimes^{ll} \cdots \otimes^{ll} 1,$$

which we need to apply the explicit form of $\varphi$ from (6.1.6) to calculate

$$\begin{aligned}
& n(\mathrm{id}_{VL} \otimes \varphi_{n-1}) \, b' \, (\mathrm{id}_{VL} \otimes \psi_n \, \mathrm{Alt}_n)(u \otimes_A X_1 \wedge \cdots \wedge X_n) = \\
&= \frac{1}{(n-1)!} \sum_{\sigma \in P(n)} \operatorname{sign} \sigma \big( u X_{\sigma(1)} \otimes^{rl} X_{\sigma(2)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \\
& \qquad - u \otimes^{rl} (\Delta^{n-1}_\ell X_{\sigma(1)})(\Delta^{n-1}_\ell X^+_{\sigma(2)}) \cdots (\Delta_\ell X^-_{\sigma(n-2)} X^+_{\sigma(n-1)}) \cdot \\
& \qquad \qquad \cdot (1 \otimes^{ll} \cdots \otimes^{ll} 1 \otimes^{ll} X^-_{\sigma(n-1)} X_{\sigma(n)}) \big). \\
&= \frac{1}{(n-1)!} \sum_{\sigma \in P(n)} \operatorname{sign} \sigma \big( u X_{\sigma(1)} \otimes^{rl} X_{\sigma(2)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \\
& \qquad - \sum_{i=1}^{n} u \otimes^{rl} (1 \otimes^{ll} \cdots \otimes^{ll} 1 \otimes^{ll} \underset{i}{X_{\sigma(1)}} \otimes^{ll} 1 \otimes^{ll} \cdots \otimes^{ll} 1) \cdot \\
& \qquad \qquad \cdot \varphi_{n-1} \psi_{n-1} (X_{\sigma(2)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)}) \big) \\
&= \frac{1}{(n-1)!} \sum_{\sigma \in P(n)} \operatorname{sign} \sigma \big( u X_{\sigma(1)} \otimes^{rl} X_{\sigma(2)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \\
& \qquad - \sum_{i}^{n-1} (-1)^i u \otimes^{rl} X_{\sigma(1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(i)} X_{\sigma(i+1)} \otimes^{ll} \cdots \otimes^{ll} X_{\sigma(n)} \big),
\end{aligned}$$

hence the right hand side again. By routine homological algebra, the induced map $\overline{\mathrm{Alt}}_* = (\psi \, \mathrm{Alt})_*$ obtained by applying the functor $A_\partial \otimes_{VL} -$ induces the isomorphism (6.4.1) in homology.

Part (*ii*): clearly, $VL$ is cocommutative over commutative $A$ with equal source and target maps; with Theorem 6.3.3 and with part (*i*) the claim follows. □

Observe that part (*i*) of the preceding theorem is a generalisation of the classical Chevalley-Eilenberg theorem, cf. e.g. [CarE, Thm. 7.1] or [Lo1, Thm. 3.3.2].

## 6.5 Example: Jet Spaces

In this section we calculate the dual Hopf-cyclic homology for the jet spaces $JL$ of a Lie-Rinehart algebra $(A, L)$, where $L$ is finitely generated $A$-projective of constant rank. Again, the outcome is in a certain sense dual to the result in the previous subsection. Write $L^* := \mathrm{Hom}_{(A,-)}(L, A)$ and, as in Section 5.6, we write $(\wedge_A L^*, d_{A,L})$ for the Lie-Rinehart cochain complex (with values in $A$). Furthermore, recall from (6.1.2) that the dual Hopf-cyclic module is given as $JL^\natural_\partial = \{C^A_n JL\}_{n \geq 0}$, where the tensor product in $C^A_n JL = JL^{{}_A \otimes n}$ reads (cf. (6.1.1))

$$JL \,{}_A\!\otimes JL = {}_\blacktriangleright JL \otimes_{A^{\mathrm{op}}} JL_\triangleleft = JL \otimes_k JL / \mathrm{span}_k \{ \phi t^\ell_{JL} a \otimes_k \phi' - \phi \otimes t^\ell_{JL} a \phi', \ a \in A \}.$$

This makes sense since $A$ is commutative.

**6.5.1 Theorem** *Let $(A, L)$ be a Lie-Rinehart algebra where, as an $A$-module, $L$ is finitely generated projective of constant rank. There is a natural morphism of mixed complexes*

$$F : \big(C^A_\bullet JL, b, B\big) \to \big(\wedge^\bullet_A L^*, 0, d_{A,L}\big),$$

*defined in degree $n$ by*

$$F : (\phi^1 {}_A\otimes \cdots {}_A\otimes \phi^n)(X_1 \wedge \cdots \wedge X_n) := (-1)^n (S_{JL}\phi^1 \wedge \cdots \wedge S_{JL}\phi^n)(X_1, \ldots, X_n).$$

*This induces isomorphisms*

$$HH_\bullet(JL) \simeq \wedge^\bullet_A L^*,$$
$$HP_\bullet(JL) \simeq \prod_{i\geq 0} H^{\bullet+2i}(L, A),$$

*where the left hand side refers to the dual Hopf-cyclic homology groups.*

PROOF: This statement is very much the dual of Theorem 5.5.7. First of all, the dual of the PBW isomorphism yields $JL \simeq \hat{S}_A L^*$ as commutative algebras. Similar to Lemma 4.3.2, there is a canonical isomorphism

$$C^A_n JL \simeq \varprojlim_p \operatorname{Hom}_A\big((VL^{\otimes^{ll} n})_{\leq p}, A\big),$$

induced by the map

$$(\phi^1 \otimes_k \cdots \otimes_k \phi^n)(u_1 \otimes_k \cdots \otimes_k u_n) = (S_{JL}\phi^1)(u_1) \cdots (S_{JL}\phi^n)(u_n).$$

The antipode $S_{JL}$ here comes into play to go from $VL^{\otimes^{rr} n}$ to $VL^{\otimes^{ll} n}$, so as to give a sense to the map $F$. Since $JL$ is a commutative algebra, it maps the Hochschild differential $b$ to zero. Clearly, $F$ is a morphism of $A$-modules, where $A$ acts on $C^A_\bullet JL$ by multiplication by $t^\ell_{JL} a,\ a \in A$, on the first component. Therefore we can localise with respect to a maximal ideal $\mathfrak{m} \subset A$ to prove that $F$ is a quasi-isomorphism. Since $L$ is $A$-projective, $L_\mathfrak{m}$ is free of rank $r$ over $A_\mathfrak{m}$ and we choose a basis $e_i \in L_\mathfrak{m},\ e^i \in L^*_\mathfrak{m},\ i = 1, \ldots, r$. The Koszul resolution

$$0 \longleftarrow A_\mathfrak{m} \xleftarrow{\ \epsilon\ } JL_\mathfrak{m} \xleftarrow{\ \partial'\ } JL_\mathfrak{m} \otimes_{A_\mathfrak{m}} L^*_\mathfrak{m} \xleftarrow{\ \partial'\ } JL_\mathfrak{m} \otimes_{A_\mathfrak{m}} \wedge^2_{A_\mathfrak{m}} L^*_\mathfrak{m} \xleftarrow{\ \partial'\ } \cdots$$

is a resolution of $A_\mathfrak{m}$ in the category of left $JL$-modules with differential

$$\partial'(\phi \otimes \omega) = \sum_{i=1}^r e^i\phi \otimes \iota_{e_i}\omega.$$

The natural map $JL_\mathfrak{m} \otimes_{A_\mathfrak{m}} \wedge^\bullet_{A_\mathfrak{m}} L^*_\mathfrak{m} \to \operatorname{Bar}_\bullet(JL_\mathfrak{m})$ given by

$$\phi \otimes_{A_\mathfrak{m}} \alpha_1 \wedge \cdots \wedge \alpha_n := \phi \otimes_{A_\mathfrak{m}} (\alpha_1 \circ \operatorname{pr}) \wedge \cdots \wedge (\alpha_n \circ \operatorname{pr}),$$

is a morphism of complexes, as one easily checks. Since $S_{JL}(\alpha \circ \operatorname{pr}) = -\alpha \circ \operatorname{pr}$ for $\alpha \in L^*$, the map $\operatorname{id} \otimes F_\mathfrak{m} : \operatorname{Bar}_\bullet(JL_\mathfrak{m}) \to JL_\mathfrak{m} \otimes_{A_\mathfrak{m}} \wedge^\bullet_{A_\mathfrak{m}} L^*_\mathfrak{m}$ is a right inverse and induces the morphism $F$ when taking the tensor product $A_\mathfrak{m} \otimes_{JL_\mathfrak{m}} -$ on both sides. This proves the first claim.

As for the second, recall the standard $A$-ring cyclic structure $JL^\natural_A := \{B^A_n JL\}_{n\geq 0}$ for $JL$ as in §1.2.4. Notice that since $JL$ is commutative, one simply has $B^A_n JL \simeq C^A_{n+1} JL$ and the map to invariants $\Psi_\epsilon : C^A_n(JL) \to C^A_{n+1}(JL)$ (see Subsection 6.1.1) is a morphism of cyclic modules. Explicitly, when restricted to $L^*$ this map is given by

$$\begin{aligned}
\Psi_\epsilon(\phi^1 {}_A\otimes \cdots {}_A\otimes \phi^n)&(X_1 \otimes \cdots \otimes X_{n+1}) = \\
&= (S_{JL}\phi^1_{(1)})(X_1) \cdots (S_{JL}\phi^n_{(1)})(X_n) \nabla^\ell_{X_{n+1}}(\phi^1_{(2)} \cdots \phi^n_{(2)})(1) \\
&= \sum_{i=1}^n \big((S_{JL}\phi^1)(X_1) \cdots \widehat{(S_{JL}\phi^i)(X_i)} \cdots (S_{JL}\phi^n)(X_n)\big) \cdot \\
&\qquad \cdot (S_{JL}\phi^i_{(1)})(X_i)\big(\epsilon_{VL}(X_{n+1}\phi^i_{(2)}(1)) - \phi^i_{(2)}(X_{n+1})\big).
\end{aligned}$$

Since the cyclic structure on $C^A_{\bullet+1}(JL)$ depends only on the structure of $JL$ as a commutative algebra, it is well-known that the morphism

$$\phi^1 {}_A\!\otimes \cdots {}_A\!\otimes \phi^{n+1} \mapsto \phi^{n+1} d_{A,L}\phi^1 \wedge \cdots \wedge d_{A,L}\phi^n$$

induces a morphism of mixed complexes $\big(C^A_\bullet(JL)[1], b, B\big) \to \big(\wedge^\bullet_A L^*, 0, d_{A,L}\big)$ (cf. Example 1.1.10(*ii*) for a similar consideration). Composing this morphism with $\Psi_\epsilon$ as above, one finds exactly the map stated in the theorem. This proves that $F$ intertwines the $B$-operator with the Lie-Rinehart coboundary $d_{A,L}$. Since we already know that this map is a quasi-isomorphism on the level of Hochschild homology, the $SBI$-sequence implies that it is a quasi-isomorphism on the level of cyclic homology. This proves the theorem. □

## 6.6 Example: Convolution Algebras

In this section we compute the dual Hopf-cyclic homology for the example of the convolution algebra $\mathcal{C}^\infty_c(G)$ for an étale groupoid $s, t : G \rightrightarrows G_0$ (see Section 4.4); we will see that it coincides with groupoid homology, see Theorem 6.6.4. To this end, let us first recall some material we need for the latter.

**6.6.1 Nerve of a Groupoid** For a groupoid $G \rightrightarrows G_0$, denote by

$$G_n = \{(g_1, \ldots, g_n) \in G^{\times n} \mid s(g_i) = t(g_{i+1}), 1 \leq i \leq n-1\}$$

for $n \geq 1$ the space of strings $\vec{g}$ of $n$ composable arrows,

$$\vec{g} = (\cdot \xleftarrow{g_1} \cdot \xleftarrow{g_2} \cdot \ldots \cdot \xleftarrow{g_n} \cdot).$$

This is consistent with the notation for $G_2$ and $G_1$ already introduced. The *nerve* of a groupoid is the simplicial space

$$G_\bullet : \ \ldots \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} G_2 \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} G_1 \ \rightrightarrows G_0 \ ,$$

formed by the family $G_\bullet := \{G_n\}_{n\geq 0}$, where $G_0$ is the base manifold as before, and the face operators $d_i : G_n \to G_{n-1}$ are given by

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 1 \leq i \leq n-1, \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n, \end{cases} \tag{6.6.1}$$

whereas $d_0, d_1 : G_1 \to G_0$ are given by source and target map, respectively. Moreover, the degeneracy operators $s_i : G_n \to G_{n+1}$ are

$$s_i(g_1, \ldots, g_n) = \begin{cases} (1_{t(g_1)}, g_1, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_i, 1_{s(g_i)}, g_{i+1}, \ldots, g_n) & \text{if } 1 \leq i \leq n. \end{cases} \tag{6.6.2}$$

Finally, for all $n \geq 2$ define an operator $t_n : G_n \to G_n$,

$$t_n(g_1, \ldots, g_n) = ((g_1 g_2 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1}), \tag{6.6.3}$$

and set $t_1(g) = g^{-1}$ and $t_0 = \mathrm{id}_{G_0}$. Then it is easy to see that the set of operators $(d_\bullet, s_\bullet, t_\bullet)$ defines a cyclic structure on $G_\bullet$.

**6.6.2 Bar Complex for Groupoids** Let $G \rightrightarrows G_0$ be an étale groupoid and assume that $\mathcal{F} \in \mathrm{Sh}(G_0)$ is a $c$-soft $G$-sheaf (cf. [Br]). Consider the map $\tau_n : G_n \to G_0$, $\vec{g} \mapsto t(g_1)$ for $n \geq 1$, where $\vec{g} = (\cdot \xleftarrow{g_1} \cdot \xleftarrow{g_2} \cdot \ldots \cdot \xleftarrow{g_n} \cdot)$ and set $\tau_0 := \mathrm{id}_{G_0}$. Observe that for all $n \geq 0$ the pull-back sheaves $\mathcal{F}^n := \tau_n^{-1}\mathcal{F}$ on $G_n$ are again $c$-soft since $\tau_n$ is étale. The family $\{\Gamma_c(G_n, \mathcal{F}^n)\}_{n\geq 0}$ of groups of compactly supported sections form a simplicial abelian group

$$B_\bullet(G, \mathcal{F}) : \qquad \ldots \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \Gamma_c(G_2, \mathcal{F}^2) \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \Gamma_c(G_1, \mathcal{F}^1) \rightrightarrows \Gamma_c(G_0, \mathcal{F}^0) \ ,$$

with simplicial operators defined as follows: for the face maps $d_i : G_n \to G_{n-1}$ from (6.6.1) of the nerve, one obtains the isomorphism $\mathcal{F}^n \to d_i^{-1}\mathcal{F}^{n-1}$, the stalk of which at $\vec{g}$ is given by $\mathcal{F}^n_{\vec{g}} = \mathcal{F}_{t(g_1)} \to \mathcal{F}_{t(g_1)} =$

$d_i^{-1}\mathcal{F}_{\vec{g}}^{n-1}$ for $i \neq 0$, hence by the identity; for $i = 0$, however, the stalk at $\vec{g}$ is $\mathcal{F}_{\vec{g}}^n = \mathcal{F}_{t(g_1)} \to \mathcal{F}_{s(g_1)} = d_0^{-1}\mathcal{F}_{\vec{g}}^{n-1}$, which is the (right) action $R$ by $g_1$. Analogously, one has the isomorphisms $\mathcal{F}^n \to s_i^{-1}\mathcal{F}^{n+1}$, given for all $i$ by the identity map. The face and degeneracy operators on $B_\bullet(G,\mathcal{F})$ now read

$$\begin{aligned} d_i(u \mid g_1,\ldots,g_n) &= \begin{cases} (ug_1 \mid g_2,\ldots,g_n) & \text{if } i=0,\\ (u \mid g_1,\ldots,g_ig_{i+1},\ldots,g_n) & \text{if } 1 \leq i \leq n-1,\\ (u \mid g_1,\ldots,g_{n-1}) & \text{if } i=n, \end{cases}\\ s_i(u \mid g_1,\ldots,g_n) &= \begin{cases} (u \mid 1_{t(g_1)}, g_1,\ldots,g_n) & \text{if } i=0,\\ (u \mid g_1,\ldots,g_i,1_{s(g_i)},g_{i+1},\ldots,g_n) & \text{if } 1 \leq i \leq n. \end{cases} \end{aligned} \tag{6.6.4}$$

A similar argument using (6.6.3) shows that there is an isomorphism $\mathcal{F}^n \to t_n^{-1}\mathcal{F}^n$ with stalk at $\vec{g}$ given by $\mathcal{F}_{\vec{g}}^n = \mathcal{F}_{t(g_1)} \to \mathcal{F}_{s(g_n)} = t_n^{-1}\mathcal{F}_{\vec{g}}^n$, i.e. the right action $R$ by $g_1 \cdots g_n$. One correspondingly defines

$$t_n(u \mid g_1,\ldots,g_n) = (ug_1\cdots g_n \mid (g_1\cdots g_n)^{-1}, g_1,\ldots,g_{n-1}) \tag{6.6.5}$$

as a cyclic operator on $B_\bullet(G,\mathcal{F})$. For later computations, we remind that our notation (cf. §1.5.3) explicitly reads here

$$t_n : \Gamma_c(G_n,\mathcal{F}^n) \to \Gamma_c(G_n,\mathcal{F}^n), \quad ((t_n,R)_*u)(\vec{g}) = \sum_{\vec{g}=t_n(\vec{g}')} u(\vec{g}')g_1'\cdots g_n'$$

for $\vec{g}' = (g_1',\ldots,g_n') \in G_n$.

**6.6.3 Lemma** [Cr1, Lem. 3.2.9] *If $\mathcal{F}$ is a c-soft $G$-sheaf, the cyclic space $B_\bullet(G,F)$ with the operators (6.6.4) and (6.6.5) computes the homology groups $HH_\bullet(G,\mathcal{F})$, $HC_\bullet(G,\mathcal{F})$ and $HP_\bullet(G,\mathcal{F})$.*

Here we only presented a simplified version sufficient for our needs; see however [Cr1] and [CrMoe2] for full details and generality introducing cyclic groupoids and cyclic sheaves.

Let us now turn to dual Hopf-cyclic homology. We will frequently need the following $n$-fold generalisations of the isomorphisms introduced in (4.4.1),

$$\begin{aligned} \Omega^n_{s,t} &: \overbrace{\mathcal{C}^\infty_c(G) \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \mathcal{C}^\infty_c(G)}^{n \text{ times}} \xrightarrow{\simeq} \mathcal{C}^\infty_c(\overbrace{G^s{\times}^t_{G_0} \cdots {}^s{\times}^t_{G_0} G}^{n \text{ times}}) = \mathcal{C}^\infty_c(G_n),\\ \Omega^n_{t,t} &: \mathcal{C}^\infty_c(G) \otimes^{ll}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{ll}_{\mathcal{C}^\infty(G_0)} \mathcal{C}^\infty_c(G) \xrightarrow{\simeq} \mathcal{C}^\infty_c(G^t{\times}^t_{G_0} \cdots {}^t{\times}^t_{G_0} G) = \mathcal{C}^\infty_c(G^n),\\ \Omega^n_{s,s} &: \mathcal{C}^\infty_c(G) \otimes^{rr}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rr}_{\mathcal{C}^\infty(G_0)} \mathcal{C}^\infty_c(G) \xrightarrow{\simeq} \mathcal{C}^\infty_c(G^s{\times}^s_{G_0} \cdots {}^s{\times}^s_{G_0} G),\\ \Omega^n_{t,s} &: \mathcal{C}^\infty_c(G) \otimes^{lr}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{lr}_{\mathcal{C}^\infty(G_0)} \mathcal{C}^\infty_c(G) \xrightarrow{\simeq} \mathcal{C}^\infty_c(G^t{\times}^s_{G_0} \cdots {}^t{\times}^s_{G_0} G) \end{aligned}$$

all given by the formula

$$\Omega^n_{\cdot,\cdot}(u_1 \otimes^{\cdot\cdot}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{\cdot\cdot}_{\mathcal{C}^\infty(G_0)} u_n)(g_1,\ldots,g_n) = u_1(g_1)u_2(g_2)\cdots u_n(g_n), \tag{6.6.6}$$

for $u_1,\ldots,u_n \in \mathcal{C}^\infty_c(G)$ and $(g_1,\ldots,g_n)$ in the respective pull-back $G^\cdot{\times}^\cdot_{G_0} \cdots {}^\cdot{\times}^\cdot_{G_0} G$. One can also decompose

$$\Omega^n_{\cdot,\cdot} = \Omega^2_{\cdot,\cdot}(\mathrm{id} \otimes \Omega^{n-1}_{\cdot,\cdot}),$$

where in this case $\Omega^2_{\cdot,\cdot}$ has the obvious meaning as an isomorphism

$$\Omega^2_{\cdot,\cdot} : \mathcal{C}^\infty_c(G) \otimes^{\cdot\cdot}_{\mathcal{C}^\infty(G_0)} \mathcal{C}^\infty_c(\overbrace{G^\cdot{\times}^\cdot_{G_0} \cdots {}^\cdot{\times}^\cdot_{G_0} G}^{n-1 \text{ times}}) \xrightarrow{\simeq} \mathcal{C}^\infty_c(\overbrace{G^\cdot{\times}^\cdot_{G_0} \cdots {}^\cdot{\times}^\cdot_{G_0} G}^{n \text{ times}}).$$

We hope that the notation by the same symbol does not create too much confusion; the individual meaning will be clear from the context. We also remind the reader of the possibility of 'mixing' these maps, as in (4.4.3). The space of interest for dual cyclic homology is then

$$\mathcal{C}^\infty_c(G)^\natural_\partial := \{C_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_c(G)\}_{n\geq 0}$$

where

$$C^n_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_c(G) = \mathcal{C}^\infty_c(G) \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \mathcal{C}^\infty_c(G), \tag{6.6.7}$$

again $n$ times in degree $n$.

**6.6.4 Theorem** *For any étale groupoid $G \rightrightarrows G_0$ over a compact manifold $G_0$, the set of simplicial and cyclic operators $(d_\bullet, s_\bullet, t_\bullet)$ on the nerve $G_\bullet$ makes $(\mathcal{C}^\infty_c(G_\bullet), d_{\bullet+}, s_{\bullet+}, t_{\bullet+})$ a cyclic vector space, which is isomorphic to the Hopf-cyclic module $\mathcal{C}^\infty_c(G)^\natural_\partial$. Hence*

$$\begin{array}{rcl} HH^\partial_\bullet(\mathcal{C}^\infty_c(G)) & \simeq & HH_\bullet(G, \mathcal{C}^\infty_G), \\ HC^\partial_\bullet(\mathcal{C}^\infty_c(G)) & \simeq & HC_\bullet(G, \mathcal{C}^\infty_G), \\ HP^\partial_\bullet(\mathcal{C}^\infty_c(G)) & \simeq & HP_\bullet(G, \mathcal{C}^\infty_G). \end{array}$$

PROOF: Since $G$ is étale, note first that the bar complex $B_\bullet(G, \mathcal{C}^\infty_G)$ is formed here by the pull-back sheaves $\tau_n^{-1}\mathcal{C}^\infty_G \simeq \mathcal{C}^\infty_{G_n}$. In degree $n$ one has $B_n(G, \mathcal{C}^\infty_G) \simeq \Gamma_c(G_n, \mathcal{C}^\infty_{G_n}) = \mathcal{C}^\infty_c(G_n)$. Hence the fact that $(\mathcal{C}^\infty_c(G), d_{\bullet+}, s_{\bullet+}, t_{\bullet+})$ is a cyclic vector space follows from the general considerations in §6.6.2, see §1.5.3 for the notation. To show that it is isomorphic to the Hopf-cyclic module $\mathcal{C}^\infty_c(G)^\natural_\partial$, it only remains to prove that

$$\begin{array}{rcl} d_{i+}\,\Omega_{s,t} & = & \Omega_{s,t}\,\tilde{d}_i, \\ s_{i+}\,\Omega_{s,t} & = & \Omega_{s,t}\,\tilde{s}_i, \\ t_{n+}\,\Omega_{s,t} & = & \Omega_{s,t}\,\tilde{t}_n, \end{array}$$

for all $0 \leq i \leq n$ and in all degrees $n$, where $(\tilde{d}_\bullet, \tilde{s}_\bullet, \tilde{t}_\bullet)$ denote in this proof the Hopf-cyclic operators from (6.1.3)–(6.1.4) for $\mathcal{C}^\infty_c(G)^\natural_\partial$. For $u^1 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^n \in C_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_c(G)$ and $(g_1, \ldots, g_{n-1}) \in G_{n-1}$, we compute

$$\begin{array}{l} (d_{0+}\,\Omega^n_{s,t}(u^1 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^n))(g_1, \ldots, g_{n-1}) \\ \quad = \displaystyle\sum_{\{g_0 \in G \mid s(g_0) = t(g_1)\}} \Omega^n_{s,t}(u^1 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^n)(g_0, g_1, \ldots, g_{n-1}) \\ \quad = \displaystyle\sum_{\{g_0 \in G \mid s(g_0) = t(g_1)\}} u^1(g_0)u^2(g_1)\cdots u^n(g_{n-1}) \\ \quad = \partial u^1(1_{t(g_1)})u^2(g_1)\cdots u^n(g_{n-1}) \\ \quad = (\partial u^1 * u^2)(g_1)\cdots u^n(g_{n-1}) \\ \quad = (\Omega^{n-1}_{s,t}\,\tilde{d}_0(u^1 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^n))(g_1, \ldots, g_{n-1}), \end{array}$$

and likewise for all remaining face and degeneracy operators. As far as the cyclic operator is concerned, one has

$$\begin{array}{l} (t_{n+}\,\Omega^n_{s,t}(u^1 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^n))(g_1, \ldots, g_n) \\ \quad = \displaystyle\sum_{t_n(g'_1, \ldots, g'_n) = (g_1, \ldots, g_n)} \!\!\cdots\!\! \sum u^1(g'_1)u^2(g'_2)\cdots u^n(g'_n) \\ \quad = u^1(g_2)\cdots u^{n-1}(g_n)u^n((g_1\cdots g_n)^{-1}), \end{array}$$

since in the sum one has $(g'_1 \cdots g'_n)^{-1} = g_1$, and moreover $g'_1 = g_2, \ldots, g'_{n-1} = g_n$; hence $g'_n = (g'_1 \cdots g'_{n-1})^{-1}g_1^{-1} = (g_1 \cdots g_{n-1})^{-1}$. On the other hand,

$$\begin{array}{l} (\Omega^n_{s,t}\,\tilde{t}_n(u^1 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^n))(g_1, \ldots, g_n) \\ = \Omega^n_{s,t}(S^{-1}(u^1_{(2)} * \cdots * u^{n-1}_{(2)} * u^n) \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^1_{(1)} \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^{n-1}_{(1)})(g_1, \ldots, g_n) \\ = (u^1_{(2)} * \cdots * u^{n-1}_{(2)} * u^n)(g_1^{-1})u^1_{(1)}(g_2)\cdots u^{n-1}_{(1)}(g_n) \\ = \displaystyle\sum_{g_1^{-1} = g'_1 \cdots g'_n} u^1_{(2)}(g'_1)\cdots u^{n-1}_{(2)}(g'_{n-1})u^n(g'_n)u^1_{(1)}(g_2)\cdots u^{n-1}_{(1)}(g_n) \\ = u^1(g_2)\cdots u^{n-1}(g_n)u^n((g_1\cdots g_n)^{-1}), \end{array}$$

by the left coproduct (4.4.6), which dictates $g'_{j-1} = g_j$ for all $2 \leq j \leq n$, whereas for the last element one has $g'_n = (g_1 g'_1 \cdots g'_{n-1})^{-1}$. □

Taking into account that $\mathcal{C}^\infty_c(G)$ is a cocommutative Hopf algebroid over $\mathcal{C}^\infty(G_0)$ (see Section 4.4), Theorem 6.3.3 immediately yields:

**6.6.5 Corollary** *For each étale groupoid $G$ over compact base, one has*

$$HC_n(G, \mathcal{C}_G^\infty) \simeq \bigoplus_{i\geq 0} HH_{2n-i}(G, \mathcal{C}_G^\infty).$$

We end this subsection by showing that the Hopf-Galois maps (6.1.5) and (6.1.6) are dual to certain maps defined on the groupoid level. Recall that for the coordinate ring $U = k[G]$ of an algebraic semigroup, the Hopf-Galois map $\beta$ from (2.2.1) is dual to the map

$$G \times G \to G \times G, \quad (g,h) \mapsto (g, gh),$$

which is bijective if and only if $G$ is a group. For groupoids, one obtains similar maps: making use of the maps $\Omega_{s,t}$, $\Omega_{t,t}$ and denoting (as before) $G_n = G\,{}^s\!\times^t_{G_0} \cdots {}^s\!\times^t_{G_0} G$ ($n$ times) and $G^n = G\,{}^t\!\times^t_{G_0} \cdots {}^t\!\times^t_{G_0} G$ ($n$ times), one finds

**6.6.6 Proposition** *The vector space isomorphisms (6.1.5) and (6.1.6) between $C_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G)$ from (6.6.7) and $C^n_{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G)$ from (5.7.2) are dual to the following diffeomorphisms on groupoids:*

$$\tilde\varphi_n : G_n \to G^n, \quad (g_1, \ldots, g_n) \mapsto (g_1, g_1g_2, \ldots, g_1g_2\cdots g_n),$$

*with inverse*

$$\tilde\psi_n : G^n \to G_n, \quad (\tilde g_1, \ldots, \tilde g_n) \mapsto (\tilde g_1, \tilde g_1^{-1}\tilde g_2, \ldots, \tilde g_{n-1}^{-1}\tilde g_n).$$

*More precisely, in each degree $n$ one has the commutative diagram*

$$\begin{tikzcd}
C_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G) \arrow[r, "\Omega^n_{s,t}"] \arrow[d, shift right, "\varphi_n"'] & \mathcal{C}_c^\infty(G_n) \arrow[d, shift right, "\tilde\varphi_{n+}"'] \\
C^n_{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G) \arrow[r, "\Omega^n_{t,t}"'] \arrow[u, shift right, "\psi_n"'] & \mathcal{C}_c^\infty(G^n). \arrow[u, shift right, "\tilde\psi_{n+}"']
\end{tikzcd}$$

PROOF: For $u^1, \ldots, u^n \in \mathcal{C}_c^\infty(G)$, $(\tilde g_1, \ldots, \tilde g_n) \in G^n$, we clearly have

$$\begin{aligned}
(\tilde\varphi_{n+}\,\Omega^n_{s,t}(u^1 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^n))(\tilde g_1, \ldots, \tilde g_n) &= \\
&= \sum_{\tilde\varphi_n(g_1,\ldots,g_n)=(\tilde g_1,\ldots,\tilde g_n)} \!\!\!\cdots \sum u^1(g_1)\cdots u^n(g_n) \\
&= \sum_{(g_1,\ldots,g_n)=\tilde\psi_n(\tilde g_1,\ldots,\tilde g_n)} \!\!\!\cdots \sum u^1(g_1)\cdots u^n(g_n) \\
&= u^1(\tilde g_1)u^2(\tilde g_1^{-1}\tilde g_2)\cdots u^n(\tilde g_{n-1}^{-1}\tilde g_n),
\end{aligned}$$

but also

$$\begin{aligned}
&(\Omega^n_{t,t}\,\varphi_n(u^1 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u^n))(\tilde g_1, \ldots, \tilde g_n) = \\
&= u^1_{(1)}(\tilde g_1)(u^1_{(2)} * u^2_{(1)})(\tilde g_2)\cdots(u^1_{(n)} * u^2_{(n-1)} * \cdots * u^{n-1}_{(2)}u^n)(\tilde g_n) \\
&= u^1_{(1)}(\tilde g_1) \sum_{\tilde g_2 = \tilde g_{2_1}\tilde g_{2_2}} u^1_{(2)}(\tilde g_{2_1})u^2_{(1)}(\tilde g_{2_2}) \cdots \\
&\qquad \cdots \sum_{\tilde g_n = \tilde g_{n_1}\cdots\tilde g_{n_n}} u^1_{(n)}(\tilde g_{n_1})u^2_{(n-1)}(\tilde g_{n_2})\cdots u^{n-1}_{(2)}(\tilde g_{n_{n-1}})u^n(\tilde g_{n_n}) \\
&= u^1(\tilde g_1)u^2(\tilde g_1^{-1}\tilde g_2)\cdots u^n(\tilde g_{n-1}^{-1}\tilde g_n).
\end{aligned}$$

This is seen as follows: as a first step, by the higher coproducts only elements with $\tilde g_1 = \tilde g_{2_1} = \ldots = \tilde g_{n_1}$ do not disappear; as a second step, one finds $\tilde g_{2_2} = \tilde g_1^{-1}\tilde g_2$, and only elements with $\tilde g_1^{-1}\tilde g_2 = \tilde g_{3_2} = \ldots = \tilde g_{n_2}$ do not vanish. Hence $\tilde g_{3_3} = (\tilde g_{3_1}\tilde g_{3_2})^{-1}\tilde g_3 = (\tilde g_1\tilde g_1^{-1}\tilde g_2)^{-1}\tilde g_3 = \tilde g_2^{-1}\tilde g_3$. By induction on $n$ steps one concludes that the non-vanishing elements have $\tilde g_{j_j} = \tilde g_{j-1}^{-1}\tilde g_j$ for all $j = 2, \ldots, n$ which are the arguments in the first components of the respective left coproducts. The same diagram can, of course, analogously be verified in the opposite direction using $\psi_n$ and $\tilde\psi_n$. $\square$

### 6.6.1 Invariants for the Convolution Algebra

In this subsection we are going to explicitly show how the dual Hopf-cyclic module $\mathcal{C}_c^\infty(G)_\partial^\natural$ can be obtained by restricting the standard cyclic $\mathcal{C}^\infty(G_0)$-ring structure of $\mathcal{C}_c^\infty(G)^\natural_{\mathcal{C}^\infty(G_0)}$ (given by (1.2.3)) to invariants, cf. (2.4.1). As in §1.2.4, set

$$\mathcal{C}_c^\infty(G)^\natural_{\mathcal{C}^\infty(G_0)} = \{B_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G)\}_{n\geq 0},$$

where in degree $n \geq 0$

$$B_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G) = C_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G)\otimes_{\mathcal{C}^\infty(G_0)\otimes\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty(G_0) = C_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G)\otimes_{\mathcal{C}^\infty(G_0)}.$$

In Theorem 6.6.4 we saw that dual Hopf-cyclic operators correspond to the respective cyclic operators on the nerve of $G$ as given in (6.6.1)–(6.6.3). The cyclic operators (1.2.3) on $\mathcal{C}_c^\infty(G)^\natural_{\mathcal{C}^\infty(G_0)}$, on the other hand, correspond to a different set of cyclic operators given on a certain subspace of the nerve. Let us introduce this subspace first:

**6.6.7 Burghelea Spaces** [Bu] For an (étale) groupoid $G \rightrightarrows G_0$, for $n \geq 0$ define

$$B_n := \{(g_0, g_1, \ldots, g_n) \in G^{\times(n+1)} \mid t(g_i) = s(g_{i-1}) \text{ for } 1 \leq i \leq n, \text{ and } t(g_0) = s(g_n)\}, \qquad (6.6.8)$$

the space of *closed* strings of $n+1$ composable arrows; note that $B_n \subset G_{n+1}$. The space $B_0 = \{g \in G | s(g) = t(g)\}$ is called the *space of loops* in $G$. The family $B_\bullet := \{B_n\}_{n\geq 0}$, which we will call the *Burghelea space*, can be turned into a simplicial space by defining face and degeneracy operators $d'_i : B_n \to B_{n-1}$, $s'_i : B_n \to B_{n+1}$, respectively, by

$$\begin{aligned} d'_i(g_0, g_1, \ldots, g_n) &= \begin{cases} (g_0, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 \leq i \leq n-1, \\ (g_n g_0, g_1, \ldots, g_{n-1}) & \text{if } i = n, \end{cases} \\ s'_i(g_0, g_1, \ldots, g_n) &= \begin{cases} (g_0, \ldots, g_i, 1_{t(g_{i+1})}, g_{i+1}, \ldots, g_n) & \text{if } 0 \leq i \leq n-1, \\ (g_0, \ldots, g_n, 1_{s(g_n)}) & \text{if } i = n. \end{cases} \end{aligned} \qquad (6.6.9)$$

Together with the cyclic operator $t'_n : B_n \to B_n$ given by

$$t'_n(g_0, \ldots, g_n) = (g_n, g_0, \ldots, g_{n-1}), \qquad (6.6.10)$$

it is easy to see that the set of operators $(d'_\bullet, s'_\bullet, t'_\bullet)$ defines a cyclic structure on $B_\bullet$. In particular, one has a natural inclusion of the nerve $G_n$:

$$i : G_n \hookrightarrow B_n, \quad (g_1, \ldots, g_n) \mapsto ((g_1 g_2 \cdots g_n)^{-1}, g_1, \ldots, g_n), \qquad (6.6.11)$$

which is compatible with the sets of operators $(d_\bullet, s_\bullet, t_\bullet)$ from (6.6.1)–(6.6.3) and $(d'_\bullet, s'_\bullet, t'_\bullet)$ (i.e. with the simplicial and cyclic structures, respectively).

**6.6.8 Proposition** *The set of simplicial and cyclic operators $(d'_\bullet, s'_\bullet, t'_\bullet)$ on the Burghelea space $B_\bullet$ makes $(\mathcal{C}_c^\infty(B_\bullet), d'_{\bullet+}, s'_{\bullet+}, t'_{\bullet+})$ a cyclic vector space. In each degree $n \geq 0$, one has*

$$B_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}_c^\infty(G) \simeq \mathcal{C}_c^\infty(B_n).$$

*In particular, $(\mathcal{C}_c^\infty(B_\bullet), d'_{\bullet+}, s'_{\bullet+}, t'_{\bullet+})$ is isomorphic to the standard $\mathcal{C}^\infty(G_0)$-ring cyclic module $\mathcal{C}_c^\infty(G)^\natural_{\mathcal{C}^\infty(G_0)}$ from §1.2.4, with operators given in (1.2.3).*

PROOF: The fact that $(\mathcal{C}_c^\infty(B_\bullet), d'_{\bullet+}, s'_{\bullet+}, t'_{\bullet+})$ is a cyclic module follows from our considerations below. It can be also shown by either a direct verification of the simplicial relations plus the additional ones (1.1.3), (1.1.4), (1.1.5) for a cyclic module, or by deducing it from the cyclicity of $B_\bullet$ and accounting for the fact that the operation of fibre summing (1.5.1) is associative in a sense. Explicitly, for any $n \geq 0$ and $u \in \mathcal{C}_c^\infty(B_n)$, $(g_0, \ldots, g_{n-1}) \in B_{n-1}$,

$$\begin{aligned} (d'_{0+}(t'_{n+}u))(g_0, \ldots, g_{n-1}) &= \sum_{d'_0(g'_0, \ldots, g'_n) = (g_0, \ldots, g_{n-1})} \!\!\cdots\!\! \sum (t'_{n+}u)(g'_0, \ldots, g'_n) \\ &= \sum_{g_0 = g'_0 g'_1} u(g'_1, g_1, \ldots, g_{n-1}, g'_0) \\ &= \sum_{(g'_n g'_0, g'_1, \ldots, g'_{n-1}) = (g_0, \ldots, g_{n-1})} \!\!\cdots\!\! \sum u(g'_0, \ldots, g'_n) \\ &= \sum_{d'_0\, t'_n(g'_0, \ldots, g'_n) = (g_0, \ldots, g_{n-1})} \!\!\cdots\!\! \sum u(g'_0, \ldots, g'_n) \\ &= ((d'_{0+}\, t'_{n+})u)(g_0, \ldots, g_{n-1}), \end{aligned}$$

and similarly for all other relations. Hence the cyclicity of $\mathcal{C}^\infty_{\rm c}(B_\bullet)$ follows, as could have been expected, from the cyclicity of $B_\bullet$. As for the second part of the Proposition, write

$$\begin{aligned} B_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G) &= \overbrace{\mathcal{C}^\infty_{\rm c}(G)\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\cdots\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G)}^{n+1 \text{ times}}\otimes_{\mathcal{C}^\infty(G_0)\otimes\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty(G_0) \\ &\simeq \mathcal{C}^\infty_{\rm c}(\underbrace{G^s\times^t_{G_0}\cdots{}^s\times^t_{G_0}G}_{n \text{ times}})\otimes^{rl,lr}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G), \end{aligned}$$

where on the first $n$ factors the isomorphism $\Omega^{n-1}_{s,t}$ was used, and the last tensor product has the obvious meaning of simultaneously balancing with respect to $\otimes^{rl}_{\mathcal{C}^\infty(G_0)}$ and $\otimes^{lr}_{\mathcal{C}^\infty(G_0)}$. The notation $\otimes^{rl,lr}_{\mathcal{C}^\infty(G_0)}$ suggests that a map $\Omega^2_{s,t\,;\,t,s}$ acting simultaneously as $\Omega^2_{s,t}$ as well as $\Omega^2_{t,s}$ by still the same formula (6.6.6) gives an isomorphism, so that

$$B_{n+1}^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G)\simeq\mathcal{C}^\infty_{\rm c}((G^s\times^t_{G_0}\cdots{}^s\times^t_{G_0}G)^{s;t}\times^{t;s}_{G_0}G)\simeq\mathcal{C}^\infty_{\rm c}(B_n).$$

Conceptually a repetition of what was implicit at the beginning of this proof, it remains to show that

$$\begin{aligned} d'_{i+}\,\Omega^2_{s,t\,;\,t,s}(\Omega^n_{s,t}\otimes\mathrm{id}) &= \Omega^2_{s,t\,;\,t,s}(\Omega^{n-1}_{s,t}\otimes\mathrm{id})\,\tilde{d}'_i, \\ s'_{i+}\,\Omega^2_{s,t\,;\,t,s}(\Omega^n_{s,t}\otimes\mathrm{id}) &= \Omega^2_{s,t\,;\,t,s}(\Omega^{n+1}_{s,t}\otimes\mathrm{id})\,\tilde{s}'_i, \\ t'_{n+}\,\Omega^2_{s,t\,;\,t,s}(\Omega^n_{s,t}\otimes\mathrm{id}) &= \Omega^2_{s,t;\,t,s}(\Omega^n_{s,t}\otimes\mathrm{id})\,\tilde{t}'_n, \end{aligned}$$

for all $0\le i\le n$ and all degrees $n$; in this proof, $(\tilde{d}'_\bullet,\tilde{s}'_\bullet,\tilde{d}'_\bullet)$ are the standard $\mathcal{C}^\infty(G_0)$-ring cyclic operators from (1.2.3). For example, for $u^0\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\cdots\otimes^{rl}_{\mathcal{C}^\infty(G_0)}u^{n-1}\otimes^{rl,lr}_{\mathcal{C}^\infty(G_0)}u^n\in B_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G)$ and $(g_0,\ldots,g_{n-1})\in B_{n-1}$, one has

$$\begin{aligned} &(d'_{n+}\,\Omega^2_{s,t\,;\,t,s}(\Omega^n_{s,t}\otimes\mathrm{id})(u^0\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\cdots\otimes^{rl}_{\mathcal{C}^\infty(G_0)}u^{n-1}\otimes^{rl,lr}_{\mathcal{C}^\infty(G_0)}u^n))(g_0,\ldots,g_{n-1}) \\ &= \sum_{d'_n(g'_0,\ldots,g'_n)=(g_0,\ldots,g_{n-1})}\!\!\cdots\sum u^0(g'_0)\cdots u^n(g'_n) \\ &= \sum_{g_0=g'_ng'_0}u^0(g'_0)u^1(g_1)\cdots u^{n-1}(g_{n-1})u^n(g'_n) \\ &= \Omega^2_{s,t;t,s}(\Omega^n_{s,t}\otimes\mathrm{id})((u^n{*}u^0)\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\cdots\otimes^{rl}_{\mathcal{C}^\infty(G_0)}u^{n-2}\otimes^{rl,lr}_{\mathcal{C}^\infty(G_0)}u^{n-1}))(g_0,\ldots,g_{n-1}) \\ &= \Omega^2_{s,t\,;\,t,s}(\Omega^n_{s,t}\otimes\mathrm{id})(\tilde{d}'_n(u^0\otimes^{rl}_{\mathcal{C}^\infty(G_0)}\cdots\otimes^{rl}_{\mathcal{C}^\infty(G_0)}u^{n-1}\otimes^{rl,lr}_{\mathcal{C}^\infty(G_0)}u^n))(g_0,\ldots,g_{n-1}), \end{aligned}$$

and much the same way for all remaining faces, degeneracies and the cyclic operator $t$. □

**6.6.9 Comodule Structures and Invariants** For the consideration of invariants (cf. Subsection 6.1.1) one wants to consider comodule structures over the enveloping algebra of the base algebra. Due to the form of the left and right coproducts in the present example, this is particularly simple; we now show how $\mathcal{C}^\infty_{\rm c}(G)$ can be seen as a coalgebra over $\mathcal{C}^\infty(G_0)^{\rm e}=\mathcal{C}^\infty(G_0)\otimes\mathcal{C}^\infty(G_0)$. As a coproduct, one needs a map

$$\Delta:\mathcal{C}^\infty_{\rm c}(G)\to\mathcal{C}^\infty_{\rm c}(G)\otimes^{ll,rr}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G),$$

since $s^\ell=t^\ell=s^r=t^r$, and as in Proposition 6.6.8 one infers the existence of an isomorphisms $\Omega_{t,t\,;\,s,s}:\mathcal{C}^\infty_{\rm c}(G)\otimes^{ll,rr}_{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\rm c}(G)\to\mathcal{C}^\infty_{\rm c}(G^{t;s}\times^{t;s}_{G_0}G)$ (that is, to the space of compactly supported smooth functions over the space of pairs $(g,g')$ that have not only identical sources but also equal targets). Evidently, the non-vanishing elements of both left and right coproduct in (4.4.6) are *already* of this form and one only needs to modify the image space. Hence we can take the same formula, and set for the coproduct of $\mathcal{C}^\infty_{\rm c}(G)$ over $\mathcal{C}^\infty(G_0)^{\rm e}$

$$\Delta':=\Omega^2_{t,t\,;\,s,s}\,\Delta:\mathcal{C}^\infty_{\rm c}(G)\to\mathcal{C}^\infty_{\rm c}(G^{t;s}\times^{t;s}_{G_0}G),\quad(\Delta'u)(g,g')=\begin{cases}u(g) & \text{if } g=g',\\ 0 & \text{otherwise.}\end{cases}$$

Again, introducing the map $d:G\to G^{t;s}\times^{t;s}_{G_0}G,\ g\mapsto(g,g)$ one may write $\Delta'=d_+$ or $\Delta=\Omega^{-1}_{t,t\,;\,s,s}d_+$. The associated counit $\epsilon'$ to this coalgebra structure is given as the 'intersection' of the left and right counits: using the Sweedler components $\Delta_\ell u=u_{(1)}\otimes^{ll}_{\mathcal{C}^\infty(G_0)}u_{(2)}$ of the left coproduct, define

$$\epsilon':\mathcal{C}^\infty_{\rm c}(G)\to\mathcal{C}^\infty(G_0)\hat{\otimes}\mathcal{C}^\infty(G_0),\quad u\mapsto\partial u_{(1)}\hat{\otimes}\,\epsilon u_{(2)}.$$

However, it will be also clear in a moment that there is an equivalent expression which uses the right coproduct. If the tensor product $\hat{\otimes}$ is a topological (e.g. projective) one, which allows the identification of $\mathcal{C}^\infty(G_0)\hat{\otimes}\mathcal{C}^\infty(G_0)$ with $\mathcal{C}^\infty(G_0 \times G_0)$ in some sense, we write for $u \in \mathcal{C}^\infty_{\mathrm{c}}(G),\ x, y \in G_0$,

$$\begin{aligned}\epsilon'(u)(x,y) = \partial u_{(1)}(x)\epsilon u_{(2)}(y) &= \sum_{s(g)=x}\sum_{t(g')=y} u_{(1)}(g)u_{(2)}(g') \\ &= \sum_{y \overset{g}{\leftarrow} x} u(g) = \sum_{s(g)=x,\, t(g)=y} u(g).\end{aligned}$$

In words, the last term involves the sum over all arrows from $x$ to $y$. To prove that $(\mathcal{C}^\infty_{\mathrm{c}}(G), \Delta', \epsilon')$ indeed fulfills the identities of a comonoid over $\mathcal{C}^\infty(G_0)^{\mathrm{e}}$, observe firstly that there is a twisted coassociativity between $\Delta'$ and both $\Delta_\ell$ and $\Delta_r$, respectively, analogous to (2.6.2). Then one has, denoting $\Delta' u = u^{(1)'} \otimes^{ll,rr}_{\mathcal{C}^\infty(G_0)} u^{(2)'}$,

$$\begin{aligned}((\mathrm{id} \otimes \epsilon')\Delta' u)(g) &= (\epsilon(u^{(2)'}_{(2)})u^{(1)'}\partial(u^{(2)'}_{(1)}))(g) \\ &= \sum_{t(g')=t(g)}\sum_{s(g'')=s(g)} u_{(2)}(g')u^{(1)'}_{(1)}(g)u^{(2)'}_{(1)}(g'') = u(g),\end{aligned}$$

hence $(\mathrm{id} \otimes \epsilon')\Delta' = \mathrm{id}$ as desired, and similarly for all remaining comonoid identities. We write $\mathcal{C}^\infty_{\mathrm{c}}(G)'$ instead of $\mathcal{C}^\infty_{\mathrm{c}}(G)$ whenever we refer to this coalgebra structure over $\mathcal{C}^\infty(G_0)^{\mathrm{e}}$.

Now consider the space $M := C^{\mathcal{C}^\infty(G_0)}_{n+1}\mathcal{C}^\infty_{\mathrm{c}}(G)$. For $u_0, \ldots, u_n \in \mathcal{C}^\infty_{\mathrm{c}}(G)$ define ${}_M\Delta : M \to M \otimes^{ll,rr}_{\mathcal{C}^\infty(G_0)} \mathcal{C}^\infty_{\mathrm{c}}(G)$ by

$${}_M\Delta(u_0 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u_n) := u_0^{(1)'} \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u_n^{(1)'} \otimes^{ll,rr}_{\mathcal{C}^\infty(G_0)} u_0^{(2)'}u_1^{(2)'}\cdots u_n^{(2)'}.$$

It is a straightforward calculation to see that ${}_M\Delta$ is a right $\mathcal{C}^\infty_{\mathrm{c}}(G)$-comodule structure over $\mathcal{C}^\infty(G_0)^{\mathrm{e}}$. In particular, one can easily check

$$\begin{aligned}&\Omega^2_{t,t\,;\,s,s}(\Omega^n_{s,t} \otimes \mathrm{id})_M\Delta(u_0 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u_n)(g_0, \ldots, g_n, g) \\ &\quad = u_0^{(1)'}(g_0)\cdots u_n^{(1)'}(g_n)(u_0^{(2)'} * \cdots * u_n^{(2)'})(g) \\ &\quad = \begin{cases} u_0(g_0)\cdots u_n(g_n) & \text{if } g = g_0 \cdots g_n, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence for $u \in \mathcal{C}^\infty_{\mathrm{c}}(G_{n+1})$ we write

$$\begin{aligned}{}_M\Delta' := \Omega^2_{t,t\,;\,s,s}(\Omega^n_{s,t} \otimes \mathrm{id})\, {}_M\Delta : \mathcal{C}^\infty_{\mathrm{c}}(G_{n+1}) \to \mathcal{C}^\infty_{\mathrm{c}}(G_{n+1}\, {}^{t;s}{\times}^{t;s}_{G_0} G),& \\ {}_M\Delta' u(g_0, \ldots, g_n, g) = \begin{cases} u(g_0, \ldots, g_n) & \text{if } g = g_0 g_1 \cdots g_n, \\ 0 & \text{otherwise.}\end{cases}&\end{aligned}$$

We finally mention that a left $\mathcal{C}^\infty_{\mathrm{c}}(G)$-comodule structure on $\mathcal{C}^\infty(G_0)$ over $\mathcal{C}^\infty(G_0)^{\mathrm{e}}$ is simply given by the injection (4.4.5), i.e.,

$$\Delta_{\mathcal{C}^\infty(G_0)} : \mathcal{C}^\infty(G_0) \to \mathcal{C}^\infty_{\mathrm{c}}(G),\ f \mapsto \tilde{f}, \quad \tilde{f}(g) = \begin{cases} f(x) & \text{if } g = 1_x \text{ for some } x \in G_0, \\ 0 & \text{otherwise.}\end{cases}$$

We now have all ingredients to define the space of invariants of the complex in question. In each degree $n \geq 0$, set

$$B^{\mathcal{C}^\infty(G_0)}_n \mathcal{C}^\infty_{\mathrm{c}}(G) \supset \mathrm{Inv}\, B^{\mathcal{C}^\infty(G_0)}_n \mathcal{C}^\infty_{\mathrm{c}}(G) := C^{\mathcal{C}^\infty(G_0)}_{n+1}\mathcal{C}^\infty_{\mathrm{c}}(G) \Box_{\mathcal{C}^\infty_{\mathrm{c}}(G)'} \mathcal{C}^\infty(G_0).$$

Explicitly, this means

$$\begin{aligned}0 &= \Omega^2_{t,t\,;\,s,s}(\Omega^n_{s,t} \otimes \mathrm{id})({}_M\Delta \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_{\mathcal{C}^\infty(G_0)}) \\ &\qquad\qquad (u_0 \otimes^{rl}_{\mathcal{C}^\infty(G_0)} \cdots \otimes^{rl}_{\mathcal{C}^\infty(G_0)} u_n \otimes_{\mathcal{C}^\infty(G_0)^{\mathrm{e}}} f)(g_0, \ldots, g_n, g) \\ &= \begin{cases} f(t(g))u_0(g_0)\cdots u_n(g_n) - u_0(g_0)\cdots u_n(g_n)f(x) & \text{if } g = g_0 \cdots g_n = 1_x \text{ for } x \in G_0, \\ -u_0(g_0)\cdots u_n(g_n)f(s(g_n)) & \text{otherwise.}\end{cases}\end{aligned}$$

Here $f \in \mathcal{C}^\infty(G_0)$, and in the last factor we took the tensor product over $\mathcal{C}^\infty(G_0)^{\mathrm{e}}$. While the lower expression in the last line never vanishes in non-trivial cases, the upper one does if $1_x = g = g_0 \cdots g_n$, that is, if $g$ is a closed string that is a unit, i.e. an $n+1$-tuple $(g_0,\dots,g_n)\in B_n$ for which $g_0\cdots g_n = 1_x$. We reformulate this with the aid of Proposition 6.6.8: an invariant element in degree $n$ is a function $u\in\mathcal{C}^\infty_{\mathrm{c}}(B_n)$ with

$$({}_M\Delta' u)(g_0,\dots,g_n,g) = \begin{cases} u(g_0,\dots,g_n) & \text{if } g = 1_x \text{ for some } x\in G_0,\\ 0 & \text{otherwise,}\end{cases}$$

if ${}_M\Delta$ is restricted from $\mathcal{C}^\infty_{\mathrm{c}}(G_{n+1})$ to $\mathcal{C}^\infty_{\mathrm{c}}(B_n)$. Note that ${}_M\Delta$ remains well-defined, since

$$\begin{aligned}(f*u_0)(g_0)u_1(g_1)\cdots u_n(g_n) &= f(t(g_0))u_0(g_0)\cdots u_n(g_n)\\ &= u_0(g_0)\cdots u_n(g_n)f(s(g_n)) = u_0(g_0)\cdots(u_n*f)(g_n)\end{aligned}$$

for closed strings, and that ${}_M\Delta$ on $\mathcal{C}^\infty_{\mathrm{c}}(B_n)$ takes an analogous form as in (6.6.9), but now seen as map $\mathcal{C}^\infty_{\mathrm{c}}(B_n)\to\mathcal{C}^\infty_{\mathrm{c}}(B_n{}^s\!\times^s_{G_0}G)$.

The following Theorem is basically a summary of what has been proven before. It explicitly connects dual Hopf-cyclic homology for convolution algebras over étale groupoids to the standard $\mathcal{C}^\infty(G_0)$-ring homology (cf. §1.2.4) by restriction (or injection) of the respective spaces. More precisely,

**6.6.10 Theorem** *The subset of invariants of $\mathcal{C}^\infty_{\mathrm{c}}(B_n)$ in each degree $n$ is isomorphic to $\mathcal{C}^\infty_{\mathrm{c}}(G_n)$, and the injection $i: G_n\hookrightarrow B_n$ from (6.6.11) induces an injection $i_+:\mathcal{C}^\infty_{\mathrm{c}}(G_n)\hookrightarrow\mathcal{C}^\infty_{\mathrm{c}}(B_n)$, which is compatible with the respective cyclic structures. Hence one has a commutative diagram*

$$\begin{array}{ccc} C_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\mathrm{c}}(G) & \longrightarrow & B_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\mathrm{c}}(G)\\ {\scriptstyle(d_\bullet,s_\bullet,t_\bullet)}\big\downarrow & & \big\downarrow{\scriptstyle(d_\bullet,s_\bullet,t_\bullet)}\\ C_{n\pm\{0,1\}}^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\mathrm{c}}(G) & \longrightarrow & B_{n\pm\{0,1\}}^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\mathrm{c}}(G),\end{array}$$

*where on the left-hand side one has the Hopf-cyclic operators from (6.1.3), (6.1.4) and on the right-hand side one has the standard operators from (1.2.3) for the $\mathcal{C}^\infty(G_0)$-ring $\mathcal{C}^\infty_{\mathrm{c}}(G)$.*

PROOF: The first statement follows from the simple observation that if $u\in\operatorname{Inv}B_n^{\mathcal{C}^\infty(G_0)}\mathcal{C}^\infty_{\mathrm{c}}(G)$, then $u\in\mathcal{C}^\infty_{\mathrm{c}}(B_n)$ with

$$\begin{aligned}\operatorname{supp}u\subset\{(g_0,\dots,g_n)\in B_n \mid g_0\cdots g_n=1_x\} &= \{(g_0,\dots,g_n)\in B_n\mid g_0=(g_1\cdots g_n)^{-1}\}\\ &= i(G_n),\end{aligned}$$

hence $u$ may be identified with a function in $\mathcal{C}^\infty_{\mathrm{c}}(G_n)$. To continue, one only needs to apply Proposition 6.6.8 and Theorem 6.6.4, which allows to equivalently prove the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{C}^\infty_{\mathrm{c}}(G_n) & \longrightarrow & \mathcal{C}^\infty_{\mathrm{c}}(B_n)\\ {\scriptstyle(d_{\bullet+},s_{\bullet+},t_{\bullet+})}\big\downarrow & & \big\downarrow{\scriptstyle(d'_{\bullet+},s'_{\bullet+},t'_{\bullet+})}\\ \mathcal{C}^\infty_{\mathrm{c}}(G_{n\pm\{0,1\}}) & \longrightarrow & \mathcal{C}^\infty_{\mathrm{c}}(B_{n\pm\{0,1\}})\end{array}$$

in each degree $n\geq 0$; this time, the sets of operators are the fibre sums of the operators (6.6.1)–(6.6.2) on $G_\bullet$ and (6.6.9)–(6.6.10) on $B_\bullet$, respectively. Now, since $i$ respects the cyclic structures of $G_\bullet$ and $B_\bullet$, so does the fibre sum $i_+$. As an example, for $u\in\mathcal{C}^\infty_{\mathrm{c}}(G_n)$ we verify

$$\begin{aligned} i_+t_{n+}u(g_0,\dots,g_n) &= \begin{cases}\sum\cdots\sum_{t_n(g'_1,\dots,g'_n)=(g_1,\dots,g_n)}u(g'_1,\dots,g'_n) & \text{if } g_0^{-1}=g_1g_2\cdots g_n,\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} u(g_2,\dots,g_n,(g_1g_2\cdots g_n)^{-1}) & \text{if } g_0^{-1}=g_1g_2\cdots g_n,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

and

$$\begin{aligned} t'_{n+}i_+u(g_0,\dots,g_n) &= \sum_{(g'_n,g'_0,\dots,g'_{n-1})=(g_0,g_1,\dots,g_n)}\!\!\cdots\sum i_+u(g'_0,\dots,g'_n)\\ &= \begin{cases} u(g_2,\dots,g_n,g_0) & \text{if } g_1^{-1}=g_2\cdots g_ng_0,\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

which is seen to coincide with the expression above. □

# Chapter 7

# Duality and Products in Algebraic (Co)Homology Theories

## 7.1 Introduction

Most classical (co)homology theories of algebraic objects such as groups, or Lie, Lie-Rinehart or associative algebras can be realised as

$$H^\bullet(X, M) := \mathrm{Ext}^\bullet_U(A, M), \quad H_\bullet(X, N) := \mathrm{Tor}^U_\bullet(N, A) \tag{7.1.1}$$

for an augmented ring $X = (U, A)$ (i.e. a ring with a distinguished left module) that is functorially attached to a given object. The cohomology coefficients are left $U$-modules $M$ and those in homology are right $U$-modules $N$.

Our aim here is to clarify the origin and interplay of multiplicative structures and dualities between such (co)homology groups, and to provide a unified treatment of results by Van den Bergh on Hochschild (co)homology [VdB] and by Huebschmann on Lie-Rinehart (co)homology [Hue3]. The key concept involved is that of a left Hopf algebroid ($\times_A$-Hopf algebra) introduced by Schauenburg [Schau2], cf. Section 2.2.

The main results can be summarised as follows:

**7.1.1 Theorem** *For any $A$-biprojective left Hopf algebroid $U$ there is a functor*

$$\otimes : U\text{-}\mathbf{Mod} \times U^{\mathrm{op}}\text{-}\mathbf{Mod} \to U^{\mathrm{op}}\text{-}\mathbf{Mod}$$

*that for $M \in U\text{-}\mathbf{Mod}$, $N \in U^{\mathrm{op}}\text{-}\mathbf{Mod}$ and $m, n \geq 0$ induces natural products*

$$\frown : \mathrm{Ext}^m_U(A, M) \times \mathrm{Tor}^U_n(N, A) \to \mathrm{Tor}^U_{n-m}(M \otimes N, A).$$

*If $A \in U\text{-}\mathbf{Mod}$ admits a finitely generated projective resolution of finite length and there exists $d \geq 0$ with $\mathrm{Ext}^m_U(A, U) = 0$ for $m \neq d$, then there is a canonical element*

$$[\omega] \in \mathrm{Tor}^U_d(A^*, A), \quad A^* := \mathrm{Ext}^d_U(A, U)$$

*such that for $m \geq 0$ and $M \in U\text{-}\mathbf{Mod}$ with $\mathrm{Tor}^A_q(M, A^*) = 0$ for $q > 0$*

$$\cdot \frown [\omega] : \mathrm{Ext}^m_U(A, M) \to \mathrm{Tor}^U_{d-m}(M \otimes A^*, A)$$

*is an isomorphism.*

This theorem will be proven in Section 7.3.

As we saw in Chapter 2 above, left $A$-bialgebroids and left Hopf algebroids over $A$ generalise bialgebras and Hopf algebras to possibly noncommutative base algebras $A$. Besides Hopf algebras, both the universal enveloping algebra $VL$ of a Lie-Rinehart algebra $(A, L)$ and the enveloping algebra $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$ of an associative algebra $A$ are left Hopf algebroids over $A$, see Subsections 4.2.1 and 4.1.1.

For any left $A$-bialgebroid $U$, the base algebra $A$ carries a left $U$-action and the category $U\text{-}\mathbf{Mod}$ of left $U$-modules is monoidal with unit object $A$, cf. Subsection 2.3.1. But it is only for left Hopf algebroids over

$A$ that one has a canonical operation $\otimes$ as in Theorem 7.1.1, which turns $U^{\mathrm{op}}$-$\mathbf{Mod}$ into a module category over $(U\text{-}\mathbf{Mod}, \otimes, A)$ (Lemma 7.2.8).

Any left Hopf algebroid carries two left and two right actions of the base algebra that all commute with each other. The biprojectivity assumed in Theorem 7.1.1 refers to the projectivity of two particular of these, see Section 7.1.1. Under this condition, we can use the elegant formalism of suspended monoidal categories from [Sua] to define products for $M, N \in U$-$\mathbf{Mod}$ and $P \in U^{\mathrm{op}}$-$\mathbf{Mod}$ by means of

$$\smile : H^m(X, M) \times H^n(X, N) \to H^{m+n}(X, M \otimes N),$$

$$\frown : H^n(X, N) \times H_p(X, P) \to H_{p-n}(X, N \otimes P),$$

where once again we use the abbreviations from (7.1.1) (cf. Sections 7.2.2 and 7.2.5).

In the last part of Theorem 7.1.1, $A^* = H^d(X, U) = \mathrm{Ext}^d_U(A, U)$ is a right $U$-module via right multiplication in $U$, and if we define the functor

$$\hat{} : U\text{-}\mathbf{Mod} \to U^{\mathrm{op}}\text{-}\mathbf{Mod}, \quad M \mapsto \hat{M} := M \otimes A^*,$$

then the statement can be rewritten as an isomorphism

$$H^m(X, M) \simeq H_{\dim(X)-m}(X, \hat{M}), \quad \dim(X) := \mathrm{proj.dim}_U(A)$$

given as in topology by the cap product with the fundamental class $[\omega] \in H_{\dim(X)}(\hat{A})$ which corresponds under the duality to $\mathrm{id}_A \in H^0(A) = \mathrm{Hom}_U(A, A)$. For $M = A$ this simply means that the $H^\bullet(A)$-module $H_\bullet(A^*)$ is free with generator $[\omega]$.

Theorem 7.1.1 is well known in group and Lie algebra (co)homology [Ha, Bie]. For $U = A \otimes_k A^{\mathrm{op}}$ it reduces to Van den Bergh's result [VdB], which has stimulated a lot of recent research, see e.g. [BroZ, Dol, Gi, LauRi]. Note that we do not need Van den Bergh's invertibility assumption about $A^*$, which says that $\hat{}$ is an equivalence. However, it is satisfied for many well-behaved algebras [ibid.] and implies the condition $\mathrm{Tor}^A_q(M, A^*) = 0$ for arbitrary $A$-bimodules $M$ (since invertible bimodules are finitely generated projective as one-sided modules from either side). For Lie-Rinehart algebras, Theorem 7.1.1 is due to Huebschmann [Hue3], and we find the general setting helpful, for example, to understand the different roles of left and right modules that were observed by Huebschmann (cf. 4.2.5). As was shown in [loc. cit.], the conditions of Theorem 7.1.1 are satisfied whenever $L$ is finitely generated projective (of constant rank) over $A$, and $A^*$ coincides as an $A$-module with $\Lambda^d_A L$ and is in particular projective, so also here we have $\mathrm{Tor}^A_q(M, A^*) = 0$ for arbitrary $(A, L)$-modules $M$.

So both these examples and the applications in homological algebra clearly demonstrate the relevance of the intermediate concept of a left Hopf algebroid.

One could generalise Theorem 7.1.1 to differentially graded left Hopf algebroids, sheaves thereof, or suitable abstract monoidal categories. One also could drop the condition $\mathrm{Ext}^n_U(A, U) = 0$ for $n \neq d$ the assumption that $\mathrm{Tor}^A_q(M, A^*) = 0$. Then one obtains an isomorphism $\mathrm{RHom}_U(A, M) \simeq (M \otimes^{\mathrm{L}}_A \mathrm{RHom}_U(A, U)) \otimes^{\mathrm{L}}_U A$ for a bounded below chain complex $M$ over $U$-$\mathbf{Mod}$.

### 7.1.1 Some Conventions

Let $U$ be a left bialgebroid over $A$ with structure maps as in Definition 2.1.2, and consider the categories $U$-$\mathbf{Mod}$ and $U^{\mathrm{op}}$-$\mathbf{Mod}$. Using the forgetful functor $U$-$\mathbf{Mod} \to A^{\mathrm{e}}$-$\mathbf{Mod}$, we regard, as in Subsection 2.3.1, any $U$-module $M$ also as an $(A, A)$-bimodule with actions

$$a \triangleright m \triangleleft b := \eta(a \otimes_k b)m, \qquad a, b \in A, m \in M. \tag{7.1.2}$$

Similarly, every right $U$-module $N$ is also an $A$-bimodule via

$$a \blacktriangleright m \blacktriangleleft b := n\eta(b \otimes_k a), \qquad a, b \in A, n \in N, \tag{7.1.3}$$

although the category $U^{\mathrm{op}}$-$\mathbf{Mod}$ for a left bialgebroid is usually not monoidal, in contrast to $U$-$\mathbf{Mod}$. A useful abbreviation in this section will be the following:

**7.1.2 Definition** If $U$ is a left $A$-bialgebroid and $M, N \in U$-$\mathbf{Mod}$ are left $U$-modules, we denote the left $U$-module $M \otimes_A N$ with $U$-action (2.3.2) by $M \otimes N$.

As before, the notations from (7.1.2) and (7.1.3) apply in particular to $U$ itself; let us repeat that as the default case we consider $U$ as an $A^{\mathrm{e}}$-module using $a \triangleright u \triangleleft b$, and otherwise we write e.g. ${}_{\blacktriangleright}U_{\triangleleft}$ to denote which actions are considered.

Since this will be repeatedly a necessary technical condition, we define:

**7.1.3 Definition** For an $A^{\mathrm{e}}$-algebra $U$ we call $M \in U\text{-}\mathbf{Mod}$ *$A$-biprojective* if both ${}_{\triangleright}M \in A\text{-}\mathbf{Mod}$ and $M_{\triangleleft} \in A^{\mathrm{op}}\text{-}\mathbf{Mod}$ are projective modules.

## 7.2 Multiplicative Structures

### 7.2.1 $\mathcal{D}^{-}(U)$ as a Suspended Monoidal Category

For any ring $U$, we denote the derived category of bounded above cochain complexes of left $U$-modules by $\mathcal{D}^{-}(U)$. As usual, we identify any $M \in U\text{-}\mathbf{Mod}$ with a complex in $\mathcal{D}^{-}(U)$ concentrated in degree 0, and identify any bounded below chain complex $P_{\bullet}$ with a bounded above cochain complex by putting $P^n := P_{-n}$.

If $U$ is an $A$-biprojective left $A$-bialgebroid, then any projective $P \in U\text{-}\mathbf{Mod}$ is $A$-biprojective. Hence the monoidal structure of $U\text{-}\mathbf{Mod}$ extends to a monoidal structure on $\mathcal{D}^{-}(U)$ with unit object still given by $A$ and product being the total tensor product $\otimes^{\mathrm{L}} = \otimes^{\mathrm{L}}_A$ (the $A$-biprojectivity of $U$-projectives is needed for example to have [W, Lemma 10.6.2]).

Together with the shift functor $T : \mathcal{D}^{-}(U) \to \mathcal{D}^{-}(U)$, $(TC)^n = C^{n+1}$, $\mathcal{D}^{-}(U)$ becomes what in [Sua] is called a suspended monoidal category. This just means that for all $C, D \in \mathcal{D}^{-}(U)$, the canonical isomorphisms

$$TC \otimes^{\mathrm{L}} D \simeq T(C \otimes^{\mathrm{L}} D) \simeq C \otimes^{\mathrm{L}} TD$$

given by the obvious renumbering make the diagrams

$$\begin{tikzcd} A \otimes^{\mathrm{L}} TC \arrow[r] \arrow[d] & TC \arrow[dl] \\ T(A \otimes^{\mathrm{L}} C) & \end{tikzcd} \qquad \begin{tikzcd} TC \otimes^{\mathrm{L}} A \arrow[r] \arrow[d] & TC \arrow[dl] \\ T(C \otimes^{\mathrm{L}} A) & \end{tikzcd}$$

commutative, whilst making the diagram

$$\begin{tikzcd} TC \otimes^{\mathrm{L}} TD \arrow[r] \arrow[d] & T(C \otimes^{\mathrm{L}} TD) \arrow[d] \\ T(TC \otimes^{\mathrm{L}} D) \arrow[r] & T^2(C \otimes^{\mathrm{L}} D) \end{tikzcd}$$

anti-commutative (commutative up to a sign $-1$).

### 7.2.2 The Products $\smile$ and $\circ$

As a special case of the constructions from [Sua], for any $A$-biprojective left $A$-bialgebroid $U$ and $L, M, N \in U\text{-}\mathbf{Mod}$ we define the cup product

$$\smile : \mathrm{Ext}^m_U(A, M) \times \mathrm{Ext}^n_U(A, N) \to \mathrm{Ext}^{m+n}_U(A, M \otimes N)$$

and the classical Yoneda product

$$\circ : \mathrm{Ext}^m_U(N, M) \times \mathrm{Ext}^n_U(L, N) \to \mathrm{Ext}^{m+n}_U(L, M).$$

The latter is just the composition of morphisms in $\mathcal{D}^{-}(U)$ if one identifies

$$\mathrm{Ext}^n_U(L, N) \simeq \mathrm{Hom}_{\mathcal{D}^{-}(U)}(L, T^n N),$$

and

$$\mathrm{Ext}^m_U(N, M) \simeq \mathrm{Hom}_{\mathcal{D}^{-}(U)}(N, T^m M) \simeq \mathrm{Hom}_{\mathcal{D}^{-}(U)}(T^n N, T^{m+n} M).$$

The former is obtained as follows: given

$$\varphi \in \mathrm{Ext}^m_U(A, M) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}(A, T^m M),$$

$$\psi \in \mathrm{Ext}^n_U(A, N) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}(A, T^n N),$$

one defines $\varphi \smile \psi$ as the composition

$$\begin{aligned} & A \simeq A \otimes A \\ \xrightarrow{\varphi\otimes\psi} \quad & T^m M \otimes^{\mathrm{L}} T^n N \simeq T^m(M \otimes^{\mathrm{L}} T^n N) \simeq T^{m+n}(M \otimes^{\mathrm{L}} N) \\ \longrightarrow \quad & T^{m+n}(M \otimes N), \end{aligned}$$

where the last map is the augmentation $M \otimes^{\mathrm{L}} N \to H^0(M \otimes^{\mathrm{L}} N) \simeq \mathrm{Tor}^A_0(M, N) \simeq M \otimes N$, or rather $T^{m+n}$ applied to this morphism in $\mathcal{D}^-(U)$.

A straightforward extension of Theorem 1.7 from [Sua] now gives:

**7.2.1 Theorem** *If $U$ is an $A$-biprojective left $A$-bialgebroid, then we have*

$$\psi \circ \varphi = \varphi \smile \psi = (-1)^{mn} \psi \smile \varphi, \quad \varphi \in \mathrm{Ext}^m_U(A, A), \psi \in \mathrm{Ext}^n_U(A, M)$$

*as elements of* $\mathrm{Ext}^{m+n}_U(A, M) \simeq \mathrm{Ext}^{m+n}_U(A, A \otimes M) \simeq \mathrm{Ext}^{m+n}_U(A, M \otimes A)$.

*In particular, through either of the products* $\mathrm{Ext}_U(A, A)$ *becomes a graded commutative algebra over the commutative subring* $\mathrm{Hom}_U(A, A)$.

PROOF: This is proven exactly as in [Sua]. For the reader's convenience we include one of the diagrams involved. The unlabeled arrows are canonical maps coming from the suspended monoidal structure.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varphi"'] & A \otimes A \arrow[d, "\mathrm{id}\otimes\varphi"'] \arrow[dr, "\psi\otimes\mathrm{id}"] & \\
T^m A \arrow[r] \arrow[dr] \arrow[dd, "\mathrm{id}"'] & A \otimes T^m A \arrow[d] \arrow[dr, "\psi\otimes\mathrm{id}"] & T^n M \otimes^{\mathrm{L}} A \arrow[d, "\mathrm{id}\otimes\varphi"] \\
 & T^m(A \otimes A) \arrow[dl] \arrow[d, "T^m(\psi\otimes\mathrm{id})"'] & T^n M \otimes^{\mathrm{L}} T^m A \arrow[dl] \arrow[d] \\
T^m A \arrow[d, "T^m(\psi)"'] & T^m(T^n M \otimes^{\mathrm{L}} A) \arrow[dl] \arrow[d] & T^n(M \otimes^{\mathrm{L}} T^m A) \arrow[dl] \\
T^{m+n} M & T^{m+n}(M \otimes^{\mathrm{L}} A) \arrow[l] &
\end{tikzcd}$$

The morphism $\psi \circ \varphi \in \mathrm{Hom}_{\mathcal{D}^-(U)}(A, T^{m+n} M)$ is the path going straight down from $A$ to $T^{m+n}M$, and $\psi \smile \varphi$ is the one which goes clockwise round the whole diagram. All faces of the diagram commute except the lower right square, which introduces a sign $(-1)^{mn}$, so we get $\psi \circ \varphi = (-1)^{mn} \psi \smile \varphi$. The other identity is shown with a similar diagram. □

### 7.2.3 Tensoring Projectives

This paragraph is a small excursus about the projectivity of the tensor product of two projective objects of a monoidal category. For example, $U \otimes U \in U$-$\mathbf{Mod}$ is not necessarily projective even for a bialgebra $U$ over a field $A = k$ (so the $A$-projectivity of $U$ or the exactness of $\otimes$ does not help). Here is a simple example (for a detailed study of examples of categories of Mackey functors see [Lew]):

**7.2.2 Example** Consider the bialgebra $U = \mathbb{C}[a,b,c]$ over $A = k = \mathbb{C}$, with

$$\Delta(a) = a \otimes a, \quad \Delta(b) = a \otimes b + b \otimes c, \quad \Delta(c) = c \otimes c,$$

$$\varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad \varepsilon(c) = 1.$$

Geometrically, this is the coordinate ring of the complex algebraic semigroup $G$ of upper triangular $2 \times 2$-matrices, and $\Delta$ and $\varepsilon$ are dual to the semigroup law $G \times G \to G$ and the embedding of the identity matrix into $G$.

We prove that $U \otimes U \in U$-$\mathbf{Mod}$ is not projective by considering the fibres of the semigroup law $G \times G \to G$. The fibre over a generic and hence invertible element is 3-dimensional, but over 0 it is 4-dimensional, and this will imply our claim. We can use for example [Mat, Theorem 19 on p. 79]:

**7.2.3 Theorem** *Let $U \subset V$ be a flat extension of commutative Noetherian rings, $\mathfrak{p} \subset V$ a prime ideal, and $\mathfrak{q} := U \cap \mathfrak{p}$. Then*

$$\dim(V_\mathfrak{p}) = \dim(U_\mathfrak{q}) + \dim(V_\mathfrak{p} \otimes_U U(\mathfrak{q})),$$

*where $\dim$ denotes the Krull dimension of a ring, $V_\mathfrak{p}$ is the localisation of $V$ at $\mathfrak{p}$ and $U(\mathfrak{q}) := U_\mathfrak{q}/\mathfrak{q}U_\mathfrak{q}$ is the residue field of the localisation $U_\mathfrak{q}$.*

Apply this to our example $U \simeq \Delta(U) \subset V := U \otimes U$: let $\mathfrak{p}$ be the ideal of $V$ generated by $a \otimes_\mathbb{C} 1$, $1 \otimes_\mathbb{C} a$, $b \otimes_\mathbb{C} 1$, $1 \otimes_\mathbb{C} b$, $c \otimes_\mathbb{C} 1$, $1 \otimes_\mathbb{C} c$. Geometrically, $V$ is the coordinate ring of $\mathbb{C}^6$ and $V_\mathfrak{p}$ is the local ring in 0, so $\dim(V_\mathfrak{p}) = 6$. Since $1 \notin \mathfrak{p}$, $\mathfrak{q} = U \cap \mathfrak{p}$ is proper, and it contains the ideal generated by $\Delta(a) = a \otimes_\mathbb{C} a$, $\Delta(b) = a \otimes_\mathbb{C} b + b \otimes_\mathbb{C} c$, $\Delta(c) = c \otimes_\mathbb{C} c$, which is maximal in $U$, so $\mathfrak{q} \subset U$ is the ideal generated by $a, b, c$, and $U_\mathfrak{q}$ is the local ring of $\mathbb{C}^3$ at 0 with $\dim(U_\mathfrak{q}) = 3$. The field $U(\mathfrak{q})$ is obviously $\mathbb{C}$, and we can write $V_\mathfrak{p} \otimes_U U(\mathfrak{q})$ also as $V_\mathfrak{p}/\Delta(\mathfrak{q})V_\mathfrak{p}$. Since $\Delta(\mathfrak{q})V_\mathfrak{p}$ is contained in the ideal $\mathfrak{r}$ generated in $V_\mathfrak{p}$ by the elements $a \otimes_\mathbb{C} 1, 1 \otimes_\mathbb{C} c$, we have $\dim(V_\mathfrak{p}/\Delta(\mathfrak{q})V_\mathfrak{p}) \geq \dim(V_\mathfrak{p}/\mathfrak{r})$. Now $V_\mathfrak{p}/\mathfrak{r}$ is the local ring of $\mathbb{C}^4 \subset \mathbb{C}^6$ at 0 and hence $\dim(V_\mathfrak{p}/\mathfrak{r}) = 4$. In total, we obtain the strict inequality $3 + \dim(V_\mathfrak{p}/\Delta(\mathfrak{q})V_\mathfrak{p}) \geq 3 + 4 = 7 > 6$, and hence $V$ is not flat over $U$ and in particular not projective.

For left Hopf algebroids the situation is, however, much simpler: notice that for any left $A$-bialgebroid $U$ and $M \in U$-$\mathbf{Mod}$

$${}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} M_\triangleleft := U \otimes_k M/\mathrm{span}\{a \blacktriangleright u \otimes_k m - u \otimes_k m \triangleleft a \,|\, u \in U, a \in A, m \in M\}$$

is a left $U$-module by left multiplication on the first factor. Just as for $M = U$, there is a Galois map

$$\beta_M : {}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} M_\triangleleft \to U \otimes M, \quad u \otimes_{A^{\mathrm{op}}} m \mapsto u_{(1)} \otimes_A u_{(2)} m,$$

and we have:

**7.2.4 Lemma** *For any left $A$-bialgebroid $U$, the generalised Galois map $\beta_M$ is a morphism of $U$-modules. If $U$ is a left Hopf algebroid over $A$, then $\beta_M$ is bijective.*

PROOF: The $U$-linearity of $\beta_M$ immediately follows from the fact that $\Delta : U \to U \times_A U \subset U \otimes_A U$ is a homomorphism of algebras over $A^{\mathrm{e}}$. Furthermore, if $\beta$ is a bijection, then so is $\beta_M$ since we can identify $\beta_M$ with $\beta \otimes_U \mathrm{id}_M$, and then the inverse is simply given by $\beta_M^{-1}(u \otimes_A m) = u_+ \otimes_{A^{\mathrm{op}}} u_- m$. □

Using this one now obtains:

**7.2.5 Theorem** *If $U$ is a left Hopf algebroid over $A$ and $U_\triangleleft \in A^{\mathrm{op}}$-$\mathbf{Mod}$ is projective, then $P \otimes Q \in U$-$\mathbf{Mod}$ is projective for all projectives $P, Q \in U$-$\mathbf{Mod}$.*

PROOF: By assumption, any projective module over $U$ is also projective over $A^{\mathrm{op}}$, and if $\varphi : R \to S$ is any ring map, then $S \otimes_R \cdot : R$-$\mathbf{Mod} \to S$-$\mathbf{Mod}$ maps projectives to projectives. This shows that ${}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} U_\triangleleft$ and hence (Lemma 7.2.4) $U \otimes U$ is projective. Since $\otimes = \otimes_A$ commutes with arbitrary direct sums, $P \otimes Q$ is projective for all projectives $P, Q$. □

**7.2.6 Corollary** *If $U$ is as in Theorem 7.2.5 and $P \in \mathcal{D}^-(U)$ is a projective resolution of $A \in U$-$\mathbf{Mod}$, then so is $P \otimes P := \mathrm{Tot}(P_\bullet \otimes P_\bullet) = P \otimes^{\mathrm{L}} P$.*

This leads to the traditional construction of $\smile$, given for $A = k$ in [CarE, Chapter XI]: one fixes a projective resolution $P$ of $A$, and by the above, $\mathrm{Ext}_U(A, M \otimes N)$ is the total (co)homology of the double (cochain) complex

$$C^2_{mn} := \mathrm{Hom}_U(P_m \otimes P_n, M \otimes N).$$

Then $\smile$ is given as the composition of the canonical map

$$\begin{aligned} & \bigoplus_{m+n=p} \mathrm{Ext}^m_U(A, M) \otimes_k \mathrm{Ext}^n_U(A, N) \\ \simeq \quad & \bigoplus_{m+n=p} H^m(\mathrm{Hom}_A(P_\bullet, M)) \otimes_k H^n(\mathrm{Hom}_A(P_\bullet, N)) \\ \to \quad & H^p(\bigoplus_{m+n=\bullet} \mathrm{Hom}_A(P_m, M) \otimes_k \mathrm{Hom}_A(P_n, N)) = H^p(\mathrm{Tot}(C^1_{\bullet\bullet})), \end{aligned}$$

where $C^1_{mn} := \mathrm{Hom}_U(P_m, M) \otimes_k \mathrm{Hom}_U(P_n, N)$, with the map

$$H(\mathrm{Tot}(C^1_{\bullet\bullet})) \to H(\mathrm{Tot}(C^2_{\bullet\bullet})) \simeq \mathrm{Ext}_U(A, M \otimes N)$$

that is induced by the morphism of double complexes

$$C^1_{mn} \ni \varphi \otimes_k \psi \mapsto \{x \otimes y \mapsto \varphi(x) \otimes \psi(y)\} \in C^2_{mn}.$$

For the sake of completeness, let us finally remark that—as for $A = k$—one can in particular use the bar complex to obtain a canonical resolution (cf. Theorem 6.2.3, where this is formulated for the case of a Hopf algebroid):

**7.2.7 Lemma** (the bar complex revisited) *For any left $A$-bialgebroid $U$, the complex of left $U$-modules*

$$\mathrm{Bar}_n\, U := ({}_{\blacktriangleright} U_{\triangleleft})^{\otimes_{A^{\mathrm{op}}} n+1}, \quad v(u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n) := vu_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n,$$

*whose boundary map is given by*

$$\begin{aligned} b' : u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n \quad \mapsto \quad & \sum_{i=0}^{n-1} (-1)^i u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_i u_{i+1} \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n \\ & +(-1)^n u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} \varepsilon(u_n) \blacktriangleright u_{n-1}, \end{aligned}$$

*is a contractible resolution of $A \in U$-$\mathbf{Mod}$, with augmentation*

$$\varepsilon : \mathrm{Bar}_0\, U = U \to A =: \mathrm{Bar}_{-1}\, U.$$

*If $U_{\triangleleft} \in A^{\mathrm{op}}$-$\mathbf{Mod}$ is projective, then $\mathrm{Bar}_n\, U \in U$-$\mathbf{Mod}$ is projective.*

PROOF: All claims are straightforward: there is a contracting homotopy

$$s : \mathrm{Bar}_n\, U \to \mathrm{Bar}_{n+1}\, U, \quad u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n \mapsto 1 \otimes_{A^{\mathrm{op}}} u_0 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} u_n, \quad n \geq 0,$$

$$s : A = \mathrm{Bar}_{-1}\, U \to U = \mathrm{Bar}_0\, U, \quad a \mapsto \eta(a \otimes 1),$$

and the projectivity of $\mathrm{Bar}_n\, U$ follows as in the proof of Theorem 7.2.5. □

### 7.2.4 The Functor $\otimes : U$-$\mathbf{Mod} \times U^{\mathrm{op}}$-$\mathbf{Mod} \to U^{\mathrm{op}}$-$\mathbf{Mod}$

Now we introduce the functor $\otimes$ mentioned in Theorem 7.1.1.

**7.2.8 Lemma** *Let $U$ be a left Hopf algebroid over $A$ and let $M \in U$-$\mathbf{Mod}$, $P \in U^{\mathrm{op}}$-$\mathbf{Mod}$ be left and right $U$-modules, respectively. Then the formula*

$$(m \otimes_A p)u := u_- m \otimes_A pu_+ \qquad u \in U,\ m \in M, p \in P, \tag{7.2.1}$$

*defines a right $U$-module structure on the tensor product*

$$M \otimes_A P := M \otimes_k P / \mathrm{span}\{m \triangleleft a \otimes_k p - m \otimes_k a \blacktriangleright p \,|\, a \in A\}. \tag{7.2.2}$$

*If $N$ is any other (left) $U$-module, then the canonical isomorphism*

$$(M \otimes N) \otimes_A P \simeq M \otimes_A (N \otimes_A P) \tag{7.2.3}$$

*of $A$-bimodules is also an isomorphism in $U^{\mathrm{op}}$-$\mathbf{Mod}$. Finally, the tensor flip*

$$(M \otimes_A P) \otimes_U N \to P \otimes_U (N \otimes_A M), \quad m \otimes_A p \otimes_U n \mapsto p \otimes_U n \otimes_A m$$

*is an isomorphism of $k$-modules.*

PROOF: To show firstly that (7.2.1) is well-defined over $A$, we compute

$$\begin{aligned} \big(m \otimes_A (a \blacktriangleright p)\big)u &= u_- m \otimes_A p\eta(1 \otimes a)u_+ = u_- m \otimes_A p(u_+ \triangleleft a) \\ &= (a \blacktriangleright u_-)m \otimes_A pu_+ = u_-\big(\eta(1 \otimes a)m\big) \otimes_A pu_+ \\ &= \big((m \triangleleft a) \otimes_A p\big)u, \end{aligned}$$

where (2.2.6) and the action properties were used. Together with (7.2.2) this also proves well-definedness of (7.2.1) with respect to the presentation of $u_+ \otimes_{A^{\mathrm{op}}} u_-$. With the help of (2.2.9), one immediately sees that for $u, v \in U$ we have

$$\big(m \otimes_A p\big)(uv) = (uv)_- m \otimes_A p(uv)_+ = v_- u_- m \otimes_A pu_+ v_+ = \big((m \otimes_A p)u\big)v,$$

since $P$ and $M$ were right and left $U$-modules, respectively. As a conclusion, $M \otimes_A P \in U^{\mathrm{op}}$-$\mathbf{Mod}$. Equation (7.2.3) is a direct consequence of the associativity of the tensor product of $A$-bimodules and of (2.2.8).

For the last part one has to check that the flip is well-defined: we have

$$\begin{aligned} \eta(1 \otimes a)m \otimes_A p \otimes_U n \quad &\mapsto \quad p \otimes_U n \otimes_A \eta(1 \otimes a)m = p \otimes_U \eta(1 \otimes a)(n \otimes_A m) \\ &= \quad p\eta(1 \otimes a) \otimes_U (n \otimes_A m), \end{aligned}$$

which is what $m \otimes_A p\eta(1 \otimes a) \otimes_U n$ is mapped to. Secondly, we have

$$\begin{aligned} m \otimes_A p \otimes_U un \quad &\mapsto \quad p \otimes_U un \otimes_A m = p \otimes_U u_{+(1)}n \otimes_A u_{+(2)}u_- m \\ &= \quad p \otimes_U u_+(n \otimes_A u_- m) = pu_+ \otimes_U n \otimes_A u_- m, \end{aligned}$$

which is what $u_- m \otimes_A pu_+ \otimes_U n = (m \otimes_A p)u \otimes_U n$ is mapped to. □

**7.2.9 Definition** We denote the $U^{\mathrm{op}}$-module constructed above by $M \otimes P$.

Thus an unadorned $\otimes$ refers from now on either to the monoidal product on $U$-$\mathbf{Mod}$ or to the action of $U$-$\mathbf{Mod}$ on $U^{\mathrm{op}}$-$\mathbf{Mod}$ just defined. For example, (7.2.3) would now simply be written as $(M \otimes N) \otimes P \simeq M \otimes (N \otimes P)$.

**7.2.10 Example** Let $(A, L)$ be a Lie-Rinehart algebra, $M$ a left and $N$ a right $VL$-module, respectively (or, in the terminology of [Hue1, Hue3], left and right $(A, L)$-modules, see 4.2.5). Using (4.2.4), one obtains the right $VL$-module structure on $M \otimes_A N$ from formula (2.4) in [Hue3, p. 112]:

$$(m \otimes_A n)X = m \otimes_A nX - Xm \otimes_A n, \qquad m \in M,\ n \in N,\ X \in L.$$

If we again assume that $U$ is $A$-biprojective, then the above results extend directly to the derived category $\mathcal{D}^-(U^{\mathrm{op}})$: we obtain a functor

$$\otimes^{\mathrm{L}} = \otimes^{\mathrm{L}}_A : \mathcal{D}^-(U) \times \mathcal{D}^-(U^{\mathrm{op}}) \to \mathcal{D}^-(U^{\mathrm{op}}),$$

and for all $M, N \in \mathcal{D}^-(U)$, $P \in \mathcal{D}^-(U^{\mathrm{op}})$ we have canonical isomorphisms

$$(M \otimes^{\mathrm{L}} N) \otimes^{\mathrm{L}} P \simeq M \otimes^{\mathrm{L}} (N \otimes^{\mathrm{L}} P), \quad (M \otimes^{\mathrm{L}} P) \otimes^{\mathrm{L}}_U N \simeq P \otimes^{\mathrm{L}}_U (N \otimes^{\mathrm{L}} M). \tag{7.2.4}$$

### 7.2.5 The Products $\frown$ and $\bullet$

These products are dual to $\smile$ and $\circ$. There is a Yoneda one

$$\bullet : \mathrm{Ext}^m_U(L, M) \times \mathrm{Tor}^U_n(N, L) \to \mathrm{Tor}^U_{n-m}(N, M)$$

which exists for any ring $U$ and $L, M \in U\text{-}\mathbf{Mod}$, $N \in U^{\mathrm{op}}\text{-}\mathbf{Mod}$: an element

$$\varphi \in \mathrm{Ext}^m_U(L, M) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}(L, T^m M)$$

defines a morphism in $\mathcal{D}^-(\mathbb{Z})$,

$$N \otimes^{\mathrm{L}}_U L \to N \otimes^{\mathrm{L}}_U T^m M, \quad x \otimes_U y \mapsto x \otimes_U \varphi(y),$$

and $\varphi \bullet \cdot$ is the induced map in (co)homology

$$\begin{aligned} & \mathrm{Tor}^U_n(N, L) \simeq H^{-n}(N \otimes^{\mathrm{L}}_U L) \\ \xrightarrow{H^{-n}(\mathrm{id}\otimes\varphi)} \quad & H^{-n}(N \otimes^{\mathrm{L}}_U T^m M) \simeq H^{m-n}(N \otimes^{\mathrm{L}}_U M) \simeq \mathrm{Tor}^U_{n-m}(N, M). \end{aligned}$$

For $M \in U\text{-}\mathbf{Mod}, N \in U^{\mathrm{op}}\text{-}\mathbf{Mod}$ as before, the cap product

$$\frown : \mathrm{Ext}^m_U(A, M) \times \mathrm{Tor}^U_n(N, A) \to \mathrm{Tor}^U_{n-m}(M \otimes N, A)$$

involves the functor $\otimes$ from the previous paragraph, so for this we want $U$ to be an $A$-biprojective left Hopf algebroid over $A$ again. Similarly as for $\bullet$,

$$\varphi \in \mathrm{Ext}^m_U(A, M) \simeq \mathrm{Hom}_{\mathcal{D}^-(U)}(A, T^m M)$$

defines a morphism in $\mathcal{D}^-(k)$,

$$\begin{aligned} & N \otimes^{\mathrm{L}}_U A \simeq N \otimes^{\mathrm{L}}_U (A \otimes A) \\ \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\varphi} \quad & N \otimes^{\mathrm{L}}_U (A \otimes^{\mathrm{L}} T^m M) \simeq N \otimes^{\mathrm{L}}_U (T^m A \otimes^{\mathrm{L}} M) \simeq (M \otimes^{\mathrm{L}} N) \otimes^{\mathrm{L}}_U T^m A \\ \longrightarrow \quad & (M \otimes N) \otimes^{\mathrm{L}}_U T^m A, \end{aligned}$$

where the last $\simeq$ in the second line is induced by the tensor flip as in the derived version (7.2.4) of Lemma 7.2.8, and the morphism from the second to the third line is similarly as in the definition of $\smile$ induced by the morphism $M \otimes^{\mathrm{L}} N \to M \otimes N$ in $\mathcal{D}^-(U^{\mathrm{op}})$ that takes zeroth cohomology. Passing now to cohomology, we obtain $\varphi \frown \cdot : \mathrm{Tor}^U_n(N, A) \to \mathrm{Tor}_{n-m}(M \otimes N, A)$.

More explicitly, if $P \in \mathcal{D}^-(U)$ is a projective resolution of $A$, then $\frown$ is induced by the morphism

$$B^1_{ij} \ni n \otimes_U (x \otimes_A y) \mapsto \{\varphi \mapsto (\varphi(y) \otimes_A n) \otimes_U x\} \in B^2_{ij}$$

from the double complex

$$B^1_{ij} := N \otimes_U (P_i \otimes_A P_j),$$

whose total homology is $\mathrm{Tor}^U(N, A)$, to the double complex

$$B^2_{ij} := \mathrm{Hom}_k(\mathrm{Hom}_U(P_j, M), (M \otimes N) \otimes_U P_i),$$

whose homology has a natural map to $\mathrm{Hom}_k(\mathrm{Ext}_U(A, M), \mathrm{Tor}^U(M \otimes N, A))$.

In direct analogy with Theorem 7.2.1 we obtain:

**7.2.11 Theorem** *If $U$ is an $A$-biprojective left Hopf algebroid over $A$, then we have*

$$\varphi \bullet (x \otimes_U y) = \varphi \frown (x \otimes_U y), \quad \varphi \in \mathrm{Ext}^m_U(A, A), x \otimes_U y \in N \otimes^{\mathrm{L}}_U A$$

*as elements of $N \otimes^{\mathrm{L}}_U A \simeq (A \otimes N) \otimes^{\mathrm{L}}_U A$.*

# 7.3 Duality and the Proof of Theorem 7.1.1

## 7.3.1 The Underived Case

In the special case that $A$ is finitely generated projective itself, Theorem 7.1.1 reduces to standard linear algebra. We go through this case first since it is both instructive and will be used in the proof of the general case. For the reader's convenience we include full proofs.

**7.3.1 Lemma** *Let $U$ be a ring, $A \in U$-$\mathbf{Mod}$ be finitely generated projective, and let $A^*$ be $\mathrm{Hom}_U(A, U)$ with its canonical $U^{\mathrm{op}}$-module structure.*

1. *$A^*$ is finitely generated projective, and if $e_1, \ldots, e_n$ are generators of $A$, then there exist generators $e^1, \ldots, e^n \in A^*$ with*
$$\sum_i e^i(a)e_i = a, \quad \sum_i e^i\alpha(e_i) = \alpha$$
*for all $a \in A$ and $\alpha \in A^*$. The element*
$$\omega := \sum_i e^i \otimes e_i \in A^* \otimes_U A$$
*is independent of the choice of the generators $e_i, e^j$.*

2. *For all $U^{\mathrm{op}}$-modules $M$, the assignment*
$$\delta(m \otimes a)(\alpha) := m\alpha(a), \quad m \in M, a \in A, \alpha \in A^*$$
*uniquely extends to an isomorphism of abelian groups*
$$\delta : M \otimes_U A \to \mathrm{Hom}_{U^{\mathrm{op}}}(A^*, M).$$

3. *One has $(A^*)^* \simeq A$ and $A^* \otimes_U M \simeq \mathrm{Hom}_U(A, M)$ for $M \in U$-$\mathbf{Mod}$.*

4. *The map $\hat{}: \mathrm{Hom}_U(A, A) \to \mathrm{Hom}_{U^{\mathrm{op}}}(A^*, A^*)$, $\hat{\varphi}(\alpha) := \alpha \circ \varphi$ is a ring anti-isomorphism (with respect to the composition $\circ$).*

PROOF: Since $A$ is projective, there is a splitting $\iota : A \to U^n$ of
$$\pi : U^n \to A, \quad (u_1, \ldots, u_n) \mapsto \sum_i u_i e_i.$$
Hence $U^n \simeq A \oplus A_\perp$ for some $A_\perp \in U$-$\mathbf{Mod}$. Dually, this gives $A^* \oplus (A_\perp)^* = (U^n)^* \simeq U^n$, whence $A^*$ is finitely generated projective. The $e^i$ can be defined as the composition of $\iota$ with the projection of $U^n$ on the $i$-th summand. This proves the first parts of 1. For 2. just note that
$$\mathrm{Hom}_{U^{\mathrm{op}}}(A^*, M) \ni \varphi \mapsto \sum_i \varphi(e^i) \otimes e_i \in M \otimes_U A$$
inverts $\delta$. Since $\omega = \delta^{-1}(\mathrm{id}_{A^*})$, it does indeed not depend on the choice of generators. 3. now follows from 1. and 2. For 4., we note that
$$\hat{\varphi}(\alpha) = \alpha \circ \varphi = \sum_i e^i\alpha(\varphi(e_i)) = \delta(\sum_i e^i \otimes \varphi(e_i))(\alpha),$$
that is, we have $\hat{\varphi} = \delta(\sum_i e^i \otimes \varphi(e_i))$. Thus $\hat{}$ is the composition of the isomorphism $\mathrm{Hom}_U(A, A) \to A^* \otimes_U A$ from 3. with the isomorphism $\delta$. Finally, we have $(\widehat{\varphi \circ \psi})(\alpha) = \alpha \circ \varphi \circ \psi = \hat{\psi}(\alpha \circ \varphi) = \hat{\psi}(\hat{\varphi}(\alpha)) = (\hat{\psi} \circ \hat{\varphi})(\alpha)$. □

As in the introduction, let us abbreviate in the situation of this theorem
$$H^0(M) := \mathrm{Hom}_U(A, M), \quad H_0(N) := N \otimes_U A$$
for $M \in U$-$\mathbf{Mod}$, $N \in U^{\mathrm{op}}$-$\mathbf{Mod}$, and call $\omega \in H_0(A^*)$ the fundamental class of $(U, A)$. Then, for $M = A$, claim 3. says that we have an isomorphism
$$\cdot \bullet \omega : H^0(A) \to H_0(A^*), \quad \varphi \mapsto \sum_i e^i \otimes \varphi(e_i). \tag{7.3.1}$$
Using Lemma 7.2.8 we can upgrade this to the underived case of Theorem 7.1.1:

**7.3.2 Lemma** *Let $U$ be a left Hopf algebroid over $A$ and assume $A$ is finitely generated projective as a $U$-module. Then the cap product with the fundamental class $\omega \in H_0(A^*) = A^* \otimes_U A$ defines for all $M \in U$-$\mathbf{Mod}$ an isomorphism*

$$\cdot \frown \omega : H^0(M) \to H_0(M \otimes A^*).$$

PROOF: We have $\varphi \frown \omega = \sum_i (\varphi(1) \otimes_A e^i) \otimes_U e_i$, and Lemma 7.2.8 identifies

$$H_0(M \otimes A^*) = (M \otimes A^*) \otimes_U A \simeq A^* \otimes_U (A \otimes M) \simeq A^* \otimes_U M.$$

In this chain of identifications, $\varphi \frown \omega$ is mapped to

$$\varphi \frown \omega \quad \mapsto \quad \sum_i e^i \otimes_U (e_i \otimes_A \varphi(1)) \mapsto \sum_i e^i \otimes_U (e_i \varphi(1)) = \sum_i e^i \otimes_U \varphi(e_i)$$

which is identified with $\varphi$ under the isomorphism $\mathrm{Hom}_U(A, M) \simeq A^* \otimes_U M$ given by $\varphi \mapsto \sum_i e^i \otimes_U \varphi(e_i)$, as in (7.3.1). The claim follows. □

### 7.3.2 The Derived Case

It remains to throw in some homological algebra to obtain Theorem 7.1.1 in general. To shorten the presentation, we define:

**7.3.3 Definition** A module $A$ over a ring $U$ is *perfect* if it admits a finite resolution by finitely generated projectives. We call such a module a *duality module* if there exists $d \geq 0$ such that $\mathrm{Ext}^n_U(A, U) = 0$ for all $n \neq d$. In this case we abbreviate $A^* := \mathrm{Ext}^d_U(A, U)$ and call $d$ the *dimension* of $A$.

The main remaining step is to prove a derived version of Lemma 7.3.1. One could use a result of Neeman by which $A \in U$-$\mathbf{Mod}$ is perfect if and only if $\mathrm{Hom}_U(A, \cdot)$ commutes with direct sums [Ke, N], or the Ischebeck spectral sequence, which degenerates at $E^2$ if $A$ is a duality module [Isch, Kr, Sk]. However, we include a more elementary and self-contained proof.

**7.3.4 Theorem** *Let $A \in U$-$\mathbf{Mod}$ be a duality module of dimension $d$.*

1. *The projective dimension of $A \in U$-$\mathbf{Mod}$ is $d$.*
2. *$A^*$ is a duality module of the same dimension $d$.*
3. *If $P_\bullet \to A$ is a finitely generated projective resolution of length $d$, then $P^*_{d-\bullet} = \mathrm{Hom}_U(P_{d-\bullet}, U)$ is a finitely generated projective resolution of $A^*$ and for all $U^{\mathrm{op}}$-modules $M$ the canonical isomorphism*

$$\delta : M \otimes_U P_i \to \mathrm{Hom}_U(P^*_i, M), \quad m \otimes_U p \mapsto \{\alpha \mapsto m\alpha(p)\}$$

*induces a canonical isomorphism*

$$\mathrm{Tor}^U_i(M, A) \to \mathrm{Ext}^{d-i}_{U^{\mathrm{op}}}(A^*, M).$$

4. *There is a canonical isomorphism $(A^*)^* \simeq A$.*

PROOF: Let $P_\bullet \to A$ be a finitely generated projective resolution of finite length $m \geq 0$ (which exists since $A$ is perfect). Then the (co)homology of

$$0 \to P^*_0 \to \ldots \to P^*_m \to 0, \quad P^*_n = \mathrm{Hom}_U(P_n, U)$$

is $\mathrm{Ext}^\bullet_U(A, U)$, so by assumption we have $m \geq d$, and the above complex is exact except at $P^*_d$ where the homology is $A^* = \mathrm{Ext}^d_U(A, U)$. Furthermore, all the $P^*_n$ are finitely generated projective since the $P_n$ are (Lemma 7.3.1).

Let $\pi_i$ be the map $P^*_i \to P^*_{i+1}$ and put $K := \ker \pi_{d+1}$. By construction,

$$0 \to K \to P^*_{d+1} \to \ldots \to P^*_m \to 0 \tag{7.3.2}$$

is exact. If one compares this exact sequence with the sequence

$$\ldots \to 0 \to 0 \to P^*_m \to P^*_m \to 0$$

using Schanuel's lemma (see [McCRob, 7.1.2]), one obtains that $K$ is projective.

The exactness of $P^*_\bullet$ at $P^*_{d+1}$ gives $K = \operatorname{im}\pi_d$, and by the projectivity of $K$, the map $\pi_d : P^*_d \to K \subset P^*_{d+1}$ splits so that $P^*_d \simeq K \oplus K_\perp$, $K_\perp := \ker \pi_d$. In particular, both $K$ and $K_\perp$ are finitely generated.

It follows from all this that the complex

$$0 \to P^*_0 \to \ldots \to P^*_{d-1} \to K_\perp \to 0 \tag{7.3.3}$$

is a finitely generated projective resolution of $A^*$: since $\operatorname{im}\pi_{d-1} \subset P^*_d$ is contained in $\ker \pi_d = K_\perp$, it is still exact at $P^*_{d-1}$, and the homology at $K_\perp$ is the homology of $P^*_\bullet$ at $P^*_d$, that is, $A^*$.

Since (7.3.2) is a finitely generated projective resolution of 0 and as a complex $P^*_{d-\bullet}$ is a direct sum of (7.3.3) and (a shift of) (7.3.2), we also know that $\operatorname{Ext}^\bullet_{U^{\mathrm{op}}}(A^*, M)$ is the (co)homology of $\operatorname{Hom}_U(P^*_{d-\bullet}, M)$ for any $M \in U^{\mathrm{op}}\text{-}\mathbf{Mod}$. By Lemma 7.3.1, this is isomorphic to $M \otimes_U P_{d-\bullet}$ as a chain complex via the isomorphism given in 3., and the homology of this complex is $\operatorname{Tor}^U_{d-\bullet}(M, A)$. This proves 3. The special case $M = U$ implies the remaining claims. □

Finally, assume that in the situation of the above theorem, $U$ is an $A$-biprojective left Hopf algebroid over $A$. Since $P$ is a projective resolution, we have $M \otimes_U P \simeq M \otimes^{\mathrm{L}}_U P$ and $\operatorname{Hom}_U(P^*, M) \simeq \operatorname{RHom}_U(P^*, M)$, and $\delta$ gives an isomorphism between the two. The *fundamental class* is defined to be

$$\omega := \delta^{-1}(\mathrm{id}_{A^*}) \in A^* \otimes^{\mathrm{L}}_U A \simeq P^* \otimes_U A \simeq A^* \otimes_U P,$$

and Theorem 7.3.4 immediately gives:

**7.3.5 Corollary** *If $e_1, \ldots, e_n$ and $\tilde e^1, \ldots \tilde e^n$ are generators of $A$ and of $A^*$, respectively, then there are $e^1, \ldots, e^n \in P^*_0$ and $\tilde e_1, \ldots, \tilde e_n \in P_d$ such that*

$$\omega = \sum_i e^i \otimes_U e_i = \sum_i \tilde e^i \otimes_U \tilde e_i,$$

*and $\delta$ is given by the Yoneda product $\cdot \bullet \omega$.*

Theorem 7.1.1 follows now as in the underived case (Lemma 7.3.2), working with $\operatorname{RHom}_U(A, M)$ and $(M \otimes^{\mathrm{L}} A^*) \otimes^{\mathrm{L}}_U A$ instead of $H^0(M) = \operatorname{Hom}_U(A, M)$ and $H_0(M \otimes A^*) = (M \otimes A^*) \otimes_U A$: using Theorem 7.2.11 and (7.2.4) one gets

$$\begin{aligned}(M \otimes^{\mathrm{L}} A^*) \otimes^{\mathrm{L}}_U A &\simeq A^* \otimes^{\mathrm{L}}_U (A \otimes^{\mathrm{L}} M) \simeq A^* \otimes^{\mathrm{L}}_U M \\ &\simeq P^* \otimes^{\mathrm{L}}_U M \simeq \operatorname{RHom}_U(P, M) \\ &\simeq \operatorname{RHom}_U(A, M),\end{aligned}$$

where we hide the reindexing of the complexes for the sake of better readability (so $P^*$ stands for $P^*_{d-\bullet}$, and both $\operatorname{RHom}_U(P, M)$ and $\operatorname{RHom}_U(A, M)$ are reindexed in the same way). This leads to a convergent spectral sequence

$$\operatorname{Tor}^U_p(\operatorname{Tor}^A_q(M, A^*), A) \Rightarrow \operatorname{Ext}^{d-p-q}_U(A, M),$$

and under the last assumption of Theorem 7.1.1 (i.e., $\operatorname{Tor}^A_q(M, A^*) = 0$ for $q > 0$) this spectral sequence degenerates to the claimed isomorphism.

# Appendix

This appendix contains a collection of well-known facts repeatedly used throughout the text.

**A.1.1 Reminder on the Behaviour of the Functors Hom and $\otimes$ on Bimodules** The material in this paragraph is standard (confer e.g. [CarE, M]) but still may be of some help to maintain a certain overview in the abundance of module structures in the preceding chapters. Compare the conventions at the end of the Introduction (page 9) for notation.

Let $R, S, T$ be any three rings.

(*i*) If two *left* $R$-modules ${}_RM$, ${}_RN$ also happen to additionally carry $S$-actions and $T$-actions from left or right, respectively, the space $\mathrm{Hom}_{(R,-)}(M,N)$ carries the following explicit (bi)module structures. For any $f \in \mathrm{Hom}_{(R,-)}(M,N),\ m \in M,\ s \in S,\ t \in T$ one has:

$$\begin{array}{rcllcl} ({}_RM_S, {}_RN_T) & \Longrightarrow & {}_S[\mathrm{Hom}_{(R,-)}(M,N)]_T, & (sft)(m) & := & [f(ms)]t \\ ({}_{R-S}M, {}_{R-T}N) & \Longrightarrow & {}_T[\mathrm{Hom}_{(R,-)}(M,N)]_S, & (tfs)(m) & := & t[f(sm)]. \end{array}$$

(*ii*) For the situation $M_R, N_R$ of two *right* $R$-modules that are also equipped with additional left or right $S$-actions and $T$-actions, respectively, the space $\mathrm{Hom}_{(-,R)}(M,N)$ carries the following explicit (bi)module structures. For each $g \in \mathrm{Hom}_{(-,R)}(M,N),\ m \in M,\ s \in S, t \in T$ one has:

$$\begin{array}{rcllcl} ({}_SM_R, {}_TN_R) & \Longrightarrow & {}_T[\mathrm{Hom}_{(-,R)}(M,N)]_S, & (tgs)(m) & := & t[g(sm)] \\ (M_{R-S}, N_{R-T}) & \Longrightarrow & {}_S[\mathrm{Hom}_{(-,R)}(M,N)]_T, & (sgt)(m) & := & [f(ms)]t. \end{array}$$

On tensor products of bimodules, one has the following bimodule structures:

$$({}_SM_R, {}_RN_T) \Longrightarrow {}_S[M \otimes_R N]_T, \quad s(m \otimes n)t := sm \otimes nt, \tag{A.1.4}$$

which may be generalised to $n$ modules $M^1, \ldots, M^n$ by

$$({}_SM^1_R, \ldots, {}_RM^n_T) \Longrightarrow {}_S[M^1 \otimes_R \ldots \otimes_R M^n]_T, \quad s(m_1 \otimes \ldots \otimes m_n)t := sm_1 \otimes \ldots \otimes m_n t.$$

**A.1.2 The Dual Basis Lemma** Unlike a free module, a projective module may not have a basis, but it always has a 'projective coordinate system' with similar properties. (see e.g. [AnFu]).

**A.1.3 Definition** Let $R$ be any ring, $P$ an $R$-module, and $I$ an index set. A pair of indexed sets $\{e_i\}_{i\in I} \in P$ and $\{e^i\}_{i\in I} \in \mathrm{Hom}_R(P,R)$ is called a *dual basis* for $P$ in case for all $x \in P$

(*i*) $e^i(x) = 0$ for almost all $i \in I$,

(*ii*) $x = \sum_I e^i(x)e_i$.

**A.1.4 Lemma** (*i*) *$P$ is projective over $R$ iff it has a dual basis.*

(*ii*) *$P$ is finitely generated projective iff there exist $e_1, \ldots, e_n \in P$ (*a generating set*) and $e^1, \ldots, e^n \in \mathrm{Hom}_R(P,R)$ such that for each $x \in P$*

$$x = \sum_{i=1}^{n} e^i(x)e_i.$$

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# Index

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