Glimpses on Grothendieck toposes in the perspective of AI

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Prehistory of Grothendieck toposes:

The notion of Grothendieck toposes depends on
two key notions which had been introduced in 1942 or 1943.

- The notion of "sheaf" was introduced by Jean Leray.

  Remarks :

  - Leray's sheaves were linear sheaves.
  - They were defined in the context of topological spaces.
  - Leray's initial definition has been modified by Henri Cartan

- The notion of "category" and the associated definitions of "functor" and "natural transformation" of functors were introduced by Samuel Eilenberg and Saunders MacLane.

Categories for cohomology:

- Grothendieck's 1957 "Tohoku" paper
  "Sur quelques points d'algèbre homologique"
  identified general properties of linear categories
  which allow to define cohomology of "left-exact" functors :

  These categories have to :

  - be "abelian categories",
  - verify extra conditions ensuring that they possess "injective resolutions".

- This paper gave two families of categories verifying those conditions :

  - The categories of linear sheaves on topological spaces.
  - The categories of linear "diagrams",
    i.e. linear "presheaves" on essentially small categories $\mathcal{C}$,
    i.e. contravariant (or covariant) functors $\mathcal{C} \to \mathrm{Modules}$.

From cohomology to sites and toposes:

- Grothendieck identified the most general context which allows to define "sheaves".

  This is the context of "site" :

  (essentially) small category $\mathcal{C}$ $+$ "topology" J on $\mathcal{C}$

  $\|$

  coherent notion of "covering" of an object X by families of morphisms $X_i \xrightarrow{x_i} X$

- He showed that for any site $(\mathcal{C}, J)$, the categories of linear sheaves on $(\mathcal{C}, J)$ verify the properties needed to define cohomology.
- He had the idea of considering not only linear sheaves but set-valued sheaves on sites $(\mathcal{C}, J)$.
- He called "topos" a category equivalent to some $\widehat{\mathcal{C}}_J =$ category of set-valued sheaves on a site $(\mathcal{C}, J)$.

Why the name "topos" ?

- Grothendieck meant that the notion of topos is the most general mathematical notion to which intuitions of topological or geometric nature still apply.
- For him, topology should become the "science of toposes".

Indeed :

- Any topological space X defines the topos
  $\mathcal{E}_X =$ topos of set-valued sheaves on X
  and can be recovered from $\mathcal{E}_X$ (up to "soberification").
- The notion of continuous map generalises to the notion of topos morphism $(\mathcal{E}' \xrightarrow{f} \mathcal{E}) =$ pair of adjoint functors$(\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$.
- The notion of point generalises to the notion of
  point of a topos $\mathcal{E}$ = topos morphism $\mathcal{S}\mathrm{et} \to \mathcal{E}$.

Toposes and geometry :

- Classical notions of geometric spaces X (ex : algebraic varieties, differential or analytic manifolds,...) give rise to

  associated toposes $\mathcal{E}_X$ endowed with
  structure "inner rings" $\mathcal{O}_X =$ ring object in $\mathcal{E}_X$.

- Classical notions of geometric maps $X \xrightarrow{p} S$ give rise to

  morphisms of toposes $(p^*, p_*) : \mathcal{E}_X \to \mathcal{E}_S$ completed with
  morphisms of ring objects $p^*\mathcal{O}_S \to \mathcal{O}_X$ or $\mathcal{O}_S \to p_*\mathcal{O}_X$.

- The notions of
  - ringed topos (or, even more generally,
    topos endowed with an "inner structure"),
  - morphism between ringed (or structured) toposes,

  are much more general.

Toposes and invariants :

- In general, invariants of topological or geometric spaces can be defined in terms of their associated toposes (or structured toposes).
- As the notion of topos (or structured topos) is hugely more general than classical notions of space,

  and as sites and associated toposes can be defined in most mathematical contexts,

  this allows to associate

  topologically-inspired or geometrically-inpired invariants

  to most mathematical contexts.

Illustrations :

– Cohomology invariants (well-defined for all toposes).

– Homotopy invariants (well-defined for most toposes).

Deepening the understanding :

- Grothendieck insists that toposes have creative power

  as defining and studying invariants in terms of toposes
  - not only makes them applicable in the most diverse contexts,
  - but also leads to develop their theory in completely new directions and ultimately makes them much more powerful.

Illustration :

- The definition of cohomology in terms of sites and toposes allowed to define completely new cohomology invariants

  such as étale or cristalline cohomology spaces.

- Trying to understand Serre and Poincaré duality in terms of toposes led Grothendieck to discover the "six operations formalism"

  consisting in six types of functors $(f^*, f_*, \otimes, \mathcal{H}om, f_!, f^!)$ between derived categories of sheaves.

- Most deep results in arithmetic algebraic geometry of the last 60 years were obtained using these.

The role of toposes as classifying spaces :

Theorem (Diaconescu). –
For any site $(\mathcal{C}, J)$ *there is a first-order (geometric) theory* $\mathbb{T}_{\mathcal{C},J}$ *such that, for any topos* $\mathcal{E}$*,*

$$\mathrm{Geom}(\mathcal{E}, \widehat{\mathcal{C}}_J) = \{\textit{category of topos morphisms } \mathcal{E} \to \widehat{\mathcal{C}}_J\}$$

is naturally equivalent to

$$\mathbb{T}_{\mathcal{C},J}\text{-}\mathrm{mod}(\mathcal{E}) = \{\textit{category of } \mathcal{E}\textit{-valued models of } \mathbb{T}_{\mathcal{C},J}\}.$$

Theorem (Grothendieck, Hakim, Lawvere, Joyal, Makkai, Reyes, . . .). –
For any first-order (geometric) theory $\mathbb{T}$*, there is a topos* $\mathcal{E}_{\mathbb{T}}$ *(called the "classifying topos" of* $\mathbb{T}$*) such that, for any topos* $\mathcal{E}$*,*

$$\mathbb{T}\text{-}\mathrm{mod}(\mathcal{E}) = \{\textit{category of } \mathcal{E}\textit{-valued models of } \mathbb{T}\}$$

is naturally equivalent to

$$\mathrm{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) = \{\textit{category of topos morphisms } \mathcal{E} \to \mathcal{E}_{\mathbb{T}}\}.$$

Remark. –
In particular $\mathbb{T}$-mod$(\mathrm{Set}) = \{$category of set-valued models of $\mathbb{T}\}$
is equivalent to $\mathrm{pt}(\mathcal{E}_{\mathbb{T}}) = \{$category of points of $\mathcal{E}_{\mathbb{T}}\}$.

Toposes as bridges :

- The theory of "classifying toposes" was introduced in the $1970's$ but little developped in the following decades.
  For instance, most algebraic geometers used to work with
  $\begin{cases} - & \text{topos cohomology,} \\ - & \text{classifying schemes,} \end{cases}$
  never heard about "classifying toposes" of first-order theories.
- This theory has been reborn when, beginning in 2009, Olivia Caramello introduced and developped her theory of "toposes as bridges".

Principles :

- A bridge is an equivalence between toposes
  $$\mathcal{E}_{\mathbb{T}}, \mathcal{E}_{\mathbb{T}'}, \widehat{\mathcal{C}}_J, \widehat{\mathcal{C}'}_{J'}, \dots$$
  associated with different theories $\mathbb{T}, \mathbb{T}'$ or sites $(\mathcal{C}, J), (\mathcal{C}', J')$,...
- Consider particular invariants of toposes.
- Compute or express these invariants in terms of different presentations of the topos by theories of sites.

My first encounter with AI engineers :

- Just by chance, in July 2017,
  I was invited to give a talk
  at a workshop on "application of mathematics" organized by Huawei.
- I decided to talk about toposes,
  because this subject was interesting for me,
  even though I thought that
  it could not be really interesting for engineers.
- I mentionned in particular
  - the relationship of images (formulated as sites) and language (formulated as theories) through equivalences of toposes,
  - the fact that
    Topos Theory generalises Galois Theory.
- This was enough to arise the interest of some engineers working on IT and AI, especially Jean-Claude Belfiore.

The rising interest of some engineers for Grothendieck toposes :

- After July 2017,
  Jean-Claude Belfiore began to study Topos Theory.
- He discovered the work of Pierre Baudot and Daniel Bennequin (2015) interpreting "Shannon entropy" in terms of cohomology of some toposes.
- Belfiore and Bennequin began to develop theories of
  - Deep Neural Networks (DNN),
  - Semantic Information.
- I moved to Huawei in September 2021 because of the interest for toposes of Belfiore and other engineers.
- I was joined a year later by a young mathematician, Aurélien Sagnier.
  He began in January to give a lecture courses on toposes (which anybody can follow online).
- Our expanding team of engineers and mathematicians follows the work of Olivia Caramello and her PhD students (and post-doc) at the newly created "Istituto Grothendieck".
- We run an open seminar at the "Lagrange Center" in Paris.

How is it possible that engineers working in IT and AI get interested in toposes ?

- Engineers are more interested in general theories with a wide spectrum of possible applications
  than in very specialized developments.
  $\rightsquigarrow$ Just as PDE's or Measure Theory,
  Topos theory is very general and can a priori be applied in a great diversity of situations.
- Many engineers accept abstraction as long as the relationship with the concrete is not lost.
  $\rightsquigarrow$ Toposes are very abstract but are related to the concrete through their presentations by sites or theories.
- AI engineers look for mathematical theories which can formalize
  - basic elements of their practice,
  - basic operations of human intellect.

  $\rightsquigarrow$ Grothendieck toposes are good candidates for the formalization of several such "basic elements" and "basic operations".

Generality and diversity of toposes :

- The two equivalent definitions of toposes

  - as categories equivalent to some $\widehat{\mathcal{C}}_J$,
  - as categories which have the same constructive categorical properties as Set,

  are most general, as well as those of associated geometric notions (morphism, point, subtopos, open or closed embedding, $\cdots$).
- Key invariants of toposes (ex. cohomology, homotopy) can be defined for all toposes or for wide classes of toposes.
- On the other hand, toposes can be associated with completely different mathematical objects :
  - topological spaces,
  - groups, monoids, categories,
  - measures on a space (in such a way that a *measure* can be interpreted as a topology on a category associated with the underlying space),
  - ...
- Situations are unified in a dynamic way when they define sites or theories whose associated toposes are equivalent or share similar properties (e.g. are equivalent to presheaf toposes $\widehat{\mathcal{C}}$).

Abstraction and concreteness :

- The duality of toposes and their presentations by sites $(\mathcal{C}, J)$ or theories $\mathbb{T}$ allows to consider a mathematical situation at two different levels :
  - the "concrete" level represented by $(\mathcal{C}, J)$ or $\mathbb{T}$,
  - the "abstract" level incarnated in $\widehat{\mathcal{C}}_J$ or $\mathcal{E}_{\mathbb{T}}$.

- In this way, the two fundamental operations of
  - abstracting,
  - going down to the concrete,

  are formalized as mathematical operations which makes then amenable to
  - reasoning,
  - computation.

- This provides a possible theoretical framework for modeling the relationship of
  - mathematical exploration,
  - computing with machines.

Extraction of an essence :

- For Grothendieck, a path in two steps

$$\left\{\begin{array}{c}\text{given}\\ \text{mathematical}\\ \text{context}\end{array}\right\} \overset{(1)}{\to} \left\{\begin{array}{c}\text{definition}\\ \text{of a site}\\ (\mathcal{C}, J)\end{array}\right\} \overset{(2)}{\to} \left\{\begin{array}{c}\text{associated}\\ \text{topos}\\ \widehat{\mathcal{C}}_J\end{array}\right\}$$

is meant to extract an "essence" of the given context.

Remarks :
- – Part (1) is not automatic.
  It is subtle and delicate.
- – Part (2) just follows a universal process.

- It has to be composed with a third step

$$\left\{\begin{array}{c}\text{topos}\\ \hline \mathcal{E} \cong \widehat{\mathcal{C}}_J\end{array}\right\} \overset{(3)}{\to} \{\text{invariants}\}$$

which consists in
- – choosing particular invariants,
- – expressing or computing them.

- This is a general process for producing and expressing "meaningful" invariants of given mathematical contexts.

Sketches and extrapolation :

- Any category $\mathcal{C}$ endowed with a topology J is related to the associated topos $\widehat{\mathcal{C}}_J$ by a canonical functor

$$\ell : \mathcal{C} \to \widehat{\mathcal{C}}_J$$

which allows to consider $\widehat{\mathcal{C}}_J$ as some kind of "completion" of $\mathcal{C}$.

- The topos $\widehat{\mathcal{C}}_J$ consists in everything that can be "extrapolated" from $\mathcal{C}$ according to a process governed by rules encoded in the topology J.

- In the reverse direction, $(\mathcal{C}, J)$ can be considered as a "sketch" of the topos $\mathcal{E} \cong \widehat{\mathcal{C}}_J$.

- This provides a mathematical model for the pair of most natural operations of the intellect : $\begin{cases} - & \text{extrapolation,} \\ - & \text{sketching.} \end{cases}$

The relationship of syntax and semantics :

- For any first-order (geometric) theory $\mathbb{T}$, its classifying topos

  $\mathcal{E}_{\mathbb{T}}$ incarnates the "semantics"

  or "expressive content" of the theory $\mathbb{T}$.

  Indeed, "points" of $\mathcal{E}_{\mathbb{T}}$ correspond to "models" of $\mathbb{T}$.

- It is constructed as a "completion" or "extrapolation"

$$\mathcal{C}_{\mathbb{T}} \to \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} = \mathcal{E}_{\mathbb{T}}$$

  from $\mathcal{C}_{\mathbb{T}}$ = "syntactic category" of $\mathbb{T}$ consisting in

  "words" and "sentences" in the vocabulary of $\mathbb{T}$,
  related by "verbs" which obey
  the "grammar rules"(i.e. axioms) of $\mathbb{T}$.

- This provides a mathematical model for the relationship between
  - syntax (formalized languages),
  - semantics (expressive contents).

The relationship of images and language :

- For any site or "sketch" $(\mathcal{C}, J)$, an equivalence

$$\widehat{\mathcal{C}}_J \cong \mathcal{E}_{\mathbb{T}}$$

means that $\mathbb{T}$ provides a "linguistic description" of the "topological content" of the "sketch" $(\mathcal{C}, J)$.

- For any first-order (geometric) theory $\mathbb{T}$, an equivalence

$$\widehat{\mathcal{C}}_J \cong \mathcal{E}_{\mathbb{T}}$$

means that $(\mathcal{C}, J)$ provides a "topological sketch" of the "expressive content" of the "language" $\mathbb{T}$.

- This provides a mathematical model for the relationship between the linguistic and geometric abilities of the intellect :

$$\text{language} \underset{\text{imagination}}{\overset{\text{description}}{\rightleftarrows}} \text{images}$$

Topos Theory as point-free geometry :

- The notion of point of a topos is well-defined
  but sites and associated toposes are not defined in terms of points,
  unlike all classical notions of space.

- This corresponds much better to the way the intellect sees images :
  we never see points !

- It is also natural to think that points do not exist in the physical world.

⇝ This raises two questions :

- Can the theory of sites and toposes
  inspire new techniques
  for representing and processing images ?
- Could it inspire new modelisations of the physical world ?

From meaningless information to semantic information ?

- Information technology usually processes numbers and numerical functions.
- Numbers and functions are very convenient because they are amenable to
  - computation,
  - approximation.
- But numbers and functions are meaningless.
- In the last years, some scientists began to think that "semantic information" should take a geometric or topological form.
- A model for that is provided by Grothendieck's "sheaf-function" correspondence, which consists in "lifting"

functions on points	$\longmapsto$	linear sheaves on sites,
operations on such functions	$\longmapsto$	"six operations" on complexes of linear sheaves.