Grothendieck toposes as mathematics for future AI: illustration by the problem of image representation

Laurent Lafforgue

(Huawei Paris Research Center, Boulogne-Billancourt, France)

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The problem of representing images as mathematical objects:

- Any computer storage or processing of images is necessarily based on a mathematical model of the nature of images.

- Classically, an image is a collection of "pixels" (= measures of intensity of light) indexed by a finite set of plane points consisting in pairs of coordinates.

- The implicit mathematical model:

  - image = numerical function(s) defined on a plane area,
  - plane area = continuous set of points which can be discretized,
  - plane point = pair of coordinates.

Objections to the classical mathematical model of images:

- For our mind, an image
  is not at all a numerical function:

  — Intensity of light is not perceived
  in numerical terms.

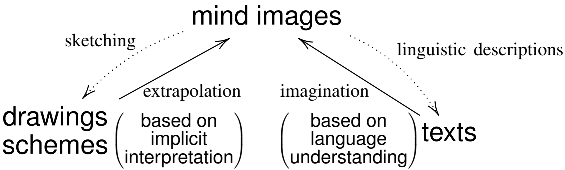
  — A plane area does not consist
  in points.

  — In fact, we see plane areas
  but we never see points.
  In our perception, points do not exist.

  — Our mind doesn't perceive coordinates:
  space and images are perceived
  in a much more vague way.

The double human expression of mind images:

- On the one hand, art representations:
  drawings, colored drawings, paintings, sketches, schemes, $\cdots$
- On the other hand, linguistic descriptions:
  describing a landscape or any type of environment with words,
  even telling a story,
  any piece of litterature
  any type of writing
  any type of speech
  } always describes a mind image.
- The basic diagram of mind images and their expressions:



A mathematical model of mind images and their expressions: sites, Grothendieck toposes, theories

- A mathematical model for art representations: sites

  A site consists in $\begin{cases} - & \text{a category } \mathcal{C}, \\ - & \text{a Grothendieck topology } J \text{ on } \mathcal{C}. \end{cases}$

  A category consists in $\begin{cases} - & \text{a list of pieces or locations,} \\ - & \text{a list of “oriented itineraries”} \\ & A \to B \text{ between pieces } A, B, \cdots \\ - & \text{a composition law for itineraries} \\ & (A \to B \to C) \Rightarrow (A \to C)\,. \end{cases}$

  A Grothendieck topology J on a category $\mathcal{C}$ consists in a building principle which allows to reconstruct “more complex pieces” A from related “simpler” pieces $A_i \to A$.
- A Grothendieck topos $\mathcal{E}$ is a category which, in a perfectly precise sense, is fully complete.
- Toposes as completions of categories:

  A topology J on a category $\mathcal{C}$ defines a topos completion

  $$\mathcal{C} \xrightarrow{\ell} \widehat{\mathcal{C}}_J \qquad \text{(so that } J = \text{“extrapolation” principle).}$$

- A mathematical model for texts: theories

  A "first-order geometric" theory consists in
  - a vocabulary,
  - grammar rules.

  Elements of vocabulary are
  - piece or location names,
  - itinerary names (associated with a pair of piece names),
  - relation names (associated with a finite family of piece names).

  Grammar rules take the form of implications $\varphi \vdash \psi$ between "geometric" formulas $\varphi, \psi, \cdots$ = "sentences" in the given vocabulary and the logical symbols $\top, \wedge, \bot, \vee, \exists$.

- Any such theory $\mathbb{T}$ defines a "syntactic" site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$:

  pieces of $\mathcal{C}_{\mathbb{T}}$ = formulas φ = sentences in the vocabulary of $\mathbb{T}$,

  itineraries of $\mathcal{C}_{\mathbb{T}}$ = implications $\varphi \vdash \psi$ which are provable from the grammar rules,

  topology of $\mathcal{C}_{\mathbb{T}}$ = principle for reducing a proof to a combination of local proofs.

- There is an associated "classifying" topos $\mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}$.

- A mathematical model for the double expressions of mind images:

Grothendieck toposes $\mathcal{E}$

expressed as $\widehat{\mathcal{C}}_J$ or $\mathcal{E}_{\mathbb{T}}$

infinitely many sketches by sites $(\mathcal{C}, J)$

infinitely many descriptions by theories $\mathbb{T}$

completion of $\mathcal{C}$ defined by J

completion $\mathcal{E}_{\mathbb{T}}$ of $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$

sites $(\mathcal{C}, J)$

first-order geometric theories $\mathbb{T}$

- A mathematical model of

drawings, schemes $\xleftarrow{\text{sketching}}$ / $\xrightarrow{\text{extrapolation}}$ mind images

extrapolation (based on interpretation)

is:

sites $(\mathcal{C}, J)$ $\xleftarrow{\text{sketching}}$ / $\longrightarrow$ toposes $\mathcal{E}$

completion $\mathcal{C} \to \widehat{\mathcal{C}}_J = \mathcal{E}$

(based on J = topology = extrapolation principle on $\mathcal{C}$)

- A mathematical model of

text (= syntactic datum) $\xleftarrow{\text{linguistic description}}$ / $\xrightarrow[\text{imagination (understanding)}]{}$ mind image (= semantic content)

is:

theory $\mathbb{T}$ $\xleftarrow{\text{syntactic description}}$ / $\xrightarrow[\text{semantic incarnation}]{}$ toposes $\mathcal{E}$

semantic incarnation
$\mathbb{T} \mapsto \mathcal{E}_{\mathbb{T}} = \mathcal{E}$
of the "meaningfull content" of $\mathbb{T}$

Sketching of images, naming functors and interpretation topologies:

- What we need for a point-free (i.e. pixel-free) topos-inspired representation of images is:

  A general theory of images $\mathbb{T}$ which is rich enough, so that any natural image (usually of a 3-dim object or environment) can be sketched as a (usually finite) category

  $$\mathcal{C} \text{ consisting in } \begin{cases} \text{pieces,} \\ \text{relations (e.g. position relations),} \end{cases}$$

  endowed with a "naming functor"

  $$\begin{array}{rcl} N : \mathcal{C} & \longrightarrow & \mathcal{C}_{\mathbb{T}}\,, \\ \text{pieces } A, B, \cdots & \longmapsto & \text{appropriate names or description sentences,} \\ (A \to B) & \longmapsto & \text{implications provable from the grammar rules of } \mathbb{T}. \end{array}$$

- Then the "naming functor" N would induce from $J_{\mathbb{T}}$ = topology of $\mathcal{C}_{\mathbb{T}}$ a canonical topology J = "extrapolation principle" of $\mathcal{C}$ characterized by a square of itineraries of toposes:

$$\begin{array}{ccc} \widehat{\mathcal{C}} & \xrightarrow{N_*} & \widehat{\mathcal{C}_{\mathbb{T}}} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{C}}_J & \longrightarrow & \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} = \mathcal{E}_{\mathbb{T}} \end{array}$$

General theory and singular descriptions:

- Suppose that

$$\begin{cases} - & \text{we have defined a rich enough theory of images } \mathbb{T}, \\ - & \text{a natural image is sketched as a category } \mathcal{C} \\ & \text{endowed with a "naming functor"} \quad N : \mathcal{C} \longrightarrow \mathcal{C}_{\mathbb{T}}\,, \end{cases}$$

inducing

$$\begin{cases} - & \text{an "interpretation topology" } J \text{ on } \mathcal{C}, \\ - & \text{an itinerary of toposes} \quad \widehat{\mathcal{C}}_J \xrightarrow{N_*} \mathcal{E}_{\mathbb{T}}\,. \end{cases}$$

- Then:
  - there is a canonical factorization of the itinerary N_*

$$\widehat{\mathcal{C}}_J \underset{\text{surjective}}{\twoheadrightarrow} \mathrm{Im}(N_*) \underset{\text{embedding}}{\hookrightarrow} \mathcal{E}_{\mathbb{T}}\,,$$

  - the subtopos $\quad \mathrm{Im}(N_*) \hookrightarrow \mathcal{E}_{\mathbb{T}}$

    is the "classifying topos" of a theory

    $\mathbb{T}'$ consisting in $\begin{cases} \text{the same vocabulary as } \mathbb{T}, \\ \text{more "grammar rules"}, \end{cases}$

    which can be considered

    a specific description

    of the particular image we are considering.

Constructing spaces of image descriptions?

- Is it possible to parametrize image descriptions by points of some space?

  Key remark: Such a space should have a continuous structure as, for us, natural images move and transform.

- If $\mathbb{T}$ is a "theory of images", rich enough to describe natural images, the problem becomes:

Question. –

(1) *Naive form: Is there a "space" whose points parametrize subtoposes of $\mathcal{E}_{\mathbb{T}}$?*

(2) *More precise unambiguous well-posed form: Is there a topos $\mathcal{J}$ such that, for any topos $\mathcal{E}$, subtoposes of the product topos $\mathcal{E} \times \mathcal{E}_{\mathbb{T}}$ correspond to topos itineraries $\mathcal{E} \to \mathcal{J}$?*

Deep learning as a relativization process?

- Suppose that we have defined a "theory of images" $\mathbb{T}$ rich enough to allow representing natural images by categories $\mathcal{C}$ endowed with a "naming functor"
$$N : \mathcal{C} \longrightarrow \mathcal{C}_{\mathbb{T}}$$
inducing a topos itinerary
$$\widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}_{\mathbb{T}} \,.$$
- A process of information extraction could be constructed as a sequence of surjective topos itineraries
$$\mathcal{E}_{\mathbb{T}} = \mathcal{E}_0 \twoheadrightarrow \mathcal{E}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}_k$$
whose steps $\mathcal{E}_i \twoheadrightarrow \mathcal{E}_{i+1}$
would gradually extract more and more abstract information.

General remarks. –

(i) *A topos $\mathcal{E}$ endowed with a topos itinerary $\mathcal{E} \to \mathcal{B}$ is called a "relative topos" over the "base topos" $\mathcal{B}$.*

(ii) *It can be presented as classifying "$\mathcal{B}$-based theories" (= theories parametrized by points of $\mathcal{B}$).*

(iii) *If only for that reason, a topos itinerary $\mathcal{E} \to \mathcal{B}$ always has meaning.*