IV. Quotients of presheaf type theories

Reminder on presheaf type theories:

- A first-order geometric theory $\mathbb{T}$ is said to be "presheaf type" if its classifying topos $\mathcal{E}_{\mathbb{T}}$ admits a presentation
$$\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$
as the topos of presheaves on an essentially small category
$$\mathcal{C} \xrightarrow{\ell} \mathcal{E}_{\mathbb{T}} \, .$$
- If $\mathcal{C}$ and $\mathcal{D}$ are two essentially small categories, the category of equivalences
$$\widehat{\mathcal{D}} \xrightarrow{\sim} \widehat{\mathcal{C}}$$
is equivalent to that of equivalences
$$\mathrm{Kar}(\mathcal{D}) \longrightarrow \mathrm{Kar}(\mathcal{C}).$$
- In particular, if $\mathcal{C}$ and $\mathcal{D}$ are Karoubi-complete, equivalences
$$\widehat{\mathcal{D}} \xrightarrow{\sim} \widehat{\mathcal{C}}$$
correspond to equivalences
$$\mathcal{D} \xrightarrow{\sim} \mathcal{C} \, .$$

Reminder of semantic and syntactic presentations of classifying toposes of presheaf type theories:

- If $\mathbb{T}$ is a presheaf type theory, the two categories

$$\begin{cases} \mathcal{M} = \mathbb{T}\text{-mod}(\mathrm{Set})_{\mathrm{fp}} \,, \\ \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}} = \text{full subcategory of "irreducible" formulas} \\ \qquad \text{in } \mathcal{C}_{\mathbb{T}} = \text{geometric syntactic category of } \mathbb{T}, \end{cases}$$

  are essentially small and Karoubi-complete.

- We have two canonical presentations

$$\begin{array}{ccc} \widehat{\mathcal{M}^{\mathrm{op}}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}} \,, \\ \widehat{\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}} \,. \end{array}$$

- The induced equivalence

$$\mathcal{M}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$$

  associates with any finitely presentable model an irreducible formula which presents this model, and the reverse equivalence

$$\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}} \xrightarrow{\sim} \mathcal{M}^{\mathrm{op}}$$

  associates with any irreducible geometric formula a model defined by this formula.

The special case of Cartesian theories:

- If $\mathbb{T}$ is a Cartesian theory
  (in particular, if $\mathbb{T}$ is an empty theory,
  or algebraic
  or "Horn"),
  we have a presentation

  $$\widehat{\mathcal{C}_{\mathbb{T}}^{\mathrm{car}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

  as the topos of presheaves on
  $\mathcal{C}_{\mathbb{T}}^{\mathrm{car}}$ = Cartesian syntactic theory of $\mathbb{T}$
  (consisting of $\mathbb{T}$-Cartesian formulas).
- The category $\mathcal{C}_{\mathbb{T}}^{\mathrm{car}}$ is Cartesian, so Karoubi-complete.

Corollary. –
If $\mathbb{T}$ is a Cartesian theory, we have:

(i) *The geometric $\mathbb{T}$-Cartesian formulas in the signature Σ of $\mathbb{T}$ are the irreducible geometric formulas.*

(ii) *There is identity of the two categories*

$$\mathcal{C}_{\mathbb{T}}^{\mathrm{car}} = \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}} .$$

The correspondence between quotient theories and topologies:

- We consider a theory $\mathbb{T}$ of presheaf type, endowed with a presentation

$$\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \,.$$

- The induced functor

$$\ell : \mathcal{C} \longrightarrow \mathcal{E}_{\mathbb{T}}$$

  factorizes into

$$\mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}} \hookrightarrow \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{E}_{\mathbb{T}} \,.$$

- Hence a fully faithful functor

$$\mathcal{C} \hookrightarrow \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$$

  and

  $\begin{cases} \text{any object } X \text{ of } \mathcal{C} \text{ defines an irreducible formula } \varphi_X, \\ \text{any irreducible formula comes from an idempotent of an object of } \mathcal{C}. \end{cases}$

- The morphisms between two objects X and Y of $\mathcal{C}$

$$X \xrightarrow{u} Y$$

  correspond to $\mathbb{T}$-provably functional formulas

$$\theta_u(\vec{x},\vec{y}) : \varphi_X(\vec{x}) \longrightarrow \varphi_Y(\vec{y}) \,.$$

From topologies to quotient theories:

Proposition. –
Consider a presentation $\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$
with the induced fully faithful functor

$$\begin{cases} \mathcal{C} & \hookrightarrow & \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}\,, \\ X & \longmapsto & \varphi_X\,, \\ (X \xrightarrow{u} Y) & \longmapsto & (\theta_u : \varphi_X \to \varphi_Y)\,. \end{cases}$$

Consider a topology J *over* $\mathcal{C}$
generated by some covering families of the form

$$(X_i \xrightarrow{u_i} X)_{i \in I}\,.$$

Then adding to the axioms of $\mathbb{T}$ *the axioms*

$$\varphi_X(\vec{x}) \vdash_{\vec{x}} \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \theta_{u_i}(\vec{x}_i, \vec{x})$$

defines a quotient theory $\mathbb{T}_J$ *of* $\mathbb{T}$
such that the equivalence $\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$
induces an equivalence

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_J}\,.$$

From quotient theories to topologies:

Proposition. – *Consider a presentation* $\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$ *with the induced fully faithful functor*

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{C}_{\mathbb{T}}\,, \\ X & \longmapsto & \varphi_X \end{array}$$

whose image is dense for the topology $J_{\mathbb{T}}$ *of* $\mathcal{C}_{\mathbb{T}}$.
Consider a quotient theory $\mathbb{T}'$ *of* $\mathbb{T}$
defined by adjoining to the axioms of $\mathbb{T}$ *some axioms* $\quad \varphi \vdash_{\vec{x}} \psi$.
For each such additional axiom $\varphi \vdash_{\vec{x}} \psi$, *proceed as follows:*

- *Consider the monomorphism of* $\mathcal{C}_{\mathbb{T}}$
$$(\varphi \wedge \psi)(\vec{x}) \hookrightarrow \varphi(\vec{x})\,.$$
- *Cover* $\varphi(\vec{x})$ *by a* $J_{\mathbb{T}}$*-covering family of images of objects of* $\mathcal{C}$
$$\varphi_{X_i} \longrightarrow \varphi(\vec{x})\,, \quad i \in I\,.$$
- *Cover each fiber product in* $\mathcal{C}_{\mathbb{T}}$
$$(\varphi \wedge \psi) \times_{\varphi} \varphi_{X_i}$$
by a $J_{\mathbb{T}}$*-covering family of images of objects of* $\mathcal{C}$
$$\varphi_{X_{i,k}} \longrightarrow (\varphi \wedge \psi) \times_{\varphi} \varphi_{X_i}\,, \quad k \in K_i\,.$$

Then the topology J' *of* $\mathcal{C}$ *which corresponds to* $\mathbb{T}'$
is generated by the covering families $\quad (X_{i,k} \longrightarrow X_i)_{k \in K_i}$.

The special case of expressing axioms in irreducible terms:

Lemma. – *Let* $\mathbb{T}$ *be a geometric theory of presheaf type. Then any implication relation between geometric formulas* $\varphi \vdash_{\vec{x}} \psi$ *is* $\mathbb{T}$*-provably equivalent to a family of axioms of the form*

$$\varphi_i \vdash_{\vec{x}_i} \bigvee_{k \in K_i} (\exists \, \vec{x}_k) \, \theta_{i,k}(\vec{x}_k, \vec{x}_i)$$

where

$$\begin{cases} \bullet \quad \text{each } \varphi_i = \varphi_i(\vec{x}_i) \text{ is an irreducible formula,} \\ \bullet \quad \text{each } \varphi_{i,k}(\vec{x}_k) \xrightarrow{\theta_{i,k}} \varphi_i(\vec{x}_i) \\ \quad\;\; \text{is a } \mathbb{T}\text{-provably functional formula} \\ \quad\;\; \text{relating two irreducible formulas } \varphi_{i,k} \text{ and } \varphi_i. \end{cases}$$

Proof. – This is the special case $\mathcal{C} = \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$ of the previous proposition.
We choose a $J_{\mathbb{T}}$-covering of φ by irreducible formulas

$$\varphi_i = \varphi_i(\vec{x}_i)$$

then a $J_{\mathbb{T}}$-covering of each fiber product in $\mathcal{C}_{\mathbb{T}}$

$$(\varphi \wedge \psi) \times_{\varphi} \varphi_i$$

by irreducible formulas $\qquad \varphi_{i,k} = \varphi_{i,k}(\vec{x}_k)\,.$

The syntactic expression of the correspondence between quotient theories and topologies:

Corollary. –
Let $\mathbb{T}$ be a geometric theory of presheaf type,
with the induced representation of its classifying topos

$$\widehat{\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \, .$$

Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$
defined by adjoining to the axioms of $\mathbb{T}$ some axioms of the form

$$\varphi \vdash_{\vec{x}} \bigvee_{k \in K} (\exists \, \vec{x}_k) \, \theta_k(\vec{x}_k, \vec{x})$$

where the

$$\theta_k : \varphi_k = \varphi_k(\vec{x}_k) \longrightarrow \varphi(\vec{x}) = \varphi$$

are morphisms of $\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$.
Then the topology J' of $\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$ which corresponds to $\mathbb{T}'$
is generated by the covering families

$$(\varphi_k \xrightarrow{\theta_k} \varphi)_{k \in K} \, .$$

Characterization of models of quotient theories:

Corollary. – *Consider a presentation* $\widehat{\mathcal{C}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$ *with the induced fully faithful functor*

$$\begin{cases} \mathcal{C} & \hookrightarrow & \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}\,, \\ X & \longmapsto & \varphi_X\,, \\ (X \xrightarrow{u} Y) & \longmapsto & (\theta_u : \varphi_X \to \varphi_Y)\,. \end{cases}$$

Consider a topology J *of* $\mathcal{C}$ *generated by covering families of the form*

$$(X_i \xrightarrow{u_i} X)_{i \in I}\,,$$

and the quotient theory $\mathbb{T}_J$ *of* $\mathbb{T}$ *which corresponds to* J.
Then a model M *of* $\mathbb{T}$ *in a topos* $\mathcal{E}$ *is a model of* $\mathbb{T}_J$ *if and only if, for any such covering family*

$$(X_i \xrightarrow{u_i} X)_{i \in I}\,,$$

the associated morphism of $\mathcal{E}$

$$\coprod_{i \in I} M\varphi_{X_i} \xrightarrow{\coprod_{i \in I} M\theta_{u_i}} M\varphi_X$$

is an epimorphism.

Interpretations of formulas and model morphisms:

Theorem. –
Let $\mathbb{T}$ be a presheaf type theory, with the induced presentation

$$\widehat{\mathcal{M}^{\mathrm{op}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \qquad \text{for } \mathcal{M} = \mathbb{T}\text{-mod}(\mathrm{Set})_{\mathrm{fp}}.$$

Let M be an object of $\mathcal{M}$ presented by an irreducible formula φ. Then for any model N of $\mathbb{T}$ in a topos $\mathcal{E}$, we can write

$$N\varphi = \mathcal{H}om\,(p^* M, N)$$

where

- *$N\varphi$ is the object of $\mathcal{E}$ which interprets the formula φ in the model N,*
- *$p^* M$ is the model of $\mathbb{T}$ in $\mathcal{E}$ deduced from M by the topos morphism*
$$p = (p^*, p_*) : \mathcal{E} \longrightarrow \mathrm{Set}\,,$$
- *$\mathcal{H}om(p^* M, N)$ is the object of $\mathcal{E}$ characterized by the property that, for any object E of $\mathcal{E}$*
$$\mathrm{Hom}(E, \mathcal{H}om(p^* M, N)) = \mathrm{Hom}(p_E^* \, p^* M, p_E^* N)$$
if p_E^* denotes the pull-back component of the canonical morphism of toposes
$$p_E : \mathcal{E}/E \longrightarrow \mathcal{E}\,.$$

Application to a semantic characterization of quotient theories:

Corollary. –
Let $\mathbb{T}$ be a presheaf type theory, with the induced presentation

$$\widehat{\mathcal{M}^{\rm op}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \, .$$

Let J be a topology over $\mathcal{M}^{\rm op}$ which defines a quotient theory $\mathbb{T}_J$ of $\mathbb{T}$.
Let $\mathcal{E}$ be a topos endowed with the morphism $p = (p^*, p_*) : \mathcal{E} \longrightarrow \mathrm{Set} \, .$
Then a model N of $\mathbb{T}$ in $\mathcal{E}$ is a model of $\mathbb{T}_J$ if and only if it is

"J-homogeneous"

in the sense that, for any J-covering family of $\mathcal{M}^{\rm op}$

$$(M_i \longrightarrow M)_{i \in I} \, ,$$

the induced morphism of $\mathcal{E}$

$$\coprod_{i \in I} \mathcal{H}om\,(p^* M_i, N) \longrightarrow \mathcal{H}om\,(p^* M, N)$$

is an epimorphism.

Remark. – It suffices to consider a family of J-covering families $(M_i \longrightarrow M)_{i \in I}$ which generates the topology J of $\mathcal{M}^{\rm op}$.

To prove the theorem: relative toposes

Proposition. – *Let E be an object of a topos $\mathcal{E}$. Then:*

(i) *The relative category $\mathcal{E}/E$ whose*

- *objects are the morphisms $E' \to E$ of $\mathcal{E}$,*
- *morphisms are the commutative triangles of $\mathcal{E}$*

$$\begin{array}{ccc} E'_1 & \longrightarrow & E'_2 \\ & \searrow \quad \swarrow & \\ & E & \end{array}$$

is a topos.

(ii) *The functor*

$$\begin{array}{cccc} p_E^* : & \mathcal{E} & \longrightarrow & \mathcal{E}/E\,, \\ & E' & \longmapsto & E' \times E \end{array}$$

is the pull-back component of a topos morphism

$$p_E : \mathcal{E}/E \longrightarrow \mathcal{E}\,.$$

Sketch of proof:

(i) If $\mathcal{E}$ is the topos of sheaves on a site $(\mathcal{C}, J)$,

$\mathcal{E}/E$ is identified with the topos of sheaves on the "category of elements" of E

$\int E = \mathcal{C}/E$ endowed with the topology induced by J.

(ii) Indeed, p_E^* respects both arbitrary colimits and (arbitrary) limits.

To prove the theorem: sheaves of functor morphisms

Proposition. – *Let $\mathcal{C}$ be a small category.*
Consider two functors with values in a topos $\mathcal{E}$

$$F_1, F_2 : \mathcal{C} \rightrightarrows \mathcal{E}\,.$$

Then the contravariant functor

$$\begin{array}{rcl} \mathcal{E}^{\rm op} & \longrightarrow & \mathrm{Set}\,, \\ E & \longmapsto & \mathrm{Hom}(p_E^* \circ F_1, p_E^* \circ F_2) \end{array}$$

is representable by an object of $\mathcal{E}$ denoted $\mathcal{H}om(F_1, F_2)\,.$

Proof sketch:

- For any objects E_1, E_2 of $\mathcal{E}$, the contravariant functor
$$E \longmapsto \mathrm{Hom}(p_E^* E_1, p_E^* E_2) = \mathrm{Hom}_{\mathcal{E}}(E_1 \times E, E_2)$$
is representable by an object of $\mathcal{E}$ $\mathcal{H}om(E_1, E_2)\,.$
- The functor under consideration is representable by a subobject of
$$\prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathcal{H}om(F_1(X), F_2(X))$$
because the conditions that define functors with values in $\mathcal{E}$ are local for the canonical topology of $\mathcal{E}$.

To prove the theorem: sheaves of model morphisms

Corollary. – *Let $\mathbb{T}$ be a first-order geometric theory. Let M_1, M_2 be two models of $\mathbb{T}$ in a topos $\mathcal{E}$. Then the contravariant functor*

$$\begin{array}{ccc} \mathcal{E}^{\rm op} & \longrightarrow & {\rm Set}\,, \\ E & \longmapsto & {\rm Hom}(p_E^* M_1, p_E^* M_2) \end{array}$$

is representable by an object of $\mathcal{E}$ denoted

$$\mathcal{H}om(M_1, M_2)\,.$$

Proof. – Let's choose a presentation

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}\,.$$

Then the category of models of $\mathbb{T}$ in a topos $\mathcal{E}$ is related to that of functors from $\mathcal{C}$ to $\mathcal{E}$ by a fully faithful functor

$$\mathbb{T}\text{-mod}(\mathcal{E}) \hookrightarrow [\mathcal{C}, \mathcal{E}]$$

whose image is made up of flat and J-continuous functors.

To prove the theorem: a topos-theoretic Yoneda lemma

Lemma. – *Consider a contravariant functor from a small category* $\mathcal{C}$ *to a topos* $\mathcal{E}$

$$F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{E} \,.$$

Let $p = (p^*, p_*) : \mathcal{E} \to \mathrm{Set}$ *be the canonical morphism of toposes. Then we have for any object* X *of* $\mathcal{C}$*, of associated presheaf*

$$y(X) = \mathrm{Hom}(\bullet, X) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \,,$$

a canonical isomorphism in $\mathcal{E}$

$$\mathcal{H}om\,(p^* \circ y(X), F) \xrightarrow{\sim} F(X) \,.$$

Proof. –

- It suffices to show that there exists a canonical bijection

$$\mathrm{Hom}(p^* \circ y(X), F) \xrightarrow{\sim} \mathrm{Hom}(1_{\mathcal{E}}, F(X)) \,.$$

- As $1_{\mathcal{E}} = p^* 1_{\mathrm{Ens}}$ and p^* admits the right-adjoint $p_* : \mathcal{E} \to \mathrm{Set}$, we are reduced to showing that there exists a canonical bijection

$$\mathrm{Hom}(y(X), p_* \circ F) \xrightarrow{\sim} \mathrm{Hom}(1_{\mathrm{Ens}}, (p_* \circ F)(X)) = p_* \circ F(X) \,.$$

  This is Yoneda's lemma.

End of the proof of the theorem:

- We consider a theory $\mathbb{T}$ of presheaf type, with the presentation
  $$\widehat{\mathcal{M}^{\rm op}} \longrightarrow \mathcal{E}_{\mathbb{T}}$$
  and the canonical equivalence
  $$\mathcal{C}_{\mathbb{T}}^{\rm ir} \xrightarrow{\sim} \mathcal{M}^{\rm op}$$
  which associates to any irreducible formula φ a model M_φ that it presents.
- The models N of $\mathbb{T}$ in a topos $\mathcal{E}$ endowed with $p = (p^*, p_*) : \mathcal{E} \to \mathrm{Set}$ can be seen as the flat functors
  $$\begin{array}{rcccc} N & : & \mathcal{C}_{\mathbb{T}}^{\rm ir} \xrightarrow{\sim} \mathcal{M}^{\rm op} & \longrightarrow & \mathcal{E}\,, \\ & & (\varphi \leftrightarrow M_\varphi) & \longmapsto & N\varphi\,. \end{array}$$
- We deduce from the "topos-theoretic Yoneda lemma" that there exists, for any such flat functor $N : \mathcal{C}_{\mathbb{T}}^{\rm ir} \xrightarrow{\sim} \mathcal{M}^{\rm op} \to \mathcal{E}$ and any object $(\varphi \leftrightarrow M_\varphi)$ of $\mathcal{C}_{\mathbb{T}}^{\rm ir} \xrightarrow{\sim} \mathcal{M}^{\rm op}$, a canonical isomorphism of $\mathcal{E}$
  $$\mathcal{H}om\,(p^* \circ M_\varphi, N) \xrightarrow{\sim} N\varphi\,.$$
- In other words, if an object M of $\mathcal{M}$ is defined by a formula φ, the interpretation of φ in a model N of $\mathbb{T}$ in $\mathcal{E}$ identifies with
  $$\mathcal{H}om\,(p^* M, N)\,.$$

Three examples of quotient theories of the theory of commutative rings:

Program:

- Introduce the algebraic theory of commutative rings.
- Present its classifying topos.
- Introduce the three quotient theories that are
  - the theory of local rings,
  - the theory of integral domains,
  - the theory of commutative fields.
- Calculate the presentations of the classifying toposes of these theories derived from that of the classifying topos of commutative ring theory.
- Compute another presentation of the classifying topos
  of the theory of commutative fields,
  derived from its presentation as
  a quotient theory of another algebraic theory,
  the theory of Von Neumann regular commutative rings.

Commutative ring theory:

Definition. –

(i) *The signature Σ_a of commutative ring theory consists of*

- *a sort A,*
- *five function symbols*

$$\begin{array}{cccccl} + & : & AA & \longrightarrow & A & \text{(addition)}, \\ \cdot & : & AA & \longrightarrow & A & \text{(multiplication)}, \\ 0 & : & & \longrightarrow & A & \text{(unit element of addition)}, \\ 1 & : & & \longrightarrow & A & \text{(unit element of multiplication)}, \\ - & : & A & \longrightarrow & A & \text{(switch to the opposite)}. \end{array}$$

(ii) *The theory $\mathbb{T}_a$ of commutative rings is defined in the signature Σ_a by the following axioms:*

- *Associativity :* $\top \vdash (a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$
  $\top \vdash (a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$
- *Commutativity:* $\top \vdash a_1 + a_2 = a_2 + a_1$
  $\top \vdash a_1 \cdot a_2 = a_2 \cdot a_1$
- *Neutrality:* $\top \vdash 0 + a = a$
  $\top \vdash 1 \cdot a = a$
- *Opposition:* $\top \vdash a + (-a) = 0$

The classifying topos of commutative ring theory:

Observation. – *The theory $\mathbb{T}_a$ of commutative rings is an algebraic theory.*

Corollary. – *The classifying topos $\mathcal{E}_{\mathbb{T}_a}$ of $\mathbb{T}_a$ takes the form*

$$\begin{cases} \widehat{\mathcal{M}^{\mathrm{op}}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}_a}\,, \\ \widehat{\mathrm{Sch}_{\mathrm{apf}}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}_a}\,, \\ \widehat{\mathcal{C}^{\mathrm{ir}}_{\mathbb{T}_a}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}_a}\,, \\ \widehat{\mathcal{C}^{\mathrm{car}}_{\mathbb{T}_a}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}_a} \end{cases}$$

where

- $\mathcal{M} = \mathbb{T}_a\text{-mod}(\mathrm{Set})_{\mathrm{pf}}$ *is the category of commutative rings of finite presentation,*
- $\mathcal{M}^{\mathrm{op}} = \mathrm{Sch}_{\mathrm{apf}}$ *is the category of affine schemes of finite presentation,*
- $\mathcal{C}^{\mathrm{ir}}_{\mathbb{T}_a} \hookrightarrow \mathcal{C}_{\mathbb{T}_a} = \mathcal{C}^{\mathrm{geo}}_{\mathbb{T}_a}$ *is the full subcategory of the geometric syntactic category $\mathcal{C}_{\mathbb{T}_a}$ made up of irreducible formulas,*
- $\mathcal{C}^{\mathrm{car}}_{\mathbb{T}_a}$ *is the Cartesian syntactic category of $\mathbb{T}_a$, equivalent to $\mathcal{C}^{\mathrm{ir}}_{\mathbb{T}_a}$.*

Local ring theory:

Definition. – *The theory of local rings* $\mathbb{T}_\ell$ *is the quotient theory of that of commutative rings* $\mathbb{T}_a$ *defined by adding the coherent axiom*

$$(\exists\, b)(b \cdot (a_1 + a_2) = 1) \vdash_{a_1, a_2} (\exists\, b_1)(b_1 \cdot a_1 = 1) \vee (\exists\, b_2)(b_2 \cdot a_2 = 1)\,.$$

Observation. –

- The three formulas

$$(\exists\, b)\,(b \cdot (a_1 + a_2) = 1)\,,$$
$$(\exists\, b_1)\,(b_1 \cdot a_1 = 1)\,,$$
$$(\exists\, b_2)\,(b_2 \cdot a_2 = 1)$$

  are Cartesian, hence irreducible.
- This is still the case with the two formulas

$$(\exists\, b)(b \cdot (a_1 + a_2) = 1) \wedge (\exists\, b_1)(b_1 \cdot a_1 = 1)\,,$$

$$(\exists\, b)(b \cdot (a_1 + a_2) = 1) \wedge (\exists\, b_2)(b_2 \cdot a_2 = 1)\,.$$

Affine schemes presented by irreducible formulas:

Lemma. –

(i) *The irreducible formulas*

$$(\exists\, b_1)\,(b_1 \cdot a_1 = 1) \qquad \textit{and} \qquad (\exists\, b_2)\,(b_2 \cdot a_2 = 1)$$

present the affine open subscheme

$$\mathrm{Spec}(\mathbb{Z}[X, X^{-1}])$$

complement of point 0 *in the affine line*

$$\mathrm{Spec}(\mathbb{Z}[X])\,.$$

(ii) *The irreducible formula*

$$(\exists\, b)(b \cdot (a_1 + a_2) = 1)$$

presents the affine open subscheme

$$\mathrm{Spec}(\mathbb{Z}[X, Y][(X + Y)^{-1}])$$

complement of the line of equation $X + Y = 0$ *in the affine plane*

$$\mathrm{Spec}(\mathbb{Z}[X, Y])\,.$$

(iii) *The irreducible formulas*

$$(\exists\, b)(b \cdot (a_1 + a_2) = 1) \wedge (\exists\, b_1)\,(b_1 \cdot a_1 = 1) \quad \textit{and}$$

$$(\exists\, b)(b \cdot (a_1 + a_2) = 1) \wedge (\exists\, b_2)\,(b_2 \cdot a_2 = 1)$$

present the affine open subschemes

$$\mathrm{Spec}(\mathbb{Z}[X, Y][X^{-1}][(X + Y)^{-1}]) \quad \textit{and} \quad \mathrm{Spec}(\mathbb{Z}[X, Y][Y^{-1}][(X + Y)^{-1}])\,.$$

The classifying topos of local ring theory:

Proposition. –

(i) *The classifying topos of the theory $\mathbb{T}_\ell$ of local rings can be presented in the form*

$$\widehat{(\mathrm{Sch}_{\mathrm{apf}})}_{J_\ell} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_\ell}$$

where J_ℓ is the smallest topology of $\mathrm{Sch}_{\mathrm{apf}}$ for which the monomorphisms

$$\mathrm{Spec}(\mathbb{Z}[X,Y][X^{-1}][(X+Y)^{-1}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X,Y][(X+Y)^{-1}]),$$

$$\mathrm{Spec}(\mathbb{Z}[X,Y][Y^{-1}][(X+Y)^{-1}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X,Y][(X+Y)^{-1}])$$

form a covering.

(ii) *This topology J_ℓ on $\mathrm{Sch}_{\mathrm{apf}}$ (= category of affine schemes of finite presentation) is none other than the Zariski topology.*

Note. – According to the "Grothendieck comparison lemma", we have an equivalence

$$\widehat{(\mathrm{Sch}_{\mathrm{apf}})}_{\mathrm{Zar}} \xrightarrow{\sim} \widehat{(\mathrm{Sch}_{\mathrm{pf}})}_{\mathrm{Zar}}.$$

Verification of the generation of the Zariski topology:

- The topology J_ℓ is contained in the Zariski topology.
  We have to prove the inclusion in the reverse direction.
- We recall that for any element a of a commutative ring A,
  $$A_a = A[X]/(a \cdot X - 1)$$
  is the ring deduced from A by formally inverting the element a.
- As J satisfies the stability axiom,
  we see that for any elements a, b of a ring A,
  the two monomorphisms
  $$\mathrm{Spec}(A_{(a+b)\cdot a}) \hookrightarrow \mathrm{Spec}(A_{(a+b)}) \quad \text{and} \quad \mathrm{Spec}(A_{(a+b)\cdot b}) \hookrightarrow \mathrm{Spec}(A_{(a+b)})$$
  form a covering.
- By stability and transitivity, we deduce that
  for any elements $a_1, \cdots, a_n$ of a ring A, the family of n morphisms
  $$\mathrm{Spec}(A_{(a_1+\cdots+a_n)\cdot a_i}) \hookrightarrow \mathrm{Spec}(A_{(a_1+\cdots+a_n)}) \,,\ 1 \leq i \leq n \,,$$
  forms a covering.
- If there exist elements $b_1, \cdots, b_n \in A$ such that $a_1 \cdot b_1 + \cdots + a_n \cdot b_n = 1$,
  the morphisms $\mathrm{Spec}(A_{a_i \cdot b_i}) \hookrightarrow \mathrm{Spec}(A)$ form a covering, so also the
  $$\mathrm{Spec}(A_{a_i}) \hookrightarrow \mathrm{Spec}(A) \,.$$

The theory of integral domains:

Definition. –
The theory of integral domains $\mathbb{T}_i$
is the quotient theory of that of commutative rings $\mathbb{T}_a$
defined by adding the coherent axiom

$$a \cdot b = 0 \vdash a = 0 \vee b = 0 \,.$$

Observation. –

- The three formulas in the variables a and b

$$\begin{aligned} a \cdot b &= 0 \,, \\ a &= 0 \,, \\ b &= 0 \end{aligned}$$

  are Cartesian, hence irreducible.
- The reverse implications

$$a = 0 \vdash a \cdot b = 0 \,,$$

$$b = 0 \vdash a \cdot b = 0$$

  are $\mathbb{T}_a$-provable.

Affine schemes presented by irreducible formulas:

Lemma. –
The irreducible formulas in the variables a and b

$$\begin{aligned} a \cdot b &= 0\,, \\ a &= 0\,, \\ b &= 0 \end{aligned}$$

present the closed affine subschemes

$$\begin{aligned} \mathrm{Spec}(\mathbb{Z}[X,Y]/(X\cdot Y)) &\hookrightarrow \mathrm{Spec}(\mathbb{Z}[X,Y])\,, \\ \mathrm{Spec}(\mathbb{Z}[X,Y]/(X)) &\hookrightarrow \mathrm{Spec}(\mathbb{Z}[X,Y])\,, \\ \mathrm{Spec}(\mathbb{Z}[X,Y]/(Y)) &\hookrightarrow \mathrm{Spec}(\mathbb{Z}[X,Y]) \end{aligned}$$

of the affine plane

$$\mathrm{Spec}(\mathbb{Z}[X,Y])\,.$$

The classifying topos of the theory of integral domains:

Proposition. –

(i) *The classifying topos of the theory $\mathbb{T}_i$ of integral domains can be presented in the form*

$$\widehat{(\mathrm{Sch}_{\mathrm{apf}})}_{J_i} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_i}$$

where J_i is the smallest topology of $\mathrm{Sch}_{\mathrm{apf}}$ for which the closed immersions

$$\mathrm{Spec}(\mathbb{Z}[X,Y]/(X)) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X,Y]/(X\cdot Y)),$$
$$\mathrm{Spec}(\mathbb{Z}[X,Y]/(Y)) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X,Y]/(X\cdot Y))$$

form a covering.

(ii) *A sieve C on an object of $\mathrm{Sch}_{\mathrm{apf}}$*

$$\mathrm{Spec}(A)$$

is J_i-covering if and only if there exist elements $a_1,\ldots,a_n \in A$ such that

$$a_1 \cdots a_n = 0$$

and that the closed embeddings

$$\mathrm{Spec}(A/(a_i)) \hookrightarrow \mathrm{Spec}(A), \quad 1 \le i \le n,$$

are elements of the sieve C.

Identification of the generated topology:

- By stability of J_i, we see that for any elements a_1, a_2 of a commutative ring A such that $a_1 \cdot a_2 = 0\,,$ the closed immersions
$$\mathrm{Spec}(A/(a_1)) \hookrightarrow \mathrm{Spec}(A) \quad \text{and} \quad \mathrm{Spec}(A/(a_2)) \hookrightarrow \mathrm{Spec}(A)$$
form a J_i-covering.
- By transitivity of J_i, we deduce that for any elements $a_1, \cdots, a_n$ of a commutative ring A such that $a_1 \cdots a_n = 0\,,$ the family of closed embeddings
$$\mathrm{Spec}(A/(a_i)) \hookrightarrow \mathrm{Spec}(A)\,, \quad 1 \le i \le n \qquad \text{forms a } J_i\text{-covering.}$$
- It remains to prove that the sieves C of the objects $\mathrm{Spec}(A)$ of $\mathrm{Sch}_{\mathrm{apf}}$ which satisfy the condition of (ii) form a topology.
The axioms of maximality and stability are clearly verified.
For transitivity, it suffices to note that if elements of a ring A
$$a_1, \cdots, a_n \quad \text{and} \quad a_{i,j}\,,\ 1 \le i \le n\,,\ 1 \le j \le k_i\,,$$
satisfy the conditions
$$a_1 \cdots a_n = 0 \quad \text{and} \quad \prod_{1 \le j \le k_i} a_{i,j} \in a_1 \cdot A\,,\ 1 \le i \le n\,,$$
then one also has
$$\prod_{1 \le i \le n} \prod_{1 \le j \le k_i} a_{i,j} = 0\,.$$

The theory of commutative fields:

Definition. –
The theory of commutative fields $\mathbb{T}_c$
is the quotient theory of that of commutative rings $\mathbb{T}_a$
defined by adding the coherent axiom

$$\top \vdash_a a = 0 \vee (\exists\, b) \cdot (b \cdot a = 1)\,.$$

Observation. –
The two formulas in the variable a

$$a = 0\,,$$

$$(\exists\, b)(b \cdot a = 1)$$

are Cartesian, hence irreducible.

Affine schemes presented by irreducible formulas:

Lemma. –
The *irreducible formulas* *in the variable a*

$$a = 0\,,$$

$$(\exists\, b)(a \cdot b = 1)$$

present the closed subscheme

$$0 : \mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X])$$

and the open subscheme

$$\mathrm{Spec}(\mathbb{Z}[X, X^{-1}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X])\,,$$

which are the closed point 0 *of the affine line*
and its complementary open subscheme.

The classifying topos of the theory of commutative fields:

Proposition. –

(i) *The classifying topos of the theory $\mathbb{T}_c$ of commutative fields can be presented in the form*

$$\widehat{(\mathrm{Sch}_{\mathrm{apf}})}_{J_c} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_c}$$

where J_c is the smallest topology of $\mathrm{Sch}_{\mathrm{apf}}$ for which the two closed and open embeddings

$$0 \quad : \quad \mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X]),$$
$$\mathrm{Spec}(\mathbb{Z}[X, X^{-1}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X])$$

form a covering.

(ii) *A sieve C on an object of $\mathrm{Sch}_{\mathrm{apf}}$*

$$\mathrm{Spec}(A)$$

is J_c-covering if and only if there exist elements $a_1, \cdots, a_n \in A$, such that, for any subset $I \subseteq \{1, \cdots, n\}$ and if A_I denotes the ring deduced from A by cancelling elements a_i, $i \in I$, and inverting elements a_j, $j \notin I$, the locally closed morphism

$$\mathrm{Spec}(A_I) \hookrightarrow \mathrm{Spec}(A)$$

is an element of C.

Identification of the generated topology:

- By stability of J_c,
  we see that for any element a of a commutative ring A,
  the pair of closed and open embeddings

$$\mathrm{Spec}(A/(a)) \hookrightarrow \mathrm{Spec}(A) \quad \text{and} \quad \mathrm{Spec}(A_a) \hookrightarrow \mathrm{Spec}(A)$$

  forms a J_i-covering.
- By transitivity of J_c, we deduce that
  for any elements $a_1, \cdots, a_n$ of a commutative ring A,
  the family of associated locally closed embeddings

$$\mathrm{Spec}(A_I) \hookrightarrow \mathrm{Spec}(A)\,, \quad I \subseteq \{1, \cdots, n\}\,,$$

  forms a J_c-covering.
- It remains to check that the sieves C of the objects $\mathrm{Spec}(A)$ of $\mathrm{Sch}_{\mathrm{apf}}$
  which satisfy the conditions of (ii) form a topology.
  Indeed, the axioms of maximality, stability and transitivity
  are clearly verified.

A remark on topologies:

Lemma. –
On the category $\mathrm{Sch}_{\mathrm{apf}}$ of affine schemes of finite presentation, one has:

(i) *The topology $J_\ell = \mathrm{Zar}$ is subcanonical.*

(ii) *The topology J_i is not subcanonical.*

(iii) *A fortiori, the topology $J_c \supseteq J_i$ is not subcanonical.*

Proof. –

(i) Indeed, for any Zariski open covering of an affine scheme

$$U_i \hookrightarrow X \,, \qquad 1 \le i \le n \,,$$

the datum of an affine scheme morphism $X \to Y$ is equivalent to the datum of a family of morphisms $U_i \to Y$, $1 \le i \le n$, which match on the $U_i \cap U_j$.

(ii) The morphism

$$\mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X]/(X^2))$$

forms a J_i-covering but it has no section.

(iii) Indeed, the topology J_c contains the topology J_i.

Looking for an alternative presentation:

Program:
Define a first-order geometric theory $\mathbb{T}_c'$ such that:

- The theory $\mathbb{T}_c'$ is "syntactically equivalent" to the theory of commutative fields $\mathbb{T}_c$ (in the sense that there exists an equivalence between their geometric syntactic categories

$$\mathcal{C}_{\mathbb{T}_c'} \xrightarrow{\sim} \mathcal{C}_{\mathbb{T}_c} \text{).}$$

- The theory $\mathbb{T}_c'$ is naturally written as a quotient of a theory $\mathbb{T}_r$ of presheaf type.
- If $\mathcal{M}_r = \mathbb{T}_r\text{-mod}(\text{Set})_{\text{pf}}$ with the canonical equivalence

$$\widehat{\mathcal{M}_r^{\text{op}}} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_r} \,,$$

the unique topology J_c on $\mathcal{M}_r^{\text{op}}$ such that

$$\widehat{(\mathcal{M}_r^{\text{op}})}_{J_c} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_c'}$$

is subcanonical.

Syntactic equivalence and semantic equivalence:

We have the two notions of equivalence:

Definition. – *Two first-order geometric theories* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *are said*

(i) *"syntactically equivalent" if there exists an equivalence between their geometric syntactic categories*

$$\mathcal{C}_{\mathbb{T}_1} \xrightarrow{\sim} \mathcal{C}_{\mathbb{T}_2} ,$$

(ii) *"semantically equivalent" (or "Morita-equivalent") if there exists an equivalence between their classifying toposes*

$$\mathcal{E}_{\mathbb{T}_1} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_2} .$$

Remarks. –

- Two theories which are syntactically equivalent are a fortiori semantically equivalent.
- Indeed, the syntactic topology $J_{\mathbb{T}}$ of a geometric syntactic category $\mathcal{C}_{\mathbb{T}}$ is induced by the categorical structure of $\mathcal{C}_{\mathbb{T}}$:
  A family of morphisms $\theta_i : \varphi_i \to \varphi$ of $\mathcal{C}_{\mathbb{T}}$ is $J_{\mathbb{T}}$-covering if and only if it is globally epimorphic.

Von Neumann's theory of regular commutative rings:

Definition. –

(i) *The signature* Σ_r *of the theory of regular commutative rings consists of the signature* Σ_a *of the theory of commutative rings supplemented by*

- *an additional function symbol*

$$(\bullet)^+ : A \to A \quad \text{(switch to pseudo-inverse).}$$

(ii) *The theory* $\mathbb{T}_r$ *of regular commutative rings is defined in the signature* Σ_r *by the axioms of the theory* $\mathbb{T}_a$ *of commutative rings supplemented by the two additional axioms:*

- *Pseudo-inversion:* $\top \vdash_a a^+ \cdot a \cdot a = a$
- *Involution :* $\top \vdash_a (a^+)^+ = a$

A theory syntactically equivalent to field theory:

Definition. – *Let $\mathbb{T}_c'$ be the quotient theory of $\mathbb{T}_r$ defined by the same axiom which defines the theory of commutative fields $\mathbb{T}_c$ as a quotient of that of commutative rings $\mathbb{T}_a$*

$$\top \vdash_a a = 0 \vee (\exists\, b)(b \cdot a = 1)\,.$$

Lemma. – *The theories $\mathbb{T}_c$ and $\mathbb{T}_c'$ are syntactically equivalent.*

Proof. –

- The formula in the two variables a and b of sort A
$$(a = 0 \wedge b = 0) \vee (b \cdot a = 1)$$
is $\mathbb{T}_c$-provably functional from $\top(a)$ to $\top(b)$.
- Associating with the $\mathbb{T}_c'$-provably functional formula of Σ_r
$$(b = a^+) : \top(a) \to \top(b)$$
the $\mathbb{T}_c$-provably functional formula of Σ_a
$$(a = 0 \wedge b = 0) \vee (b \cdot a = 1) : \top(a) \to \top(b)$$
defines an equivalence of categories
$$\mathcal{C}_{\mathbb{T}_c'} \xrightarrow{\sim} \mathcal{C}_{\mathbb{T}_c}\,.$$

The classifying topos of the theory of regular commutative rings:

Observation. –
Von Neumann's theory $\mathbb{T}_r$ of regular commutative rings
is an algebraic theory.

Corollary. – *The classifying topos $\mathcal{E}_{\mathbb{T}_r}$ of $\mathbb{T}_r$ can be presented in the forms*

$$\begin{cases} \widehat{\mathcal{M}_r^{\mathrm{op}}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}_r}\,, \\ \widehat{\mathcal{C}_{\mathbb{T}_r}^{\mathrm{ir}}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}_r}\,, \\ \widehat{\mathcal{C}_{\mathbb{T}_r}^{\mathrm{car}}} & \xrightarrow{\sim} & \mathcal{E}_{\mathbb{T}_r} \end{cases}$$

where

- $\mathcal{M}_r = \mathbb{T}_r\text{-mod}(\mathrm{Set})_{\mathrm{pf}}$
  is the category of regular commutative rings of finite presentation,
- $\mathcal{C}_{\mathbb{T}_r}^{\mathrm{ir}} \hookrightarrow \mathcal{C}_{\mathbb{T}_r}$
  is the full subcategory of the geometric syntactic category $\mathcal{C}_{\mathbb{T}_r}$ consisting of irreducible formulas,
- $\mathcal{C}_{\mathbb{T}_r}^{\mathrm{car}}$ *is the Cartesian syntactic category of $\mathbb{T}_r$, equivalent to $\mathcal{C}_{\mathbb{T}_r}^{\mathrm{ir}}$.*

Regular rings and commutative rings:

Proposition. – *Consider the forgetting functor of the operation* $(\bullet)^+$ *of regular rings*

$$\begin{array}{rcl} \mathbb{T}_r\text{-mod}(\mathrm{Set}) & \longrightarrow & \mathbb{T}_a\text{-mod}(\mathrm{Set})\,, \\ (A,(\bullet)^+) & \longmapsto & A\,. \end{array}$$

Then:

(i) *For any regular commutative ring* $(A,(\bullet)^+)$*, the diagonal morphism*

$$A \longrightarrow \prod_{p\in\mathrm{Spec}(A)} A/p \hookrightarrow \prod_{p\in\mathrm{Spec}(A)} \mathrm{Frac}(A/p)$$

is injective, and each of its components

$$A \longrightarrow A/p \hookrightarrow \mathrm{Frac}(A/p) = \kappa_p$$

transforms the operation $(\bullet)^+$ *of* A *into the operation*

$$\begin{array}{rrcl} (\bullet)^+ : & \kappa_p & \longrightarrow & \kappa_p\,, \\ & k & \longmapsto & k^+ = \begin{cases} 0 & \text{if} \quad k = 0\,, \\ k^{-1} & \text{if} \quad k \neq 0\,. \end{cases} \end{array}$$

(ii) *This forgetting functor admits for left-adjoint the functor*

$$\begin{array}{rcl} \mathbb{T}_a\text{-mod}(\mathrm{Set}) & \longrightarrow & \mathbb{T}_r\text{-mod}(\mathrm{Set})\,, \\ B & \longmapsto & B^+ \end{array}$$

where B^+ *is the smallest subring of* $\prod_{q\in\mathrm{Spec}(B)} \kappa_q$ *which*

- *contains the image of* B *by the diagonal morphism* $B \to \prod_{q\in\mathrm{Spec}(B)} \kappa_q$,
- *is respected by the product of the involutions* $(\bullet)^+ : \kappa_q \to \kappa_q$.

(iii) *The composite functor*

$$\mathbb{T}_r\text{-mod}(\mathrm{Set}) \longrightarrow \mathbb{T}_a\text{-mod}(\mathrm{Set}) \longrightarrow \mathbb{T}_r\text{-mod}(\mathrm{Set})\,, \quad (A,(\bullet)^+) \longmapsto A \longmapsto A^+$$

is canonically isomorphic to the identity functor.

Identification of the structure of regular rings:

Prove (i). For this, consider a regular ring $(A, (\bullet)^+)$.

- For any element a of A, one has
$$a^+ \cdot a^2 = a$$
and therefore, for any integer $n \geq 2$,
$$(a^+)^{n-1} \cdot a^n = a\,.$$
It follows that any nilpotent element of A is zero, and therefore that
$$A \longrightarrow \prod_{p \in \mathrm{Spec}(A)} A/p$$
is an embedding.
- For any prime ideal $p \in \mathrm{Spec}(A)$ and any element $a \in A$, the images $\overline{a}$ and $\overline{a}^+$ of a and a^+ in
$$A/p \hookrightarrow \mathrm{Frac}(A/p) = \kappa_p$$
satisfy the equations
$$\overline{a}^+ \cdot \overline{a}^2 = \overline{a} \qquad \text{and} \qquad (\overline{a}^+)^2 \cdot \overline{a} = \overline{a}^+$$
which imply
$$\begin{cases} \overline{a}^+ = 0 & \text{if} \quad \overline{a} = 0\,, \\ \overline{a}^+ = \overline{a}^{-1} & \text{if} \quad \overline{a} \neq 0\,. \end{cases}$$

Construction of a left-adjoint:

Prove (ii). For this, consider

- a regular commutative ring $(A, (\bullet)^+)$,
- a commutative ring B,
- the smallest subring B^+ of $\prod_{q \in \mathrm{Spec}(B)} \kappa_q$ which

$$\begin{cases} \text{contains the image of } B \text{ by } B \to \prod_q \kappa_q, \\ \text{is respected by the product of the involutions } (\bullet)^+ : \kappa_q \to \kappa_q. \end{cases}$$

- Any morphism of regular rings $B^+ \to (A, (\bullet)^+)$ induces a morphism of commutative rings $B \to A$.
- Conversely, let us consider a morphism of commutative rings

$$u : B \longrightarrow A\,.$$

It induces a map $u^{-1} : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(B)$ and, for any prime ideal $p \in \mathrm{Spec}(A)$ sent to $u^{-1}(p) = q \in \mathrm{Spec}(B)$, a commutative square:

$$\begin{array}{ccc} B/q & \longrightarrow & A/p \\ \downarrow & & \downarrow \\ \kappa_q & \longrightarrow & \kappa_p \end{array}$$

The pull-back of $A \hookrightarrow \prod_{p \in \mathrm{Spec}(A)} A/p \hookrightarrow \prod_p \kappa_p$ in the product $\prod_{q \in \mathrm{Spec}(B)} \kappa_q$ is a $(\bullet)^+$-stable subring which contains the image of B, so it contains B^+.

Full faithfulness of the forgetting functor:

Prove (iii).

For any regular commutative ring $(A, (\bullet)^+)$, the embedding

$$A \hookrightarrow \prod_{p \in \mathrm{Spec}(A)} A/p \hookrightarrow \prod_{p \in \mathrm{Spec}(A)} \kappa_p$$

identifies A with the smallest subring

$$A^+ \hookrightarrow \prod_{p \in \mathrm{Spec}(A)} \kappa_p$$

which $\begin{cases} \text{contains the image of } A, \\ \text{is respected by the product of the involutions } (\bullet)^+ : \kappa_p \to \kappa_p. \end{cases}$

Corollary. – *The forgetting functor of the operation* $(\bullet)^+$

$$\begin{array}{ccc} \mathbb{T}_r\text{-mod}(\mathrm{Set}) & \longrightarrow & \mathbb{T}_a\text{-mod}(\mathrm{Set}), \\ (A, (\bullet)^+) & \longmapsto & A \end{array}$$

is fully faithful.

Transport of finitely presented models:

Corollary. – *The left-adjoint functor of the forgetting functor*

$$\begin{array}{rcl} \mathbb{T}_a\text{-mod}(\mathrm{Set}) & \longrightarrow & \mathbb{T}_r\text{-mod}(\mathrm{Set})\,, \\ B & \longmapsto & B^+ \end{array}$$

induces a functor

$$\mathrm{Sch}_{\mathrm{apf}} = \mathbb{T}_a\text{-mod}(\mathrm{Set})^{\mathrm{op}}_{\mathrm{pf}} \longrightarrow \mathbb{T}_r\text{-mod}(\mathrm{Set})^{\mathrm{op}}_{\mathrm{pf}}$$

which associates to any model of $\mathbb{T}_a$ presented by a $\mathbb{T}_a$-Cartesian formula φ of Σ_a

$$\mathrm{Spec}(B_\varphi)$$

a model of $\mathbb{T}_r$ presented by φ seen as a $\mathbb{T}_r$-Cartesian formula of Σ_r.

Proof. – *Indeed, for any regular commutative ring $(A, (\bullet)^+)$, the set*

$$\mathrm{Hom}(B_\varphi^+, (A, (\bullet)^+))$$

identifies by adjunction with

$$\mathrm{Hom}(B_\varphi, A)$$

and so with the interpretation of the formula φ in A

$$A\varphi\,.$$

Regular rings presented by irreducible formulas:

Corollary. –
The irreducible formulas in the variable a

$$a = 0\,,$$

$$(\exists\, b)(a \cdot b = 1)$$

present the transforms by the left-adjoint functor

$$\begin{array}{rcl} \mathrm{Sch}_{\mathrm{apf}} = \mathbb{T}_a\text{-mod}(\mathrm{Set})^{\mathrm{op}}_{\mathrm{pf}} & \longrightarrow & \mathbb{T}_r\text{-mod}(\mathrm{Set})^{\mathrm{op}}_{\mathrm{pf}} = \mathrm{Sch}_{\mathrm{arpf}} \\ \mathrm{Spec}(B) & \longmapsto & \mathrm{Spec}(B^+) \end{array}$$

of the closed embedding

$$0 : \mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X])$$

and of the open embedding

$$\mathrm{Spec}(\mathbb{Z}[X, X^{-1}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[X])\,.$$

An alternative presentation of the classifying topos of field theory:

Corollary. –

The classifying topos of the theory $\mathbb{T}_c$ of commutative fields can be presented in the form

$$(\widehat{\mathrm{Sch}_{\mathrm{arpf}}})_{J_c} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_c}$$

where J_c is the smallest topology of the full subcategory

$$\mathbb{T}_r\text{-mod}(\mathrm{Set})_{\mathrm{pf}}^{\mathrm{op}} = \mathrm{Sch}_{\mathrm{arpf}} \hookrightarrow \mathrm{Sch}_{\mathrm{apf}} = \mathbb{T}_a\text{-mod}(\mathrm{Set})_{\mathrm{pf}}^{\mathrm{op}}$$

for which the two morphisms

$$\mathrm{Spec}(\mathbb{Z}^+) \longrightarrow \mathrm{Spec}(\mathbb{Z}[X]^+)\,,$$

$$\mathrm{Spec}(\mathbb{Z}[X, X^{-1}]^+) \longrightarrow \mathrm{Spec}(\mathbb{Z}[X]^+)$$

form a covering.

Identification of the generated topology:

Corollary. –

A sieve C on an object

$$(\mathrm{Spec}(A), (\bullet)^+)$$

of

$$\mathrm{Sch}_{\mathrm{arpf}} = \mathbb{T}_r\text{-mod}(\mathrm{Set})^{\mathrm{op}}_{\mathrm{pf}}$$

is J_c-covering if and only if

there exist elements $a_1, \cdots, a_n \in A$ such that,

for any subset $I \subseteq \{1, \cdots, n\}$

and if A_I denotes the ring deduced from A by

cancelling elements a_i, $i \in I$, and inverting elements a_j, $j \notin I$,

the morphism induced by adjunction

$$\mathrm{Spec}(A_I^+) \longrightarrow (\mathrm{Spec}(A), (\bullet)^+)$$

is an element of C.

Decomposition of regular commutative rings:

Lemma. – *For any element a of a regular commutative ring A, one has*

- *A/aA and $A_a = A[X]/(a \cdot X - 1)$ are regular,*
- *the morphism $A \to A/aA \times A_a$ is an isomorphism.*

Proof. –
For any prime ideal $p \in \mathrm{Spec}(A)$, the image of a in A/p is

$$\begin{cases} 0 & \text{if} \quad p \in \mathrm{Spec}(A/aA)\,, \\ \text{invertible} & \text{if} \quad p \in \mathrm{Spec}(A_a)\,. \end{cases}$$

The equations $a^+ \cdot a^2 = a$ and $(a^+)^2 \cdot a = a^+$
then imply that the image of $a^+ \cdot a$ in A/p is

$$\begin{cases} 0 & \text{if} \quad p \in \mathrm{Spec}(A/aA)\,, \\ 1 & \text{if} \quad p \in \mathrm{Spec}(A_a)\,. \end{cases}$$

Hence the conclusion.

Corollary. –
On the category of regular affine schemes of finite presentation

$$\mathrm{Sch}_{\mathrm{arpf}} = \mathbb{T}_r\text{-mod}(\mathrm{Set})^{\mathrm{op}}_{\mathrm{pf}}\,,$$

the topology J_c is subcanonical.