First-order provability and generation of Grothendieck topologies

I. Toposes as bridges between geometric forms and linguistic descriptions

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General plan :

Grothendieck topologies,

sheaves, toposes

and points

The notion of site as a formalisation of the idea of "geometric shape":

For us in this presentation,
a "geometric shape"
will be a "site":

Definition. –
A "site" is a pair $(\mathcal{C}, J)$ consisting of

- *a category $\mathcal{C}$ assumed to be "small" or "essentially small",*
- *a "Grothendieck topology" J on the category $\mathcal{C}$.*

The notion of category:

Definition. – *A category $\mathcal{C}$ consists in a triple datum of*

$$\begin{cases} \bullet \quad \text{a collection } \mathrm{Ob}(\mathcal{C}) \text{ whose elements are called the } \underline{\text{objects}} \text{ of } \mathcal{C}, \\ \bullet \quad \text{for any pair of objects } X, Y, \text{ a collection } \mathrm{Hom}(X,Y) \\ \quad\ \text{whose elements } f \text{ are called the } \underline{\text{morphisms}}\ f : X \to Y \text{ from } X \text{ to } Y, \\ \bullet \quad \text{for any triplet of objects } X, Y, Z \text{ a } \underline{\text{law of composition}} \text{ of morphisms} \\ \qquad\qquad \begin{array}{rcl} \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) & \longrightarrow & \mathrm{Hom}(X,Z)\,, \\ (f,g) & \longmapsto & g \circ f\,, \end{array} \end{cases}$$

verifying:

$$\begin{cases} \bullet \quad \text{this law is } \underline{\text{associative}}, \text{ in the sense that for any sequence of morphisms} \\ \quad\ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T, \quad \text{we have} \quad h \circ (g \circ f) = (h \circ g) \circ f, \\ \bullet \quad \text{every object } X \text{ is associated with a morphism } \underline{\text{“identity”}}\ \mathrm{id}_X : X \to X, \\ \quad\ \text{such that, for any morphism } f : X \to Y \text{ [resp. } g : Y \to X\text{]}, \\ \quad\ \text{we have} \quad f \circ \mathrm{id}_X = f \qquad \text{[resp. } \mathrm{id}_X \circ g = g\text{].} \end{cases}$$

Remarks. –

(i) A category $\mathcal{C}$ is said to be "locally small"
if, for all objects X, Y of $\mathcal{C}$,
$\mathrm{Hom}(X, Y)$ is a set.

(ii) A category $\mathcal{C}$ is said to be "small"
if it is locally small
and if its objects form a set $\mathrm{Ob}(\mathcal{C})$.

(iii) In a category $\mathcal{C}$,
a morphism $f : X \to Y$ is an "isomorphism"
if there is a morphism (necessarily unique)

$$f^{-1} : Y \longrightarrow X$$

such that

$$f^{-1} \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}_Y \,.$$

Categories as a mathematical environment:

- As a general rule, mathematical objects of a certain kind and the transformations of these objects respecting this common nature form a category.
- First example: the category Set

  of sets ($=$ objects)

  and of applications ($=$ morphisms).
- From this example,

  an infinite variety of derived examples:

  Any "structure type" $\mathbb{T}$ defines

  a "category of set-based models of $\mathbb{T}$"

  $$\mathbb{T}\text{-mod}(\mathrm{Set})$$

  of which

  – the objects are the sets endowed with a structure of type $\mathbb{T}$,

  – the morphisms are the maps which respect structures of type $\mathbb{T}$.

Categories of transitions from one environment to another:

Definition. – *Any pair of categories $\mathcal{C}, \mathcal{D}$ defines a category*

$$[\mathcal{C}, \mathcal{D}]$$

of which

- *the objects are the "functors"*

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

  i.e. applications

$$\begin{cases} \mathrm{Ob}(\mathcal{C}) \ni X & \longmapsto & F(X) \in \mathrm{Ob}(\mathcal{D})\,, \\ \mathrm{Hom}(X, Y) \ni f & \longmapsto & F(f) \in \mathrm{Hom}(F(X), F(Y))\,, \end{cases}$$

  such that

$$\begin{cases} F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \text{ for any } X \in \mathrm{Ob}(\mathcal{C})\,, \\ F(g \circ f) = F(g) \circ F(f) \text{ for any } X \xrightarrow{f} Y \xrightarrow{g} Z\,, \end{cases}$$

- *the morphisms $F \to G$ are the "natural transformations" from a functor F to a functor G, i.e. the applications*

$$\mathrm{Ob}(\mathcal{C}) \ni X \longmapsto (\alpha_X : F(X) \to G(X) \in \mathrm{Hom}(F(X), G(Y))$$

  such that, for any morphism $f : X \to Y$ of $\mathcal{C}$,

$$\alpha_Y \circ F(f) = G(F) \circ \alpha_X\,.$$

Remarks. –

(i) Every category $\mathcal{C}$ has an "identity functor"

which is
$$\mathrm{id}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$$
$$\begin{cases} X & \longmapsto & X \,, \\ (X \xrightarrow{f} Y) & \longmapsto & (X \xrightarrow{f} Y) \,. \end{cases}$$

(ii) The small categories and the functors between them form a locally small category denoted

$$\mathrm{Cat} \,.$$

(iii) A functor $F : \mathcal{C} \to \mathcal{D}$ is called an "equivalence of categories" if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that

$$\begin{cases} G \circ F \cong \mathrm{id}_{\mathcal{C}} \text{ in } [\mathcal{C}, \mathcal{C}] \,, \\ F \circ G \cong \mathrm{id}_{\mathcal{D}} \text{ in } [\mathcal{D}, \mathcal{D}] \,. \end{cases}$$

(iv) A category is said to be "essentially small" if it is equivalent to a small category.

Opposite categories and contravariant functors:

Definition. – *Any category $\mathcal{C}$ defines an "opposite category" $\mathcal{C}^{\mathrm{op}}$ by*

$$\begin{cases} \bullet \quad \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})\,, \\ \bullet \quad \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)\,, \\ \bullet \quad \begin{cases} \qquad g \circ f & \longleftrightarrow & f \circ g & \qquad \textit{for} \\ \left(\begin{array}{c} X \xrightarrow{f} Y \xrightarrow{g} Z \\ \textit{in } \mathcal{C}^{\mathrm{op}} \end{array} \right) & \longleftrightarrow & \left(\begin{array}{c} X \xleftarrow{f} Y \xleftarrow{g} Z \\ \textit{in } \mathcal{C} \end{array} \right). \end{cases} \end{cases}$$

Note. – We always have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

Definition. – *For all categories $\mathcal{C}, \mathcal{D}$, we define:*

(i) *A "contravariant functor" from $\mathcal{C}$ to $\mathcal{D}$ is a functor*

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}\,.$$

(ii) *The category of contravariant functors from $\mathcal{C}$ to $\mathcal{D}$ is*

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{D}]\,.$$

Note. – We always have $[\mathcal{C}, \mathcal{D}^{\mathrm{op}}] = [\mathcal{C}^{\mathrm{op}}, \mathcal{D}]^{\mathrm{op}}$.

The Yoneda embedding and presheaves:

Yoneda's lemma. –

(i) *If $\mathcal{C}$ is a locally small category, we have the "Yoneda functor"*

$$\begin{array}{rcl} y : \mathcal{C} & \longrightarrow & [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] = \widehat{\mathcal{C}}\,, \\ X & \longmapsto & \mathrm{Hom}(\bullet, X) = \left\{ \begin{array}{rcl} Y & \mapsto & \mathrm{Hom}(Y, X)\,, \\ (Y_1 \xrightarrow{f} Y_2) & \mapsto & [\mathrm{Hom}(Y_2, X) \xrightarrow{\bullet \circ f} \mathrm{Hom}(Y_1, X)]\,. \end{array} \right. \end{array}$$

(ii) *This functor is "fully faithful" in the sense that, for all objects X, X', the application $y : \mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}_{\widehat{\mathcal{C}}}(y(X), y(X'))$ is one-to-one.*

Definition. – *If $\mathcal{C}$ is a small (or essentially small) category:*

(i) *Contravariant functors*

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

are called the "presheaves" on $\mathcal{C}$.

(ii) *The category $\widehat{\mathcal{C}}$ is called the "topos of presheaves" on $\mathcal{C}$.*

Combinatorial generation of categories:

Observation. – *Any category $\mathcal{C}$ can be presented as generated by*

- *a collection $\mathrm{Ob}(\mathcal{C})$ of objects X,*
- *a collection $\mathcal{F}$ of arrows $X \xrightarrow{f} Y$,*
- *a collection of equality relations*

$$f_n \circ \cdots \circ f_1 = f'_{n'} \circ \cdots \circ f'_1$$

between formal composites of arrow strings of $\mathcal{F}$

$$X = X_0 \xrightarrow{f_1} X_1 \cdots \xrightarrow{f_n} X_n = Y$$

and

$$X = X'_0 \xrightarrow{f'_1} X'_1 \cdots \xrightarrow{f'_{n'}} X'_{n'} = Y\,.$$

Note. – Any such relationship $f_n \circ \cdots \circ f_1 = f'_{n'} \circ \cdots \circ f'_1$ generates others by composition on the left or on the right.

Note. – Any small category is a filtering colimit ($=$ inductive limit) of categories which are "finitely presented" (i.e. generated by finite sets of objects, arrows and relations).

An important remark for us:

The transition to a combinatorial representation of a category is generally impractical.

- Suppose for example that
$$\mathcal{C} = \mathbb{T}\text{-mod}(\mathrm{Set})$$
is the category of sets endowed with a type of algebraic structure $\mathbb{T}$. Then :
  - Any object of $\mathcal{C}$ can be defined in terms of generators and relations (and it is a filtering colimit of objects defined in terms of finite families of generators and relations).
  - On the other hand, the morphisms between such objects
$$X \longrightarrow Y$$
consist of structure-preserving applications of type $\mathbb{T}$.
Knowing them comes down to solving equations.
It is generally not possible.

Ex: If $\mathbb{T}$ is the "commutative ring" structure type,
$$\begin{aligned}&\mathrm{Hom}(\mathbb{Z}[T_1, \cdots, T_n]/\{P_i(T_1, \cdots, T_n), 1 \leq i \leq k\}, A)\\ &= \{(a_1, \cdots, a_n) \in A^n \mid P_i(a_1, \cdots, a_n) = 0, 1 \leq i \leq k\}.\end{aligned}$$

The notion of Grothendieck topology:

We can give two definitions

- in terms of sieves,
- in terms of covering families of morphisms.

Definition. –

(i) *Let X be an object of a category $\mathcal{C}$.*
A "sieve" over X is a collection C of morphisms to target X

$$U \xrightarrow{u} X$$

such that:

$$\left\{ \begin{array}{l} \text{For any element } U \xrightarrow{u} X \text{ of } C \text{ and any morphism } V \xrightarrow{v} U \text{ of } \mathcal{C}, \\ \text{the composite } u \circ v : V \to U \to X \text{ is an element of } C. \end{array} \right.$$

(ii) *The inverse image by a morphism $X' \xrightarrow{x} X$ of $\mathcal{C}$ of a sieve C on X is the sieve on X'* $\quad x^* C = \{U' \xrightarrow{u} X' \mid x \circ u \in C\}$.

Remarks. –

- Any category equivalence $F : \mathcal{C} \to \mathcal{D}$ identifies sieves on an object X of $\mathcal{C}$ to sieves on the object $F(X)$ of $\mathcal{D}$.
- If $\mathcal{C}$ is small or essentially small, the sieves on an object X of $\mathcal{C}$ form a set.

Definition 1. –
A "Grothendieck topology" J on a category $\mathcal{C}$ is an application

$$\mathrm{Ob}(\mathcal{C}) \ni X \longmapsto J(X) = \textit{collection of sieves over } X$$

which satisfies the following three axioms:

$$\begin{cases}
\text{(Maximality)} & \text{For any object } X \text{ of } \mathcal{C}, \\
& \text{its "maximal sieve"} \\
& \text{(consisting of all morphisms to target } X\text{)} \\
& \text{is an element of } J(X). \\
\\
\text{(Stability)} & \text{For any morphism } X' \xrightarrow{x} X \text{ of } \mathcal{C} \\
& \text{and any } C \in J(X), \\
& \text{then } x^* C \in J(X'). \\
\\
\text{(Transitivity)} & \text{For all sieves } C, C' \text{ on an object } X \text{ of } \mathcal{C} \\
& \text{such that } \ C \in J(X) \\
& \text{and } \qquad u^* C' \in J(U), \quad \forall\, (U \xrightarrow{u} X) \in C, \\
& \text{then } \qquad C' \in J(X).
\end{cases}$$

Ordering on topologies and generated topologies:

Lemma. – *Let $\mathcal{C}$ be an essentially small category. Then :*

(i) *Grothendieck topologies J of $\mathcal{C}$ form a set ordered by the inclusion relation.*

(ii) *For any family of topologies J_i, $i \in I$, on $\mathcal{C}$, their intersection*

$$\bigwedge_{i\in I} J_i$$

is a topology on $\mathcal{C}$.

(iii) *For any family of sieves C_i on objects X_i of $\mathcal{C}$, there exists a smallest topology J on $\mathcal{C}$ which contains all the sieves C_i. This is the topology generated by C_i, $i \in I$.*

Remarks. –

- A union of topologies J_i, $i \in I$, is not a topology (it satisfies stability but not transitivity). But it generates a topology $\bigvee\limits_{i\in I} J_i$.
- One can show that for all topologies J and J_i, $i \in I$, on $\mathcal{C}$, we have
$$J \wedge \Big(\bigvee_{i\in I} J_i\Big) = \bigvee_{i\in I} (J \wedge J_i)\,.$$

Definition 2. – *A Grothendieck topology J on $\mathcal{C}$ (essentially small) is a property (called "property to be J-covering") of families of morphisms with the same target*

$$(U_i \xrightarrow{u_i} X)_{i\in I}\,,$$

which satisfies the following three axioms:

- *(Maximality)* All morphisms $X \xrightarrow{\mathrm{id}_X} X$ are J-covering.
- *(Stability)* For any morphism $X' \xrightarrow{x} X$ of $\mathcal{C}$ and any J-covering family $(U_i \xrightarrow{u_i} X)_{i\in I}$ of X, there is a J-covering family $(U'_j \xrightarrow{u'_j} X')_{j\in I'}$ of X' such as each composite $U'_j \xrightarrow{u'_j} X' \xrightarrow{x} X$ factors through at least one $U_i \xrightarrow{u_i} X$.
- *(Transitivity)* If $(U_i \xrightarrow{u_i} X)_{i\in I}$ is J-covering, then:
  - Any family $(W_k \xrightarrow{w_k} X)$ through which the $U_i \xrightarrow{u_i} X$ factorize is J-covering.
  - If for all $i \in I$, $(V_{i,j} \xrightarrow{v_{i,j}} U_i)_{j\in J_i}$ is a J-covering family, the family of composites

$$(V_{i,j} \xrightarrow{v_{i,j}} U_i \xrightarrow{u_i} X)_{i\in I, j\in J_i}$$

is J-covering.

Relationship between the two definitions:

Observation. – *In a category* $\mathcal{C}$, *any family of morphisms with the same target* X

$$(U_i \xrightarrow{u_i} X)_{i \in I}$$

generates a sieve over X *which is*

$$\{U \xrightarrow{u} X \mid \exists\, i \in I, \exists\,(U \xrightarrow{u'} U_i), u = u_i \circ u'\}.$$

Lemma. – *On an essentially small category* $\mathcal{C}$, *the two definitions of topologies correspond by:*

(i) *A Grothendieck topology* $J : X \mapsto J(X)$ *of* $\mathcal{C}$ *defines a notion of* J*-covering family*

$$(U_i \xrightarrow{u_i} X)_{i \in I}$$

$=$ *family whose generated sieve is an element of* $J(X)$.

(ii) *A notion of* J*-covering family* $(U_i \xrightarrow{u_i} X)_{i \in I}$ *defines a Grothendieck topology* J *of which the sieves are those generated by* J*-covering families.*

Combinatorial generation of topologies:

Consider an essentially small category $\mathcal{C}$.
We start from an arbitrary application

$\mathrm{Ob}(\mathcal{C}) \ni X \longmapsto J_0(X) =$ set of sieves over X.

Problem. – *Construct the topology J on $\mathcal{C}$ generated by J_0.*

- Step 1: Replace $X \mapsto J_0(X)$ by
$$X \longmapsto J_1(X) = J_0(X) \cup \{\text{maximal sieve over } X\}.$$
- Step 2: Replace $X \mapsto J_1(X)$ by
$$X \longmapsto J_2(X) = \bigcup_{S \in \mathrm{Ob}(\mathcal{C})} \bigcup_{\substack{(X \xrightarrow{s} S) \\ \in \mathrm{Hom}(X,S)}} \{s^* C \mid C \in J_1(S)\}.$$
- Step 3:
$J(X)$ is the set of sieves that contain a family of composites
$$U_k \xrightarrow{u_k} U_{k-1} \xrightarrow{u_{k-1}} U_{k-2} \longrightarrow \cdots \longrightarrow U_1 \xrightarrow{u_1} U_0 = X$$
where, for each fixed i, $0 \leq i < k$, and each fixed U_i, the morphisms
$$U_{i+1} \xrightarrow{u_{i+1}} U_i$$
generate a sieve over U_i which is an element of $J_2(U_i)$.

Proof that J is a Grothendieck topology:

- Maximality: $J(X) \supseteq J_2(X) \supseteq J_1(X) \ni$ maximal sieve over X.
- Transitivity: A composite of "multicomposites" (of the form of step 3) of lengths k and k' is a "multicomposite" of length $k + k'$.
- Stability: We consider a morphism $x : X' \to X$ and a sieve C of $J(X)$ which contains a family of composites
$$U_k \xrightarrow{u_k} U_{k-1} \longrightarrow \cdots \longrightarrow U_1 \xrightarrow{u_1} U_0 = X$$
as in step 3. Then we construct by induction on i, $0 \leq i < k$, families of morphisms
$$U'_{i+1} \xrightarrow{u'_{i+1}} U'_i \qquad \text{with}$$

$$\begin{cases} - \quad U'_0 = X' , \\ - \quad \text{for fixed } i \text{ and } U'_i, \text{ the sieve generated by the } U'_{i+1} \to U'_i \\ \quad\;\; \text{is in } J_2(U'_i), \\ - \quad \text{each } U'_{i+1} \to U'_i \text{ fits in a commutative square of the form:} \\ \qquad \begin{array}{ccc} U'_{i+1} & \xrightarrow{u'_{i+1}} & U'_i \\ \downarrow & & \downarrow \\ U_{i+1} & \xrightarrow{u_{i+1}} & U_i \end{array} \end{cases}$$

Remarks on the generation of Grothendieck topologies:

On the spawning process:

- We therefore start from a family of subsets
$$J(X) \subset \Omega(X) = \{\text{set of sieves over } X\}, \quad X \in \mathrm{Ob}(\mathcal{C}).$$
- Step 1 of adding maximal sieves is trivial.
- Step 2 is to make the family "symmetric"
in the sense of stable by applications
$f^* : \Omega(Y) \to \Omega(X)$, $(X \xrightarrow{f} Y) =$ morphism of $\mathcal{C}$.
- Step 3 consists in making the family "stable by composition".
What is remarkable is that the family remains symmetric.

On the importance of this process:

We will see later that there is an equivalence between

- the general problem of provability
of any "geometric" assertion
in any "first-order geometric" theory,
- the general problem of determining whether
a sieve C over an object X of a small site $(\mathcal{C}, J)$
belongs or not to the topology of $\mathcal{C}$ generated
by J and a family of sieves C_i over objects X_i of $\mathcal{C}$.

Looking for algorithms:

Problem. – *Under what conditions is there an algorithm which determines whether a sieve C over an object X of a small category $\mathcal{C}$ belongs to the topology generated by a topology J of $\mathcal{C}$ and a family of sieves C_i, $i \in I$, over objects X_i of $\mathcal{C}$?*

Remarks. –

- The category $\mathcal{C}$ can be
  - defined by a list of generators and relations,
  - defined as the "syntactic geometric category" (or "coherent", or "regular", or "Cartesian") of a theory $\mathbb{T}$, introduced later. As we will see,

    $\begin{cases} \text{its objects are the "geometric" formulas} \\ \text{(or "coherent", "regular", "Cartesian") written in the language of } \mathbb{T}, \\ \text{its morphisms are the relations between these formulas} \\ \text{which satisfy the property of being "}\mathbb{T}\text{-provably functional".} \end{cases}$
- The topology J on $\mathcal{C}$ can be
  - defined as the family of sieves which satisfy certain properties,
  - defined as the family of sieves that contain a family of morphisms which satisfies certain properties.

The notion of sheaf on a site:

Definition. –
Let be a site $(\mathcal{C}, J) = \begin{cases} \text{(essentially) small category } \mathcal{C}, \\ \text{Grothendieck topology } J \text{ on } \mathcal{C}. \end{cases}$

(i) *A sheaf on $(\mathcal{C}, J)$ is a presheaf*

$$F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

such that, for any object X of $\mathcal{C}$ and any sieve $C \in J(X)$, the map

$$F(X) \longrightarrow \varprojlim_{(U \xrightarrow{u} X) \in C} F(U) = \left\{ (s_u \in F(U))_{(U \xrightarrow{u} X) \in C} \;\middle|\; s_{u'} = F(v)(s_u), \forall \begin{pmatrix} \begin{tikzcd} U' \arrow[rr, "v"] \arrow[dr, "u'"'] & & U \arrow[dl, "u"] \\ & X & \end{tikzcd} \end{pmatrix} \right\}$$

is a bijection.

(ii) *A sheaf morphism on $(\mathcal{C}, J)$*

$$F_1 \longrightarrow F_2$$

is a presheaf morphism,
that is, a morphism of the category

$$\widehat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \,.$$

The notion of sheaf topos:

Definition. –

Consider a site $(\mathcal{C}, J)$.

We call sheaf topos on $(\mathcal{C}, J)$

the full subcategory

$$\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

consisting

$$\begin{cases} \bullet & \text{of sheaves on } (\mathcal{C}, J), \\ \bullet & \text{of their morphisms (as presheaves).} \end{cases}$$

Note. –

Just like $\widehat{\mathcal{C}}$,

the category $\widehat{\mathcal{C}}_J$ is locally small.

The sheafification functor:

Proposition. – *Let $(\mathcal{C}, J)$ be a site. The embedding functor*

$$j_* : \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

admits a "left adjoint"

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J \qquad (= \text{functor of sheafification})$$

which respects

- *not only arbitrary colimits,*
- *but also finite limits.*

Remarks. –

(i) Thus, there is a natural transformation

$$\mathrm{id}_{\widehat{\mathcal{C}}} \longrightarrow j_* \circ j^*$$

of functors $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$, such that for any presheaf P and any sheaf F, the induced application

$$\mathrm{Hom}_{\widehat{\mathcal{C}}_J}(j^* P, F) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathcal{C}}}(j_* \circ j^* P, j_* F) \to \mathrm{Hom}_{\widehat{\mathcal{C}}_J}(P, j_* F)$$

is one-to-one.

(ii) We also have a natural isomorphism of functor $\widehat{\mathcal{C}}_J \to \widehat{\mathcal{C}}_J$

$$j^* \circ j_* \xrightarrow{\sim} \mathrm{id}_{\widehat{\mathcal{C}}_J} .$$

Construction of the sheafification functor:

Lemma. –

Consider a site $(\mathcal{C}, J)$.

The sheafification functor

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J$$

is constructed as the composite functor

$$P \longmapsto j^* P = (P^+)^+$$

where

$$P \longmapsto P^+$$

is the functor that associates with any presheaf P the presheaf P^+ *defined by*

$$X \longmapsto P^+(X) = \underbrace{\varinjlim_{C \in J(X)}}_{\substack{\text{filtering colimit on}\\ \text{the ordered set of}\\ \text{sieves } C \in J(X)}} \; \underbrace{\varprojlim_{(U \xrightarrow{u} X) \in C}}_{\substack{\text{(projective) limit on}\\ \text{the elements of } C}} P(U)\,.$$

Sheafification and Grothendieck topologies:

Observation. – *In a locally small category $\mathcal{C}$, the sieves C over an object X are the sub-presheaves*

$$C \hookrightarrow \mathrm{Hom}(\bullet, X) = y(X) \qquad in \quad \widehat{\mathcal{C}}\,.$$

Theorem. – *Consider a full subcategory*

$$\mathcal{E} \xhookrightarrow{j_*} \widehat{\mathcal{C}}$$

such that j_* admits a left adjoint

$$j^* : \widehat{\mathcal{C}} \longrightarrow \mathcal{E}$$

which respects arbitrary colimits and finite limits. For any object X of $\mathcal{C}$, let

$$J_{\mathcal{E}}(X)$$

be the set of sieves C on X such that

$$j^*C \hookrightarrow j^*\mathrm{Hom}(\bullet, X) \quad \textit{is an isomorphism of } \mathcal{E}.$$

Then :

(i) *$J_{\mathcal{E}}$ is a Grothendieck topology.*

(ii) *$j_* : \mathcal{E} \hookrightarrow \widehat{\mathcal{C}}$ factors into an equivalence*

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_{J_{\mathcal{E}}}\,.$$

Subtoposes and Grothendieck topologies:

Definition. – *Let $\mathcal{C}$ be an (essentially) small category.*
We call subtoposes of $\widehat{\mathcal{C}}$
the equivalence classes of full subcategories

$$\mathcal{E} \overset{j_*}{\hookrightarrow} \widehat{\mathcal{C}}$$

such that j_* admits a left adjoint $j^* : \widehat{\mathcal{C}} \to \mathcal{E}$
which respects arbitrary colimits and finite limits.

Corollary. – *The two maps*

$$J \longmapsto (\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J, \widehat{\mathcal{C}}_J \overset{j_*}{\hookrightarrow} \widehat{\mathcal{C}}),$$

$$(\widehat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E}, \mathcal{E} \overset{j_*}{\hookrightarrow} \widehat{\mathcal{C}}) \longmapsto J_{\mathcal{E}}$$

define two inverse bijections (which reverse the order relations) between

- *the ordered set of Grothendieck topologies J on $\mathcal{C}$,*
- *the subtoposes of $\widehat{\mathcal{C}}$.*

Note. – In particular, the subtoposes of $\widehat{\mathcal{C}}$ form a set.

Canonical functor and sheaf representation:

Definition. – *Consider a site* $(\mathcal{C}, J)$.
We call "canonical functor" the composite functor

$$\ell : \mathcal{C} \underset{\substack{\| \\ \text{Yoneda}}}{\xrightarrow{\;y\;}} \widehat{\mathcal{C}} \underset{\substack{\| \\ \text{sheafification}}}{\xrightarrow{\;j^*\;}} \widehat{\mathcal{C}}_J \,.$$

Lemma. – *Any object* F *of* $\widehat{\mathcal{C}}_J$ *is written as the colimit*

$$F = \varinjlim_{(X,x)\in\int F} \ell(X)$$

indexed by the "elements category of F*"*

$$\int F$$

of which

$$\begin{cases} \bullet \quad \text{the objects are } (X,x),\ X \in \mathrm{Ob}(\mathcal{C}),\ x \in F(X), \\ \bullet \quad \text{the morphisms } (X,x) \to (Y,y) \text{ are the morphisms of } \mathcal{C} \\ \qquad\qquad\qquad\qquad X \xrightarrow{f} Y \\ \quad \text{such that} \qquad\qquad x = F(f)(y). \end{cases}$$

Note. – Thus, $\widehat{\mathcal{C}}_J$ appears as a sort of "completion" of $\mathcal{C}$.
Hence the notation $\widehat{\mathcal{C}}_J$.

Subcanonical topologies:

Proposition. – *Consider a site* $(\mathcal{C}, J)$*.*
The following properties are equivalent:

(1) *The canonical functor*

$$\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$$

is "fully faithful", which means that the applications

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(\ell(X), \ell(Y))$$

are bijections.

(2) *For any object* X *of* $\mathcal{C}$*, the presheaf*

$$y(X) = \mathrm{Hom}(\bullet, X) \qquad \textit{is a } J\textit{-sheaf.}$$

(3) *One has*

$$J \subseteq J_c$$

where J_c *is the "canonical topology" of* $\mathcal{C}$
for which a sieve C *over an object* X *is in* $J_c(X)$ *if*

- *for any morphism* $X' \xrightarrow{x} X$ *and any object* Y*, the map*
  $$\mathrm{Hom}(X', Y) \longrightarrow \varprojlim_{(U' \to X') \in x^* C} \mathrm{Hom}(U', Y)$$
  is a bijection.

The intrinsic notion of topos:

Definition. –

A category $\mathcal{E}$ is called a <u>topos</u>

if it admits an <u>equivalence</u>

with the <u>category of sheaves</u> on a site $(\mathcal{C}, J)$

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}\,.$$

Note. –

For any topos $\mathcal{E}$,

there is an <u>infinite</u> collection

(so big that it's not even a set)

of <u>different sites</u> $(\mathcal{C}, J)$ and equivalences

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}\,.$$

Generating representations of a topos:

Grothendieck's comparison lemma. –

Consider a site $(\mathcal{C}, J)$.

Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$

which is "dense" in the sense that

$$\left\{ \bullet \quad \begin{array}{l} \text{any object of } \mathcal{C} \text{ admits a } J\text{-cover} \\ \text{consisting in objects of } \mathcal{C}'. \end{array} \right.$$

Let J' be the topology of $\mathcal{C}'$ induced by the topology J of $\mathcal{C}$.

Then the functor of presheaf restriction

$$\begin{array}{ccc} \widehat{\mathcal{C}} & \longrightarrow & \widehat{\mathcal{C}}', \\ (P : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}) & \longmapsto & (\mathcal{C}'^{\mathrm{op}} \hookrightarrow \mathcal{C}^{\mathrm{op}} \xrightarrow{P} \mathrm{Set}) \end{array}$$

induces an equivalence of categories

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \widehat{\mathcal{C}}'_{J'}.$$

Toposes as "pastiches" (Grothendieck) of the category of sets:

Theorem. –
A category $\mathcal{E}$ is a topos
if and only if it shares
the following properties of the category of sets:

(0) *$\mathcal{E}$ is locally small.*

(1) *$\mathcal{E}$ admits arbitrary (projective) limits, in particular*

- *a terminal object $1_{\mathcal{E}} = 1$,*
- *products $\prod_{i \in I} E_i$,*
- *fiber products $X \times_S Y$ of diagrams* $\begin{array}{ccc} & & Y \\ & & \downarrow y \\ X & \xrightarrow{x} & S \end{array}$

  characterized by the property that, for any object Z,

$$\mathrm{Hom}(Z, X \times_S Y) = \left\{ \text{set of commutative squares } \begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow y \\ X & \xrightarrow{x} & S \end{array} \right\}.$$

(2) *$\mathcal{E}$ admits arbitrary colimits (= inductive limits), in particular,*

$$\left\{\begin{array}{l}
\bullet \quad \text{an initial object } \emptyset_{\mathcal{E}} = \emptyset, \\
\bullet \quad \text{sums } \coprod_{i \in I} E_i, \\
\bullet \quad \text{amalgamated sums } X \coprod_Z Y \text{ of diagrams } \begin{array}{ccc} Z & \xrightarrow{y} & Y \\ {\scriptstyle x}\downarrow & & \\ X & & \end{array} \\
\quad\ \ \text{characterized by the property that, for any object } S, \\
\quad\ \ \mathrm{Hom}(X \coprod_Z Y, S) = \left\{ \text{set of commutative squares } \begin{array}{ccc} Z & \xrightarrow{y} & Y \\ {\scriptstyle x}\downarrow & & \downarrow \\ X & \longrightarrow & S \end{array} \right\}.
\end{array}\right.$$

(3) *For any morphism $X \to S$ of $\mathcal{E}$, the fiber product functor with X over S*

$$X \times_S \bullet$$

respects arbitrary colimits.

(4) *In $\mathcal{E}$, the functors of filtering colimits respect finite limits.*

(5) *For all objects X and Y of $\mathcal{E}$, their sum*

$$X \coprod Y$$

is disjoint in the sense that

$$X \times_{X \coprod Y} Y = \emptyset\,.$$

(6) *For a morphism of $\mathcal{E}$*

$$X \xrightarrow{u} Y$$

to be a isomorphism, it suffices that it be both

$$\begin{cases} \bullet & \text{a "monomorphism"} \\ & \text{(in the sense that, for any object } Z, \\ & \mathrm{Hom}(Z,X) \xrightarrow{u\circ\bullet} \mathrm{Hom}(Z,Y) \text{ is injective),} \\ & \\ \bullet & \text{and an "epimorphism"} \\ & \text{(in the sense that, for any object } Z, \\ & \mathrm{Hom}(Y,Z) \xrightarrow{\bullet\circ u} \mathrm{Hom}(X,Z) \text{ is injective).} \end{cases}$$

(7) *For any object X of $\mathcal{E}$, its "subobjects"*
(= equivalence classes of monomorphisms $X' \hookrightarrow X$)
form a set.

(8) *For any object X of $\mathcal{E}$,*
its "quotients" (= equivalence classes of epimorphisms $X \twoheadrightarrow X'$)
form a set.

(9) *For any object X of $\mathcal{E}$, the two applications*

$$\begin{array}{rcl} (X \twoheadrightarrow X') & \longmapsto & R = X \times_{X'} X\,, \\ (R \hookrightarrow X \times X) & \longmapsto & X \coprod_R X \end{array}$$

define two inverse bijections between

$$\left\{ \begin{array}{l} \bullet \quad \textit{the set of quotients of } X, \\ \\ \bullet \quad \textit{the set of equivalence relations on } X, \\ \quad\;\; \textit{i.e. subobjects} \\ \qquad\qquad R \hookrightarrow X \times X \\ \quad\;\; \textit{such that, for any object } Z \textit{ of } \mathcal{E}, \textit{ the subset} \\ \quad\;\; \mathrm{Hom}(Z,R) \hookrightarrow \mathrm{Hom}(Z,X) \times \mathrm{Hom}(Z,X) \\ \quad\;\; \textit{is an equivalence relation on } \mathrm{Hom}(Z,X). \end{array} \right.$$

Moreover, any morphism $X \xrightarrow{u} Y$ has an image

$$X \twoheadrightarrow \mathrm{Im}(u) \hookrightarrow Y$$

defined as the quotient of X by the equivalence relation

$$R = X \times_Y X \hookrightarrow X \times X\,.$$

(10) *There are families of objects in $\mathcal{E}$*

$$(X_i)_{i\in I}$$

which are "separating" in the sense that,
for all objects X and Y of $\mathcal{E}$,
the composition map

$$\mathrm{Hom}(X, Y) \longrightarrow \prod_{i\in I} \prod_{u\in\mathrm{Hom}(X_i,X)} \mathrm{Hom}(X_i, Y)$$

is injective.

Representation of toposes as sheaf categories:

Theorem (Giraud). – *Consider a topos $\mathcal{E}$.*
Let a small full subcategory $\mathcal{C}$ of $\mathcal{E}$ whose objects form a separating family of $\mathcal{E}$.
Let J be the Grothendieck topology of $\mathcal{C}$ for which a family of morphisms

$$(X_i \longrightarrow X)_{i \in I}$$

is covering if and only if the morphism of $\mathcal{E}$

$$\coprod_{i \in I} X_i \longrightarrow X$$

is an epimorphism.
Then the two functors

$$\begin{array}{rcl} E & \longmapsto & (\mathrm{Hom}(\bullet, E) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}), \\ \mathcal{E} & \longrightarrow & \widehat{\mathcal{C}}, \end{array}$$

and

$$\begin{array}{rcl} \widehat{\mathcal{C}}_J & \longrightarrow & \mathcal{E}, \\ F & \longmapsto & \varinjlim_{(X,x) \in \int F} X \ (= \textit{colimit calculated in } \mathcal{E}) \end{array}$$

define two inverse equivalences between $\mathcal{E}$ and $\widehat{\mathcal{C}}_J$.

Towards the notion of point of a topos:

Associate to any topological space X

$$\begin{cases} \bullet & \text{the category } O_X \text{ of its open subsets,} \\ \bullet & \text{the canonical topology } J_X \text{ on } O_X, \\ \bullet & \text{the topos } \mathcal{E}_X \text{ of sheaves on the site } (O_X, J_X). \end{cases}$$

Then:

Proposition. –

(i) *Any element $x \in X$ defines a pair of adjoint functors*

$$(\mathcal{E}_X \xrightarrow{x^*} \mathrm{Set}, \mathrm{Set} \xrightarrow{x_*} \mathcal{E}_X)$$

such that x^* respects finite limits.

(ii) *More generally, any continuous application $T \xrightarrow{x} X$ defines a pair of adjoint functors*

$$(\mathcal{E}_X \xrightarrow{x^*} \mathcal{E}_T, \mathcal{E}_T \xrightarrow{x_*} \mathcal{E}_X)$$

such that x^* respects finite limits.

(iii) *In the special case where $T \overset{x}{\hookrightarrow} X$ is a subspace, the induced functor*

$$x_* : \mathcal{E}_T \hookrightarrow \mathcal{E}_X$$

is fully faithful.

The notion of point of a topos:

Definition. – *Let $\mathcal{E}$ be a topos.*

(i) *We call point of $\mathcal{E}$ any pair of adjoint functors*

$$(\mathcal{E} \xrightarrow{p^*} \mathrm{Set}, \mathrm{Set} \xrightarrow{p_*} \mathcal{E})$$

such that p^* respects finite limits.

(ii) *More generally, we call*

point of $\mathcal{E}$ with values in a topos $\mathcal{E}'$ (or topos morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$) any pair of adjoint functors

$$f = (\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

such that f^* respects finite limits.

(iii) *Such a morphism of toposes $\mathcal{E}' \to \mathcal{E}$*

$$f = (\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

is called an "embedding" if its push-forward component

$$f_* : \mathcal{E}' \longrightarrow \mathcal{E}$$

is fully faithful.
A subtopos of $\mathcal{E}$ is an equivalence class of embeddings

$$\mathcal{E}' \hookrightarrow \mathcal{E} \ .$$

The categories of points of a topos:

Definition. – *Let $\mathcal{E}$ be a topos.*

(i) *Given two points of $\mathcal{E}$ with values in a topos $\mathcal{E}'$*

$$f = (f^*, f_*) \quad \textit{and} \quad g = (g^*, g_*)\,,$$

we call morphism from f to g

$$f \longrightarrow g$$

the datum of a natural transformation

$$f^* \longrightarrow g^*$$

or, which amounts to the same by adjunction,

$$g_* \longrightarrow f_*\,.$$

(ii) *For any topos $\mathcal{E}'$, we denote*

$$[\mathcal{E}', \mathcal{E}]_T$$

the category of points of $\mathcal{E}$ with values in $\mathcal{E}'$.
If $\mathcal{E}' = \mathrm{Set}$, it is also denoted

$$\mathrm{pt}(\mathcal{E})$$

and called the point category of $\mathcal{E}$.

Topological points and topos-theoretic points:

Proposition. –
Let X be a topological space which is "sober".

(i) *The map*

$$\begin{array}{rcl} X & \longrightarrow & \mathrm{pt}(\mathcal{E}_X)\,, \\ x & \longmapsto & (\mathcal{E}_X \xrightarrow{x^*} \mathrm{Set}, \mathrm{Set} \xrightarrow{x_*} \mathcal{E}_X) \end{array}$$

is a bijection from X
to the isomorphism classes of points of $\mathcal{E}_X$.

(ii) *More generally, for any topological space T, the map*

$$(T \xrightarrow{x} X) \longmapsto (\mathcal{E}_X \xrightarrow{x^*} \mathcal{E}_T, \mathcal{E}_T \xrightarrow{x_*} \mathcal{E}_X)$$

is a bijection
from the set of continuous maps $T \to X$
to isomorphism classes of topos morphisms

$$\mathcal{E}_T \longrightarrow \mathcal{E}_X\,.$$

Duality of presentations and evaluations of objects in a topos :

Let $\mathcal{E}$ be a topos.

Presentations. – *Any site* $(\mathcal{C}, J)$ *with an equivalence*

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$$

which extends a functor $\ell : \mathcal{C} \to \mathcal{E}$*,*
allows to present any object E *of* $\mathcal{E}$ *in the form*

$$E = \varinjlim_{(S,s)\in\int F} \ell(S)$$

where F *is the* J*-sheaf on* $\mathcal{C}$

$$F = \mathrm{Hom}(\ell(\bullet), E)\,.$$

Evaluations. –
For any point (x^*, x_*) *of* $\mathcal{E}$ *[resp. any point with values in a topos* $\mathcal{E}'$*],*
the functor

$$x^* : \mathcal{E} \longrightarrow \mathrm{Set} \qquad [\textit{resp.} \quad x^* : \mathcal{E} \longrightarrow \mathcal{E}'\ ]$$

transforms objects of $\mathcal{E}$ *into sets [resp. into objects of* $\mathcal{E}'$*]*
and respects all structures that are expressed in terms of
arbitrary colimits and finite limits.

A nonlinear analogue of the Fourier transform and wavelet decompositions?

$\mathcal{E}$ = topos	$\longleftrightarrow$	$\mathcal{E}$ = function space of $X = \mathbb{R}, \mathbb{R}^n, \cdots$
(x^*, x_*) = point of $\mathcal{E}$	$\longleftrightarrow$	point x of X
(x^*, x_*) = point of $\mathcal{E}$ with values in a topos $\mathcal{E}'$	$\longleftrightarrow$	parameter point of X $t : T \to X$
functor of evaluation $x^* : \mathcal{E} \to \mathrm{Set}$ or $\mathcal{E}'$	$\longleftrightarrow$	evaluation of functions $f \mapsto f(x)$ or $f \circ t$
choice of a presentation by a site $(\mathcal{C}, J)$ $\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$	$\longleftrightarrow$	choice of an orthogonal basis (ex: Fourier characters, wavelet basis)
transform of E object of $\mathcal{E}$ into $F = \mathrm{Hom}(\ell(\bullet), E)$ object of $\widehat{\mathcal{C}}_J$	$\longleftrightarrow$	Fourier transform or wavelet transform
formula of presentation $E = \varinjlim_{(S,s) \in \int F} \ell(S)$	$\longleftrightarrow$	Fourier decomposition or decomposition into wavelets

Possible applications to nonlinear signals and codes?

- Can we interpret
  - the objects E of a topos $\mathcal{E}$ as signals?
  - the presentations of a topos $\mathcal{E}$ by small sites $(\mathcal{C}, J)$
    $$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$$
    as codes?
  - the associated representations of the objects E of $\mathcal{E}$
    $$E = \varinjlim_{(S,s) \in \int \mathrm{Hom}(\ell(\bullet), E)} \ell(S)$$
    as encodings?
- If so, can we define conditions ensuring that a code
  $$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$$
  is effective?

Note. – The more a topology J on a small category $\mathcal{C}$ moves away from the "discrete" topology (for which the only covering sieves are the maximal sieves), the more the representations

$$E = \varinjlim_{(S,s) \in \int \mathrm{Hom}(\ell(\bullet), E)} \ell(S)$$

are redundant.

Lecture I.2

Linguistic descriptions of points

and first-order geometric theories

The notion of classifying space:

Naive definition. –
A space (topological, differential, analytic, algebraic . . .) X is said to be "classifying" if its points parameterize mathematical structures of a certain type.

Examples. –

- Projective spaces $\mathbb{P}^n$:
  points $\leftrightarrow$ lines of the standard linear space of dimension $n+1$.
- Hilbert spaces $\mathrm{Hilb}(n)$:
  points $\leftrightarrow$ closed subschemes of the projective space $\mathbb{P}^n$.
- Modular varieties $\mathcal{M}_g$:
  points $\leftrightarrow$ algebraic (smooth projective) curves of genus g.
- Jacobian variety Jac_X of a curve X:
  points $\leftrightarrow$ invertible vector bundles of degree 0 on X.
- Modular varieties $\mathcal{A}_g$:
  points $\leftrightarrow$ abelian varieties (principally polarized) of dimension g.

Announcement: All toposes are classifying.

Theorem. – *Any presentation of a topos $\mathcal{E}$ by a small site $(\mathcal{C}, J)$*

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$$

defines a "first-order (geometric) theory"

$$\mathbb{T} = \mathbb{T}_{\mathcal{C},J}$$

and equivalences of categories

$$\mathrm{pt}(\mathcal{E}) \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathrm{Set}) = \textit{category of set-based models of } \mathbb{T},$$

$$\begin{array}{ccc} [\mathcal{E}', \mathcal{E}]_T & \xrightarrow{\sim} & \mathbb{T}\text{-mod}(\mathcal{E}') \\ \| & & \| \\ \textit{category of points} & & \textit{category of models of } \mathbb{T} \\ \textit{of } \mathcal{E} \textit{ with values in } \mathcal{E}' & & \textit{in the topos } \mathcal{E}'. \end{array}$$

Note. – We shall see that the language of $\mathbb{T} = \mathbb{T}_{\mathcal{C},J}$ consists of

$$\begin{cases} \text{“sorts”} = \text{list of objects of } \mathcal{C}, \\ \text{“function symbols”} = \text{list of morphisms of } \mathcal{C}. \end{cases}$$

Reverse announcement:

Any "geometric" theory is classified by a topos.

Theorem. – *Any "first-order geometric" theory* $\mathbb{T}$ *defines:*

(i) *a "functor of models" of* $\mathbb{T}$ *in toposes* $\mathcal{E}$

$$\begin{array}{rcl} \mathcal{E} & \longmapsto & \textit{category } \mathbb{T}\text{-mod}\,(\mathcal{E})\,, \\ (\mathcal{E}' \xrightarrow{f} \mathcal{E}) & \longmapsto & \textit{functor } f^* : \mathbb{T}\text{-mod}\,(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}\,(\mathcal{E}'), \end{array}$$

(ii) *a "classifying topos"* $\mathcal{E}_{\mathbb{T}}$ *characterized up to equivalence by a system of equivalences of categories*

$$[\mathcal{E}, \mathcal{E}_{\mathbb{T}}]_T \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathcal{E})\,.$$

Note. – Thus, any topos $\mathcal{E}$ admits

– an infinite variety of "geometric" presentations

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}\,,$$

– an infinite variety of "linguistic descriptions"

$$\mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}\,.$$

Diaconescu's equivalence:

Proposition. – *Consider a site* $(\mathcal{C}, J)$. *For any topos* $\mathcal{E}$, *the functor*

$$\begin{array}{ccc} [\mathcal{E}, \widehat{\mathcal{C}}_J]_T & \longrightarrow & [\mathcal{C}, \mathcal{E}]\,, \\ x = (\widehat{\mathcal{C}}_J \xrightarrow{x^*} \mathcal{E}, \mathcal{E} \xrightarrow{x_*} \widehat{\mathcal{C}}_J) & \longmapsto & (\mathcal{C} \xrightarrow{\rho} \mathcal{E}) = (\mathcal{C} \xrightarrow{\ell} \widehat{\mathcal{C}}_J \xrightarrow{x^*} \mathcal{E}) \end{array}$$

is an equivalence from the category of topos morphisms

$$x : \mathcal{E} \longrightarrow \widehat{\mathcal{C}}_J$$

to the category of functors

$$\rho : \mathcal{C} \longrightarrow \mathcal{E}$$

which are

(A) "flat" in the sense that the induced functor

$$\begin{array}{cccc} \widehat{\rho} & : & \widehat{\mathcal{C}} & \longrightarrow & \mathcal{E}\,, \\ & & P & \longmapsto & \varinjlim\limits_{(S,s) \in \int P} \rho(S) \end{array}$$

respects finite limits,

(B) "J-continuous" in the sense that any J-covering family of $\mathcal{C}$

$$(X_i \longrightarrow X)_{i \in I}$$

is transformed by ρ *into an epimorphism of* $\mathcal{E}$

$$\coprod_{i \in I} \rho(X_i) \longrightarrow \rho(X)\,.$$

Proof sketch:

- For any functor $\rho : \mathcal{C} \to \mathcal{E}$, the functor

$$\begin{array}{rccl} \widehat{\rho} & : & \widehat{\mathcal{C}} & \longrightarrow & \mathcal{E}\,, \\ & & P & \longmapsto & \varinjlim\limits_{(S,s)\in\int P} \rho(S) \end{array}$$

  is the unique extension of ρ which respects arbitrary colimits.

- If $x^* : \widehat{\mathcal{C}}_J \to \mathcal{E}$ respects colimits and

$$\rho = x^* \circ \ell : \mathcal{C} \xrightarrow{\ell} \widehat{\mathcal{C}}_J \xrightarrow{x^*} \mathcal{E}\,,$$

  then

$$\widehat{\rho} \cong x^* \circ j^* : \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J \xrightarrow{x^*} \mathcal{E}\,.$$

- In this case, x^* respects finite limits
  if and only if $\widehat{\rho}$ respects finite limits.

- For any functor $\rho : \mathcal{C} \to \mathcal{E}$, the functor

  $\widehat{\rho} : \widehat{\mathcal{C}} \to \mathcal{E}$ factors through $\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$
  if and only if ρ is J-continuous.

- A functor $\widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}$ admits a right adjoint
  if and only if it respects arbitrary colimits.

The theory of flat and J-continuous functors:

Observation. –
Let $\mathcal{C}$ be a small category.
To consider a flat and J-continuous functor
with values in a topos $\mathcal{E}$
is equivalent to consider a structure consisting

$$\begin{cases} \bullet & \text{of objects } MA \text{ of } \mathcal{E} \text{ indexed by the objects } A \text{ of } \mathcal{C}, \\ \bullet & \text{of morphisms } MA \xrightarrow{Mf} MB \text{ of } \mathcal{E} \\ & \text{indexed by the morphisms } f : A \to B \text{ of } \mathcal{C}, \end{cases}$$

and which satisfies

$$\begin{cases} \bullet & \text{the axioms ensuring that it is a functor,} \\ \bullet & \text{the axioms ensuring that this functor is flat,} \\ \bullet & \text{the axioms ensuring that this functor is } J\text{-continuous.} \end{cases}$$

We shall formalise these axioms,
which will introduce the notion of "first-order geometric theory".

The language of diagrams:

Definition. –

Let $\mathcal{C}$ be a small category (or more generally a quiver).

The language of $\mathcal{C}$-diagrams

is the "signature" (or "first-order language") $\Sigma_{\mathcal{C}}$

consisting of

- *"sorts" A which correspond to the objects of $\mathcal{C}$,*
- *"function symbols" $f : A \to B$*

  which correspond to the arrows $A \xrightarrow{f} B$ of $\mathcal{C}$.

Definition. –

The <u>*category*</u> *of* $\Sigma_{\mathcal{C}}$*-*<u>*structures*</u> *of a topos* $\mathcal{E}$

$$\Sigma_{\mathcal{C}}\text{-str}(\mathcal{E})$$

has for <u>*objects*</u> *the applications*

$$M = \begin{cases} \underline{\textit{sort}}\ A\ (= \textit{object of}\ \mathcal{C}) & \longmapsto \quad MA = \underline{\textit{object}}\ \textit{of}\ \mathcal{E}, \\ \underline{\textit{symbol}}\ (A \to B) & \longmapsto \quad (MA \xrightarrow{Mf} MB) = \underline{\textit{morphism}}\ \textit{of}\ \mathcal{E}, \end{cases}$$

and for morphisms $M \to N$ *the maps*
u : *sort* $A \longmapsto$ *morphism* $(MA \xrightarrow{u_A} NA)$ *of* $\mathcal{E}$
such that all the following squares <u>*commute*</u>*:*

$$\begin{tikzcd} MA \arrow[r, "u_A"] \arrow[d, "Mf"'] & NA \arrow[d, "Nf"] \\ NB \arrow[r, "u_B"] & NB \end{tikzcd}$$

Functor theory:

Definition. –
Let $\mathcal{C}$ be a small category.
The theory $\mathbb{T}_{\mathcal{C}}$ of functors on $\mathcal{C}$
consists of the language $\Sigma_{\mathcal{C}}$ completed by the axioms

$$\begin{cases} \top \vdash_{x^A} \mathrm{id}_A(x^A) = x^A \text{ for any sort } A \text{ of } \Sigma_{\mathcal{C}}, \\ \top \vdash_{x^A} (g \circ f)(x^A) = g(f(x^A)) \text{ for all function symbols} \\ \qquad\qquad A \xrightarrow{f} B \text{ and } B \xrightarrow{g} C \text{ of } \Sigma_{\mathcal{C}}. \end{cases}$$

Lemma. – *For any topos $\mathcal{E}$, the category of functors*

$$\mathcal{C} \longrightarrow \mathcal{E}$$

identifies with the category of models of $\mathbb{T}_{\mathcal{C}}$ in $\mathcal{E}$

$$\mathbb{T}_{\mathcal{C}}\text{-mod}(\mathcal{E}),$$

defined as the full subcategory of

$$\Sigma_{\mathcal{C}}\text{-str}(\mathcal{E})$$

consisting of $\Sigma_{\mathcal{C}}$-structures M in $\mathcal{E}$ which satisfy the axioms of $\mathbb{T}_{\mathcal{C}}$.

Interpretation of axioms:

- Each x^A is a formal variable assigned to a sort A.
- Each symbol $\vdash_{x^A}$ has the meaning of an implication
$$\varphi \vdash_{x^A} \psi$$
between formulas φ, ψ in the variable x^A.
- Each formula φ or ψ in the variable x^A defines for any structure M in a topos $\mathcal{E}$ a subobject $M\varphi$ or $M\psi$ of the object MA.
Such a structure M satisfies an axiom of the form
$$\varphi \vdash_{x^A} \psi$$
if the two subobjects $M\varphi$ and $M\psi$ of MA verify
$$M\varphi \subseteq M\psi\,.$$
- The symbol $\top$ stands for "truth".
In a variable x^A, it defines for any structure M the total subobject $M\top = MA$ of MA.
A structure M satisfies an axiom of the form
$$\top \vdash_{x^A} \varphi$$
if the subobject $M\varphi$ of MA is equal to MA.

- For any function symbol $f : A \to B$, the formula

$$f(x^A)$$

is interpreted in a structure M as the morphism

$$MA \xrightarrow{Mf} MB\,.$$

- For all function symbols $f : A \to B$ and $g : B \to C$,
the term obtained by substitution of a function with a variable

$$g(f(x^A))$$

is interpreted in a structure M as the composite morphism

$$MA \xrightarrow{Mf} MB \xrightarrow{Mg} MC\,.$$

- Thus, a structure M satisfies the axiom

$$\top \vdash_{x^A} \ \mathrm{id}_A(x^A) = x^A$$

if and only if $\qquad M\,\mathrm{id}_A = \mathrm{id}_{MA}$.

- Similarly, a structure M satisfies the axiom

$$\top \vdash_{x^A} \ (g \circ f)(x^A) = g(f(x^A))$$

if and only if $\qquad M(g \circ f) = Mg \circ Mf$.

Flat functor theory:

Definition. – *Let $\mathcal{C}$ be a small category.*
The theory $\mathbb{T}_{\mathcal{C}}^{p}$ of flat functors on $\mathcal{C}$ consists in the theory $\mathbb{T}_{\mathcal{C}}$ of functors on $\mathcal{C}$ supplemented with the following axioms:

(A1) Axiom without free variable

$$\top \vdash \bigvee_{A = \text{sort}} (\exists\, x^A)\, \top(x^A)\,.$$

(A2) The family of axioms indexed by pairs of sorts A, B

$$\top \vdash_{x^A, y^B} \bigvee_{\substack{Z \,=\, \text{sort} \,=\, \text{object of } \mathcal{C} \\ (f,g) \in \mathrm{Hom}(Z,A) \times \mathrm{Hom}(Z,B)}} (\exists\, z^Z)(x^A = f(z^Z) \wedge y^B = g(z^Z))\,.$$

(A3) The family of axioms indexed by pairs of morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$ of $\mathcal{C}$

$$f(x^A) = g(x^A) \vdash_{x^A} \bigvee_{\substack{Z \,=\, \text{sort} \\ h \in \mathrm{Hom}(Z,A) \\ \text{such that } f \circ h = g \circ h}} (\exists\, z^Z)(x^A = h(z^Z))\,.$$

Theorem. –
For any topos $\mathcal{E}$, the category of flat functors

$$\mathcal{C} \longrightarrow \mathcal{E}$$

identifies with the category of models of $\mathbb{T}^p_{\mathcal{C}}$ in $\mathcal{E}$

$$\mathbb{T}^p_{\mathcal{C}}\text{-mod}(\mathcal{E})$$

defined as the full subcategory of

$$\mathbb{T}_{\mathcal{C}}\text{-mod}(\mathcal{E})$$

made up of models M of $\mathbb{T}_{\mathcal{C}}$ in $\mathcal{E}$
which satisfy the complementary axioms
(A1), (A2) and (A3) of $\mathbb{T}^p_{\mathcal{C}}$.

Note. –
The theory $\mathbb{T}^p_{\mathcal{C}}$ has the same language $\Sigma_{\mathcal{C}}$ as $\mathbb{T}_{\mathcal{C}}$
but has more axioms.
We say that $\mathbb{T}^p_{\mathcal{C}}$ is a quotient theory of $\mathbb{T}_{\mathcal{C}}$.

Interpretation of axioms:

- The "free variables" of a formula are those on which no quantifier applies. So the formula
$$(\exists\, x^A)\ \top(x^A)$$
has no free variable, and the formula
$$(\exists\, z^Z)\ (x^A = h(z^Z))$$
has x^A for unique free variable.

- Each formula φ or ψ without free variable defines for any structure M in a topos $\mathcal{E}$ a subobject $M\varphi$ or $M\psi$ of the terminal object $1 = 1_{\mathcal{E}}$ of $\mathcal{E}$. Such a structure M satisfies an axiom of the form
$$\varphi \vdash \psi$$
if the two subobjects $M\varphi$ and $M\psi$ of $1 = 1_{\mathcal{E}}$ verify
$$M\varphi \subseteq M\psi\,.$$

- In particular, if φ is a formula without free variable, a structure M satisfies the axiom
$$\top \vdash \varphi$$
if the subobject $M\varphi$ of $1 = 1_{\mathcal{E}}$ is equal to $1 = 1_{\mathcal{E}}$.

- The existential quantifier $\exists$ in one or more variables has the meaning of an image by the projection defined by forgetting this or these variables.
  If for example φ is a formula into free variables x^A, (x^A, z^Z) or (x^A, y^B, z^C), interpreted in a structure M of a topos $\mathcal{E}$ as a subobject

$$M\varphi \hookrightarrow MA\,,\ M\varphi \hookrightarrow MA \times MZ\,,\ \text{or } M\varphi \hookrightarrow MA \times MB \times MZ\,,$$

  then the formula $(\exists\, x^A)M\varphi$ or $(\exists\, z^Z)M\varphi$ is interpreted as the image subobject of $M\varphi$ in

$$1 = 1_{\mathcal{E}}\,, \quad MA \quad \text{or} \quad MA \times MB$$

  by the canonical projection morphism

$$MA \to 1\,, \quad MA \times MZ \to MA \quad \text{or} \quad MA \times MB \times MZ \to MA \times MB\,.$$

- The symbol $\wedge$ has the meaning of a finite conjunction and is interpreted as a finite intersection of subobjects.
If for example φ and ψ are two formulas
in free variables (x^A, y^B, z^Z),
interpreted in a structure M of a topos $\mathcal{E}$ as two subobjects

$$M\varphi \hookrightarrow MA \times MB \times MZ\,, \text{ and } M\psi \hookrightarrow MA \times MB \times MZ\,,$$

the formula $\varphi \wedge \psi$ is interpreted as
their intersection, i.e. their fiber product.

- The symbol $\vee$ or $\bigvee$ has the meaning of a finite or infinite disjunction and is interpreted as an arbitrary union of subobjects.
If for example the φ_i, $i \in I$, are formulas
in variables (x^A, y^B), x^A or without free variable,
interpreted in a structure M of a topos $\mathcal{E}$ as subobjects

$$M\varphi_i \hookrightarrow MA \times MB\,, \quad M\varphi_i \hookrightarrow MA \quad \text{or} \quad M\varphi_i \hookrightarrow 1 = 1_{\mathcal{E}}\,,$$

the formula $\bigvee\limits_{i \in I} \varphi_i$ is interpreted as
their union, i.e. the image of the morphism

$$\coprod_{i \in I} M\varphi_i \to MA \times MB\,, \quad \coprod_{i \in I} M\varphi_i \to MA \quad \text{or} \quad \coprod_{i \in I} M\varphi_i \to 1\,.$$

- Ultimately, a structure M satisfies axiom (A1)
$$\top \vdash \bigvee_{A = \text{sort}} (\exists\, x^A)\, \top(x^A)$$
if $1 = 1_{\mathcal{E}}$ is the <u>union of images of the morphisms</u>
$$MA \longrightarrow 1\,.$$
- It satisfies the axioms of (A2)
$$\top \vdash_{x^A, y^B} \bigvee_{\substack{Z = \text{sort} \\ (f,g) \in \mathrm{Hom}(Z,A) \times \mathrm{Hom}(Z,B)}} (\exists\, z^Z)(x^A = f(z^Z) \wedge y^B = g(z^Z))$$
if $MA \times MB$ is the <u>union of images of the morphisms</u>
$$MZ \xrightarrow{Mf \times Mg} MA \times MB$$
indexed by diagrams $A \xleftarrow{f} Z \xrightarrow{g} B$ of $\mathcal{C}$.
- For any pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$, it satisfies the axiom of (A3)
$$f(x^A) = g(x^A) \vdash_{x^A} \bigvee_{\substack{Z = \text{sort} \\ (Z \xrightarrow{h} A) \text{ such that } f \circ h = g \circ h}} (\exists\, z^Z)(x^A = h(z^Z))$$
if the subobject of MA defined by the equation $Mf = Mg$
is the <u>union of images of the morphisms</u> $\quad MZ \xrightarrow{Mh} MA$
indexed by morphisms $h : Z \rightarrow A$ of $\mathcal{C}$ such that $f \circ h = g \circ h$.

The theory of flat and J-continuous functors:

Definition. –

Let $(\mathcal{C}, J)$ be a small site.

The theory $\mathbb{T}_{\mathcal{C},J}$ of flat and J-continuous functors on $\mathcal{C}$ consists in the theory $\mathbb{T}_{\mathcal{C}}^{p}$ of flat functors on $\mathcal{C}$ completed with the following family of axioms:

(B) *For any J-covering family of morphisms of $\mathcal{C}$*

$$A_i \xrightarrow{f_i} A\,, \quad i \in I\,,$$

the axiom

$$\top \vdash_{x^A} \bigvee_{i \in I} (\exists\, x_i^{A_i})(x^A = f_i(x_i^{A_i}))\,.$$

Corollary (of the previous theorem). –

For any topos $\mathcal{E}$, the category of flat and J-continuous functors

$$\mathcal{C} \longrightarrow \mathcal{E}$$

identifies with the category of models of $\mathbb{T}_{\mathcal{C},J}$ in $\mathcal{E}$

$$\mathbb{T}_{\mathcal{C},J}\text{-mod}(\mathcal{E})$$

defined as the full subcategory of

$$\mathbb{T}_{\mathcal{C}}^{p}\text{-mod}(\mathcal{E})$$

made up of models M of $\mathbb{T}_{\mathcal{C}}^{p}$ in $\mathcal{E}$
which satisfy the complementary axioms (B) of $\mathbb{T}_{\mathcal{C},J}$.

Note. –

The theory $\mathbb{T}_{\mathcal{C},J}$ has the same language $\Sigma_{\mathcal{C}}$ as $\mathbb{T}_{\mathcal{C}}$ or $\mathbb{T}_{\mathcal{C}}^{p}$
but has more axioms.
So $\mathbb{T}_{\mathcal{C},J}$ is a quotient theory of $\mathbb{T}_{\mathcal{C}}$ or $\mathbb{T}_{\mathcal{C}}^{p}$.

Interpretation of axioms:

- Consider a flat functor from $\mathcal{C}$ to a topos $\mathcal{E}$,
  i.e. a model M in $\mathcal{E}$ of the theory $\mathbb{T}_{\mathcal{C}}^{p}$.
  For a J-covering family of morphisms of $\mathcal{C}$

$$(A_i \xrightarrow{f_i} A)_{i \in I}\,,$$

  to say that M satisfies the axiom

$$\top \vdash_{x^A} \bigvee_{i \in I} (\exists\, x_i^{A_i})(x^A = f_i(x_i^{A_i}))$$

  means that MA is the union of the images of the morphisms

$$MA_i \xrightarrow{Mf_i} MA\,, \quad i \in I\,,$$

  in other words that the morphism

$$\coprod_{i \in I} MA_i \longrightarrow MA$$

  is an epimorphism.
- This means exactly that this flat functor

$$\mathcal{C} \longrightarrow \mathcal{E}$$

  is J-continuous.

The general notion of first-order language:

Definition. –
A "signature" (or "first-order language") Σ consists of

- *a (finite or infinite) family of "sorts" (i.e. of "object names")*

$$A, B, C, \cdots$$

- *a (finite or infinite) family of "function symbols"*

$$f : A_1 \cdots A_n \longrightarrow B \qquad (\textit{with } n \geq 0)$$

  which go from a finite sequence of sorts $A_1 \cdots A_n$ to a sort B,

- *a (finite or infinite) family of "relation symbols"*

$$R \rightarrowtail A_1 \cdots A_n \qquad (\textit{with } n \geq 0)$$

  between finite sequences of sorts $A_1 \cdots A_n$.

The notion of Σ-structure:

Definition. –
Let Σ be a signature.
A Σ-structure in a topos $\mathcal{E}$ is a triple mapping which associates

- *to any sort A of Σ an object MA of $\mathcal{E}$,*
- *to any function symbol $f : A_1 \cdots A_n \to B$ of Σ a morphism of $\mathcal{E}$ of the form*

$$Mf : MA_1 \times \cdots \times MA_n \longrightarrow MB$$

$$\textit{(and} \quad Mf : 1 = 1_{\mathcal{E}} \longrightarrow MB \quad \textit{if} \quad n = 0),$$

- *to any relation symbol $R \rightarrowtail A_1 \cdots A_n$ of Σ a subobject in $\mathcal{E}$ of the form*

$$MR \hookrightarrow MA_1 \times \cdots \times MA_n$$

$$\textit{(and} \quad MR \hookrightarrow 1 = 1_{\mathcal{E}} \quad \textit{if} \quad n = 0).$$

The notion of morphism of Σ-structures:

Definition. – *Let Σ be a signature.*
A morphism between two Σ-structures M, N in a topos $\mathcal{E}$

$$u : M \longrightarrow N$$

is a mapping

sort $A \mapsto$ morphism $u_A : MA \to NA$ of $\mathcal{E}$, such that:

- *for all function symbol $f : A_1 \cdots A_n \to B$ of Σ, the square*

$$\begin{tikzcd} MA_1 \times \cdots \times MA_n \arrow[r] \arrow[d, "u_{A_1} \times \cdots \times u_{A_n}"'] & MB \arrow[d, "u_B"] \\ NA_1 \times \cdots \times NA_n \arrow[r] & NB \end{tikzcd}$$

is commutative,

- *for all relation symbol $R \rightarrowtail A_1 \cdots A_n$ of Σ,*
we have a commutative square:

$$\begin{tikzcd} MR \arrow[r, hook] \arrow[d] & MA_1 \times \cdots \times MA_n \arrow[d, "u_{A_1} \times \cdots \times u_{A_n}"] \\ NR \arrow[r, hook] & NA_1 \times \cdots \times NA_n \end{tikzcd}$$

Functors of Σ-structures:

Lemma. – *Let Σ be a signature.*

(i) *For any topos $\mathcal{E}$, the Σ-structures in $\mathcal{E}$ and their morphisms constitute a (locally small) category*

$$\Sigma\text{-str}(\mathcal{E})\,.$$

(ii) *Any topos morphism $\mathcal{E}' \to \mathcal{E}$*

$$f = (\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

induces a pair of adjoint functors

$$f^* : \Sigma\text{-str}(\mathcal{E}) \longrightarrow \Sigma\text{-str}(\mathcal{E}')$$

and

$$f_* : \Sigma\text{-str}(\mathcal{E}') \longrightarrow \Sigma\text{-str}(\mathcal{E})\,.$$

Proof. –
Indeed, f^* and f_* respect finite limits,
in particular finite products and monomorphisms.

The constituent elements of geometric formulas:

Definition. –
Consider a signature Σ.
The constituent elements of the "geometric formulas" of Σ are:

- *the sorts, the function symbols and the relation symbols of Σ,*
- *formal variables $x^A, y^B, \cdots$*
  each of which is assigned to a sort $A, B, \cdots$ of Σ,
- *the equality relation $=$,*
- *the symbol $\top$ ("truth")*
  and the symbol for finite conjunction $\wedge$,
- *the existential quantifier $\exists$,*
- *the symbol $\perp$ ("falsity")*
  and the symbols of finitary disjunction $\vee$ or infinitary disjunction $\bigvee$.

Note. – A variable x^A of a formula is said

- "tied" if subject to a quantifier $\exists$,
- "free" otherwise.

Contexts and interpretations of geometric formulas:

Definition. –
Let φ be a geometric formula of a signature Σ.
A "context" of φ is a finite family of variables

$$\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$$

which contains all free variables of φ
(which are therefore in finite number).

Interpretation of formulas:

A geometric formula φ of signature Σ in a context

$$\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$$

is meant to define,
for any Σ-structure M in a topos $\mathcal{E}$,
a subobject

$$M\varphi = M\varphi(x_1^{A_1} \cdots x_n^{A_n}) \hookrightarrow MA_1 \times \cdots \times MA_n \,.$$

Geometric theories and their models:

Definition. – *A "first-order geometric theory" $\mathbb{T}$ consists of*

- *a signature Σ,*
- *a family of implications between geometric formulas of Σ*

$$\varphi \vdash_{\vec{x}} \psi \quad \textit{in contexts} \quad \vec{x} = x_1^{A_1} \cdots x_n^{A_n} \,,$$

called the "axioms" of Σ.

Definition. – *Consider a geometric theory $\mathbb{T}$ of signature Σ.*

(i) *A "model" of $\mathbb{T}$ in a topos $\mathcal{E}$ is a Σ-structure M of $\mathcal{E}$ such that, for any axiom of $\mathbb{T}$*

$$\varphi \vdash_{\vec{x}} \psi \quad \textit{in a context} \quad \vec{x} = x_1^{A_1} \cdots x_n^{A_n} \,,$$

we have

$$M\varphi \subseteq M\psi$$

as subobjects of $MA_1 \times \cdots \times MA_n$.

(ii) *The models of $\mathbb{T}$ in a topos $\mathcal{E}$ form a full subcategory*

$$\mathbb{T}\text{-mod}(\mathcal{E}) \quad \textit{of} \quad \Sigma\text{-str}(\mathcal{E}) \,.$$

First step in the development of geometric formulas: the terms

Definition. – *A <u>"term"</u> in a signature Σ is an entry of the form*

$$y^B \qquad \textit{(\underline{variable} of sort } B\textit{)}$$

or $t(x_1^{A_1} \cdots x_n^{A_n})$ (with values of sort B) which is deduced from y^B by a <u>finite succession of variable substitutions</u>

$$x_i^{A_i}$$

by expressions $f(y_1^{A'_1} \cdots y_m^{A'_m})$ formed from

$$\begin{cases} (f : A'_1 \cdots A'_m \to A_i) = \underline{\textit{function symbol}} \textit{ of } \Sigma, \\ y_1^{A'_1} \cdots y_m^{A'_m} = \textit{ new variables.} \end{cases}$$

Interpretation in a Σ-structure M of a topos $\mathcal{E}$:

- A <u>term</u> $t(x_1^{A_1} \cdots x_n^{A_n})$ with values of a sort B <u>interprets</u> as a <u>morphism</u> $Mt : MA_n \times \cdots \times MA_n \longrightarrow MB$.
- The variable y^B is interpreted as $\mathrm{id} : MB \to MB$.
- The <u>substitution</u> of a variable $x_i^{A_i}$ by $f(y_1^{A'_1} \cdots y_m^{A'_m})$ is interpreted as <u>composition with</u> $Mf : MA'_1 \times \cdots \times MA'_m \to MA_i$.

Second step for creating geometric formulas: the atomic formulas

Definition. – *An "atomic formula" of signature Σ is*

(1) *a relation formula $R(x_1^{A_1} \cdots x_n^{A_n})$ for a relation symbol $R \rightarrowtail A_1 \cdots A_n$ of Σ,*

(2) *an equality formula $x^B = y^B$ for a sort B of Σ,*

(3) *a formula derived from (1) or (2) by substitution of certain variables by terms.*

Interpretation in a Σ-structure M of a topos $\mathcal{E}$:

(1) is interpreted as the relational subobject

$$MR \hookrightarrow MA_1 \times \cdots \times MA_n\,.$$

(2) is interpreted as the diagonal subobject

$$MB \hookrightarrow MB \times MB\,.$$

(3) The substitution of a variable $x_i^{A_i}$ by a term $t(y_1^{A'_1} \cdots y_m^{A'_m})$ is interpreted as the pull-back (= fiber product) of subobjects by the morphism $Mt : MA'_1 \times \cdots \times MA'_m \to MA_i$.

Third step in the development of geometric formulas: Horn formulas

Definition. – *A* *<u>"Horn formula"</u>* *of signature* Σ *is of the form*

(1) $\top(\vec{x})$ *(<u>"truth"</u> in a context* $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$*)* ,

(2) $\varphi_1 \wedge \cdots \wedge \varphi_k$

for <u>atomic formulas</u> $\varphi_1, \cdots, \varphi_k$ *in the same context* $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$.

Interpretation in a Σ-structure M of a topos $\mathcal{E}$:

(1) The formula $\top(\vec{x})$ in a context $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$
interprets as the <u>total subobject</u>

$$MA_1 \times \cdots \times MA_n \quad \text{of} \quad MA_1 \times \cdots \times MA_n\,.$$

(2) A conjunction $\varphi_1 \wedge \cdots \wedge \varphi_k$
is interpreted as <u>the intersection</u>

$$M(\varphi_1 \wedge \cdots \wedge \varphi_k) = M\varphi_1 \cap \cdots \cap M\varphi_k$$

of the subobjects $M\varphi_1, \cdots, M\varphi_k$ of $MA_1 \times \cdots \times MA_n$.

Fourth step in the development of geometric formulas: regular formulas

Definition. – *A "regular formula" of signature* Σ *is*

(1) *a Horn formula,*

(2) *a formula in a context* $\vec{x} = (x_1^{A_1} \cdots x_n^{A_n})$ *of the form*

$$\varphi(\vec{x}) = (\exists\, \vec{y})\, \psi(\vec{x}, \vec{y})$$

for a Horn formula ψ *in a context* $(\vec{x}, \vec{y}) = (x_1^{A_1} \cdots x_n^{A_n} y_1^{B_1} \cdots y_k^{B_k})$.

Interpretation in a Σ**-structure** M **of a topos** $\mathcal{E}$**:**

A formula of the form (2)

$$\varphi(\vec{x}) = (\exists\, \vec{y})\, \psi(\vec{x}, \vec{y})$$

interprets as the subobject

$$M\varphi \hookrightarrow MA_1 \times \cdots \times MA_n$$

which is the image of the subobject

$$M\psi \hookrightarrow MA_1 \times \cdots \times MA_n \times MB_1 \times \cdots \times MB_k$$

by the projection morphism

$$MA_1 \times \cdots \times MA_n \times MB_1 \times \cdots \times MB_k \longrightarrow MA_1 \times \cdots \times MA_n\,.$$

Fifth step in the development of geometric formulas: coherent or geometric formulas

Definition. – *A "coherent [resp. "geometric"] formula" of signature* Σ *is*

(1) $\perp(\vec{x})$ *("falsity" in a context* $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$*)* ,

(2) *a finite [resp. infinite] disjunction*

$$\varphi_1 \vee \cdots \vee \varphi_k \qquad \textit{[resp.} \quad \bigvee_{i \in I} \varphi_i \textit{]}$$

of regular formulas φ_i *in the same context* $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$.

Interpretation in a Σ**-structure** M **of a topos** $\mathcal{E}$**:**

(1) The formula $\perp(\vec{x})$ in a context $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$ is interpreted as the initial subobject i.e. empty subobject

$$\emptyset \hookrightarrow MA_1 \times \cdots \times MA_n \,.$$

(2) A finite or infinite disjunction of regular formulas φ_i in a context $\vec{x} = x_1^{A_1} \cdots x_n^{A_n}$ is interpreted as the finite or infinite union of subobjects

$$M\varphi_i \hookrightarrow MA_1 \times \cdots \times MA_n$$

i.e. as the image of the morphism

$$\coprod_i M\varphi_i \longrightarrow MA_1 \times \cdots \times MA_n \,.$$

A note on the order of elaboration of geometric formulas:

Fact. – *It is part of the "inference rules" of "geometric logic" that:*

- *For all geometric formulas* φ *of context* $\vec{x}$ *and* ψ *of context* $\vec{x}, \vec{y}$, *the formulas of context* $\vec{x}$
$$\varphi(\vec{x}) \wedge (\exists\, \vec{y})\psi(\vec{x}, \vec{y}) \quad \textit{and} \quad (\exists \vec{y})(\varphi(\vec{x}) \wedge \psi(\vec{x}, \vec{y}))$$
are provably equivalent.

- *For all geometric formulas* φ_i, $i \in I$, *of context* $\vec{x}, \vec{y}$, *the formulas of context* $\vec{x}$
$$\bigvee_{i \in I} (\exists\, \vec{y})\varphi_i(\vec{x}, \vec{y}) \quad \textit{and} \quad (\exists\, \vec{y})(\bigvee_{i \in I} \varphi_i(\vec{x}, \vec{y}))$$
are provably equivalent.

- *For all geometric formulas* φ *and* ψ_i, $i \in I$, *of context* $\vec{x}$, *the formulas of context* $\vec{x}$
$$\varphi \wedge \bigvee_{i \in I} \psi_i \quad \textit{and} \quad \bigvee_{i \in I} (\varphi \wedge \psi_i)$$
are provably equivalent.

Consequence. – This is why, in the definition of geometric formulas, the symbols $\wedge, \exists$ and $\bigvee$ appear in some ordering.

Fragments of geometric logic:

Definition. – *A geometric theory $\mathbb{T}$ of signature Σ is called*

(i) *"algebraic" if*

$$\begin{cases} \bullet \quad \Sigma \text{ has no relation symbol,} \\ \bullet \quad \text{all the axioms of } \mathbb{T} \text{ have the form} \\ \qquad\qquad \top \vdash t_1(\vec{x}) = t_2(\vec{x}) \\ \quad\;\; \text{for pairs of terms } t_1, t_2 \text{ in the same context } \vec{x}, \end{cases}$$

(ii) *"Horn", "regular" or "coherent"*
if all the axioms of $\mathbb{T}$ are implications

$$\varphi_i \vdash \psi_i$$

between pairs of formulas φ_i, ψ_i
which are "Horn", "regular" or "coherent".

Note. –
A theory without axiom, reduced to its sole signature, is called "empty".

The Cartesian fragment:

Definition. – *Let $\mathbb{T}$ be a geometric theory of signature Σ.*

(i) *A geometric formula of Σ of context $\vec{x}$ is said to be "$\mathbb{T}$-Cartesian" if it has the form*

$$(\exists\, \vec{y})\psi(\vec{x}, \vec{y})$$

for a Horn formula $\psi(\vec{x}, \vec{y})$ of context $\vec{x}, \vec{y}$ such that the implication

$$\psi(\vec{x}, \vec{y}) \wedge \psi(\vec{x}, \vec{y}\,') \vdash \vec{y} = \vec{y}\,'$$

is provable in the theory $\mathbb{T}$.

(ii) *The theory $\mathbb{T}$ is said to be "Cartesian" if all its axioms*

$$\varphi_i \vdash \psi_i$$

are implications between $\mathbb{T}$-Cartesian formulas.

Remarks. –

(i) Any $\mathbb{T}$-Cartesian formula is regular.
So any Cartesian theory is regular.

(ii) Any Horn theory
(and a fortiori any empty theory, or any algebraic theory)
is Cartesian.

Classifying toposes,

toposes as bridges

and the equivalence between

first-order provability

and Grothendieck topologies generation

Geometric Syntactic Categories:

Definition. – *Let $\mathbb{T}$ be a geometric theory with signature Σ.*
We call "geometric syntactic category" of $\mathbb{T}$, denoted

$$\mathcal{C}_{\mathbb{T}}^{\text{geo}},$$

the category thus defined:

(i) *Its objects are the geometric formulas of Σ*

$$\varphi(\vec{x}) \quad \text{in contexts } \vec{x} = x_1^{A_1} \cdots x_n^{A_n}$$

(considered up to substitution of some variables by other variables).

(ii) *The morphisms between two geometric formulas in disjoint contexts*

$$\varphi(\vec{x}) \longrightarrow \psi(\vec{y})$$

are the formulas

$$\theta(\vec{x}, \vec{y})$$

(considered up to $\mathbb{T}$-provable equivalence)
which are "$\mathbb{T}$-provably functional" in the sense that

$$\left.\begin{array}{l} \theta(\vec{x},\vec{y}) \vdash \varphi(\vec{x}), \\ \theta(\vec{x},\vec{y}) \vdash \psi(\vec{y}), \\ \varphi(\vec{x}) \vdash (\exists\, \vec{y})\theta(\vec{x},\vec{y}), \quad \theta(\vec{x},\vec{y}) \wedge \theta(\vec{x},\vec{y}\,') \vdash \vec{y} = \vec{y}\,' \end{array}\right\} \text{ are } \mathbb{T}\text{-provable.}$$

(iii) *The composite of two morphisms*

$$\varphi(\vec{x}) \xrightarrow{\theta_1(\vec{x},\vec{y})} \psi(\vec{y}) \xrightarrow{\theta_2(\vec{y},\vec{z})} \chi(\vec{z})$$

is defined as the class of the $\mathbb{T}$-provably functional formula

$$(\exists\, \vec{y})(\theta_1(\vec{x},\vec{y}) \wedge \theta_2(\vec{y},\vec{z}))\,.$$

Remarks. –

(i) The category $\mathcal{C}_{\mathbb{T}}^{\mathrm{geo}}$ is essentially small.

(ii) It admits arbitrary finite limits.

Coherent, regular and Cartesian syntactic categories:

Definition. – *Let $\mathbb{T}$ be a geometric theory of signature Σ which is coherent [resp. regular, resp. Cartesian].*
We call coherent syntactic category of $\mathbb{T}$ [resp. regular, resp. Cartesian syntactic category]

$$\mathcal{C}_{\mathbb{T}}^{\mathrm{coh}} \quad \textit{[resp. } \mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}, \textit{ resp. } \mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}\textit{]}$$

the subcategory of $\mathcal{C}_{\mathbb{T}}^{\mathrm{geo}}$ of which

- *the objects are the coherent [resp. regular, resp. $\mathbb{T}$-Cartesian] formulas*

  $$\varphi(\vec{x})$$

  (up to substitution of variables),
- *the morphisms between such formulas in disjoint contexts*

  $$\varphi(\vec{x}) \longrightarrow \psi(\vec{y})$$

  are the equivalence classes of $\mathbb{T}$-provably functional formulas

  $$\theta(\vec{x}, \vec{y})$$

  which are coherent [resp. regular, resp. $\mathbb{T}$-Cartesian].

Remarks. –

(i) The categories $\mathcal{C}_{\mathbb{T}}^{\rm coh}$, $\mathcal{C}_{\mathbb{T}}^{\rm reg}$ or $\mathcal{C}_{\mathbb{T}}^{\rm cart}$ are small.

(ii) As the categories $\mathcal{C}_{\mathbb{T}}^{\rm geo}$, they have arbitrary finite limits.

Proposition. –
If $\mathcal{C}$ is an essentially small category
which has arbitrary finite limits,
a functor in a topos $\mathcal{E}$

$$\mathcal{C} \longrightarrow \mathcal{E}$$

is flat
if and only if it respects finite limits.

Consequence. –
This proposition applies to syntactic categories

$$\mathcal{C}_{\mathbb{T}}^{\rm geo}, \mathcal{C}_{\mathbb{T}}^{\rm coh}, \mathcal{C}_{\mathbb{T}}^{\rm reg} \quad \text{or} \quad \mathcal{C}_{\mathbb{T}}^{\rm cart}\,.$$

Subobjects and provability:

Proposition. – *Let $\mathbb{T}$ be a geometric [resp. coherent, resp. regular, resp. Cartesian] theory of signature Σ.*

Then we have in the syntactic category $\mathcal{C}_{\mathbb{T}} = \mathcal{C}_{\mathbb{T}}^{\mathrm{geo}}, \mathcal{C}_{\mathbb{T}}^{\mathrm{coh}}, \mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ or $\mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$:

(i) *For any context $\vec{x} = (x_1^{A_1} \cdots x_n^{A_n})$ of Σ, the subobjects of*

$$\top(\vec{x})$$

are the geometric [resp. coherent, regular, resp. $\mathbb{T}$-cartesian] formulas

$$\varphi(\vec{x})$$

considered up to $\mathbb{T}$-provable equivalence.

(ii) *Two such formulas considered as subobjects of $\top(\vec{x})$*

$$\varphi(\vec{x}) \quad \text{and} \quad \psi(\vec{x})$$

satisfy the inclusion relation

$$\varphi(\vec{x}) \subseteq \psi(\vec{x})$$

if and only if the implication

$$\varphi \vdash_{\vec{x}} \psi$$

is $\mathbb{T}$-provable.

Syntactic topologies:

Definition. – *Let $\mathbb{T}$ be a theory of signature Σ which is Cartesian [resp. regular, resp. coherent, resp. geometric] . Let*

$$J_{\mathbb{T}} = J_{\mathbb{T}}^{\mathrm{disc}} \quad \textit{[resp. } J_{\mathbb{T}}^{\mathrm{reg}}, \textit{ resp. } J_{\mathbb{T}}^{\mathrm{coh}}, \textit{ resp. } J_{\mathbb{T}}^{\mathrm{geo}} \textit{]}$$

be the "syntactic topology" on $\mathcal{C}_{\mathbb{T}} = \mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$ [resp. $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$, $\mathcal{C}_{\mathbb{T}}^{\mathrm{coh}}$, or $\mathcal{C}_{\mathbb{T}}^{\mathrm{geo}}$] for which a family of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$\theta_i : \varphi_i(\vec{x}_i) \xrightarrow{\theta_i(\vec{x}_i, \vec{x})} \varphi(\vec{x})\,, \quad i \in I\,,$$

is covering if and only if:

- *in the Cartesian case:* *there exists $i \in I$ such that $\mathrm{id} : \varphi(\vec{x}) \to \varphi(\vec{x})$ factors through θ_i,*
- *in the regular case: there exists $i \in I$ such that*
$$\varphi(\vec{x}) \vdash (\exists\, \vec{x}_i)\theta_i(\vec{x}_i, \vec{x}) \quad \textit{is } \mathbb{T} \textit{ -provable,}$$
- *in the coherent case:* *there exists a finite subset $\{i_1, \cdots, i_n\} \subseteq I$ such that*
$$\varphi(\vec{x}) \vdash (\exists\, \vec{x}_{i_1})\theta_{i_1}(\vec{x}_{i_1}, \vec{x}) \vee \cdots \vee (\exists\, \vec{x}_{i_n})\theta_{i_n}(\vec{x}_{i_n}, \vec{x}) \quad \textit{is } \mathbb{T}\textit{-provable ,}$$
- *in the geometric case:*
$$\varphi(\vec{x}) \vdash \bigvee_{i \in I} (\exists\, \vec{x}_i)\theta_i(\vec{x}_i, \vec{x}) \quad \textit{is } \mathbb{T}\textit{-provable.}$$

Models and flat $J_{\mathbb{T}}$-continuous functors:

Theorem. –
Let $\mathbb{T}$ be a theory of signature Σ which is geometric, coherent, regular or Cartesian.
Let $\mathcal{C}_{\mathbb{T}} = \mathcal{C}_{\mathbb{T}}^{\mathrm{geo}}, \mathcal{C}_{\mathbb{T}}^{\mathrm{coh}}, \mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ or $\mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$.
Let $\mathcal{E}$ be a topos.
Let's associate to any model M of $\mathbb{T}$ in $\mathcal{E}$ the functor

$$F_M : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{E}$$

which maps

- *any object $\varphi(\vec{x})$ of $\mathcal{C}_{\mathbb{T}}$ to the object*
  $$M\varphi(\vec{x}) \quad \text{of} \quad \mathcal{E}\,,$$
- *any morphism of $\mathcal{C}_{\mathbb{T}}$*
  $$\varphi(\vec{x}) \xrightarrow{\theta(\vec{x},\vec{y})} \psi(\vec{y})$$
  to the morphism of $\mathcal{E}$
  $$M\varphi(\vec{x}) \longrightarrow M\psi(\vec{y})$$
  whose graph is the subobject
  $$M\theta(\vec{x},\vec{y}) \hookrightarrow M\varphi(\vec{x}) \times M\psi(\vec{y})\,.$$

Then:

(i) *The functor*

$$M \longmapsto M_F$$

defines an equivalence from the category of models

$$\mathbb{T}\text{-mod}(\mathcal{E})$$

to the category of functors

$$F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{E}$$

which are flat and $J_{\mathbb{T}}$-continuous for the syntactic topology $J_{\mathbb{T}}$ on $\mathcal{C}_{\mathbb{T}}$.

(ii) *Its inverse equivalence*

$$F \longmapsto M_F$$

transforms any such flat and $J_{\mathbb{T}}$-continuous functor

$$F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{E}$$

into the model M_F which associates

- *to any sort A the object*
$$M_F A = F(\top(x^A)),$$
- *to any function symbol $f : A_1 \cdots A_n \to B$ the morphism*
$$M_F f = F(y^B = f(x_1^{A_1} \cdots x_n^{A_n})),$$
- *to any relation symbol $R \rightarrowtail A_1 \cdots A_n$ the subobject*
$$M_F R = F(R(x_1^{A_1} \cdots x_n^{A_n})) \hookrightarrow F(\top(x_1^{A_1} \cdots x_n^{A_n})) = M_F A_1 \times \cdots \times M_F A_n .$$

Corollary. – *Let $\mathbb{T}$ be a theory of signature Σ which is geometric, coherent, regular or Cartesian.*
Let its *syntactic category*

$$\mathcal{C}_{\mathbb{T}} = \mathcal{C}_{\mathbb{T}}^{\mathrm{geo}}, \mathcal{C}_{\mathbb{T}}^{\mathrm{coh}}, \mathcal{C}_{\mathbb{T}}^{\mathrm{reg}} \text{ or } \mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$$

be endowed with its *syntactic topology*

$$J_{\mathbb{T}} = J_{\mathbb{T}}^{\mathrm{geo}}, J_{\mathbb{T}}^{\mathrm{coh}}, J_{\mathbb{T}}^{\mathrm{reg}} \text{ or } J_{\mathbb{T}}^{\mathrm{disc}}\,.$$

Let $\mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}$
be the *topos of sheaves on the syntactic site* $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$.
Finally, let $M_{\mathbb{T}}$
be the *model* *of* $\mathbb{T}$ *in* $\mathcal{E}_{\mathbb{T}}$ *which corresponds to the canonical morphism*

$$\ell : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{E}_{\mathbb{T}}\,.$$

Then, for any topos $\mathcal{E}$*, the functor*

$$(f : \mathcal{E} \to \mathcal{E}_{\mathbb{T}}) = (\mathcal{E}_{\mathbb{T}} \xrightarrow{f^*} \mathcal{E}, \mathcal{E} \xrightarrow{f_*} \mathcal{E}_{\mathbb{T}}) \longmapsto f^* M_{\mathbb{T}}$$

defines an equivalence *of the* *category of topos morphisms*

$$[\mathcal{E}, \mathcal{E}_{\mathbb{T}}]_T$$

to the *category of models* *of* $\mathbb{T}$

$$\mathbb{T}\text{-mod}(\mathcal{E})\,.$$

Remarks. –

(i) In particular, we have a canonical equivalence

$$\mathrm{pt}(\mathcal{E}_{\mathbb{T}}) \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathrm{Set})\,.$$

(ii) The topos $\mathcal{E}_{\mathbb{T}}$ is called the "classifying topos" of $\mathbb{T}$.
It is characterized up to canonical equivalence by the equivalences

$$[\mathcal{E}, \mathcal{E}_{\mathbb{T}}] \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathcal{E})\,.$$

(iii) In particular, if $\mathbb{T}$ is coherent, regular or Cartesian, the topos $\mathcal{E}_{\mathbb{T}}$ considered up to canonical equivalence, does not depend on the chosen syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$.

(iv) If $\mathbb{T}$ is a Cartesian theory (especially if $\mathbb{T}$ is a empty theory, is algebraic or Horn), we can take

$$\mathcal{E}_{\mathbb{T}} = \widehat{\mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}}\,.$$

Thus, $\mathbb{T}$ is "of presheaf type" in the sense that its classifying topos is equivalent to the topos of presheaves

$$\widehat{\mathcal{C}}$$

on a small category $\mathcal{C}$.

Summary of what we already know on toposes and their multiple presentations:

- Toposes are by definition categories $\mathcal{E}$ which are equivalent to categories of sheaves on sites $(\mathcal{C}, J)$

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}\,.$$

- Any presentation

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$$

  and any choice of a full and J-dense subcategory

$$\mathcal{C}' \hookrightarrow \mathcal{C}$$

  endowed with the topology J' induced by J
  defines a new equivalence

$$\widehat{\mathcal{C}}'_{J'} \xrightarrow{\sim} \mathcal{E}\,.$$

- Any choice in a topos $\mathcal{E}$
  of a small full and separating subcategory

$$\mathcal{C} \hookrightarrow \mathcal{E}$$

  and of the topology J of $\mathcal{C}$ induced by $\mathcal{E}$ defines an equivalence

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}\,.$$

On the side of theories and their geometric expressions:

- Any "first-order geometric" theory $\mathbb{T}$
  admits a "classifying topos" $\mathcal{E}_{\mathbb{T}}$ endowed with a universal $\mathbb{T}$-model $M_{\mathbb{T}}$,
  characterized by the property that, for any topos $\mathcal{E}$, the functor
  $$\begin{array}{rcl} [\mathcal{E}, \mathcal{E}_{\mathbb{T}}]_T & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{E}) \, , \\ f & \longmapsto & f^* M_{\mathbb{T}} \end{array}$$
  is an equivalence of categories.

- This classifying topos $\mathcal{E}_{\mathbb{T}}$ can be constructed
  as the topos of sheaves on the site $(\mathcal{C}_{\mathbb{T}}^{\mathrm{geo}}, J_{\mathbb{T}}^{\mathrm{geo}})$ consisting in
  $\begin{cases} \text{geometric formulas of the signature } \Sigma \text{ of } \mathbb{T}, \\ \text{and their } \mathbb{T}\text{-provably functional relations.} \end{cases}$

- If $\mathbb{T}$ is coherent or regular, $\mathcal{E}_{\mathbb{T}}$ can also be constructed
  as the topos of sheaves on the site
  $\begin{cases} (\mathcal{C}_{\mathbb{T}}^{\mathrm{coh}}, J_{\mathbb{T}}^{\mathrm{coh}}) \text{ of coherent formulas of } \Sigma, \\ (\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}, J_{\mathbb{T}}^{\mathrm{reg}}) \text{ of regular formulas of } \Sigma. \end{cases}$

- Finally, if $\mathbb{T}$ is Cartesian, $\mathcal{E}_{\mathbb{T}}$ can also be constructed
  as the topos of presheaves on the category
  $\mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$ of $\mathbb{T}$-Cartesian formulas of Σ.

Regarding the expression of sites in terms of theories:

- Any small category $\mathcal{C}$ defines a signature $\Sigma_{\mathcal{C}}$
  - whose sorts are the objects of $\mathcal{C}$,
  - whose function symbols are the morphisms of $\mathcal{C}$,
  - and which has no relation symbol.
- Any topology J of $\mathcal{C}$ defines a geometric theory $\mathbb{T}_{\mathcal{C},J}$ of signature $\Sigma_{\mathcal{C}}$, such that, for any topos $\mathcal{E}$,

$$\mathbb{T}_{\mathcal{C},J}\text{-mod}(\mathcal{E})$$

is the category of functors

$$\mathcal{C} \longrightarrow \mathcal{E}$$

which are flat and J-continuous.

- Thus, the topos of sheaves on the site $(\mathcal{C}, J)$

$$\widehat{\mathcal{C}}_J$$

appears as the classifying topos of the geometric theory $\mathbb{T}_{\mathcal{C},J}$

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}_{\mathcal{C},J}} .$$

Geometric presentations and linguistic descriptions:

- Thus, we already know that any topos $\mathcal{E}$ admits

$$\begin{cases} - \quad \text{an } \underline{\text{infinite diversity of geometric presentations}} \\ \qquad\qquad \widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}\,, \\ - \quad \text{an } \underline{\text{infinite diversity of linguistic descriptions}} \\ \qquad\qquad \mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}\,. \end{cases}$$

- Each geometric presentation of $\mathcal{E}$
$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$ extending a functor $\ell : \mathcal{C} \to \mathcal{E}$
induces natural expressions of its objects E in the form

$$E = \varinjlim_{(X,x)\in\int F} \ell(X) \quad \text{with} \quad F = \mathrm{Hom}(\ell(\bullet), E)\,.$$

- Each linguistic description of $\mathcal{E}$

$$\mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

induces natural descriptions of its points with values in toposes $\mathcal{E}'$

$$[\mathcal{E}', \mathcal{E}]_T \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathcal{E}')\,.$$

The meanings of the passage to toposes: the geometric side

- For Grothendieck,
  the most diverse mathematical situations
  give rise to the natural definition of sites
  whose associated toposes embody

  "the essence of these situations".

- In particular,
  in any geometric situation
  or in any situation that the extraordinarily general notion of site
  allows to see and study geometrically,
  what is really meaningful
  is defined from the topos or the toposes
  associated with this situation.

- For example,
  the cohomological or homotopic invariants
  of topological spaces or manifolds
  are invariants of the toposes
  associated with these spaces or these manifolds.

- Actually,
  the general notion of topos
  was first discovered by Grothendieck
  as

  the most general setting
  in which cohomological invariants are defined.

- Similarly, according to Grothendieck and Artin-Mazur,

  the π_1
  and all homotopic invariants

  are defined in the general framework of toposes.

The meanings of the passage to toposes: the linguistic side

- By definition,
  for any "first-order geometric theory" $\mathbb{T}$,
  its classifying topos $\mathcal{E}_{\mathbb{T}}$
  represents the functor of its models
  in the sense that there exist natural equivalences of categories

$$[\mathcal{E}', \mathcal{E}_{\mathbb{T}}]_T \xrightarrow{\sim} \mathbb{T}\text{-mod}(\mathcal{E}')$$

  for any topos $\mathcal{E}'$.

- This means that the constructions

$$\mathbb{T} \longmapsto (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \longmapsto \mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}$$

  incarnate mathematically
  the passage from syntax to semantics
  in the sense of Tarski.

- Thus, two theories $\mathbb{T}_1$ and $\mathbb{T}_2$ are "Morita-equivalent" in the sense that

$$\mathcal{E}_{\mathbb{T}_1} \cong \mathcal{E}_{\mathbb{T}_2}$$

  if and only if they are "semantically equivalent" in the sense that
  their model functors are equivalent.

- This leads to especially study
  the equivalences of Morita
  between first-order geometric theories.
- Following Olivia Caramello, this also leads to
  study together theories $\mathbb{T}$
  whose classifying topos $\mathcal{E}_{\mathbb{T}}$ is of such or such particular type.
- For example, a theory $\mathbb{T}$
  is said to be "presheaf type"
  if $\mathcal{E}_{\mathbb{T}}$ is equivalent to a topos of presheaves $\widehat{\mathcal{C}}$.

$\longrightarrow$ Presheaf type theories include
algebraic theories, empty theories and Cartesian theories.

$\longrightarrow$ O. Caramello gave several series of
necessary and sufficient criteria
for a theory to be presheaf type.

- Similarly, she characterized
  theories $\mathbb{T}$ which are "Galois"
  in the sense that their topos $\mathcal{E}_{\mathbb{T}}$ is equivalent
  to the topos of continuous actions of a topological group.

The general notion of topos invariant:

- It can be a property (P)
  that can be verified or not by any topos
  and which is respected by all topos equivalences
  $$\mathcal{E}' \xrightarrow{\sim} \mathcal{E}\,.$$
- It can also be a
  covariant or contravariant functor
  from the category of toposes to a category $\mathcal{A}$

$$\begin{array}{ccccc} H & : & \text{topos } \mathcal{E} & \longmapsto & \text{object } H(\mathcal{E}) \text{ of } \mathcal{A}, \\ & & (f:\mathcal{E}' \to \mathcal{E}) & \longmapsto & \left\{ \begin{array}{ll} & f_* : H(\mathcal{E}') \to H(\mathcal{E}) \\ \text{or} & f^* : H(\mathcal{E}) \to H(\mathcal{E}') \end{array} \right\}, \\ & & \| & & \| \\ & & \begin{array}{c}\text{topos}\\ \text{morphism}\end{array} & & \text{morphism of } \mathcal{A} \end{array}$$

and which transforms
any topos equivalence $\mathcal{E}' \xrightarrow{\sim} \mathcal{E}$
into an isomorphism of $\mathcal{A}$.

Caramello's "toposes as bridges" technique: The principles

- Any topos $\mathcal{E}$ incarnates some mathematical content.
- Any presentation of $\mathcal{E}$ by a site $(\mathcal{C}, J)$

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$$

  represents a geometric point of view on this content.
- Any description of $\mathcal{E}$ by a theory $\mathbb{T}$

$$\mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$$

  represents a linguistic expression of this content.
- Varying the presentations of a topos $\mathcal{E}$ by sites or theories

  represents a mathematical incarnation

  of the multiplication of geometric or linguistic points of view

  on the mathematical content embodied by this topos.

- Choosing a relevant question
  about a mathematical content embodied by a topos $\mathcal{E}$
  is realized by considering and studying
  an invariant property (P) or an invariant H of toposes
  in the specific case of the topos $\mathcal{E}$.
- Expressing an invariant property (P) or an invariant H of toposes
  in terms of a geometric presentation by a site $(\mathcal{C}, J)$
  or of a linguistic description by a theory $\mathbb{T}$
  amounts to bring down this property or this abstract invariant
  to concrete data or statements
  directly formulated from $(\mathcal{C}, J)$ or $\mathbb{T}$.
- Bringing together expressions of an abstract property or an invariant
  in terms of different geometric presentations
  or language descriptions
  embodies mathematically the operation
  of confronting diverse points of view.
  This causes correspondences and equivalences to appear,
  most often unexpected,
  between various forms of expression of the same abstract phenomena.

The "toposes as bridges" technique: Its implementation

- Consider invariant properties (P) or invariants H of toposes (or classes of such properties or such invariants), and express them or calculate them in the terms of different types of presentation sites or description theories.

- Use already known processes allowing to display equivalences of toposes,

  and enrich the known processes

  $\left\{\begin{array}{l} \text{— by a systematic study of morphisms between toposes defined by sites or theories,} \\ \text{— by obtaining new criteria for such morphisms to be equivalences.} \end{array}\right.$

- Combine the two to make appear new correspondences or equivalences.

- Conversely, consider
  classical equivalences or correspondences of mathematics,
  and try to lift them to
  equivalences of toposes associated with sites or theories
  from which they would be deduced by expression or calculation
  of some invariants in various presentations.

- In this way, build up and gradually enrich
  a library constituted

  - of known equivalences
    between toposes associated with sites or theories,
  - of classes of invariants of toposes
    which are interesting in certain types of situations,
  - of expressions or of concrete calculations of such invariants
    in various types of geometric presentations
    or of linguistic descriptions.

- When a classical concrete equivalence is lifted to
  an abstract equivalence of toposes,
  one can consider other invariants and calculate them to obtain
  other concrete "sister" equivalences of the starting equivalence.

Remarks on the "toposes as bridges" program:

- This program is structurally math-wide.
- O. Caramello showed with application examples that her "technique of toposes as bridges" generates non-trivial and unexpected results in various fields of mathematics.

$\longrightarrow$ This is enough to justify that her program continues to be expanded and deepened by a new school of mathematics.

Which parts of this program can give rise to algorithms?

$\longrightarrow$ First and foremost, the calculation of some topos invariants classes in terms of certain classes of geometric presentations or language descriptions.
(NB: O. Caramello has proved at least one "meta-theorem" which establishes that certain classes of invariants of toposes are "computable".)

$\longrightarrow$ The constitution of "libraries" of topos equivalences and invariants of toposes?

The particular case of a basic invariant: associating to any topos the ordered set of its subtoposes

We recall:

- An embedding of toposes is a topos morphism $\mathcal{E}' \to \mathcal{E}$

$$j = (\mathcal{E} \xrightarrow{j^*} \mathcal{E}', \mathcal{E}' \xrightarrow{j_*} \mathcal{E})$$

  such that $j_* : \mathcal{E}' \to \mathcal{E}$ is a fully faithful functor.

- Two topos embeddings

$$j_1 : \mathcal{E}_1 \hookrightarrow \mathcal{E} \quad \text{and} \quad j_2 : \mathcal{E}_2 \hookrightarrow \mathcal{E}$$

  are said to be equivalent [resp. ordered in $\mathcal{E}_1 \leq \mathcal{E}_2$]
  if there exists an equivalence [resp. an embedding]

$$e : \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2 \qquad [\text{resp.} \quad \mathcal{E}_1 \hookrightarrow \mathcal{E}_2\ ]$$

  and an isomorphism of topos embeddings

$$j_2 \circ e \cong j_1 \,.$$

- For any small category $\mathcal{C}$, the two applications

$$\begin{cases} \begin{array}{c} J \\ \| \\ \text{topology on } \mathcal{C} \end{array} \longmapsto (\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J, \widehat{\mathcal{C}}_J \xhookrightarrow{j_*} \widehat{\mathcal{C}})\,, \\ \\ (\widehat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E}, \mathcal{E} \xhookrightarrow{j_*} \widehat{\mathcal{C}}) \longmapsto \begin{array}{c} J_{\mathcal{E}} = \text{ topology on } \mathcal{C} \\ \text{for which a sieve} \\ C \hookrightarrow \mathrm{Hom}(\bullet, X) \\ \text{is covering if} \\ j^* C \to j^* \mathrm{Hom}(\bullet, X) \\ \text{is an isomorphism} \end{array} \end{cases}$$

are <u>two inverse bijections</u>

reversing the relations of order, between

- <u>the ordered set of topologies</u> on $\mathcal{C}$,
- the <u>subtoposes</u> of $\widehat{\mathcal{C}}$,
  which therefore <u>form a set</u>.

Corollary. –

(i) *For any topos $\mathcal{E}$,*
its subtoposes form an ordered set.

(ii) *For any presentation of $\mathcal{E}$ by a site $(\mathcal{C}, J)$*

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}\,,$$

the set of its subtoposes identifies,
modulo reversal of the order relation,
with the ordered set of topologies J' on $\mathcal{C}$ such that

$$J' \supseteq J\,.$$

Remark. – In particular, the ordered set

$$\{J' = \text{ topology on } \mathcal{C} \mid J' \supseteq J\}$$

does not depend on the chosen presentation

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$$

of the topos $\mathcal{E}$.

Functoriality of subtoposes:

The map

topos $\mathcal{E} \longmapsto$ ordered set of subtoposes of $\mathcal{E}$

defines a topos invariant which is both covariant and contravariant

$$(\mathcal{E}' \xrightarrow{f} \mathcal{E}) \longmapsto \begin{cases} f_* : (\mathcal{E}'_1 \hookrightarrow \mathcal{E}') & \mapsto & \mathrm{Im}(\mathcal{E}'_1 \hookrightarrow \mathcal{E}' \xrightarrow{f} \mathcal{E}), \\ f^{-1} : (\mathcal{E}_1 \hookrightarrow \mathcal{E}) & \mapsto & (f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}') \end{cases}$$

where :

- The image of a topos morphism
$$f_1 : \mathcal{E}'_1 \longrightarrow \mathcal{E}$$
is the unique subtopos $\mathrm{Im}(f_1) = (\mathcal{E}_1 \hookrightarrow \mathcal{E})$
such that f_1 factors into
$$\mathcal{E}'_1 \xrightarrow{\bar{f}_1} \mathcal{E}_1 \hookrightarrow \mathcal{E}$$
where $\bar{f}_1$ is a surjective morphism of toposes
in the sense that $\bar{f}_1^*$ is a faithful functor.
- The pull-back by $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ of a subtopos $\mathcal{E}_1 \hookrightarrow \mathcal{E}$
is the unique subtopos $f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}'$
such as a topos morphism $g : \mathcal{E}'' \to \mathcal{E}'$
factors through $f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}'$ if and only if
$f \circ g$ factors through $\mathcal{E}_1 \hookrightarrow \mathcal{E}$.

A natural question in the framework of the "toposes as bridges" technique:

Question. –

Let $\mathcal{E}$ be a topos.

Consider a description of this topos

as the classifying topos

of a "first-order geometric" theory $\mathbb{T}$ of signature Σ

$$\mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \, .$$

Is it possible to describe the invariant of $\mathcal{E}$

$$\{\text{ordered set of subtoposes of } \mathcal{E}\}$$

in terms of the theory $\mathbb{T}$?

Subtoposes and quotient theories:

Theorem (O. Caramello). –
Let $\mathbb{T}$ be a geometric theory of signature Σ.
Then there exist two constructive inverse bijections between

$$\begin{cases} - & \textit{the ordered set of subtoposes of } \mathcal{E}_{\mathbb{T}}, \\ - & \textit{the ordered set of quotient theories of } \mathbb{T}, \\ & \textit{modulo equivalence.} \end{cases}$$

Meaning of words:

(i) If $\mathbb{T}$ is a geometric theory with signature Σ,
a quotient theory $\mathbb{T}'$ of $\mathbb{T}$
is a geometric theory with the same signature Σ
such that any axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$.

(ii) Two theories $\mathbb{T}_1$ and $\mathbb{T}_2$ with the same signature Σ
are said to be equivalent
if each one is a quotient of the other.

Remarks. –

- Caramello's thesis (reprinted in her book "Theories, Sites, Toposes") gives two constructive proofs
  and various applications, in particular to questions of provability.
- The empty theory of signature Σ has for classifying topos
  the topos of presheaves $\widehat{\mathcal{C}_\Sigma}$
  on the Cartesian syntactic category $\mathcal{C}_\Sigma = \mathcal{C}_\Sigma^{\mathrm{cart}}$.
  According to the theorem, any theory $\mathbb{T}$ of signature Σ
  admits a classifying topos of the form

$$\mathcal{E}_\mathbb{T} = \widehat{(\mathcal{C}_\Sigma)_{J_\mathbb{T}}}$$

  for some topology $J_\mathbb{T}$ of $\mathcal{C}_\Sigma$.
  This way of building $\mathcal{E}_\mathbb{T}$ had already been discovered
  by Michel Coste and Marie-Françoise Roy
  within the framework of "finitary" theories.

Application to provability problems:

We would like to explore to what extent
the following corollary lends itself to machine computation:

Corollary. – *Let $\mathbb{T}$ be a geometric theory of signature Σ.*
Let $(\mathcal{C}, J)$ be a presentation site for the classifying topos of $\mathbb{T}$:

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \, .$$

Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$
defined by an additional axiom of the form

$$\varphi \vdash_{\vec{x}} \psi \, .$$

Let J' be the unique topology on $\mathcal{C}$ containing J such that

$$\widehat{\mathcal{C}}_{J'} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}'} \, .$$

Then the axiom

$$\varphi \vdash_{\vec{x}} \psi$$

is provable in $\mathbb{T}$ if and only if

$$J' = J \, .$$

Remark. – For this, we look for conditions under which
the topology J' is generated over J by sieves
associated constructively with $\varphi \vdash_{\vec{x}} \psi$.