V. Operations on topologies, generation formula and applications

Let's begin by introducing the presheaf Ω of sieves on objects of an essentially small category:

Definition. – *Let $\mathcal{C}$ be an essentially small category. We denote Ω the presheaf*

$$\begin{array}{ccl} \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}\,, \\ X & \longmapsto & \Omega(X) = \textit{set of sieves of } \mathcal{C} \textit{ on } X, \\ (X' \xrightarrow{x} X) & \longmapsto & \begin{cases} x^* : \Omega(X) & \to & \Omega(X')\,, \\ \qquad\quad C & \mapsto & x^*C = \{U \xrightarrow{u} X' \mid x \circ u \in C\}. \end{cases} \end{array}$$

Remark. – The presheaf Ω is called the "subobject classifier" of $\widehat{\mathcal{C}}$ because, for any presheaf P, the map

$$(P \xrightarrow{\chi} \Omega) \longmapsto (P_\chi : X \mapsto \{p \in P(X) \mid \chi(p) = \text{ maximal sieve on } X\})$$

defines a bijection

$$\mathrm{Hom}(P, \Omega) \xrightarrow{\sim} \{\text{set of subobjects } P' \hookrightarrow P\}$$

whose reverse bijection is

$$(P' \hookrightarrow P) \longmapsto$$

$$\left(\chi : P \to \Omega, \left\{ \begin{matrix} p \in P(X) \text{ seen as a} \\ \text{morphism } \mathrm{Hom}(\bullet, X) \to P \end{matrix} \right\} \longmapsto \text{sieve } \mathrm{Hom}(\bullet, X) \times_P P' \right).$$

Subobjects of the subobject classifier:

We still consider the topos $\widehat{\mathcal{C}}$ of presheaves
on an essentially small category $\mathcal{C}$.
The endomorphisms of its subobject classifier $\quad \Omega \longrightarrow \Omega$
correspond to *subobjects* $\quad D \hookrightarrow \Omega\,.$

Lemma. – *The subobjects* $\quad D \hookrightarrow \Omega$
are the maps $\quad X \longmapsto D(X) =$ *subset of* $\Omega(X)$
which satisfy the axiom of "stability":
For any morphism $X' \xrightarrow{x} X$ *of* $\mathcal{C}$*, the map*

$$x^* : \Omega(X) \to \Omega(X') \text{ sends } D(X) \text{ into } D(X').$$

Remark. – Such a subobject $D \hookrightarrow \Omega$ is a topology on $\mathcal{C}$
if it additionally satisfies both axioms:

- Maximality: the maximal sieve on any object X of $\mathcal{C}$
is an element of the subset $D(X) \subseteq \Omega(X)$.
- Transitivity: the endomorphism which corresponds to D
$\chi : \Omega \longrightarrow \Omega$ is idempotent, i.e. $\chi \circ \chi = \chi$.

The sieve closing operation defined by a topology:

Lemma. – *For any subobject $D \hookrightarrow \Omega$, the corresponding endomorphism*

$$\chi : \Omega \longrightarrow \Omega$$

associates with any sieve C on an object X the sieve on X

$$\overline{C} = \{U \xrightarrow{u} X \mid u^* C \in D(U)\}.$$

Remarks. –

- If $D \hookrightarrow \Omega$ satisfies the axiom of maximality, we always have $C \subseteq \overline{C}$.
- In this case, D is a topology J if and only if

$$\overline{\overline{C}} = \overline{C} \quad \text{for any sieve } C.$$

- If D is a topology J, a sieve C over X is called "J-closed" if $\overline{C} = C$
  i.e. if an arrow $U \xrightarrow{u} X$
  is in C as soon as it is locally in C.
- In this case, for any sieve C, we call J-closure of C the sieve

$$\overline{C} = \{U \xrightarrow{u} X \mid u \text{ is locally in } C\}.$$

Intersections, unions and generated topologies:

The set of subobjects $D \hookrightarrow \Omega$
is endowed with the order relation defined by inclusion.
Any family of subobjects has an intersection and a union.

Lemma. – *For any family of topologies J_i, $i \in I$, on $\mathcal{C}$, seen as subobjects $J_i \hookrightarrow \Omega$, their intersection is still a topology denoted* $\bigwedge_{i \in I} J_i$.

Remark. –
On the other hand, a union of topologies is not in general a topology.

Corollary. –

(i) *For any subobject $D \hookrightarrow \Omega$, there is a smallest topology J_D over $\mathcal{C}$ which contains D. It is called the topology generated by D.*

(ii) *In particular, for any family of topologies J_i, $i \in I$, on $\mathcal{C}$, there is a smallest topology* $\bigvee_{i \in I} J_i$ *which contains all topologies J_i, $i \in I$.*

Intersections and unions of theories:

Let $\mathbb{T}_0$ be a geometric theory which admits $\widehat{\mathcal{C}}$ for classifying topos.
Then the topologies J on $\mathcal{C}$ correspond to the quotient theories of $\mathbb{T}_0$
$\mathbb{T}$ considered up to equivalence.

Reminder. – *The order relation between topologies on* $\mathcal{C}$ *corresponds to the order relation* $\mathbb{T}_1 \leq \mathbb{T}_2$*, defined by requiring that any geometric property which is* $\mathbb{T}_1$*-provable is* $\mathbb{T}_2$*-provable.*

Corollary. –

(i) *The intersection of topologies corresponds to the operation* $(\mathbb{T}_i)_{i\in I} \mapsto \bigwedge_{i\in I} \mathbb{T}_i$ *defined by requiring that a geometric property be provable in* $\bigwedge_{i\in I} \mathbb{T}_i$ *if and only if it is provable in each* $\mathbb{T}_i$, $i \in I$.

(ii) *The join of topologies corresponds to the operation* $(\mathbb{T}_i)_{i\in I} \mapsto \bigvee_{i\in I} \mathbb{T}_i$ *defined by requiring that* $\bigvee_{i\in I} \mathbb{T}_i$ *be the smallest theory in which a property is provable if it is provable in at least one of the theories* $\mathbb{T}_i$, $i \in I$.

The implication operation between topologies:

Proposition. – *For any topology J on $\mathcal{C}$, the intersection functor with J*

$$J \wedge \bullet : K \longmapsto J \wedge K$$

has a right adjoint

$$J' \longmapsto (J \Rightarrow J')\,,$$

characterized by the property that, for any topology K, we have

$$K \leq (J \Rightarrow J')$$

if and only if

$$J \wedge K \leq J'\,.$$

Remark. – In other words, if $\mathbb{T}_0$ is a theory classified by the topos $\widehat{\mathcal{C}}$, the intersection functor with any theory $\mathbb{T}$ quotient of $\mathbb{T}_0$

$$\mathbb{T} \wedge \bullet$$

has a right adjoint

$$\mathbb{T}' \longmapsto (\mathbb{T} \Rightarrow \mathbb{T}')$$

characterized by the property that, for any quotient theory $\mathbb{T}''$ of $\mathbb{T}_0$, we have

$$\mathbb{T}'' \leq (\mathbb{T} \Rightarrow \mathbb{T}')$$

if and only if

$$\mathbb{T} \wedge \mathbb{T}'' \leq \mathbb{T}'\,.$$

Distributivity of intersections and joins:

Corollary. –

(i) *For any topologies J and J_i, $i \in I$, on $\mathcal{C}$, we have*

$$J \wedge \bigvee_{i \in I} J_i = \bigvee_{i \in I} (J \wedge J_i) \, .$$

(ii) *For any topologies J and $J_1, \cdots, J_n$ on $\mathcal{C}$, we have*

$$J \vee (J_1 \wedge \cdots \wedge J_n) = (J \vee J_1) \wedge \cdots \wedge (J \vee J_n) \, .$$

Remark. – These properties carry over to the ordered set of quotient theories of a theory $\mathbb{T}_0$ classified by the topos $\widehat{\mathcal{C}}$.

Proof. –

(i) The functor $J \wedge \bullet$ admits a right adjoint so it respects colimits $\bigvee_{i \in I}$.

(ii) It suffices to consider the case $n = 2$. Then

$$\begin{aligned} (J \vee J_1) \wedge (J \vee J_2) &= (J \wedge J) \vee (J_1 \wedge J) \vee (J \wedge J_2) \vee (J_1 \wedge J_2) \\ &= J \vee (J_1 \wedge J_2) \, . \end{aligned}$$

Beginning of the construction of the adjoint: a necessary condition

Lemma. – *Let J, J' and K be three topologies on $\mathcal{C}$ such that*

$$J \wedge K \leq J' .$$

Consider

$$\begin{cases} X \text{ an object of } \mathcal{C}, \\ C \text{ a sieve on } X \text{ which belongs to } K(X), \\ (U \xrightarrow{u} X) \text{ a morphism,} \\ C' \text{ a sieve on } U \text{ element of } J(U),\ J'\text{-closed and such that} \\ \qquad u^* C \subseteq C' . \end{cases}$$

Then the sieve C' on U is maximal.

Proof. –
The sieve C' over U is J-covering.
It is also K-covering since it contains $u^*(C)$.
As $J \wedge K \leq J'$, this implies that C' is J'-covering.
As it is J'-closed, it contains $U \xrightarrow{\text{id}} U$, that is to say is maximal.

Verification that the necessary condition defines a topology:

Proposition. – *Let J and J' be two topologies on $\mathcal{C}$.*
For any object X of $\mathcal{C}$, let $D(X) \subseteq \Omega(X)$
be the set of sieves C on X such that,
for any morphism $U \xrightarrow{u} X$ and any sieve C' on U, the conditions

$$\begin{cases} C' \text{ is } J\text{-covering}, \\ C' \text{ is } J'\text{-closed}, \\ C' \text{ contains } u^*(C) \end{cases}$$

imply that C' is the maximal sieve.
Then D is a topology on $\mathcal{C}$.

Proof. –

- As the definition makes appear a quantification on all morphisms $U \xrightarrow{u} X$ of target X, D satisfies the axiom of stability, i.e. is a subobject $D \hookrightarrow \Omega$.
- D satisfies the axiom of maximality because, if C is the maximal sieve on X, its pull-back $u^*(C)$ by any $U \xrightarrow{u} X$ is the maximal sieve of U.

Verification of the transitivity axiom:

- We consider a sieve C on an object X of $\mathcal{C}$ and a sieve C' on X, element of $D(X)$, such that for any element $U' \xrightarrow{u'} X$ of C', we have $\qquad u'^*(C) \in D(U')$.
- It must be shown that this implies $\qquad C \in D(X)$.
- Let us therefore consider a morphism $U \xrightarrow{u} X$ and a sieve S on U which
$$\begin{cases} \text{is } J\text{-covering,} \\ \text{is } J'\text{-closed,} \\ \text{contains } u^*(C). \end{cases}$$
It must be shown that S is necessarily the maximal sieve.
- For any element $U' \xrightarrow{u'} U$ of $u^*(C')$, we have
$$(u \circ u')^*(C) \in D(U') \quad \text{since} \quad u \circ u' \in C'.$$
We also have that $\begin{cases} u'^*(S) \text{ is } J\text{-covering and } J'\text{-closed,} \\ \text{it contains } (u \circ u')^*(C), \end{cases}$
so $u'^*(S)$ is the maximal sieve on U' which means that $\qquad u' \in S$.
- So S is J-covering and J'-closed and it contains the sieve $u^*(C')$.
- This implies as intended that S is the maximal sieve over U.

Joyal's "left" and "right" operators:

Definition. – *In the context of a topos of presheaves* $\widehat{\mathcal{C}}$*,*
we associate with any subobject $D \hookrightarrow \Omega$
the subobjects $D^\ell \hookrightarrow \Omega$ *and* $D^r \hookrightarrow \Omega$ *defined by the formulas:*

(i) *For any object* X *of* $\mathcal{C}$*,*

$$D^\ell(X) = \left\{ C = \text{sieve on } X \,\middle|\, \begin{array}{l} \text{for any } U \xrightarrow{u} X \text{ and any } C' \in D(U) \\ \text{such that } u^*(C) \subseteq C', \\ C' \text{ is necessarily the maximal sieve} \end{array} \right\}.$$

(ii) *For any object* X *of* $\mathcal{C}$*,*

$$D^r(X) = \left\{ C = \text{sieve on } X \,\middle|\, \begin{array}{l} \text{for any } U \xrightarrow{u} X \text{ and any } C' \in D(U) \\ \text{such that } C' \subseteq u^*(C), \\ \text{we necessarily have } u \in C \end{array} \right\}.$$

Remarks. –

(i) D^ℓ and D^r satisfy the "stability" axiom
because $D^\ell(X)$ and $D^r(X)$ are defined by a condition
which includes quantization over all $U \xrightarrow{u} X$.

(ii) If $D_1 \leq D_2$, we necessarily have $D_2^\ell \leq D_1^\ell$ and $D_2^r \leq D_1^r$.

Note on references:

- Joyal didn't publish himself
  his theory of operators $\quad D \mapsto D^\ell \quad$ and $\quad D \mapsto D^r$.
- It is exposed in P. Johnstone's book
  "Sketches of an Elephant: a topos theory compendium".
- The definitions of these operators and their study are formulated by Johnstone within the framework and in the language of "elementary toposes", considered as a type of algebraic structure.
- This implies that, in this book, Joyal operators are defined by algebraic type formulas, which are written

$$\begin{aligned} D^\ell &= \forall_{\pi_1}(\pi_2^*(D) \Rightarrow \theta) \\ D^r &= \forall_{\pi_2}(\pi_1^*(D) \Rightarrow \theta) \end{aligned}$$

  where π_1, π_2 are the two projections $\Omega \times \Omega \rightrightarrows \Omega$,

  and $\theta \hookrightarrow \Omega \times \Omega$ is the equalizer of $\Omega \times \Omega \underset{\Rightarrow}{\overset{\pi_2}{\rightrightarrows}} \Omega$.
- The explanation of these definitions in terms of sieves,
  in the context of presheaf toposes,
  is given in chapter IV of the book of O. Caramello
  "Theories, Sites, Toposes".

A formula for calculating generated topologies:

Theorem. – *For any subobject* $D \hookrightarrow \Omega$ *i.e. any application*

$$\begin{array}{ccc} X & \longmapsto & D(X) \\ \| & & \| \\ \text{object of } \mathcal{C} & & \text{family of sieves on } X \end{array}$$

which satisfies the stability axiom,
the Grothendieck topology J_D *of* $\mathcal{C}$ *generated by D*
is given by the formula

$$J_D = (D^r)^\ell \, .$$

Remarks:

- The Joyal theory presented by Johnstone includes a characterization of the operator $D \longmapsto (D^r)^\ell$ in the framework of "elementary toposes".
- A proof of the equality $J_D = (D^r)^\ell$ within the framework of presheaf toposes is given in chapter IV of Caramello's book.

The main ingredients of the proof:

- We already know that the operators

$$D \longmapsto D^\ell \quad \text{and} \quad D \longmapsto D^r$$

reverse order relation,
therefore we have for any $D_1 \leq D_2$ the order relation

$$(D_1^r)^\ell \leq (D_2^r)^\ell \, .$$

- To prove the formula, it suffices to prove the three following properties:

(1) For any subobject $D \hookrightarrow \Omega$, we have the order relation $D \leq (D^r)^\ell$.

(2) For any subobject $D \hookrightarrow \Omega$, the subobject $D^\ell \hookrightarrow \Omega$ is a topology.

(3) For any topology $J \hookrightarrow \Omega$, we have the fixed point property $(J^r)^\ell = J$.

Explicitation of the composite operator of Joyal:

Combining the two defining formulas of the operators

$$D \longmapsto D^\ell \quad \text{and} \quad D \longmapsto D^r\,,$$

we obtain :

Lemma. –

For any subobject $D \hookrightarrow \Omega$
and any object X *of* $\mathcal{C}$,
$(D^r)^\ell(X)$ *is the set of sieves* C *over* X
such that, for any morphism $u : U \to X$,
a sieve C' *over* U *is the maximal sieve if*

- *it contains* $u^*(C)$,
- *for any morphism* $u' : U' \to U$
  we have $u' \in C'$
  if $u'^*(C')$ *contains at least one sieve on* U'
  which is an element of $D(U')$.

Verification of inequality (1):

Lemma. – *For any subobject $D \hookrightarrow \Omega$, we have the order relation*

$$D \leq (D^r)^\ell .$$

Proof. –

- We have to show that,
  for any object X of $\mathcal{C}$, any sieve $C \in D(X)$,
  and any morphism $u : U \to X$,
  a sieve C' on U is the maximal sieve if

  $$\begin{cases} - & \text{it contains } u^*(C) \\ - & \text{for any morphism } u' : U' \to U \\ & \text{we have } u' \in C' \\ & \text{if } u'^*(C') \text{ contains at least one sieve on } U' \\ & \text{which is an element of } D(U'). \end{cases}$$

- Indeed, for any sieve C' on U which satisfies these two conditions,
  we have for any morphism $u' : U' \to U$

  $$u'^*(C') \geq u'^* \circ u^*(C) \in D(U')$$

  hence $u' \in C'$.

- As intended, this means that C' is the maximal sieve.

Verification of property (2): each D^ℓ is a topology

Proposition. – *For any subobject $D \hookrightarrow \Omega$, the subobject*

$$D^\ell \hookrightarrow \Omega$$

is a topology.

Proof. –

- For any object X of $\mathcal{C}$, $D^\ell(X)$ is by definition the set of sieves C of X such that

$$\left\{\begin{array}{l}\text{for any } U \xrightarrow{u} X \text{ and any } C' \in D(U) \\ \text{verifying } u^*(C) \leq C', \\ C' \text{ is necessarily the maximal sieve.}\end{array}\right.$$

- D^ℓ satisfies the axiom of "maximality":
Indeed, if C is the maximal sieve over X
the order relation $u^*(C) \leq C'$ for $U \xrightarrow{u} X$
implies that C' is the maximal sieve over U.

Verification that D^ℓ satisfies the transitivity axiom:

- Consider a sieve C on an object X of $\mathcal{C}$ and a sieve $C' \in D^\ell(X)$ such that, for any $(U \xrightarrow{u} X) \in C'$, we have
$$u^*(C) \in D^\ell(U)\,.$$
- Prove that $C \in D^\ell(X)$
that is, for any morphism $u : U \to X$
and any sieve $S \in D(U)$, the order relation
$$u^*(C) \leq S$$
implies that S is the maximal sieve of U.
- We have $u^*(C') \in D^\ell(U)$.
For any morphism $(U' \xrightarrow{u'} U) \in u^*(C')$, we have
$u \circ u' \in C'$ with $u'^*(S) \in D(U')$ and $(u \circ u')^*(C) \leq u'^*(S)$.
- We deduce that $u'^*(S)$ is the maximal sieve of U', i.e. $u' \in S$.
- In other words, we have $u^*(C') \leq S$.
- As we also have by hypothesis $S \in D(U)$,
we conclude as desired that S is the maximal sieve on U.

Verification of property (3) of fixity of topologies:

Proposition. – *For any Grothendieck topology J on $\mathcal{C}$ seen as a subobject $J \hookrightarrow \Omega$, we have the fixed point property*

$$J = (J^r)^\ell \,.$$

Proof. –

- We already know that $J \leq (J^r)^\ell$.
- We are reduced to showing that for any object X of $\mathcal{C}$ and any sieve $C \in (J^r)^\ell(X)$, we necessarily have $C \in J(X)$.
- The hypothesis $C \in (J^r)^\ell(X)$ means that, for any morphism $u : U \to X$, a sieve C' on U is the maximal sieve if

$$\begin{cases} \bullet \quad \text{it contains } u^*(C), \\ \bullet \quad \text{it contains any morphism } u' : U' \to U \\ \qquad \text{such that the sieve } u'^*(C') \text{ on } U' \text{ is } J\text{-covering.} \end{cases}$$

End of the proof:

- In other words,
  the hypothesis $C \in (J^r)^\ell(X)$
  means that, for any morphism $u : U \to X$,
  a sieve C' over U is the maximal sieve if

  $$\begin{cases} \bullet \quad \text{it contains } u^*(C), \\ \bullet \quad \text{it is } J\text{-closed.} \end{cases}$$

- This also means that, for any $u : U \to X$,
  the sieve

  $$u^*(C) \quad \text{on} \quad U$$

  is J-covering.
- So we have as announced

  $$C \in J(X)\,.$$

- This completes the proof of the proposition,
  therefore also of the theorem.

The case of joins of topologies:

- We consider a family of topologies

$$J_i\,, \quad i \in I\,,$$

  on a small category $\mathcal{C}$.

- Posing for any object X of $\mathcal{C}$

$$D(X) = \bigcup_{i \in I} J_i(X) \subseteq \Omega(X)\,,$$

  we define a subobject

$$D \hookrightarrow \Omega$$

  which can be denoted simply

$$D = \bigcup_{i \in I} J_i\,.$$

- We know from the previous theorem that

$$\bigvee_{i \in I} J_i = (D^r)^\ell\,.$$

- We are going to make explicit D^r and then $(D^r)^\ell$ if

$$D = \bigcup_{i \in I} J_i\,.$$

Closed sieves with respect to a family of topologies:

Lemma. – *Suppose* $D = \bigcup_{i \in I} J_i$
for topologies J_i *on* $\mathcal{C}$.
Then, for any object X *of* $\mathcal{C}$,

$$D^r(X)$$

is the set of sieves C *on* X
which are closed with respect to each topology J_i, $i \in I$.

Proof. –

- By definition,
  $D^r(X)$ is the set of sieves C over X
  such that, for any morphism $U \xrightarrow{u} X$, we have
  $$u \in C$$
  if there exists $i \in I$ and $C' \in J_i(U)$ verifying
  $$C' \subseteq u^* C\,,$$
  i.e. $u \circ u' \in C, \forall\, (U' \xrightarrow{u'} U) \in C'$.
- This amounts to requiring that C is J_i-closed, for any $i \in I$.

Closing a sieve with respect to a family of topologies:

Proposition. – *Consider a family of topologies J_i, $i \in I$, on $\mathcal{C}$. Then:*

(i) *For any sieve C on an object X of $\mathcal{C}$ there exists a smallest sieve $\overline{C}$, called the closure of C relative to all J_i, which*

$$\begin{cases} \bullet & \text{contains } C, \\ \bullet & \text{is closed relative to each } J_i,\ i \in I. \end{cases}$$

(ii) *For any morphism $U \xrightarrow{u} X$ of $\mathcal{C}$ and any sieve C on X, we have*

$$\overline{u^*(C)} = u^*(\overline{C})\,.$$

Proof. –

(i) Any intersection of J_i-closed sieves is J_i-closed.

(ii) The sieve $u^*(\overline{C})$ contains $u^*(C)$ and it is closed relatively to each J_i, $i \in I$. So we have

$$\overline{u^*(C)} \subseteq u^*(\overline{C})\,.$$

The reverse inclusion results from the following description:

Description of the closure of a sieve:

Lemma. – *Consider a family of topologies J_i, $i \in I$, on $\mathcal{C}$.*
Consider a sieve C on an object X of $\mathcal{C}$.
Then the closure of C relative to all J_i, $i \in I$,

$$\overline{C}$$

consists of the morphisms

$$U \xrightarrow{u} X$$

such that there exists a multicomposite family

$$U_k \xrightarrow{u_k} U_{k-1} \xrightarrow{u_{k-1}} \cdots \longrightarrow U_1 \xrightarrow{u_1} U_0 = U$$

verifying the following properties:

- *For any subscript ℓ, $1 \leq \ell < k$, and any partial composite*
  $$U_\ell \xrightarrow{u_\ell} U_{\ell-1} \xrightarrow{u_{\ell-1}} \cdots \longrightarrow U_1 \xrightarrow{u_1} U_0 \,,$$
  the family of the morphisms
  $$U_{\ell+1} \xrightarrow{u_{\ell+1}} U_\ell$$
  is J_i-covering for at least one topology J_i, $i \in I$.
- *All composites*
  $$U_k \xrightarrow{u_k} U_{k-1} \xrightarrow{u_{k-1}} \cdots \longrightarrow U_1 \xrightarrow{u_1} U_0 = U \xrightarrow{u} X$$
  are elements of C.

Description of a union of topologies:

Corollary. – *Consider a family of topologies J_i, $i \in I$, on $\mathcal{C}$. Let*

$$C \longmapsto \overline{C}$$

be the sieve closing operation with respect to all J_i, $i \in I$.
Then, for any object X of $\mathcal{C}$,

$$\left(\bigvee_{i \in I} J_i\right)(X)$$

is the set of sieves C over X such that
$\overline{C}$ is the maximal sieve over X.

Proof. –

- Denoting $D = \bigcup_{i \in I} J_i$, it follows from the theorem that

$$\left(\bigvee_{i \in I} J_i\right)(X) = (D^r)^\ell(X)$$

  is the set of sieves C on X
  such that, for any morphism $U \xrightarrow{u} X$,
  $\overline{u^*(C)}$ is the maximal sieve of U.
- The conclusion follows from the formula $\overline{u^*(C)} = u^*(\overline{C})$.

The example of product categories:

- Consider small categories $\mathcal{C}_1, \cdots, \mathcal{C}_k$ endowed with topologies $J_1, \cdots, J_k$.
- We have the product category $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ whose objects are the families of objects of $\mathcal{C}_1, \cdots, \mathcal{C}_k$

  $$(X_1, \cdots, X_k)$$

  and whose morphisms

  $$(U_1, \cdots, U_k) \longrightarrow (X_1, \cdots, X_k)$$

  are the families of morphisms of $\mathcal{C}_1, \cdots, \mathcal{C}_k$

  $$(U_1 \xrightarrow{u_1} X_1, \cdots, U_k \xrightarrow{u_k} X_k).$$

- For any i, $1 \leq i \leq k$, we can endow $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ with the topology still denoted J_i for which a sieve on an object

  $$(X_1, \cdots, X_k)$$

  is covering if it contains a family of the form

  $$(\mathrm{id}_{X_1}, \cdots, \mathrm{id}_{X_{i-1}}, u_i, \mathrm{id}_{X_{i+1}}, \cdots, \mathrm{id}_{X_k})$$

  where the $U_i \xrightarrow{u_i} X_i$ form a J_i-covering family of X_i.

The notion of product topology:

We therefore consider small categories

$$\mathcal{C}_1, \cdots, \mathcal{C}_k$$

endowed with topologies

$$J_1, \cdots, J_k$$

and the product category

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$$

endowed with the induced topologies

$$J_1, \cdots, J_k \, .$$

Definition. – *With these notations, the topology*

$$J_1 \vee \cdots \vee J_k$$

on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ can be called

the "product topology" of the J_i, $1 \leq i \leq k$,

and denoted

$$J_1 \times \cdots \times J_k \, .$$

Explicitation of the product topology:

We still consider small categories $\mathcal{C}_1, \cdots, \mathcal{C}_k$
endowed with topologies $J_1, \cdots, J_k$
and the induced topologies $J_1, \cdots, J_k$ on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$.

Corollary. – *Let*

$$C \longmapsto \overline{C}$$

be the operator that associates with any sieve

C on an object $(X_1, \cdots, X_k)$ of $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$

the smallest sieve

$\overline{C}$ containing C

which is closed relative to topologies $J_1, \cdots, J_k$.
Then a sieve

C on $(X_1, \cdots, X_k)$

is covering for the product topology $J_1 \times \cdots \times J_k$ if and only if

the sieve $\overline{C}$

is maximal.

Relation with products of topological spaces:

- Consider the case where
$$\mathcal{C}_1 = O(X_1), \cdots, \mathcal{C}_k = O(X_k)$$
are the categories of non-empty open subsets of topological spaces $X_1, \cdots, X_k$.
- Then
$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$$
is the full subcategory of the category
$$O(X_1 \times \cdots \times X_k) \quad \text{of open subsets of} \quad X_1 \times \cdots \times X_k$$
consisting of objects which are products of non-empty open subsets.
- By construction of the topology of the product space
$$X_1 \times \cdots \times X_k$$
the full subcategory
$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_k \hookrightarrow O(X_1 \times \cdots \times X_k)$$
is dense.

Corollary. – *The topos of sheaves on the product space $X_1 \times \cdots \times X_k$ identifies with the topos of sheaves on the category*
$$O(X_1) \times \cdots \times O(X_k)$$
endowed with the topology induced by that of $O(X_1 \times \cdots \times X_k)$.

Product topological spaces and product topologies:

- We consider topological spaces $X_1, \cdots, X_k$
  and their categories of non-empty open subsets $O(X_1), \cdots, O(X_k)$
  with their canonical topologies $J_1, \cdots, J_k$.
- A sieve on an object

  $$U_1 \times \cdots \times U_k \quad \text{of} \quad O(X_1) \times \cdots \times O(X_k) \hookrightarrow O(X_1 \times \cdots \times X_k)$$

  is a subset $C \subseteq O(U_1) \times \cdots \times O(U_k)$
  which contains any element of $O(U_1) \times \cdots \times O(U_k)$
  smaller than an element of C.
- Such a sieve C is J_i-closed if it contains any element

  $$U_1' \times \cdots \times U_k'$$

  such that U_i' is covered by open subsets U_i''
  which verify the property that the

  $$U_1' \times \cdots \times U_{i-1}' \times U_i'' \times U_{i+1}' \times \cdots \times U_k' \qquad \text{are elements of } C.$$

- We denote $C \longmapsto \overline{C}$
  the operator that associates with any sieve C on an object $U_1 \times \cdots \times U_k$
  the smallest sieve containing C which is J_i-closed for all i, $1 \leq i \leq k$.

Comparison of the product topology and the topology of the product:

We still consider topological spaces $X_1, \cdots, X_k$
and the canonical topologies $J_1, \cdots, J_k$
on the categories of non-empty open subsets $O(X_1), \cdots, O(X_k)$.

Proposition. –

(i) *The product topology $J_1 \times \cdots \times J_k$ on*

$$O(X_1) \times \cdots \times O(X_k) \hookrightarrow O(X_1 \times \cdots \times X_k)$$

is contained in the topology induced by that of the product topological space $X_1 \times \cdots \times X_k$.

(ii) *They are equal if $k-1$ of the k spaces $X_1, \cdots, X_k$ are locally compact.*

Remark. – We recall that a topological space X
is "locally compact" if
any open subset U of X
can be written as a union of open subsets $V \subseteq U$
whose closures $\overline{V} \subseteq U$ are compact spaces.

Application to a "pointless" characterization of the topology on a product space:

Corollary. – *Consider topological spaces* $X_1, \cdots, X_k$ *which are "locally compact", except maybe one of them. Let J be the topology on the product of categories of non-empty open subsets*

$$O(X_1) \times \cdots \times O(X_k) \hookrightarrow O(X_1 \times \cdots \times X_k)$$

which is induced by the canonical topology of $X_1 \times \cdots \times X_k$*. Then a sieve*

C on an object $U_1 \times \cdots \times U_k$ *of* $O(X_1) \times \cdots \times O(X_k)$

is covering if and only if

$$\overline{C} = \left\{ \begin{array}{c} \text{smallest sieve containing } C \\ \text{which is closed relatively to} \\ \text{the canonical topology } J_i \text{ of each factor } O(X_i) \end{array} \right\}$$

is the maximal sieve.

Proof of the identity of the two topologies:

(i) It is obvious that the topology $J_1 \times \cdots \times J_k$ on $O(X_1) \times \cdots \times O(X_k)$ is contained in that induced by the embedding

$$O(X_1) \times \cdots \times O(X_k) \hookrightarrow O(X_1 \times \cdots \times X_k) .$$

(ii)

- For the reverse inclusion, it is enough to consider the case of a product of two spaces X and Y such that X is locally compact.
- It suffices to show that if C is a sieve of $X \times Y$ which is covering in the ordinary sense, then $\overline{C}$ is the maximal sieve.
- Let U be an open subset of X such that $\overline{U}$ is compact and y be an element of Y.
  For any $x \in \overline{U}$ there exists in C an object $U_x \times V_x$ with $x \in U_x$ and $y \in V_x$.
  The compact space $\overline{U}$ is covered by the U_x hence by a finite family $U_{x_1}, \cdots, U_{x_n}$.
  Posing $V_y = V_{x_1} \cap \cdots \cap V_{x_n}$, the sieve C contains the $U_{x_i} \times V_y$, so the sieve $\overline{C}$ contains $U \times V_y$.
- So $\overline{C}$ contains $U \times Y$.
- We conclude that $\overline{C}$ contains $X \times Y$ since X is a union of open subsets U such that $\overline{U}$ is compact.

A deduction theorem in geometric logic:

Let's give the following application
of the generated topology calculation formula:

Theorem (Caramello). –
Let $\mathbb{T}$ be a geometric theory of signature Σ.
Let φ and ψ be two geometric formulas without free variable in the signature Σ.
Suppose that the implication with no free variable

$$\top \vdash \psi$$

is provable in the quotient theory of $\mathbb{T}$ defined by adding the axiom

$$\top \vdash \varphi \,.$$

Then the implication

$$\varphi \vdash \psi$$

is provable in the theory $\mathbb{T}$.

Remark. – The converse is obvious.

Geometric translation of the theorem:

- We consider

  $\mathcal{C} = \mathcal{C}_{\mathbb{T}} =$ geometric syntactic category of $\mathbb{T}$,
  $J = J_{\mathbb{T}} =$ syntactic topology of $\mathcal{C}_{\mathbb{T}}$,
  $\mathcal{E} = \mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} =$ classifying topos of $\mathbb{T}$,

  equipped with the canonical functor

  $$\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J = \mathcal{E}$$

  which is fully faithful.

- The formula $\top$ defines the terminal object 1 of $\mathcal{C}$ and the formulas without free variable φ and ψ define two subobjects

  $$m \hookrightarrow 1 \quad \text{and} \quad n \hookrightarrow 1\,.$$

- The implication

  $$\varphi \vdash \psi$$

  is provable in $\mathbb{T}$
  if and only if the monomorphism

  $$m \wedge n \hookrightarrow m$$

  is J-covering.

Topological form of the deduction theorem:

- Consider an essentially small Cartesian category $\mathcal{C}$ and its terminal object 1.
- For a subobject $m \hookrightarrow 1$, denote J_m the topology of $\mathcal{C}$ that it generates: a sieve C on an object X is J_m-covering if and only if it contains the monomorphism

$$m \times_1 X \hookrightarrow X\,.$$

- We are reduced to proving:

Theorem. – *Under the above conditions, consider also a subobject $n \hookrightarrow 1$, and a topology J of $\mathcal{C}$.*

Then the monomorphism

$$m \wedge n \hookrightarrow m$$

is J-covering if (and only if) the monomorphism

$$n \hookrightarrow 1$$

is covering for the topology generated by J and J_m.

Implication operators between subobjects in toposes:

In order to prove the previous topological theorem,
we need the implication operators in toposes:

Proposition. – *Let $\mathcal{E}$ be a topos.*

(i) *For any subobjects S_1 and S_2 of an object E of $\mathcal{E}$ there exists a unique subobject*

$$(S_1 \Rightarrow S_2) \hookrightarrow E$$

characterized by the property that, for any subobject $S \hookrightarrow E$, we have

$$S \leq (S_1 \Rightarrow S_2)$$

if and only if

$$S \wedge S_1 \leq S_2 \, .$$

(ii) *For any morphism $E' \xrightarrow{e} E$ of $\mathcal{E}$ and for any subobjects S_1 and S_2 of E, we have*

$$e^{-1}(S_1 \Rightarrow S_2) = (e^{-1} S_1 \Rightarrow e^{-1} S_2)$$

denoting e^{-1} the pull-back functor

$$(S \hookrightarrow E) \longmapsto (S \times_E E' \hookrightarrow E') \, .$$

Proof of the topological theorem:

- Assume that the existence of implication operators $\Rightarrow$ is known as well as their compatibility with pull-backs.
- Let C be the sieve on the terminal object 1 of $\mathcal{C}$ consisting in morphisms $X \xrightarrow{p} 1$ such that $\ell(p)$ factors through the subsheaf

$$(\ell(m) \Rightarrow \ell(n)) \hookrightarrow \ell(1)\,.$$

- We are going to prove the following properties of the sieve C:

$$\begin{cases} (1) & \text{The monomorphism } n \hookrightarrow 1 \text{ is an element of } C. \\ (2) & \text{The sieve } C \text{ is } J\text{-closed.} \\ (3) & \text{The sieve } C \text{ is } J_m\text{-closed.} \end{cases}$$

- These properties and the generated topology calculation formula imply that, if $n \hookrightarrow 1$ is covering for the topology $J \vee J_m$, then C is the maximal sieve of the object 1 of $\mathcal{C}$.
- In other words, we have

$$\ell(m) \leq \ell(n)$$

which means as wanted that

$$m \wedge n \hookrightarrow m \qquad \text{is } J\text{-covering.}$$

Verification of the properties of the sieve of implications:

The sieve C on the terminal object 1 of $\mathcal{C}$ defined by the subsheaf

$$(\ell(m) \Rightarrow \ell(n)) \hookrightarrow \ell(1)$$

has the following properties:

(1) It contains the monomorphism $n \hookrightarrow 1$.
Indeed, we have the inclusion

$$\ell(m) \wedge \ell(n) \leq \ell(n)\,.$$

(2) It is J-closed.
This follows from the fact that it is defined by a sub-presheaf of $\ell(1)$ which is a sheaf for the topology J.

(3) It is J_m-closed.
Indeed, for any morphism $X \xrightarrow{p} 1$
such that $m \times_1 X \hookrightarrow X \xrightarrow{p} 1$ is in C,
$\ell(m \times_1 X) \hookrightarrow \ell(X)$ factorizes through

$$(\ell(m \times_1 X) \Rightarrow \ell(n \times_1 X)) \hookrightarrow \ell(X)$$

which means $\ell(m \times_1 X) \leq \ell(n \times_1 X)$
i.e. $p^*(\ell(m) \Rightarrow \ell(n)) = \ell(X)$ and $(X \xrightarrow{p} 1) \in C$.

Back to implication operators:

- The proof of the deduction theorem
  and its topological variant which implies it
  will be complete if we prove that:

  $$\begin{cases} \text{(i)} & \text{For any object } E \text{ of a topos } \mathcal{E} \text{ there exists an implication operator } \Rightarrow \\ & \text{between subobjects of } E. \\ \text{(ii)} & \text{This operator is respected by the functor} \\ & \text{of pull-backs of subobjects} \\ & \text{defined by a morphism } e : E' \to E \text{ of } \mathcal{E}. \end{cases}$$

- For (i), the intersection functor with a subobject $S_1 \hookrightarrow E$

  $$S \longmapsto S \wedge S_1$$

  admits a right adjoint

  $$S_2 \longmapsto (S_1 \Rightarrow S_2)$$

  because it respects colimits.
  Indeed, for any subobject $S_2 \hookrightarrow E$
  and if we consider the family of subobjects S' of E such that

  $$S' \wedge S_1 \leq S_2 \,,$$

  their union $(S_1 \Rightarrow S_2)$ still satisfies the inequality

  $$(S_1 \Rightarrow S_2) \wedge S_1 \leq S_2 \,.$$

Implication operators and pull-backs:

- Considering a morphism of a topos $\mathcal{E}$ $\quad e : E' \longrightarrow E$ and two subobjects S_1, S_2 of E, we must verify that

$$e^{-1}(S_1 \Rightarrow S_2) = (e^{-1} S_1 \Rightarrow e^{-1} S_2)\,.$$

- We can assume that $\quad \mathcal{E} = \widehat{\mathcal{C}}_J$ is the topos of sheaves on a site $(\mathcal{C}, J)$, therefore is written as a subtopos of a topos of presheaves

$$\mathcal{E} = \widehat{\mathcal{C}}_J \xhookrightarrow{(j^*, j_*)} \widehat{\mathcal{C}}\,.$$

- As j^* and j_* respect finite limits and $j^* \circ j_*$ identifies with the functor id of $\widehat{\mathcal{C}}_J$, we have for all subobjects S_1, S_2 of any object E of $\mathcal{E} = \widehat{\mathcal{C}}_J$

$$(S_1 \Rightarrow S_2) = j^*(j_* S_1 \Rightarrow j_* S_2)\,.$$

- This reduces the verification to the case where

$\mathcal{E} = \widehat{\mathcal{C}}$ is the topos of presheaves on $\mathcal{C}$.

- If $\mathcal{E} = \widehat{\mathcal{C}}$, we have the formula for any object X of $\mathcal{C}$

$$(S_1 \Rightarrow S_2)(X) = \left\{ \begin{matrix} x \in E(X) \mid \forall\,(U \xrightarrow{u} X) = \text{ morphism of } \mathcal{C}, \\ E(u)(x) \in S_2(U) \cup (E(U) - S_1(U)) \end{matrix} \right\}.$$