SPECIAL VALUES OF L-FUNCTIONS AND THE REFINED GAN-GROSS-PRASAD CONJECTURE

HARALD GROBNER & JIE LIN

ABSTRACT. We prove explicit rationality-results for Asai- L-functions, $L^S(s,\Pi',\mathrm{As}^{\pm})$, and Rankin-Selberg L-functions, $L^S(s,\Pi\times\Pi')$, over arbitrary CM-fields F, relating critical values to explicit powers of $(2\pi i)$. Besides determining the contribution of archimedean zeta-integrals to our formulas as concrete powers of $(2\pi i)$, it is one of the crucial advantages of our refined approach, that it applies to very general non-cuspidal isobaric automorphic representations Π' of $\mathrm{GL}_n(\mathbb{A}_F)$. As a major application, this enables us to establish a certain algebraic version of the Gan–Gross–Prasad conjecture, as refined by N. Harris, for totally definite unitary groups: This generalizes a deep result of Zhang and complements totally recent progress of Beuzard-Plessis. As another application we obtain a generalization of an important result of Harder–Raghuram on quotients of consecutive critical values, proved by them for totally real fields, and achieved here for arbitrary CM-fields F and pairs (Π,Π') of relative rank one.

CONTENTS

INTRODUCTION

Rationality for critical values. In the algebraic theory of special values of L-functions, Deligne's conjecture for critical L-values of motives is still one of the driving forces. Cut down to one line, it asserts that the critical values at $s=m\in\mathbb{Z}$ of the L-function $L(s,\mathbb{M})$ of a motive $\mathbb{M}$ can be described, up to multiplication by elements in a concrete number-field $E(\mathbb{M})$, in terms of certain *geometric* period-invariants $c^{\pm}(\mathbb{M})$ and certain explicit powers of $(2\pi i)$, [Del79, Conj. 2.8]:

$$L(m,\mathbb{M})\sim_{E(\mathbb{M})}(2\pi i)^{d(m)}c^{(-1)^m}(\mathbb{M}).$$

In this generality, Deligne's conjecture is still far open. The deeper reason for this, though, seems almost like a paradox: It tempting to believe that it is exactly the well-reduced, slender rigidity of the world of motives, which allows one to express critical values $L(m,\mathbb{M})$ by such clear and basal invariants (namely $c^{(-1)^m}(\mathbb{M})$, $E(\mathbb{M})$ and, most fundamental, $(2\pi i)^{d(m)}$), on the one hand, while

2010 *Mathematics Subject Classification.* 11F67 (Primary) 11F70, 11G18, 11R39, 22E55 (Secondary).

H.G. is supported by START-prize Y-966 of the Austrian Science Fund (FWF). J.L. was supported by the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 290766 (AAMOT).

it seems to be exactly the same (beautiful) rigidity of the world of motives, which does not leave enough argumentative room to attack Deligne's conjecture directly, on the other hand.

Yielding to this tempting belief, it is hence not surprising that it has been the (much less rigid) automorphic side – invoking the (conjectural) dictionary, hinging motives $\mathbb{M}$ over a number field F and automorphic representations Π of $\mathrm{GL}_n(\mathbb{A}_F)$, by a comparison of their L-functions – where most progress on understanding the algebraic nature of special values of L-functions has been achieved.

Indeed, there is a spectacular series of results, relating critical values $s = \frac{n-1}{2} + m$ (due to a basic shift of the argument s now in $\frac{n-1}{2} + \mathbb{Z}$) of an automorphic L-function $L(s, \Pi)$, up to multiplication by elements in a number field $E(\Pi)$ depending on Π, to certain *representation-theoretical* period invariants $p(\Pi)$ and a purely archimedean factor $p(m, \Pi_\infty)$. Obviously, interpreting Deligne's conjecture automorphically, here the period-invariant $p(\Pi)$ takes the role of $c^\pm(\mathbb{M})$, the number field $E(\Pi)$ the role of $E(\mathbb{M})$ and finally the archimedean factor $p(m, \Pi_\infty)$ the place of $(2\pi i)^{d(m)}$.

In many regards it is the latter archimedean factor $p(m, \Pi_\infty)$ (essentially the weighted sum of archimedean zeta-integrals), which turns out to be the most mysterious ingredient: In fact, over several decades it has even been unknown if it is eventually zero (which would obviously have made all automorphic rationality-theorems meaningless) until – after various important but partial results – B. Sun established the non-vanishing of $p(m, \Pi_\infty)$ in great generality in breakthrough work.

However, apart from particular cases, an explicit expression for $p(m, \Pi_\infty)$, putting it in a precise relationship with its all precise motivic counterpart $(2\pi i)^{d(m)}$ predicted by Deligne's conjecture, is yet to be found.

In this paper, we solve this intriguing problem, for Rankin-Selberg L-functions, $L^S(s, \Pi \times \Pi')$, and Asai- L-functions, $L^S(s, \Pi', \mathrm{As}^\pm)$, over arbitrary CM-fields F: We establish precise rationality-theorems, whose archimedean factors are indeed explicit powers of $(2\pi i)$. As a general rule, these powers match the power $(2\pi i)^{d(m)}$, predicted by Deligne, on the nose.

Main results I: Rationality for Rankin-Selberg L-functions with explicit archimedean factors. Our rationality-results apply to a large class of automorphic representations Π and Π'. More precisely, we let F be any CM-field and Π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, whereas $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ may even be an isobaric sum on $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, fully induced from an arbitrary number $k \geq 1$ of distinct, but again arbitrary, unitary cuspidal automorphic representations Π_i.

Let $s = \frac{1}{2} + m$ be a critical point of $L^S(s, \Pi \times \Pi')$. Clearly, in order to lay our hands on the archimedean factor (i.e., the contribution of the archimedean zeta integrals to our formulas), we have to specify our possible choices of Π_∞ and Π'_∞: If $m \neq 0$, the only condition they have to satisfy is to be conjugate self-dual with non-vanishing relative Lie algebra cohomology with respect to an irreducible algebraic coefficient module $\mathcal{E}_\mu$, respectively $\mathcal{E}_{\mu'}$, allowing a non-trivial $\mathrm{GL}_{n-1}(F \otimes_\mathbb{Q} \mathbb{R})$-intertwining $\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'} \to \mathbb{C}$.

The case $m = 0$, i.e., to obtain a rationality-result with explicit powers of $(2\pi i)$ for the central critical value $L^S(\frac{1}{2}, \Pi \times \Pi')$, is more complicated by nature and needs an additional non-vanishing assumption on the central critical value of some auxiliary representations, constructed from suitable Hecke characters, see Hyp./Conj. 4.29 and Hyp. 4.20 & 4.26. This assumption is due to the limitation of current techniques only. Indeed, in all cases considered in this paper, the aforementioned hypotheses are expected to hold in full generality. As an example, we only remark here that the latter two hypotheses, Hyp. 4.20 & 4.26, can be dropped, if $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ are sufficiently regular, i.e., the successive coordinates of μ and μ' differ at least by 2.

Here is our first main theorem, relating critical values of $L^S(s, \Pi \times \Pi')$ with explicit powers of $(2\pi i)$:

Theorem A. *Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let $\Pi' = \Pi_1 \boxplus \dots \boxplus \Pi_k$ be an isobaric automorphic representation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, fully induced from an arbitrary number $k \geq 1$ of distinct unitary cuspidal automorphic representations Π_i and write $\mathcal{G}(\omega_{\Pi'_f})$ for the Gauß-sum of its central character. Assume that Π_∞ and Π'_∞ are conjugate self-dual, cohomological with respect to an irreducible algebraic coefficient module $\mathcal{E}_\mu$, respectively $\mathcal{E}_{\mu'}$, allowing a non-trivial $\mathrm{GL}_{n-1}(F \otimes_\mathbb{Q} \mathbb{C})$-intertwining $\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'} \to \mathbb{C}$. Let $s = \frac{1}{2} + m$ be a critical point of $L^S(s, \Pi \times \Pi')$, where, if $m = 0$, we assume the auxiliary non-vanishing hypotheses Hyp./Conj. 4.29 and Hyp. 4.20 & 4.26 mentioned above.*

Then there are non-zero Whittaker periods $p(\Pi) \in \mathbb{C}^\times$ and $p(\Pi') \in \mathbb{C}^\times$, defined by a comparison of a fixed rational structure on the Whittaker model of Π_f, resp. Π'_f, with a fixed rational structure on the cohomology of Π, resp. Π', and we obtain

$$L^S(\tfrac{1}{2} + m, \Pi \times \Pi') \ \sim_{E(\Pi)E(\Pi')} (2\pi i)^{mdn(n-1) - \frac{1}{2}d(n-1)(n-2)} p(\Pi)\ p(\Pi')\mathcal{G}(\omega_{\Pi'_f})$$

which is equivariant under the natural action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. Here, "$\sim_{E(\Pi)E(\Pi')}$" means up to multiplication by an element in the number field $E(\Pi)E(\Pi')$ obtained by composing the Galois closure F^{Gal} of $F/\mathbb{Q}$ in $\bar{\mathbb{Q}}$ with the fields of rationality of Π resp. Π'.

Being able to determine the contribution of the archimedean zeta-integrals for the first time as an explicit power of $(2\pi i)$, our Thm. A may be regarded as a joint subtle refinement of [Gro17a, Thm. 1.8], [Gro-MHar16, Thm. 3.9] and [Rag16, Thm. 1.1] over general CM-fields F. We remark that the presence of F^{Gal} in our formula(s) is indispensable due to the use of our "Minimizing–Lemma", cf. Lem. 1.19: This is a useful tool, which allows to reduce relations of algebraicity very easily to fields of minimal size, as long as they contain F^{Gal}.

The Whittaker periods $p(\Pi)$ and $p(\Pi')$ mentioned in Thm. A are constructed in Prop. 1.8 in one go: Invoking several deep theorems on the nature of *non-cuspidal* automorphic cohomology, we are able to transfer the general principle of how to construct Whittaker periods, developed in [GHar83], [Mah05] and in particular in [Rag-Sha08], from cuspidal representations to general *Eisenstein representations*, i.e., (a slight generalization of) our general isobaric sums $\Pi' = \Pi_1 \boxplus \dots \boxplus \Pi_k$. This is a non-trivial step in the construction of our Whittaker periods, which is achieved in §1.5.2 (see in particular Thm. 1.12 and its Cor. 1.13) and which builds vitally on recent results in [Gro13].

As a result, we obtain a uniform generalization of the construction in [Rag-Sha08]: Our generalization applies to arbitrary Eisenstein representations (which fully cover the case of a cuspidal representation by specifying $k = 1$ in $\Pi' = \Pi_1 \boxplus \dots \boxplus \Pi_k$). As a side-effect, the proof of Thm. 1.12 shows that our generalization is in fact "sharp", i.e., the family of Eisenstein representations is the largest family of automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, for which one may construct Whittaker periods.

Main results II: Rationality for Asai L-functions with explicit archimedean factors. Thm. A above is proved by detour to Asai L-functions. In order to explain our approach, and, in fact, our second main theorem, let $\Pi' = \Pi_1 \boxplus \dots \boxplus \Pi_k$ be an Eisenstein representation of $\mathrm{GL}_n(\mathbb{A}_F)$ as above, i.e., a cohomological isobaric sum, fully induced from an arbitrary number $k \geq 1$ of distinct unitary cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$. A short moment of thought shows that Π_i is cohomological itself, if and only if $n \equiv n_i \mod 2$, cf. §1.4.3. Putting $e \in \{0,1\}$ equal to the residue class of $n - n_i \mod 2$ and $\eta : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ equal to the extension of the quadratic character ε attached to F and its maximal totally real subfield F^+ by class field theory, cf. §1.1.2, we end up with a unitary cuspidal representation $\Pi_i^{\mathsf{alg}} := \Pi_i \otimes \eta^e$, which is cohomological with respect to an algebraic coefficient system $\mathcal{E}_{\mu_i}$ in any case.

If Π' is moreover conjugate self-dual (a condition which we previously only assumed for its archimedean component), then one can show (cf. Cor. 3.4) that the Asai L-function $L^S(s, \Pi', \mathrm{As}^{(-1)^n})$ of sign $(-1)^n$ is holomorphic and non-vanishing at $s = 1$. Moreover, $s = 1$ is critical for $L(s, \Pi', \mathrm{As}^{(-1)^n})$. Our second main theorem relates this critical value $L^S(1, \Pi', \mathrm{As}^{(-1)^n})$ with an explicit power of $(2\pi i)$:

Theorem B. *Let F be any CM-field and let $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ be a cohomological isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, fully induced from an arbitrary number $k \geq 1$ of distinct conjugate self-dual cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$. If $\mathcal{E}_{\mu_i}$ is not sufficiently regular, we assume the auxiliary non-vanishing hypotheses Hyp. 4.20 & 4.26 for Π_i^{alg}. Then we have*

$$L^S(1, \Pi', \mathrm{As}^{(-1)^n}) \sim_{E(\Pi')} (2\pi i)^{dn} p(\Pi')$$

which is equivariant under the natural action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

As we adumbrated above, the proofs of Thm. A and Thm. B are in fact hinged together. Let us sketch their joint argument here (as well as the structure of our paper) before we describe various applications and consequences of Thm. A and Thm. B for the theory of special values of L-functions, in particular the vital area of research opened by the Gan–Gross–Prasad conjecture and its refinements. The reader, who prefers to get in touch with these applications first, may skip the next subsection for the moment, and directly proceed to our sections "Main applications" below.

Sketch of the proofs' main idea. As shortly explained, section 1 develops the theory of Whittaker periods for general Eisenstein representations, but also recalls the two main results of [Gro17a] and [Gro-MHar-Lap16], which are the starting-point of our investigations: These latter results, quoted here as Thm. 1.30 and Thm. 1.27, essentially match Thm. A and Thm. B, respectively, but with two very crucial differences:

(1) The contribution of archimedean zeta-integrals to the formulas for critical L-values is not determined, but left unspecified as some mysterious non-zero constants $p(m, \Pi_\infty, \Pi'_\infty)$ in Thm. 1.30, resp. $a(\Pi'_\infty)$ in Thm. 1.27.
(2) Thm. 1.27 only covers the case of cuspidal automorphic representations Π', i.e., necessarily forces $k = 1$.

In order to overcome these two problems, sections 2 and 3.1 develop precise formulas for Asai L-functions of general, conjugate self-dual isobaric sums $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ in terms of their cuspidal summands Π_i, cf. Lem. 3.3; as well as a precise relation between the Whittaker periods $p(\Pi')$ and $p(\Pi_i^{\mathsf{alg}})$ of Π' and its twisted summands Π_i^{alg}: Indeed, in Thm. 2.5 we establish the broadest possible generalization, but also a significant refinement of [Gro-MHar16], Prop. 5.3 and [Lin15b], Prop. 3.4.1, by showing that

$$p(\Pi') \sim_{E(\Pi')E(\phi)} \prod_{1 \leq i \leq k} p(\Pi_i^{\mathsf{alg}}) \prod_{1 \leq i < j \leq k} L^S(1, \Pi_i \times \Pi_j^\vee).$$

This formula is one of the keys to extend Thm. 1.27 to isobaric sums of an arbitrary number of summands $k \geq 1$, i.e., to overcome (2) above.

Section 3.2 and 4 are devoted to a solution of (1) laying the final foundation of the proof of Thm. A and Thm. B. Its guiding principle is the following observation: Since $p(m,\Pi_\infty,\Pi'_\infty)$ in Thm. 1.30 and $a(\Pi'_\infty)$ in Thm. 1.27 only depend on the archimedean components of Π, resp. Π', we may replace our global representations at hand by any suitable simpler choice of automorphic representations in Thm. 1.30 and Thm. 1.27, which have the same archimedean components Π_∞ resp. Π'_∞, and hence give rise to the same archimedean periods $p(m,\Pi_\infty,\Pi'_\infty)$ and $a(\Pi'_\infty)$; but now as part of global formulas involving only much simpler global terms.

Indeed, knowing that Π_∞ and Π'_∞ are necessarily fully induced from characters (because they are cohomological and generic), there are two obvious choices of such simpler automorphic representations on $\mathrm{GL}_n(\mathbb{A}_F)$, resp. $\mathrm{GL}_{n-1}(\mathbb{A}_F)$: One is to take n (resp. $n-1$) Hecke characters χ_i of F, chosen such that their archimedean components $\chi_{i,\infty}$ simply give the characters forming the inducing datum of Π_∞ resp. Π'_∞; and then to consider the isobaric sum of these characters χ_i. The other obvious choice is to fix an appropriate cyclic CM-extension L of F of degree n (resp. $n-1$) over F and to pick an appropriate Hecke character χ of L, whose archimedean component recovers the inducing datum of Π_∞ resp. Π'_∞; and then to perform automorphic induction from $\mathrm{GL}_1(\mathbb{A}_L)$ to $\mathrm{GL}_n(\mathbb{A}_F)$ (resp. $\mathrm{GL}_1(\mathbb{A}_L)$ to $\mathrm{GL}_{n-1}(\mathbb{A}_F)$).

These two much simpler choices of automorphic representations are precisely our "auxiliary representations" mentioned right before Thm. A above: Being constructed from Hecke characters, they enjoy the great advantage that their Rankin-Selberg L-functions and Asai L-functions can be described in terms of Hecke L-functions, whose critical L-values are under control. Reinserting this knowledge, gained by both of our two types of construction, into the period-relations, provided abstractly by Thm. 1.30 and Thm. 1.27, then yields two different explicit relations for the original unknown archimedean factors $p(m,\Pi_\infty,\Pi'_\infty)$ and $a(\Pi'_\infty)$. Determining them as concrete powers of $(2\pi i)$, finally amounts to solving a system of two equations with two variables (see Thm. 4.17 and its Cor. 4.30). The proof of Thm. A and Thm. B then follows easily, see §5, Thm. 5.2 and Thm. 5.1.

Main applications I: The refined conjecture of Gan–Gross-Prasad for unitary groups. Combining Thm. A with Thm. B yields the following result, which is both, a generalization as well as a subtle refinement of [Gro-MHar16], Cor. 6.25, with the additional asset that it avoids any reference to our global Whittaker periods $p(\Pi)$ and $p(\Pi')$:

Theorem C. *Let F be any CM-field and let Π and Π' be two cohomological conjugate self-dual automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, resp. $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, which satisfy the conditions of Thm. A and Thm. B. Then, for every critical point $\frac{1}{2}+m$ of $L(s,\Pi\times\Pi')$, we obtain*

$$\frac{L^S(\frac{1}{2}+m,\Pi\times\Pi')}{L^S(1,\Pi,\mathrm{As}^{(-1)^n})\ L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})}\sim_{E(\Pi)E(\Pi')}(2\pi i)^{mdn(n-1)-dn(n+1)/2}.$$

and this relation is equivariant under the natural action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

We believe that this result, which holds for all critical points $\frac{1}{2}+m$ of $L(s,\Pi\times\Pi')$, is interesting in its own right. Specifying $m=0$, however, we immediately obtain the relation

$$\frac{L^S(\frac{1}{2},\Pi\times\Pi')}{L^S(1,\Pi,\mathrm{As}^{(-1)^n})\ L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})}\sim_{E(\Pi)E(\Pi')}(2\pi i)^{-dn(n+1)/2}, \tag{0.1}$$

which leads us directly to the heart of an exciting, deep refinement of the global Gan–Gross-Prasad conjecture, [NHar14, Liu16], for unitary groups: Recall our arbitrary CM-field F with maximal totally real subfield F^+ and the quadratic Hecke character ε attached to the extension F/F^+.

For unitary groups $\mathscr{G}(\mathcal{V})/F^+$ and $\mathscr{G}(\mathcal{W})/F^+$, attached to a pair of Hermitian spaces $\mathcal{W} \subset \mathcal{V}$ of dimension $\dim_F(\mathcal{V}) = n > \dim_F(\mathcal{W}) = m$, the global GGP-conjecture, as most recently refined by Liu, [Liu16], predicts a very precise relationship of a quotient of L-functions, which is of the type of the left-hand-side of (0.1), and a global period integral $\mathcal{P}(\varphi, \varphi')$ of two tempered cusp forms $\varphi \in \pi$ and $\varphi' \in \pi'$ on $\mathscr{G}(\mathcal{V})(\mathbb{A}_{F^+})$, reps. $\mathscr{G}(\mathcal{W})(\mathbb{A}_{F^+})$:

$$|\mathcal{P}(\varphi,\varphi')|^2 = \frac{\Delta_{\mathscr{G}(\mathcal{V})}}{2^a} \frac{L^S(\frac{1}{2}, \pi \boxtimes \pi')}{L^S(1,\pi,\mathrm{Ad})\ L^S(1,\pi',\mathrm{Ad})} \prod_{v \in S} \alpha_v(\varphi_v, \varphi'_v). \tag{0.2}$$

Here, $\alpha_v(\varphi_v, \varphi'_v)$ are local integrals – stabilized and suitably normalized – over certain matrix coefficients, whereas $\Delta_{\mathscr{G}(\mathcal{V})}/2^a$ is a rather elementary constant, attached to the (expected) Vogan-Arthur packets of π and π' and the Gross-motives: $\Delta_{\mathscr{G}(\mathcal{V})} = \prod_{i=1}^n L(i, \varepsilon_f^i)$. The careful reader, interested in precise definitions and assertions, is referred to §6.1–§6.2 for a detailed account.

Considering the classical[1] case when $\mathcal{W}$ has codimension 1 in $\mathcal{V}$, i.e., $m = n-1$, provides the intimate link of our explicit formula for quotients of critical L-values, (0.1), and the refined GGP-conjecture, (0.2), just pronounced: If π and π' are cohomological representations of $\mathscr{G}(\mathcal{V})(\mathbb{A}_{F^+})$, reps. $\mathscr{G}(\mathcal{W})(\mathbb{A}_{F^+})$, then we may apply quadratic base change BC (unconditionally) and obtain two cohomological isobaric automorphic representations $BC(\pi)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ and $BC(\pi')$ of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, respectively. If these new representations $BC(\pi)$ and $BC(\pi')$ satisfy the conditions of our Thm. A with Thm. B above, they may take the role of Π and Π' in (0.1), and so we may replace the quotient of L-functions in (0.2) by the respective quotient of L-functions in (0.1); as moreover

$$\Delta_{\mathscr{G}(\mathcal{V})} \sim_{F^{Gal}} (2\pi i)^{dn(n+1)/2},$$

i.e., up to some algebraic number in F^{Gal}, the Gross-motives' factor $\Delta_{\mathscr{G}(\mathcal{V})}$ in (0.2) equals the inverse of the right-hand-side $(2\pi i)^{-dn(n+1)/2}$ of (0.1), we obtain the fundamental relation

$$\frac{\Delta_{\mathscr{G}(\mathcal{V})}}{2^a} \frac{L^S(\frac{1}{2}, \pi \boxtimes \pi')}{L^S(1,\pi,\mathrm{Ad})\ L^S(1,\pi',\mathrm{Ad})} \sim_{F^{Gal}} \frac{(2\pi i)^{\frac{dn(n+1)}{2}}\ L^S(\frac{1}{2}, \Pi \times \Pi')}{L^S(1,\Pi,\mathrm{As}^{(-1)^n})\ L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})} \sim_{E(\pi)E(\pi')} 1,$$

which finally enables us to show

Theorem D. *Let F be any CM-field with fixed maximal totally real subfield F^+ and let $\mathscr{G}(V)$ and $\mathscr{G}(\mathcal{W})$ be two arbitrary unitary groups over F^+ of codimension one. Let π (resp. π') be a tempered cohomological cuspidal automorphic representation of $\mathscr{G}(V)$ (resp. $\mathscr{G}(\mathcal{W})$), appearing with multiplicity one in the cuspidal spectrum. Assume that the quadratic base change $BC(\pi) = \Pi$ is a cohomological cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ and that the quadratic base change $BC(\pi') = \Pi'$ is a cohomological isobaric automorphic representation $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ fully induced from an arbitrary number $k \geq 1$ of distinct cuspidal representations.*

(i) *If the pair (Π, Π') satisfies the conditions of Thm. A and Thm. B, and if $\mathscr{G}(V)$ and $\mathscr{G}(\mathcal{W})$ are moreover totally definite, then for all decomposable smooth $E(\pi)$-rational (resp. $E(\pi')$-rational) functions $\varphi = \otimes'_v \varphi_v \in \pi$ (resp. $\varphi' = \otimes'_v \varphi'_v \in \pi'$),*

$$|\mathcal{P}(\varphi,\varphi')|^2 \sim_{E(\pi)E(\pi')} \frac{\Delta_{\mathscr{G}(V)}\ L^S(\frac{1}{2}, \pi \boxtimes \pi')}{L^S(1,\pi,\mathrm{Ad})\ L^S(1,\pi',\mathrm{Ad})} \prod_{v \in S} \alpha_v(\varphi_v, \varphi'_v) \tag{0.3}$$

where "$\sim_{E(\pi)E(\pi')}$" means up to multiplication by an element q in the number field $E(\pi)E(\pi')$, depending only on π and π'. This q is in fact independent of φ and φ'.

[1]See the original paper [Gro-Pra92], where it all started; and [NHar14], which anticipates the conjectures in [Liu16].

(ii) *If $\mathscr{G}(V)$ and $\mathscr{G}(\mathcal{W})$ are not totally definite, but the respective coefficient modules in cohomology $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ allow a non-trivial* $\mathrm{GL}_{n-1}(F\otimes_{\mathbb{Q}}\mathbb{C})$*-intertwining $\mathcal{E}_\mu\otimes\mathcal{E}_{\mu'}\to\mathbb{C}$, then the same conclusion as in (i) holds trivially for all decomposable cusp forms $\varphi=\otimes'_v\varphi_v\in\pi$ and $\varphi'=\otimes'_v\varphi'_v\in\pi'$.*

We refer to §6.4 for a proof of Thm. D. Let us emphasize its two main advantages in the context of the exciting recent literature of the GGP-conjecture:

(1) We do not assume any condition of local supercuspidality of $\pi\otimes\pi'$. In all preceding important work on the refined GGP-conjecture for unitary groups, which built on the trace formula, this assumption of supercuspidality has been indispensable due to the limitations of current state of the theory (see [Zha14b], [Beu16b]). Here we completely avoid this condition, as well as any problems connected to the use of the fundamental lemma for the Jacquet-Rallis relative trace formulae.
(2) We allow general isobaric sums for the base change of π', i.e., we do not restrict ourselves to representations lifting to cuspidal representations. This restriction has been made in [Zha14b], Thm. Thm. 1.2.(2), for instance.

It is intrinsic to our approach via relations of algebraicity that our result cannot detect the non-vanishing of the left- and right-hand-side in (0.3). If the quantities in (0.3) are non-zero, however, then our theorem asserts that both sides of (0.2) are inside the same number field $E(\pi)E(\pi')$ and hence firmly supports their conjectured equality. We expect moreover that an extension of our techniques will eventually prove (0.3) even for non-rational decomposable cusp forms φ, φ'.

Main applications II: A result of Harder–Raghuram. Our main result on period relation of critical values of Rankin–Selberg L-functions with explicit powers of $(2\pi i)$, Thm. A, also provides a direct generalization of an important result of Harder–Raghuram for pairs of automorphic representations $\Pi\otimes\Pi'$ of $\mathrm{GL}_n(\mathbb{A}_F)\times\mathrm{GL}_{n-1}(\mathbb{A}_F)$ as above.

Indeed, recapitulating their result very shortly, in [GHar-Rag17] a period, denoted $\Omega^{\varepsilon'}({}^\iota\sigma'_f)$, has been constructed and related to the ratio of consecutive critical values of Ranking–Selberg L-functions of cuspidal automorphic representations σ and σ' of $\mathrm{GL}_n(\mathbb{A}_{F^+})\times\mathrm{GL}_{n'}(\mathbb{A}_{F^+})$. Here, n is assumed to be even while n' is assumed to be odd.

In contrast to this very general theorem for consecutive quotients of critical L-values for cusp forms over totally real fields, our Thm. A implies the following general result for quotients of critical L-values of automorphic representations $\Pi\otimes\Pi'$ of $\mathrm{GL}_n(\mathbb{A}_F)\times\mathrm{GL}_{n-1}(\mathbb{A}_F)$ over arbitrary CM-fields:

Theorem E. *Let F be any CM-field and let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and $\Pi'=\Pi_1\boxplus\ldots\boxplus\Pi_k$ an isobaric automorphic representation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, fully induced from an arbitrary number $k\geq 1$ of distinct unitary cuspidal automorphic representations Π_i. Assume that Π_∞ and Π'_∞ are conjugate self-dual, cohomological with respect to an irreducible algebraic coefficient module $\mathcal{E}_\mu$, respectively $\mathcal{E}_{\mu'}$, allowing a non-trivial $\mathrm{GL}_{n-1}(F\otimes_{\mathbb{Q}}\mathbb{C})$-intertwining $\mathcal{E}_\mu\otimes\mathcal{E}_{\mu'}\to\mathbb{C}$. Let $\frac{1}{2}+m, \frac{1}{2}+\ell$ be two critical points of $L^S(s,\Pi\times\Pi')$, where, if $m\ell=0$, we assume the auxiliary non-vanishing hypotheses Hyp./Conj. 4.29 and Hyp. 4.20 & 4.26. Whenever $L^S(\frac{1}{2}+\ell,\Pi\times\Pi')$ is non-zero (e.g., if $\ell\neq 0$), we obtain*

$$\frac{L^S(\frac{1}{2}+m,\Pi\times\Pi')}{L^S(\frac{1}{2}+\ell,\Pi\times\Pi')}\sim_{E(\Pi)E(\Pi')}(2\pi i)^{d(m-\ell)n(n-1)}.$$

and this relation is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. *In particular, if* $L^S(\frac{3}{2}+m,\Pi\times\Pi')$ *is non-zero (e.g., if* $m\neq -1$*), the quotient of consecutive critical L-values satisfies*

$$(2\pi i)^{dn(n-1)}\frac{L^S(\frac{1}{2}+m,\Pi\times\Pi')}{L^S(\frac{3}{2}+m,\Pi\times\Pi')}\in E(\Pi)E(\Pi').$$

This theorem also complements earlier great achievements of Januszewski, see [Jan16], Thm. A, where an analogously explicit result has been proved (under different assumptions) for pairs of cuspidal representations (π,σ) over totally real fields.

Acknowledgements: We would like to thank Michael Harris for many very valuable discussions and his constant advice. We also thank Hang Xue and Raphaël Beuzart-Plessis for their helpful comments and for answering our questions on the Gan–Gross–Prasad conjecture.

1. Preliminaries

1.1. Number fields and Hecke characters.

1.1.1. *Number fields.* Generally, if $\mathbb{F}\subset\mathbb{C}$ is any number field, then we denote by $J_{\mathbb{F}}$ the finite set of its field embeddings $\iota:\mathbb{F}\hookrightarrow\mathbb{C}$ and by $\mathbb{F}^{Gal}$ the Galois closure of $\mathbb{F}/\mathbb{Q}$ in $\bar{\mathbb{Q}}$. More concretely, we let F be any CM-field of dimension $2d=\dim_{\mathbb{Q}}F$ and set of archimedean places $S_\infty=S(F)_\infty$. Each place $v\in S_\infty$ hence refers to a fixed pair of conjugate complex embeddings $(\iota_v,\bar\iota_v)\in J_F^2$ of F, where we will drop the subscript "v" if it is clear from the context. This fixes a choice of a CM-type $\Sigma=\{\iota_v,v\in S_\infty\}$. We write F^+ for the maximal totally real subfield of F. Its set of real places will be identified with S_∞, identifying a place v with its first component embedding $\iota_v\in\Sigma$. Again, we may drop the subscript "v" if possible. We let $\mathrm{Gal}(F/F^+)=\{1,c\}$. The ring of adeles over F (resp. over F^+) is denoted $\mathbb{A}_F$ (resp. $\mathbb{A}_{F^+}$), their respective rings of integers $\mathcal{O}_F$ (resp. $\mathcal{O}_{F^+}$).

Whenever we write L^S for an object $L=\prod_v L_v$ admitting an Euler product factorization, then we mean the partial object $L^S:=\prod_{v\notin S}L_v$ for some choice of finite set of places S of F, containing S_∞. As a general rule, if L depends on further data for which the notion of ramification is defined, we assume that S contains all such ramified places.

1.1.2. *Characters and Gauß sums.* Let χ be any Hecke character of a CM-field. Following [MHar97] p. 82, we denote its contragredient conjugate dual by $\check\chi:=\chi^{-1,c}=\bar\chi^\vee$. The normalized absolute value on $\mathbb{A}_F$ is denoted $\|\cdot\|$. We extend the quadratic Hecke character $\varepsilon:(F^+)^\times\backslash\mathbb{A}^\times_{F^+}\to\mathbb{C}^\times$, associated to F/F^+ via class field theory, to a conjugate self-dual unitary Hecke character $\eta:F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times$. At $v\in S_\infty$ we have $\eta_v(z)=z^t\bar z^{-t}$, for $z\in F_v$, where $t=t_v\in\frac{1}{2}+\mathbb{Z}$. For our results there will be no loss of generality, if we assume from now on that $t=0$, i.e., $\eta_v(z)=z^{1/2}\bar z^{-1/2}$. We may define a non-unitary algebraic Hecke character $\phi:F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times$, by $\phi:=\eta\,\|\cdot\|^{1/2}$. We then have that $\phi\phi^c=\|\cdot\|$ and $\phi_v(z)=z^1\bar z^0$ for all $v\in S_\infty$ and $z\in F_v$. Once and for all we fix a non-trivial additive character $\psi:F\backslash\mathbb{A}_F\to\mathbb{C}^\times$ as in Tate's thesis, see, e.g., [Rag-Sha08].
Let χ be an algebraic Hecke character. We define the Gauß sum of its finite part χ_f, following Weil [Wei67, VII, Sect. 7]: Let $\mathfrak{c}_\chi$ stand for the conductor ideal of χ_f and let $y=(y_v)_{v\notin S_\infty}\in\mathbb{A}_f^\times$ be chosen such that $\mathrm{ord}_v(y_v)=-\mathrm{ord}_v(\mathfrak{c}_\chi)-\mathrm{ord}_v(\mathfrak{D}_F)$. Here, $\mathfrak{D}_F$ stands for the absolute different of F, that is, $\mathfrak{D}_F^{-1}=\{x\in F: Tr_{F/\mathbb{Q}}(x\mathcal{O}_F)\subset\mathbb{Z}\}$.

The Gauß sum of χ_f with respect to y and ψ is now defined as $\mathcal{G}(\chi_f,\psi_f,y):=\prod_{v\notin S_\infty}\mathcal{G}(\chi_v,\psi_v,y_v)$, where the local Gauß sum $\mathcal{G}(\chi_v,\psi_v,y_v)$ is defined as

$$\mathcal{G}(\chi_v,\psi_v,y_v):=\int_{\mathcal{O}^\times_{F_v}}\chi_v(u_v)^{-1}\psi_v(y_vu_v)\,du_v.$$

For almost all v, we have $\mathcal{G}(\chi_v, \psi_v, y_v) = 1$, and for all v we have $\mathcal{G}(\chi_v, \psi_v, y_v) \neq 0$. (See, for example, Godement [God70, Eq. 1.22].) Note that, unlike in [Wei67], we do not normalize the Gauß sum to make it have absolute value one. For the sake of easing notation and readability we suppress its dependence on ψ and y, and denote $\mathcal{G}(\chi_f, \psi_f, y)$ simply by $\mathcal{G}(\chi_f)$.

1.2. **Algebraic groups and real Lie groups.** We let G_n, or simply G, be $G := G_n := \mathrm{GL}_n/F$. Let V_n be an n-dimensional, non-degenerate c-Hermitian space over F, $n \geq 2$, with corresponding unitary group $H := H_n := U(V_n)$ over F^+. At $v \in S_\infty$ (identified now with its first entry $v = \iota_v$), we have $H_n(F_v^+) \cong U(r_v, s_v)$ for some signature $0 \leq r_v, s_v \leq n$. If V_k is some non-degenerate F-subspace of V_n, we view $U(V_k)$ as a natural F^+-subgroup of $U(V_n)$.

If $\mathscr{G}$ is any reductive algebraic group over a number field $\mathbb{F}$, we write $\mathscr{G}_\infty = R_{\mathbb{F}/\mathbb{Q}}(\mathscr{G})(\mathbb{R})$. At $v \in S_\infty$ we denote by K_v the product of the center $Z_G(F_v)$ of $G(F_v)$ and a fixed maximal compact subgroup of $G(F_v)$ (isomorphic to the compact real unitary group $U(n)$) and we let $K_\infty := \prod_{v \in S_\infty} K_v \subset G_\infty$. Similarly, if H is any given unitary group, we let C_v be the product of the center $Z_H(F_v^+)$ of $H(F_v^+)$ and a fixed maximal compact subgroup of $H(F_v^+)$ (isomorphic to $U(r_v) \times U(s_v)$) and we let $C_\infty := \prod_{v \in S_\infty} C_v \subset H_\infty$.
Lower case gothic letters denote the Lie algebra of the corresponding real Lie group (e.g., $\mathfrak{g}_v := Lie(G(F_v))$, $\mathfrak{k}_v := Lie(K_v)$, $\mathfrak{h}_v := Lie(H(F_v^+))$, etc. ...).

1.3. **Highest weight modules and cohomological representations.** We let $\mathcal{E}_\mu$ be an irreducible finite-dimensional representation of the *real* Lie group $G_\infty = R_{F/\mathbb{Q}}(G)(\mathbb{R})$ on a complex vector-space, given by its highest weight $\mu = (\mu_v)_{v \in S_\infty}$. Throughout this paper such a representation will be assumed to be algebraic: In terms of the standard choice of a maximal torus and positivity on the corresponding set of roots, this means that $\mu_v = (\mu_{\iota_v}, \mu_{\bar\iota_v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and each component weight μ_{ι_v} and $\mu_{\bar\iota_v}$ consists of a decreasing sequence of entries $\mu_{\iota_v,j} \geq \mu_{\iota_v,j+1}$ and $\mu_{\bar\iota_v,j} \geq \mu_{\bar\iota_v,j+1}$ for all $1 \leq j \leq n-1$.

Similarly, given a unitary group $H = U(V_n)$, we let $\mathcal{F}_\lambda$ be an irreducible finite-dimensional representation of the real Lie group $H_\infty = R_{F^+/\mathbb{Q}}(U(V_n))(\mathbb{R})$ on a complex vector-space, given by its highest weight $\lambda = (\lambda_v)_{v \in S_\infty}$. Again, every such representation is assumed to be algebraic, which means that each component $\lambda_v \in \mathbb{Z}^n$. Moreover, one has $\lambda_{v,j} \geq \lambda_{v,j+1}$ for all $1 \leq j \leq n-1$ and $v \in S_\infty$.

A representation Π_∞ of G_∞ is said to be *cohomological* if there is a highest weight module $\mathcal{E}_\mu$ as above such that $H^*(\mathfrak{g}_\infty, K_\infty, \Pi_\infty \otimes \mathcal{E}_\mu) \neq 0$. Analogously, a representation π_∞ of H_∞ is said to be *cohomological* if there is a highest weight module $\mathcal{F}_\lambda$ as above such that $H^*(\mathfrak{h}_\infty, C_\infty, \pi_\infty \otimes \mathcal{F}_\lambda) \neq 0$. See [Bor-Wal00],§I, for details.

1.3.1. *An action of* $\mathrm{Aut}(\mathbb{C})$ *on finite-dimensional representations.* Let $\mathscr{G}$ be a connected reductive group over $\mathbb{Q}$ and let $\mathcal{E}$ be a finite-dimensional complex vector space on which $\mathscr{G}(\mathbb{C})$ acts by linear transformations, i.e., there is a group homomorphism $\epsilon : \mathscr{G}(\mathbb{C}) \to \mathrm{GL}(\mathcal{E})$. Given $\sigma \in \mathrm{Aut}(\mathbb{C})$, we may define a new linear action ${}^\sigma\epsilon : \mathscr{G}(\mathbb{C}) \to \mathrm{GL}({}^\sigma\mathcal{E})$ as follows: Its underlying complex vector space is ${}^\sigma\mathcal{E} := \mathcal{E} \otimes_\sigma \mathbb{C}$, (i.e., the same abelian group as the original space $\mathcal{E}$, but with a new scalar multiplication $\alpha \star_\sigma v := \sigma^{-1}(\alpha) \cdot v$) with linear action of $g \in \mathscr{G}(\mathbb{C})$ defined by

$$ {}^\sigma\epsilon(g)v := \epsilon(\sigma^{-1}(g))v. $$

Here, we view $\mathscr{G}(\mathbb{C}) \subseteq \mathrm{GL}_N(\mathbb{C})$ as being embedded into a (fixed) general linear group over $\mathbb{Q}$, whence applying σ^{-1} to the complex matrix entries of g gives rise to a well-defined element $\sigma^{-1}(g) \in \mathscr{G}(\mathbb{C})$.

As $\sigma^{-1}(g) = g$ for all $g \in \mathscr{G}(\mathbb{Q})$, this yields a σ-linear isomorphism of finite-dimensional $\mathscr{G}(\mathbb{Q})$-representations

$$\tilde{\sigma} : \mathcal{E} \xrightarrow{\sim} {}^{\sigma}\mathcal{E} \tag{1.1}$$

Obviously, if ϵ was algebraic, then so is ${}^{\sigma}\epsilon$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. If $(\epsilon, \mathcal{E})$ is furthermore an *irreducible* algebraic representation of $\mathscr{G}(\mathbb{C})$, then the collection $\{({}^{\sigma}\epsilon, {}^{\sigma}\mathcal{E}) : \sigma \in \mathrm{Aut}(\mathbb{C})\}$ of equivalence classes of the representations $({}^{\sigma}\epsilon, {}^{\sigma}\mathcal{E})$ is finite. This follows from checking the effect of σ on the highest weight of $\mathcal{E}$, which, by assumption, defines an *algebraic* character. In particular, for irreducible algebraic representations $\mathcal{E}$, the subgroup $\mathfrak{S}(\mathcal{E})$ of $\mathrm{Aut}(\mathbb{C})$ consisting of all automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ for which (1.1) is an isomorphism of $\mathscr{G}(\mathbb{Q})$-representations (i.e., linear), has finite index in $\mathrm{Aut}(\mathbb{C})$. Hence, the rationality-field of the $\mathscr{G}(\mathbb{Q})$-representation $\mathcal{E}$, $\mathbb{Q}(\mathcal{E}) := \mathbb{C}^{\mathfrak{S}(\mathcal{E})}$ is a number field.

The above construction applies in particular to irreducible algebraic representations $\mathcal{E}_\mu$ of G_∞ (resp. $\mathcal{F}_\lambda$ of H_∞) as defined in §1.3: Both ${}^{\sigma}\mathcal{E}_\mu$ and ${}^{\sigma}\mathcal{F}_\lambda$ will define a representation of G_∞ (resp. H_∞) by restriction from the respective group of complex points. As a representation of $G(F)$, ${}^{\sigma}\mathcal{E}_\mu$ may be identified with the representation $\mathcal{E}_{{}^{\sigma}\mu}$ of highest weight ${}^{\sigma}\mu = ((\mu_{\sigma^{-1}\circ\iota_v}, \mu_{\sigma^{-1}\circ\bar{\iota}_v})_{v\in S_\infty})$. The reader may be warned that the analogous assertion does not necessarily apply to the representation ${}^{\sigma}\mathcal{F}_\lambda$ of $H(F^+)$.

1.3.2. *Rational structures on algebraic representations and cohomology.* Let $\mathscr{G}$ be any reductive algebraic group over $\mathbb{Q}$ which gives rise to a connected complex Lie group $\mathscr{G}(\mathbb{C})$ and let $\epsilon : \mathscr{G}(\mathbb{C}) \to \mathrm{GL}(\mathcal{E})$ be any irreducible algebraic representation of $\mathscr{G}(\mathbb{C})$ on a finite-dimensional complex vector space $\mathcal{E}$. Then we obtain the following

Proposition 1.2. *Let L be any finite Galois extension of $\mathbb{Q}$ over which $\mathscr{G}$ splits. Then, the restricted representation $\epsilon : \mathscr{G}(\mathbb{Q}) \to \mathrm{GL}(\mathcal{E})$ is defined over the number field $L \cdot \mathbb{Q}(\mathcal{E})$.*

Let $\mathscr{C}_\infty$ be the product of the connected component of the identity of the center of $\mathscr{G}(\mathbb{R})$ and a maximal compact subgroup of $\mathscr{G}(\mathbb{R})$ (e.g., K_∞ or C_∞ from §1.2). The admissible $\mathscr{G}(\mathbb{A}_f)$-module defined by the cohomology group $H^q(S_{\mathscr{G}}, \mathcal{E})$ in degree q of

$$S_{\mathscr{G}} := \mathscr{G}(\mathbb{Q})\backslash\mathscr{G}(\mathbb{A}_\mathbb{Q})/\mathscr{C}_\infty$$

with respect to the locally constant sheaf on $S_{\mathscr{G}}$ given by $\mathcal{E}$, hence inherits a natural $L\cdot\mathbb{Q}(\mathcal{E})$-rational structure from Prop. 1.2. Moreover, for each $\sigma \in \mathrm{Aut}(\mathbb{C})$, (1.1) induces a σ-linear $\mathscr{G}(\mathbb{A}_f)$-equivariant bijection

$$\tilde{\sigma}^q : H^q(S_{\mathscr{G}}, \mathcal{E}) \xrightarrow{\sim} H^q(S_{\mathscr{G}}, {}^{\sigma}\mathcal{E}). \tag{1.3}$$

1.4. **Automorphic representations π and Π.** Throughout this paper, as a general rule, Π denotes an irreducible automorphic representation of $G(\mathbb{A}_F) = \mathrm{GL}_n(\mathbb{A}_F)$, whereas π denotes an irreducible automorphic representation of $H(\mathbb{A}_{F^+}) = U(V_n)(\mathbb{A}_{F^+})$, in the sense of [Bor-Jac79], §4, whose particular additional properties (such as being "cuspidal", "unitary", "generic", "cohomological", "a quadratic base change", etc.) will be specified at each of its occurrences.

1.4.1. *Cohomological automorphic representations.* Let $\Pi(r) := \Pi \cdot \|\det\|^r$, with Π a unitary automorphic representation of $G(\mathbb{A}_F)$ and $r \in \mathbb{R}$ (that means, $\Pi(r)$ is essentially unitary automorphic), which is cohomological with respect to $\mathcal{E}_\mu$. Suppose $\Pi(r)$ is generic at each $v \in S_\infty$, then

$$\Pi(r)_v \cong \mathrm{Ind}_{B(\mathbb{C})}^{G(\mathbb{C})}[z_1^{\ell_{v,1}+r}\bar{z}_1^{-\ell_{v,1}+r} \otimes ... \otimes z_n^{\ell_{v,n}+r}\bar{z}_n^{-\ell_{v,n}+r}],$$

where

$$\ell_{v,j} := \ell(\mu_{\iota_v}, j) := -\mu_{\iota_v,n-j+1} - r + \frac{n+1}{2} - j$$

and induction from the standard Borel subgroup $B = TN$ is unitary, cf. [Enr79, Thm. 6.1] (See also [Gro-Rag14, §5.5] for a detailed exposition). In particular, such a $\Pi(r)$ is essentially tempered at all $v \in S_\infty$. Observe that also the contrary holds: An essentially unitary automorphic representation of $G(\mathbb{A}_F)$, which has an essentially tempered and cohomological archimedean factor $\Pi(r)_\infty$ is generic at all places $v \in S_\infty$. One has

$$H^q(\mathfrak{g}_v, K_v, \Pi(r)_v \otimes \mathcal{E}_{\mu_v}) \cong \bigwedge^{q-\frac{n(n-1)}{2}} \mathbb{C}^{n-1}. \tag{1.4}$$

Slightly more general, let us also consider the automorphic twists $\Pi(r)\phi^e$, with $e \in \{0, 1\}$. Then it is easy to see from the very definition of ϕ that, $\Pi(r)$ is cohomological if and only if $\Pi(r)\phi$ is (the respective highest weight modules arise form each other by adding $-e$ to each entry of the ι_v-components, $v \in S_\infty$). Indeed, the respective cohomology groups of both the representations $(\Pi(r)\phi^e)_v$, $e = 0, 1$, are isomorphic (and hence described by (1.4).)

For $\Pi(r)$ as above, we define the notion of *infinity-type* with repsect to our chosen CM-type Σ: Abbreviating $a_{\iota,i} := \ell(\mu_\iota, i) + r$ and $a_{\bar\iota,i} := -\ell(\mu_\iota, i) + r$, the infinity-type of $\Pi(r)$ at $\iota \in \Sigma$ is the set of inducing characters $\{z^{a_{\iota,i}} \bar{z}^{a_{\bar\iota,i}}\}_{1 \le i \le n}$. Hence, our notion of infinity-type recovers the "*type à l'infini*" defined in [Clo90], §3.3, by substracting $\frac{n-1}{2}$ from each entry of the pairs $(a_{\iota,i}, a_{\bar\iota,i}) \in (\frac{n-1}{2} + \mathbb{Z})^2$.

If π is a unitary automorphic representation of $H(\mathbb{A}_{F^+}) = U(V_n)(\mathbb{A}_{F^+})$, which is tempered and cohomological with respect to $\mathcal{F}_\lambda$, then each of its archimedean component-representations π_v of $U(r_v, s_v)$ is isomorphic to one of the $\binom{n}{r_v}$ inequivalent discrete series representations of Harish-Chandra parameter $\chi_{\lambda_v+\rho_v}$, [Vog-Zuc84].

1.4.2. $\mathrm{Aut}(\mathbb{C})$*-twisted automorphic representations and attached number fields.* Let π (resp. Π) be a cohomological cuspidal automorphic representation of $H(\mathbb{A}_{F^+})$ (resp. $G(\mathbb{A}_F)$). Then the σ-linear isomorphism (1.3) together with the well-known "sandwich-property", which stacks cuspidal, interior and square-integrable automorphic cohomology, see, e.g., [Sch90], p. 11, gives rise to a σ*-twisted* square-integrable cohomological automorphic representation ${}^\sigma\pi$ (resp. ${}^\sigma\Pi$) of $H(\mathbb{A}_{F^+})$ (resp. $G(\mathbb{A}_F)$), whose finite component $({}^\sigma\pi)_f$ (resp. $({}^\sigma\Pi)_f$) allow an equivariant σ-linear isomorphism to π_f (resp. Π_f). In terms of [Wal85a] §I.1, this amounts to $({}^\sigma\pi)_f \cong {}^\sigma(\pi_f)$ (resp. $({}^\sigma\Pi)_f \cong {}^\sigma(\Pi_f)$). As prescribed by (1.3) the archimedean component ${}^\sigma\pi_\infty$ (resp. ${}^\sigma\Pi_\infty$) is unique up to L-packets (i.e., has predetermined infinitesimal character, namely the one of ${}^\sigma\mathcal{E}^\vee_\mu$, resp. ${}^\sigma\mathcal{F}^\vee_\lambda$). The σ-twists are cuspidal, if the original representation was globally generic. In particular, ${}^\sigma\Pi$ will always be cuspidal and its archimedean component ${}^\sigma\Pi_\infty$ is uniquely determined, see [Gro17a] where this is made explicit.

As a consequence, the *rationality-fields* $\mathbb{Q}(\pi_f)$ and $\mathbb{Q}(\Pi_f)$ (defined as the fixed field in $\mathbb{C}$ of all automorphisms σ, which leave the finite part of the given automorphic representation stable, cf. [Wal85a], §I.1) are finite extensions of $\mathbb{Q}$ for all cohomological cuspidal automorphic representations π and Π: For Π this is proved implicitly in [Clo90], Thm. 3.13 (and explicitly in [Gro-Rag14] Thm. 8.1), while for π the argument given in [Wal85a] Cor. I.8.3, [Gro-Rag14] Thm. 8.1 or [Gro15], Cor. 3.6 transfers verbatim. Since Π satisfies Strong Multiplicity One, it is consistent to write $\mathbb{Q}(\Pi) = \mathbb{Q}(\Pi_f)$. Moreover, $\mathbb{Q}(\Pi_f) \supseteq \mathbb{Q}(\mathcal{E}_\mu)$ for the same reason, see the proof of [Gro-Rag14] Cor. 8.7.

Let now $E(\pi_f)$ be an appropriate finite extension of $\mathbb{Q}(\pi_f)$ over which π_f is defined in the sense of [Wal85a]: That means that there is a $H(\mathbb{A}_f)$-stable $E(\pi_f)$-subspace $\pi_{E(\pi_f)} \subseteq \pi_f$ such that the natural map $\pi_{E(\pi_f)} \otimes_{E(\pi_f)} \mathbb{C} \to \pi_f$ is an isomorphism. (For the existence of such an extension $E(\pi_f)$ and $\pi_{E(\pi_f)}$ see, e.g., [Gro-Seb16], Thm. A.2.4). We will use the following abbreviations

$$\boxed{E(\pi) := \mathbb{Q}(\mathcal{F}_\lambda) \cdot E(\pi_f) \cdot F^{Gal}, \qquad \text{and} \qquad E(\Pi) := \mathbb{Q}(\Pi) \cdot F^{Gal},}$$

recalling that F^{Gal} denotes the Galois closure of $F/\mathbb{Q}$ in $\bar{\mathbb{Q}}$. By what we have just said all these fields are number fields.

1.4.3. *Eisenstein representations.* The automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, which we will mainly be considering in this paper, will be cohomological *Eisenstein representations*, i.e., cohomological automorphic representations of the form

$$\Pi'(r)\phi^e := \Pi'\|\det\|^r\phi^e, \quad r \in \mathbb{R}, e \in \{0,1\}$$

where Π' is an isobaric automorphic sum

$$\Pi' := \Pi_1 \boxplus ... \boxplus \Pi_k \cong \mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}[\Pi_1 \otimes ... \otimes \Pi_k], \tag{1.5}$$

which we assume to be fully-induced from distinct unitary cuspidal automorphic representations Π_i of general linear groups $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, $1 \leq i \leq k$. Here, we let $P = L_P N_P$ be the standard parabolic subgroup of $G_n = \mathrm{GL}_n/F$ with Levi subgroup $L = L_P$ isomorphic to $\prod_{i=1}^k \mathrm{GL}_{n_i}$. As a paradigmatic example, a cohomological automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, which is obtained by quadratic base change from a quasi-split unitary group as in [Cog-PS-Sha11], p. 122, is of the form of Π', cf. [Cog-PS-Sha11, Thm. 6.1].

Abbreviate $\tau := \Pi_1 \otimes ... \otimes \Pi_k$, so by our conventions $\tau(r)\phi^e := \Pi_1(r)\phi^e \otimes ... \otimes \Pi_k(r)\phi^e$ is the cuspidal representation of $L(\mathbb{A}_F)$, whose parabolic induction is isomorphic to $\Pi'(r)\phi^e$ by assumption. It is worth noting, however, that the isomorphism in (1.5) between the (only abstract!) global induced representation $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}[\tau(r)\phi^e]$ and the actual automorphic representation $\Pi'(r)\phi^e$ is given by computing the Eisenstein series $E_P(h,\lambda)$ attached to a K_∞-finite section $h \in \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}[\tau(r)\phi^e]$, formally defined as

$$E_P(h,\lambda)(g) := \sum_{\gamma \in P(F)\backslash G(F)} h(\gamma g)(id)\; e^{\langle \lambda, H_P(\gamma g)\rangle},$$

H_P begin the Harish-Chandra height function, $\lambda \in \check{\mathfrak{a}}_{P,\mathbb{C}} := X^*(L_P) \otimes_{\mathbb{Z}} \mathbb{C}$, followed by evaluating $E_P(h,\lambda)$ at the point $\lambda = 0$, see [Lan79], proof of Prop. 2. In fact, in [Lan79] a regularization $q(\lambda)E_P(h,\lambda)$ (the regularizing non-zero holomorphic function $q(\lambda)$ being defined as in [Mœ-Wal95], Lem. I.4.10, for instance) had to be used in order to obtain the desired isomorphism between $\Pi'(r)\phi^e$ and $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}[\tau(r)\phi^e]$. Hence our description of this latter isomorphism only follows knowing the following lemma:

Lemma 1.6. *For any K_∞-finite section $h_{r,\phi^e} \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\tau(r)\phi^e]$, the Eisenstein series $E_P(h_{r,\phi^e},\lambda)$ is holomorphic at the point of evaluation $\lambda = 0$.*

Proof. Indeed, one has

$$\begin{aligned}
E_P(h_{r,\phi^e},\lambda)(g) &= \sum_{\gamma \in P(F)\backslash G(F)} h_{r,\phi^e}(\gamma g)(id)\; e^{\langle \lambda, H_P(\gamma g)\rangle} \\
&= \sum_{\gamma \in P(F)\backslash G(F)} h_{0,\eta^e}(\gamma g)(id)\; \|\det(\gamma g)\|^{r+e/2}\; e^{\langle \lambda, H_P(\gamma g)\rangle} \\
&= \|\det(g)\|^{r+e/2} \sum_{\gamma \in P(F)\backslash G(F)} h_{0,\eta^e}(\gamma g)(id)\; e^{\langle \lambda, H_P(\gamma g)\rangle} \\
&= \|\det(g)\|^{r+e/2}\; E_P(h_{0,\eta^e},\lambda)(g)
\end{aligned}$$

for some K_∞-finite $h_{0,\eta^e} \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\tau\eta^e]$. As $\tau\eta^e$ is unitary, the latter Eisenstein series $E_P(h_{0,\eta^e},\lambda)(g)$ converges absolutely at $\lambda = 0$ as it is well-known by [Mœ-Wal95, IV.1.11]. $\square$

We denote the, finally well-defined isomorphism (of underlying $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-modules) by

$$\mathrm{Eis}_0 : \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}[\tau(r)\phi^e] \xrightarrow{\sim} \Pi'(r)\phi^e$$

$$h \mapsto E_P(h, 0).$$

It is due to this realization of $\Pi'(r)\phi^e$ in the space of cohomological automorphic forms $\mathcal{A}(G)$ that we have chosen the name Eisenstein representation. As it is obvious by its very definition, the family of Eisenstein representations contains all essentially unitary cohomological cuspidal automorphic representations $\Pi(r)$ (by letting $k = 1$, $e = 0$) and all cohomological fully-indiced isobaric sums Π' (by letting $r = e = 0$) as above. Moreover, now knowing the way we view $\Pi'(r)\phi^e$ as a subrepresentation of $\mathcal{A}(G)$ through the injection Eis_0, the arguments given in [Sha10, Prop. 7.1.3, Thm. 3.5.12 and Rem. 3.5.14] imply that $\Pi'(r)\phi^e$ is globally ψ-generic, i.e., the ψ-Fourier coefficient W^ψ does not vanish on an Eisenstein representation $\Pi'(r)\phi^e$.

We write ρ_i for the restriction $\rho_P|_{\mathrm{GL}_{n_i}(\mathbb{A}_F)}$, so we get explicitly $\rho_i = \| \det_{\mathrm{GL}_{n_i}} \|^{\frac{a_i}{2}}$, with

$$a_i = \sum_{j=1}^{k-i} n_{i+j} - \sum_{j=1}^{i-1} n_j = n_{i+1} + n_{i+2} + ... + n_k - n_1 - n_2 - ... - n_{i-1}. \tag{1.7}$$

Hence, $a_i \equiv n - n_i \mod 2$ and so ρ_i is algebraic if and only if $n \equiv n_i \mod 2$. As by assumption $\Pi'(r)\phi^e$ is cohomological, [Bor-Wal00], Thm. III.3.3, implies that $\Pi_i(r)\phi^e\rho_i = \Pi_i(r + \frac{a_i}{2})\phi^e$ are cohomological for $1 \leq i \leq k$. Hence, for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, there are well-defined pairwise different, in general non-unitary cuspidal automorphic representations ${}^\sigma(\Pi_i(r)\phi^e\rho_i)$ of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, which are cohomological with respect to the σ-permuted coefficient module of $\mathrm{GL}_{n_i,\infty}$, see §1.4.2. We abbreviate

$$(\Pi_i(r)\phi^e)^\sigma := {}^\sigma(\Pi_i(r)\phi^e\rho_i) \cdot \rho_i^{-1}.$$

Then it is proved as in [Gro17a] §1.2.5 that

$${}^\sigma\Pi'(r)\phi^e := (\Pi_1(r)\phi^e)^\sigma \boxplus ... \boxplus (\Pi_k(r)\phi^e)^\sigma$$

is an isobaric automorphic representation, which is again fully-induced from the pairwise different, in general non-unitary cuspidal automorphic representations $(\Pi_i(r)\phi^e)^\sigma$. If $\Pi'(r)\phi^e$ is cohomological with respect to $\mathcal{E}_\mu$, then ${}^\sigma\Pi'(r)\phi^e$ is cohomological with respect to ${}^\sigma\mathcal{E}_\mu$ and it satisfies $({}^\sigma\Pi'(r)\phi^e)_f \cong {}^\sigma((\Pi'(r)\phi^e)_f)$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. As ${}^\sigma\Pi'(r)\phi^e \cong \Pi'(r)\phi^e$ if and only if

$$\{(\Pi_1(r)\phi^e)^\sigma, ..., (\Pi_k(r)\phi^e)^\sigma\} = \{\Pi_1(r)\phi^e, ..., \Pi_k(r)\phi^e\}$$

the rationality field $\mathbb{Q}(\Pi'(r)\phi^e)$ of an Eisenstein representation is contained in the composition of number fields $\prod_{i=1}^k \mathbb{Q}(\Pi_i(r)\phi^e\rho_i)$ of the cohomological, cuspidal automorphic summands $\Pi_i(r)\phi^e\rho_i$, an hence a finite extension of $\mathbb{Q}$ itself.

1.5. **Whittaker periods for Eisenstein representations and critical values of automorphic L-functions.**

1.5.1. *Abstract Whittaker periods and automorphic cohomology.* Let Π_f be an irreducible admissible representation of $G(\mathbb{A}_f) = \mathrm{GL}_n(\mathbb{A}_f)$ and suppose that ${}^\sigma(\Pi_f)$ is generic for all $\sigma \in$Aut$(\mathbb{C})$. Let $W({}^\sigma\Pi_f)$ be the Whittaker model with respect to our fixed non-trivial additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $v \notin S_\infty$ let $t_{\sigma,v}$ be the unique diagonal matrix of type $t_{\sigma,v} = \mathrm{diag}(x_1, ..., x_{n-1}, 1) \in T(\mathcal{O}_{F_v})$ such that $\sigma(\psi_v(n)) = \psi_v(t_{\sigma,v}^{-1} n t_{\sigma,v})$ for all $n \in N(F_v)$ and let $t_\sigma = (t_{\sigma,v})_{v\notin S_\infty} \in G(\mathbb{A}_f)$. Then for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ the σ-linear, $G(\mathbb{A}_f)$-equivariant bijection

$$\tilde\sigma_{\Pi_f} : \mathrm{Ind}_{N(\mathbb{A}_f)}^{G(\mathbb{A}_f)}[\psi_f] \xrightarrow{\sim} \mathrm{Ind}_{N(\mathbb{A}_f)}^{G(\mathbb{A}_f)}[\psi_f]$$

$$\xi \mapsto \tilde\sigma_{\Pi_f}(\xi) : g \mapsto \sigma(\xi(t_\sigma \cdot g))$$

will map $W(\Pi_f)$ onto $W({}^\sigma\Pi_f)$ by the uniqueness of local Whittaker models. See [Rag-Sha08, §3.2] or [Gro-MHar-Lap16, §4.1].

Let $S_n := S_G = G(F)\backslash G(\mathbb{A}_F)/K_\infty$. Then $H^q(S_n, \mathcal{E}_\mu)$, the admissible $G(\mathbb{A}_f)$-module defined by the cohomology of S_n, has a natural L-rational structure over any field extension $\mathbb{C} \supseteq L \supseteq \mathbb{Q}(\mathcal{E}_\mu)$. See §1.3.2 (with $\mathscr{G} = R_{F/\mathbb{Q}}(G)$) above. Let $H^q(S_n, \mathcal{E}_\mu)_L$ be this natural L-structure and recall the σ-linear $G(\mathbb{A}_f)$-equivariant bijections $\tilde\sigma^q : H^q(S_n, \mathcal{E}_\mu) \to H^q(S_n, {}^\sigma\mathcal{E}_\mu)$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ from (1.3).

For the next proposition we remind that if $\mathrm{Hom}_{G(\mathbb{A}_f)}(W(\Pi_f), H^q(S_n, \mathcal{E}_\mu))$ is one-dimensional, then the image $\Upsilon(W(\Pi_f)) =: H^q(S_n, \mathcal{E}_\mu)(\Pi_f)$ of any homomorphism $\Upsilon \in \mathrm{Hom}_{G(\mathbb{A}_f)}(W(\Pi_f), H^q(S_n, \mathcal{E}_\mu))$ is independent of the choice of Υ and isomorphic to Π_f.

Proposition 1.8. *Let Π_f be the finite part of an irreducible admissible representation Π of $G(\mathbb{A}_F) = \mathrm{GL}_n(\mathbb{A}_F)$, such that ${}^\sigma(\Pi_f)$ is generic for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Assume that there is an irreducible algebraic coefficient module $\mathcal{E}_\mu$ such that $\mathrm{Hom}_{G(\mathbb{A}_f)}(W({}^\sigma\Pi_f), H^q(S_n, {}^\sigma\mathcal{E}_\mu))$ is one-dimensional for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, and that moreover $\tilde\sigma^q(H^q(S_n, \mathcal{E}_\mu)(\Pi_f)) = H^q(S_n, {}^\sigma\mathcal{E}_\mu)({}^\sigma\Pi_f)$. Then, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ the following hold:*

(1) *Let ${}^\sigma L := \mathbb{Q}({}^\sigma\Pi_f)\mathbb{Q}({}^\sigma\mathcal{E}_\mu)$.*

$$H^q(S_n, {}^\sigma\mathcal{E}_\mu)({}^\sigma\Pi_f)_{{}^\sigma L} := H^q(S_n, {}^\sigma\mathcal{E}_\mu)({}^\sigma\Pi_f) \cap H^q(S_n, {}^\sigma\mathcal{E}_\mu)_{{}^\sigma L}$$

defines an ${}^\sigma L$-structure on $H^q(S_n, {}^\sigma\mathcal{E}_\mu)({}^\sigma\Pi_f)$.

(2) *There are non-zero complex numbers $p({}^\sigma\Pi) = p({}^\sigma\Pi_f, {}^\sigma\Upsilon)$, depending on a chosen embedding ${}^\sigma\Upsilon \in \mathrm{Hom}_{G(\mathbb{A}_f)}(W({}^\sigma\Pi_f), H^q(S_n, {}^\sigma\mathcal{E}_\mu))$ and the chosen, fixed ${}^\sigma L$-structures on domain and target space of ${}^\sigma\Upsilon$, such that*

$$\begin{array}{ccc} W(\Pi_f) & \xrightarrow{p(\Pi)^{-1}\cdot\Upsilon} & H^q(S_n, \mathcal{E}_\mu)(\Pi_f) \\ \downarrow{\scriptstyle \tilde\sigma_{\Pi_f}} & & \downarrow{\scriptstyle \tilde\sigma^q} \\ W({}^\sigma\Pi_f) & \xrightarrow{p({}^\sigma\Pi)^{-1}\cdot{}^\sigma\Upsilon} & H^q(S_n, {}^\sigma\mathcal{E}_\mu)({}^\sigma\Pi_f) \end{array}$$

commutes.

Proof. One has ${}^\sigma(H^q(S_n, \mathcal{E}_\mu)) = H^q(S_n, {}^\sigma\mathcal{E}_\mu) = \tilde\sigma^q(H^q(S_n, \mathcal{E}_\mu))$ by the definition of ${}^\sigma\mathcal{E}_\mu$, hence also ${}^\sigma(H^q(S_n, \mathcal{E}_\mu)(\Pi_f)) = \tilde\sigma^q(H^q(S_n, \mathcal{E}_\mu)(\Pi_f))$. Since $\tilde\sigma^q(H^q(S_n, \mathcal{E}_\mu)(\Pi_f)) = H^q(S_n, {}^\sigma\mathcal{E}_\mu)({}^\sigma\Pi_f)$ by assumption, (1) follows from [Clo90], Lem. 3.2.1.
Having fixed an ${}^\sigma L$-structure on $H^q(S_n, {}^\sigma\mathcal{E}_\mu)({}^\sigma\Pi_f)$, the existence of the non-zero numbers $p({}^\sigma\Pi) = p({}^\sigma\Pi_f, {}^\sigma\Upsilon)$ as well as the commutativity the above square follows word for word as in [Rag-Sha08, Definition/Proposition 4.2.1], observing that [Jac-PS-Sha81, Thm. (4.1.(i))] is valid for every Π_v, $v \notin S_\infty$. □

Definition 1.9. Let Π be any representation of $G(\mathbb{A}_F) = \mathrm{GL}_n(\mathbb{A}_F)$ as in Prop. 1.8. We call the non-zero complex number $p(\Pi)$ the *Whittaker period attached to* Π (and some fixed choice of embedding Υ). By construction, it is well-defined only up to multiplication with elements in $L^\times$.

1.5.2. *An important family of examples - The Eisenstein representations.* Let us now make the above proposition concrete for a large family of automorphic representations. More precisely, we consider our family of Eisenstein representations

$$\Pi'(r)\phi^e = \Pi'\|\det\|^r\phi^e, \quad r \in \mathbb{R}, e \in \{0,1\}$$

as defined in §1.4.3 above. Let $\mathcal{E}_\mu$ the irreducible algebraic representation of G_∞ with respect to which $\Pi'(r)\phi^e$ is cohomological. Since $\Pi'(r)\phi^e$ is globally generic, $(\Pi'(r)\phi^e)_\infty$ is of the form described in §1.4.1 and has one-dimensional $(\mathfrak{g}_\infty, K_\infty)$-cohomology in its lowest non-vanishing degree

$$b_n := d\frac{n(n-1)}{2}$$

by (1.4). As the choice of our Whittaker period $p(\Pi'(r)\phi^e)$ all depends on the particular map Υ, the various choices made entering its definition shall now be specified. Knowing these choices explicitly will be of great significance in §2.

Firstly, we observe that the map Eis_0 from §1.4.3 is closely related to the surjective *Eisenstein construction map*, Eis_∂, which we will briefly recall here and which has been defined in all details in [Gro13], §2.4 (where it has been denoted $\mathrm{Eis}_{\mathcal{J},\{P\},\varphi_P}$). To do this, let $S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})$ is the symmetric algebra of the orthogonal complement $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$ of $\check{\mathfrak{a}}_{G,\mathbb{C}} = X^*(G) \otimes_\mathbb{Z} \mathbb{C} = \{\det^s, s \in \mathbb{C}\}$ in $\check{\mathfrak{a}}_{P,\mathbb{C}}$. It may be interpreted as the algebra of differential operators $\partial^{\underline{n}}/\partial\lambda^{\underline{n}}$, ($\underline{n} = (n_1, ..., n_{\dim \check{\mathfrak{a}}^G_P})$ being a multi-index with respect to some fixed basis of $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$) on $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$.

Furthermore, let $\mathcal{J}$ be the ideal of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\infty,\mathbb{C}})$, which annihilates the contragredient representation $\mathcal{E}^\vee_\mu$ of $\mathcal{E}_\mu$ and let φ_P be the associate class of the cuspidal automorphic representation $\tau(r)\phi^e$. A $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-module $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ has then been defined as the space of automorphic forms of $G(\mathbb{A}_F)$ supported in φ_P and annihilated by a power of $\mathcal{J}$: More precisely, it is the span of the holomorphic values at the point $\lambda = \mathrm{pr}_{\check{\mathfrak{a}}_{P,\mathbb{C}} \to \check{\mathfrak{a}}^G_{P,\mathbb{C}}}((r + e/2, ..., r + e/2)) = 0$ of the Eisenstein series $E_P(h, \lambda)$, h running through all K_∞-finite sections $h \in \mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}[\tau(r)\phi^e]$, together with all their derivatives in the parameter λ. (See [Fra-Sch98] 1.2 – 1.4 or [Gro13] 2.3 for a comprehensive account.)

Bearing these definitions in mind, the Eisenstein construction map Eis_∂ has a rather obvious definition as

$$\mathrm{Eis}_\partial : \mathrm{Ind}^{G(\mathbb{A}_F)}_{P(\mathbb{A}_F)}[\tau(r)\phi^e \otimes S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})] \twoheadrightarrow \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$$

$$h \otimes \frac{\partial^{\underline{n}}}{\partial\lambda^{\underline{n}}} \mapsto \frac{\partial^{\underline{n}}}{\partial\lambda^{\underline{n}}} E_P(h,\lambda)|_{\lambda=0}.$$

Now, in order to describe our choice of Υ, we fix a basis element of the one-dimensional vector space $H^{b_n}(\mathfrak{g}_\infty, K_\infty, W(\Pi'(r)\phi^e)_\infty \otimes \mathcal{E}_\mu)$. By [Bor-Wal00], Prop. II.3.1, this basis element is a K_∞-homomorphism $[\Pi'(r)\phi^e] : \Lambda^{b_n}\mathfrak{g}_\infty/\mathfrak{k}_\infty \to W(\Pi'(r)\phi^e)_\infty \otimes \mathcal{E}_\mu$. We view this basis homomorphism as an element in $\left(\Lambda^{b_n}(\mathfrak{g}_\infty/\mathfrak{k}_\infty)^* \otimes W(\Pi'(r)\phi^e)_\infty \otimes \mathcal{E}_\mu\right)^{K_\infty}$ and write it explicitly as

$$\sum_{\underline{i}=(i_1,...,i_{b_n})} \sum_{\alpha=1}^{\dim \mathcal{E}_\mu} X^*_{\underline{i}} \otimes \xi_{\infty,\underline{i},\alpha} \otimes e_\alpha.$$

Here, $X^*_{\underline{i}} := X^*_{i_1} \wedge ... \wedge X^*_{i_{b_n}}$ for some fixed basis $\{X_j\}$ of $\mathfrak{g}_\infty/\mathfrak{k}_\infty$, $e_\alpha := \otimes_{v\in S_\infty} e_{\alpha,v} \in E_\mu = \otimes_{v\in S_\infty} E_{\mu_v}$, such that $\{e_{\alpha,v}\}_\alpha$ defines a basis of E_{μ_v} for all $v \in S_\infty$; whereas $\xi_{\infty,\underline{i},\alpha} \in W(\Pi'(r)\phi^e)_\infty$ are Whittaker functionals chosen accordingly.

Recall that W^ψ denotes the map computing the ψ-Fourier coefficient of an element in $\Pi'(r)\phi^e$. We obtain

Proposition 1.10. *There is the following sequence of $G(\mathbb{A}_f)$-morphisms*

$$\begin{aligned}
W((\Pi'(r)\phi^e)_f) &\xrightarrow{\sim} W(\Pi'(r)\phi^e)_f \\
&\xrightarrow{\sim}_{[\Pi'(r)\phi^e]\otimes} H^{b_n}(\mathfrak{g}_\infty, K_\infty, W(\Pi'(r)\phi^e)_\infty \otimes \mathcal{E}_\mu) \otimes W(\Pi'(r)\phi^e)_f \\
&\xrightarrow{\sim} H^{b_n}(\mathfrak{g}_\infty, K_\infty, W(\Pi'(r)\phi^e) \otimes \mathcal{E}_\mu) \\
&\xrightarrow{\sim}_{(W^\psi)^{-1,b_n}} H^{b_n}(\mathfrak{g}_\infty, K_\infty, \Pi'(r)\phi^e \otimes \mathcal{E}_\mu) \\
&\xrightarrow{\sim}_{\mathrm{Eis}_0^{-1,b_n}} H^{b_n}(\mathfrak{g}_\infty, K_\infty, \mathrm{Ind}^{G(\mathbb{A}_F)}_{P(\mathbb{A}_F)}[\tau(r)\phi^e] \otimes \mathcal{E}_\mu) \\
&\xrightarrow{\sim} H^{b_n}(\mathfrak{g}_\infty, K_\infty, \mathrm{Ind}^{G(\mathbb{A}_F)}_{P(\mathbb{A}_F)}[\tau(r)\phi^e \otimes S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})] \otimes \mathcal{E}_\mu) \\
&\xrightarrow{\sim}_{\mathrm{Eis}_\partial^{b_n}} H^{b_n}(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes \mathcal{E}_\mu) \\
&\hookrightarrow H^q(S_n, \mathcal{E}_\mu)
\end{aligned}$$

the four unspecified maps being (a fixed choice of) the obvious ones.

Proof. By the preceding discussion we only need to show that the last three maps are isomorphisms, respectively injective. The obvious morphism of admissible $G(\mathbb{A}_f)$-modules

$$H^q(\mathfrak{g}_\infty, K_\infty, \mathrm{Ind}^{G(\mathbb{A}_F)}_{P(\mathbb{A}_F)}[\tau(r)\phi^e] \otimes \mathcal{E}_\mu) \to H^q(\mathfrak{g}_\infty, K_\infty, \mathrm{Ind}^{G(\mathbb{A}_F)}_{P(\mathbb{A}_F)}[\tau(r)\phi^e \otimes S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})] \otimes \mathcal{E}_\mu)$$

is surjective for all degrees q, by the Künneth rule, whereas injectivitiy in degree $q = b_n$ follows from the minimality of b_n, see [Gro-Rag14], (7.25) and (7.26) where this is made explicit. For $\mathrm{Eis}_\partial^{b_n}$ we observe that by our Lem. 1.6 the length of the filtration of the $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, defined in [Gro13], §3.1, may be chosen to be $m(\{P\}) = 0$. Hence, as established in [Gro13], Cor. 16, $H^{b_n}(\mathfrak{g}_\infty, K_\infty, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes \mathcal{E}_\mu)$ decomposes as a direct sum, which – invoking [Mœ-Wal89], II & III – in fact degenerates to one single summand, namely $H^{b_n}(\mathfrak{g}_\infty, K_\infty, \mathrm{Ind}^{G(\mathbb{A}_F)}_{P(\mathbb{A}_F)}[\tau(r)\phi^e \otimes S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})] \otimes \mathcal{E}_\mu)$. As a consequence, the Eisenstein construction map induces an isomorphism in cohomology in all degrees q, hence in particular in degree $q = b_n$. The injectivity of the last map is obvious recalling the description of $H^q(S_n, \mathcal{E}_\mu)$ as the space of automorphic cohomology, [Fra98], Thm. 18, and its fine decomposition [Fra-Sch98], Thm. 2.3. See also [Gro13], §4. □

Definition 1.11. We let $\Upsilon = \Upsilon_{\Pi'(r)\phi^e}$ be the composition of the maps in Prop. 1.10. For each $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^\sigma\Upsilon = \Upsilon_{{}^\sigma\Pi'(r)\phi^e}$ is hence a non-trivial element in $\mathrm{Hom}_{G(\mathbb{A}_f)}(W(({}^\sigma\Pi'(r)\phi^e)_f), H^{b_n}(S_n, {}^\sigma\mathcal{E}_\mu))$.

Theorem 1.12. *The space $\mathrm{Hom}_{G(\mathbb{A}_f)}(W(({}^\sigma\Pi'(r)\phi^e)_f), H^{b_n}(S_n, {}^\sigma\mathcal{E}_\mu))$ is one-dimensional and one has the equality $\tilde{\sigma}^{b_n}(H^{b_n}(S_n, \mathcal{E}_\mu)((\Pi'(r)\phi^e)_f)) = H^{b_n}(S_n, {}^\sigma\mathcal{E}_\mu)(({}^\sigma\Pi'(r)\phi^e)_f)$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.*

Proof. By §1.4.3, it is enough to show this for $\sigma = id$. As shown in [Lan79] (combine Lem. 1 and the arguments buried on p. 204–205 therein), the family of all isobaric automorphic representations of $G(\mathbb{A}_F)$ exhausts the space of subrepresentations of the space $\mathcal{A}_\mathcal{J}(G, r + e/2)$ of automorphic forms, which are annihilated by some power of $\mathcal{J}$ and transform by the character $\|\det\|^{r+e/2}$ on the right. Hence, by [Fra98], Thm. 18, the one-dimensionality of $H^{b_n}(\mathfrak{g}_\infty, K_\infty, (\Pi'(r)\phi^e)_\infty \otimes \mathcal{E}_\mu)$ together with Multiplicity One and Strong Multiplicity One for isobaric automorphic representations, cf. [Jac-Sha81b], Thm. 4.4, shows that $(\Pi'(r)\phi^e)_f$ appears precisely once as $G(\mathbb{A}_f)$-submodule of

$H^{b_n}(S_n, \mathcal{E}_\mu)$. Hence, the space $\mathrm{Hom}_{G(\mathbb{A}_f)}(W((\Pi'(r)\phi^e)_f), H^{b_n}(S_n, \mathcal{E}_\mu))$ is one-dimensional. Bearing in mind that all Eisenstein series $E_P(h, \lambda)$, h running through the K_∞-finite sections $h \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\tau(r)\phi^e]$, are holomorphic at their point of evaluation $\lambda = 0$ prescribed by the associate class φ_P of $\tau(r)\phi^e$, the second assertion follows precisely as in [Gro17a], Prop. 1.6. □

Corollary 1.13. *For each Eisenstein representation $\Pi'(r)\phi^e$ the bottom-degree Whittaker periods $p(\Pi'(r)\phi^e) = p(\Pi'(r)\phi^e, \Upsilon)$ are well-defined. In particular, putting $e = 0$, and $k = 1$, every cohomological, essentially unitary cuspidal automorphic representation Π and putting $e = r = 0$ every unitary cohomological isobaric sum Π' as in §1.4.3, satisfies the assumptions of Prop. 1.8 in its minimal, non-vanishing degree b_n and with Υ as above.*

Remark 1.14. In the case of cusp forms our construction recovers the Whittaker periods constructed in [Rag-Sha08], whereas in the case of unitary isobaric sums Π' we retrieve the periods considered in [Gro17a, §1.8.2]. Moreover, as it follows from the proof of Thm. 1.12, our family of Eisenstein representations is in fact the largest class of automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, for which a Whittaker periods may be defined in the sense of the procedure outlined in Prop. 1.8.

1.6. **Relations of algebraicity.**

Definition 1.15. Let $\mathbb{F} \subset \mathbb{C}$ be any subfield and let $x, y \in \mathbb{C}$ be two complex numbers. We say $x \sim_{\mathbb{F}} y$ if there is an $a \in \mathbb{F}$ such that $x = ay$ or $ax = y$.

Remark 1.16. Note that this relation is symmetric, but not transitive unless all sides of the relation are non zero. More precisely, if $x, y, z \in \mathbb{C}$ such that $x \sim_{\mathbb{F}} y$ and $y \sim_{\mathbb{F}} z$, then we do not have $x \sim_{\mathbb{F}} z$ in general, unless $xyz \neq 0$.

The main goal of this paper is to prove several such relations among different L-values and various periods. As we have seen in the previous sections, the automorphism group $\mathrm{Aut}(\mathbb{C})$ acts on the set of representations, hence it will also act on the set of L-values and periods. The relations that we will prove behave well under the action of $\mathrm{Aut}(\mathbb{C})$ in the following sense.

Definition 1.17. Let $\mathbb{F}, L \subset \mathbb{C}$ be two subfields. Let $x = \{x(\sigma)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ and $y = \{y(\sigma)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ be two families of complex numbers. We say $x \sim_L y$ (and this relation) *is equivariant under* $\mathrm{Aut}(\mathbb{C}/\mathbb{F})$, if either $y(\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, or if $y(\sigma) \neq 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ and the following two conditions are verified:

(1) $x(\sigma) \sim_{\sigma(L)} y(\sigma)$ for all σ;

(2) $\sigma\left(\dfrac{x(\tau)}{y(\tau)}\right) = \dfrac{x(\sigma\tau)}{y(\sigma\tau)}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{F})$ and all $\tau \in \mathrm{Aut}(\mathbb{C})$.

Remark 1.18. One can replace the first condition by requiring $x(\sigma) \sim_{\sigma(L)} y(\sigma)$ for all σ running through a choice of representatives of $\mathrm{Aut}(\mathbb{C})/\mathrm{Aut}(\mathbb{C}/\mathbb{F})$. In particular, if $\mathbb{F} = \mathbb{Q}$, instead of verifying the first condition for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, one only needs to verify it for a single fixed $\sigma_0 \in \mathrm{Aut}(\mathbb{C})$.

Lemma 1.19 (Minimizing-Lemma)**.** *Let $\mathbb{F} \subset \mathbb{C}$ be any number field and let $L \subset \mathbb{C}$ be a number field, containing $\mathbb{F}^{Gal}$. Let $x = \{x(\sigma)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ and $y = \{y(\sigma)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ be as in Def. 1.17 and suppose that $y(\sigma) \neq 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. If the complex numbers $x(\sigma)$ and $y(\sigma)$ depend only on the restriction of σ to L, then the second condition of Def. 1.17 implies the first.*

Proof. Fix $\sigma_0 \in \mathrm{Aut}(\mathbb{C})$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ fixing $\sigma_0(L)$, one has $\sigma\sigma_0 |_L = \sigma_0 |_L$. Hence $x(\sigma\sigma_0) = x(\sigma_0)$ and $y(\sigma\sigma_0) = y(\sigma_0)$ by our assumptions. Moreover, since $L \supset \mathbb{F}^{Gal}$, we know $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{F})$. By the second condition, we have:

$$\sigma\left(\frac{x(\sigma_0)}{y(\sigma_0)}\right) = \frac{x(\sigma\sigma_0)}{y(\sigma\sigma_0)} = \frac{x(\sigma_0)}{y(\sigma_0)}.$$

Therefore $\frac{x(\sigma_0)}{y(\sigma_0)} \in \sigma_0(L)$ for all σ_0 as expected.

□

Remark 1.20. The previous lemma allows us to minimize the number field $\mathbb{F}$ in the relation. This will be very useful in the proof of the main theorems. Its omnipresence in our arguments is one of the main reasons why our formulas only hold over F^{Gal}.

As a first, useful lemma, these notions imply

Lemma 1.21. *Interpreted as a family of complex numbers as in Def. 1.17, $\mathcal{G}(\varepsilon_f) \sim_{F^{Gal}} 1$ is equivariant under the action of* $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. Recall that $L(s,\varepsilon_f) = \frac{\zeta_F(s)}{\zeta_{F^+}(s)}$. Let Reg_F and Reg_{F^+} be the regulator of F and F^+ respectively. By Proposition 3.7 of [Paz14], we know $Reg_F \sim_{\mathbb{Q}} Reg_{F^+}$. Hence, denoting the absolute discriminant of F (resp. F^+) by D_F (resp. D_{F^+}), the class number formula implies that

$$L(1,\varepsilon_f) = \frac{Res_{s=1}\zeta_F(s)}{Res_{s=1}\zeta_{F^+}(s)} \sim_{\mathbb{Q}} \frac{(2\pi)^d Reg_F |D_{F^+}|^{1/2}}{Reg_{F^+}|D_F|^{1/2}} \sim_{\mathbb{Q}} \frac{(2\pi)^d |D_{F^+}|^{1/2}}{|D_F|^{1/2}}.$$

On the other hand, a classical result of Siegel, [Sie69] (revealing $L(1-m,\varepsilon_f) \in \mathbb{Q}$ for $m \geq 1$), combined with the functional equation, cf., e.g., [Bum97], §3.1, shows that $L(m,\varepsilon_f) \sim_{\mathbb{Q}} \mathcal{G}(\varepsilon_f)(2\pi i)^{md}$, for odd $m \geq 1$. Consequently, we obtain

$$\mathcal{G}(\varepsilon_f) \sim_{\mathbb{Q}} i^d \frac{|D_{F^+}|^{1/2}}{|D_F|^{1/2}}.$$

Since F^+ is totally real, we know that $|D_{F^+}|^{1/2} = \pm D_{F^+}^{1/2} \sim_{F^+,Gal} 1$. It remains to show that $|D_F|^{1/2} \sim_{F^{Gal}} i^d$. To this end, let $\alpha \in F$ be a purely imaginary element, i.e., $\bar{\alpha} = -\alpha$. Since $-2\alpha = \det\begin{pmatrix} 1 & \alpha \\ 1 & -\alpha \end{pmatrix}$, it is easy to see that $|N_{F^+/\mathbb{Q}} D_{F/F^+}|^{1/2} \sim_{\mathbb{Q}} \prod_{v\in S_\infty} |\iota_v(\alpha)|$ where D_{F/F^+} is the relative discriminant with respect to F/F^+. So, $|D_F|^{1/2} \sim_{\mathbb{Q}} \prod_{v\in S_\infty} |\iota_v(\alpha)| \cdot |D_{F^+}|^{1/2} \sim_{F^+,Gal} \prod_{v\in S_\infty} |\iota_v(\alpha)|$. We know that $\prod_{v\in S_\infty} \iota_v(\alpha)$ is an algebraic number giving rise to an extension of $\mathbb{Q}$ of degree 2. Its complex conjugate equals $(-1)^d \prod_{v\in S_\infty} \iota_v(\alpha)$. Hence if d is even, then it is real quadratic and $\prod_{v\in S_\infty} |\iota_v(\alpha)| = \pm \prod_{v\in S_\infty} \iota_v(\alpha) \sim_{F^{Gal}} 1 = i^d$; otherwise, it is imaginary quadratic and $\prod_{v\in S_\infty} |\iota_v(\alpha)| = \pm i \prod_{v\in S_\infty} \iota_v(\alpha) \sim_{F^{Gal}} i \sim_{F^{Gal}} i^d$ as expected. □

As a consequence of the above discussion, one has

$$\zeta_{F^+}(m) \sim_{F^{Gal}} (2\pi i)^{md} \text{ if } m \geq 2 \text{ is even} \tag{1.22}$$

$$L(m,\varepsilon_f) \sim_{F^{Gal}} (2\pi i)^{md} \text{ if } m \geq 1 \text{ is odd} \tag{1.23}$$

both relations being equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. Indeed, the first relation follows from applying the functional equation and the fact that $|D_{F^+}|^{1/2} \sim_{F^{Gal}} 1$, explained in the proof of Lem. 1.21, to [Sie69]; whereas the latter follows directly from Lem. 1.21 and the fact that $L(m,\varepsilon_f) \sim_{\mathbb{Q}} \mathcal{G}(\varepsilon_f)(2\pi i)^{md}$, for odd $m \geq 1$, as explained in the proof of of Lem. 1.21.

1.7. **Critical values of automorphic L-functions.** Let $\mathscr{G}$ be a connected reductive algebraic group over a number field $\mathbb{F}$. Assume that $N = N(\mathbb{F}) \geq 1$ is chosen minimal such that $\mathscr{G}$ embeds into $\mathrm{GL}_N/\mathbb{F}$. Let π be an irreducible admissible representation of $\mathscr{G}(\mathbb{A}_\mathbb{F})$ for which a completed standard L-function $L(s,\pi) = \prod_v L(s,\pi_v)$ is defined satisfying a global functional equation $L(s,\pi) = \varepsilon(s,\pi) \cdot L(1-s,\pi^\vee)$, cf. [Bor79, §IV].

Definition 1.24. A complex number $s_0 \in \frac{N-1}{2} + \mathbb{Z}$ is called *critical* for $L(s,\pi)$ if both $L(s,\pi_\infty)$ and $L(1-s,\pi^\vee_\infty)$ are holomorphic at $s = s_0$.

1.7.1. *Critical points for Rankin-Selberg L-function.* Let $\Pi(r)$ (resp. $\Pi'(s)$) be an essentially unitary, generic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ (resp. $\mathrm{GL}_{n-1}(\mathbb{A}_F)$), cohomological with respect to $\mathcal{E}_\mu$ (resp. $\mathcal{E}_{\mu'}$). For $\iota \in \Sigma$ let us write $a_{\iota,i} := \ell(\mu_\iota, i) + r$ and $b_{\iota,j} := \ell(\mu'_\iota, j) + s$ for their respective infinity-types at ι, cf. §1.4.1. We assume that

$$a_{\iota,i} + b_{\iota,j} \neq r + s \text{ for any } 1 \leq i \leq n, 1 \leq j \leq n-1 \text{ and any } \iota \in \Sigma. \tag{1.25}$$

The critical points for the Rankin-Selberg L-function $L(s, \Pi \times \Pi')$ are the half integers $\frac{1}{2} + m$ with $m \in \mathbb{Z}$ such that

$$-\min_{i,j,\iota}\{|a_{\iota,i} + b_{\iota,j} - r - s|\} < \tfrac{1}{2} + m + r + s \leq \min_{i,j,\iota}\{|a_{\iota,i} + b_{\iota,j} - r - s|\}.$$

This can be easily seen using Deligne's approach in terms of Hodge types of motives, cf. [Del79] and [MHar-Lin17]. For a direct computation involving only representation theory of real reductive groups, see [Rag16], Cor. 2.35. It is easy to see that the set of critical points is non empty. We denote it by $\mathrm{Crit}(\Pi(r) \times \Pi'(s))$. For example, if $r = s = 0$, then $\frac{1}{2}$ is always a critical point.

Remark 1.26. (1) There are no critical points for $\Pi(r) \times \Pi'(s)$ if the inequality (1.25) is not satisfied (cf. 1.7 of [MHar97]).

(2) The precise relation between highest weights and infinity-types is given in subsection 1.4.1. One can check easily that (1.25) is automatically satisfied if the *piano-hypothesis*, i.e., Hypothesis 1.29 below, holds for μ and μ'. If this is the case and if both representations $\Pi(r)$ and $\Pi'(s)$ are Eisenstein representations, cf. §1.4.3, then $\mathrm{Crit}(\Pi(r) \times \Pi'(s)) = \mathrm{Crit}({}^\sigma\Pi(r) \times {}^\sigma\Pi'(s))$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$: For this combine [Gro17a], 1.5-1.6 and [Gro-MHar16], Lem. 3.5.

1.7.2. *Critical points for the Asai L-functions.* Consider now $\Pi = \Pi(0)$, i.e., a unitary generic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, which is cohomological with respect to $\mathcal{E}_\mu$. We shall additionally assume now that Π is conjugate self-dual. Then the calculations in section 1.3 of [MHar13] show that the critical points of $L(s, \Pi, \mathrm{As}^{(-1)^n})$ (resp. $L(s, \Pi, \mathrm{As}^{(-1)^{n-1}})$ are the positive odd or non-positive even integers (resp. positive even or negative odd) m such that

$$\max_{i,j,\iota}\{a_{\iota,i} - a_{\iota,j} \mid a_{\iota,i} - a_{\iota,j} < 0\} < m \leq \min_{i,j,\iota}\{a_{\iota,i} - a_{\iota,j} \mid a_{\iota,i} - a_{\iota,j} > 0\}.$$

In particular, the integers 0 and 1 are always critical for $L(s, \Pi, \mathrm{As}^{(-1)^n})$ and never for $L(s, \Pi, \mathrm{As}^{(-1)^{n-1}})$.

1.8. Two rationality theorems revisited.

We recall now two rationality-results for critical L-values, proved in [Gro-MHar-Lap16] and [Gro17a], which are the starting point of our investigations:

Theorem 1.27 ([Gro-MHar-Lap16] Thm. 7.1)**.** *Let Π be a conjugate self-dual, cuspidal automorphic representation of $G(\mathbb{A}_F) = \mathrm{GL}_n(\mathbb{A}_F)$, which is cohomological with respect to $\mathcal{E}_\mu$. Let $p(\Pi)$ be the bottom-degree Whittaker period defined in Cor. 1.13. Then, the following holds:*

(1) *For all $\sigma \in \mathrm{Aut}(\mathbb{C}/F^{Gal})$, there exists a non-zero constant $a({}^\sigma\Pi_\infty)$, only depending on the archimedean component of ${}^\sigma\Pi$, such that*

$$\sigma\left(\frac{L^S(1, \Pi, \mathrm{As}^{(-1)^n})}{p(\Pi)a(\Pi_\infty)}\right) = \frac{L^S(1, {}^\sigma\Pi, \mathrm{As}^{(-1)^n})}{p({}^\sigma\Pi)a({}^\sigma\Pi_\infty)}.$$

(2) *Equivalently, interpreting both sides below as a family of numbers $x = \{x(\sigma)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ and $y = \{y(\sigma)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ as in Def. 1.17,*

$$L^S(1, \Pi, \mathrm{As}^{(-1)^n}) \sim_{E(\Pi)} a(\Pi_\infty)\ p(\Pi),$$

is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Remark 1.28. Theorem 1.27 has been stated in [Gro-MHar-Lap16] without reference to the Whittaker periods $p(\Pi)$. For the convenience of the reader we sketch how one obtains the above transcription, which shall be part of P. Lopez's thesis: Thm. 7.1 of [Gro-MHar-Lap16] is a consequence of Thm. 5.3 and 6.4 *ibidem*. Thm. 6.4 can be rewritten as an $\mathrm{Aut}(\mathbb{C})$-equivariant relation of the residue $\mathrm{Res}_{s=1}(L^S(s,\Pi\times\Pi^\vee))$ with an archimedean period, the bottom-degree Whittaker period $p(\Pi)$ and a top-degree version of the latter, as explained in [Gro17b], Thm. 8.5. Similarly, Thm. 5.3 in [Gro-MHar-Lap16] can be restated as an $\mathrm{Aut}(\mathbb{C}/F^{Gal})$-equivariant relation of the residue $\mathrm{Res}_{s=1}(L^S(s,\Pi,\mathrm{As}^{(-1)^{n-1}}))$, an archimedean period and the above mentioned top-degree Whittaker period. Taking the quotient of the first and the second relation gives the theorem.

For the next rationality-result we note that a pair of irreducible algebraic representations $\mathcal{E}_\mu$ of $G_{n,\infty}$ and $\mathcal{E}_{\mu'}$ of $G_{n-1,\infty}$, given by their highest weights $\mu=(\mu_v)_{v\in S_\infty}$ and $\mu'=(\mu'_v)_{v\in S_\infty}$ satisfy the *piano-hypothesis*[2] if the following holds

Hypothesis 1.29 (Piano-hypothesis). *At each archimedean place $v=(\iota_v,\bar\iota_v)\in S_\infty$,*

$$\mu_{\iota_v,1}\geq -\mu'_{\iota_v,n-1}\geq \mu_{\iota_v,2}\geq -\mu'_{\iota_v,n-2}\geq \ldots\geq -\mu'_{\iota_v,1}\geq \mu_{\iota_v,n}$$

$$\mu^\vee_{\bar\iota_v,1}\geq -\mu'^\vee_{\bar\iota_v,n-1}\geq \mu^\vee_{\bar\iota_v,2}\geq -\mu'^\vee_{\bar\iota_v,n-2}\geq \ldots\geq -\mu'^\vee_{\bar\iota_v,1}\geq \mu^\vee_{\bar\iota_v,n}.$$

Theorem 1.30 ([Gro17a] Thm. 1.8). *Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is cohomological with respect to $\mathcal{E}_\mu$ and let Π' by an isobaric automorphic representation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, as in §1.4.3, cohomological with respect to $\mathcal{E}_{\mu'}$. Let $p(\Pi)$ and $p(\Pi')$ be the bottom-degree Whittaker periods defined in Cor. 1.13. Assume that Π' has central character $\omega_{\Pi'}$ and that the highest weights $\mu=(\mu_v)_{v\in S_\infty}$ and $\mu'=(\mu'_v)_{v\in S_\infty}$ satisfy the piano-hypothesis, cf. Hypothesis 1.29. Then, the following holds:*

(1) *For every $\sigma\in\mathrm{Aut}(\mathbb{C})$ and all critical values $\frac12+m\in\mathrm{Crit}(\Pi\times\Pi')=\mathrm{Crit}({}^\sigma\Pi\times{}^\sigma\Pi')$, there exists a non-zero constant $p(m,{}^\sigma\Pi_\infty,{}^\sigma\Pi'_\infty)$, only depending on m and the archimedean components of ${}^\sigma\Pi$ and ${}^\sigma\Pi'$ such that*

$$\sigma\left(\frac{L^S(\frac12+m,\Pi\times\Pi')}{p(\Pi)\ p(\Pi')\ p(m,\Pi_\infty,\Pi'_\infty)\ \mathcal{G}(\omega_{\Pi'_f})}\right)=\frac{L^S(\frac12+m,{}^\sigma\Pi\times{}^\sigma\Pi')}{p({}^\sigma\Pi)\ p({}^\sigma\Pi')\ p(m,{}^\sigma\Pi_\infty,{}^\sigma\Pi'_\infty)\ \mathcal{G}(\omega_{{}^\sigma\Pi'_f})},$$

(2) *Equivalently, interpreting both sides below as a family of numbers $x=\{x(\sigma)\}_{\mathrm{Aut}(\mathbb{C})}$ and $y=\{y(\sigma)\}_{\mathrm{Aut}(\mathbb{C})}$ as in Def. 1.17,*

$$L^S(\tfrac12+m,\Pi\times\Pi')\ \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')} p(\Pi)\ p(\Pi')\ p(m,\Pi_\infty,\Pi'_\infty)\ \mathcal{G}(\omega_{\Pi'_f})$$

is equivariant under $\mathrm{Aut}(\mathbb{C})$.

Remark 1.31. We can replace the Gauß sum $\mathcal{G}(\omega_{\Pi'_f})$ by $\mathcal{G}(\omega_{\Pi'_f}|_{F^+})$ (see (2.13) and (2.14) below). In particular, when Π' is conjugate self-dual, $\omega_{\Pi'_f}$ is trivial on $N_{\mathbb{A}_F/\mathbb{A}_{F^+}}(\mathbb{A}_F^\times)$, so $\omega_{\Pi'_f}|_{F^+}$ is either trivial or the finite part of the quadratic character ε associate to the extension F/F^+. However, as Π' is assumed to be cohomological, necessarily $\omega_{\Pi'_f}|_{F^+}=\mathbf{1}_f$. Therefore we can savely remove the Gauß sum $\mathcal{G}(\omega_{\Pi'_f})$ in Thm. 1.30 when Π' is conjugate self-dual.

[2] This condition has been called "*interlacing-hypothesis*" in [Gro17a]. Here we prefer to call it after the more poetical picture of a piano's keys, however: The white ones ("whole-tones") taking the role of the coordinates of μ, the black keys ("half-tones") taking the role of the coordinates of μ'.

2. Period relations for isobaric sums

2.1. **Eisenstein representations and boundary cohomology.** Let $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ be an Eisenstein representation as in 1.4.3, cohomological with respect to $\mathcal{E}_\mu$. Then there is a unique Kostant representative $w \in W^P$ (cf. [Bor-Wal00] III.1.4 & Thm. III.3.3) such that $\Pi_i \rho_i$ is cohomological with respect to the irreducible algebraic coefficient module $\mathcal{E}_{\mu_w,i} := \mathcal{E}_{w(\mu+\rho)-\rho}|_{\mathrm{GL}_{n_i,\infty}}$. Since the point of evaluation of the Eisenstein series in Π' is always centered at $\lambda = 0$, see the proof of Lem. 1.6, the length of this Kostant representative is $\ell(w) = \frac{1}{2}\dim_{\mathbb{R}} N_{P,\infty}$, cf. [Bor80], Lem. 2.12 and hence minimal by [Gro13], Prop. 12.

Let $\partial_P S_G := P(F)\backslash G(\mathbb{A}_F)/K_\infty$ be the face corresponding to the parabolic subgroup $P \subseteq G$ in the Borel–Serre–compactification of S_G, cf. [Bor-Ser73], [Roh96]. It is well-known (cf. [Sch90], 7.1–7.2) that there is an isomorphism of $G(\mathbb{A}_f)$-modules

$$H^q(\partial_P S_G, \mathcal{E}_\mu) \ \xrightarrow{\sim}\ {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left[\bigoplus_{w\in W^P} H^{q-\ell(w)}(L(F)\backslash L(\mathbb{A}_F)/(K_\infty\cap L_\infty), \mathcal{E}_{\mu_w})\right]$$

"^{a}Ind" denoting un-normalized or algebraic induction. Hence the Künneth rule implies that there is furthermore an isomorphism of $G(\mathbb{A}_f)$-modules

$$I^q_{\partial_P} : H^q(\partial_P S_G, \mathcal{E}_\mu) \xrightarrow{\sim} {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left[\bigoplus_{w\in W^P}\ \bigoplus_{q_1+...+q_k=q-\ell(w)}\ \bigotimes_{i=1}^k H^{q_i}(S_{n_i}, \mathcal{E}_{\mu_w,i})\right], \tag{2.1}$$

Let $W(A_P) := N_{G(F)}(A_P(F))/L(F)$, interpreted as the set of automorphisms $A_P \xrightarrow{\sim} A_P$, which are given by conjugation by an element in $G(F)$ and which induce an isomorphism $L \xrightarrow{\sim} L$. Hence, the elements $\tilde{w} \in W(A_P)$ act naturally on the inducing representation $\tau = \Pi_1 \otimes ... \otimes \Pi_k$ of Π', by permuting its factors: $\tau^{\tilde{w}} = \Pi_{\tilde{w}^{-1}(1)} \otimes ... \otimes \Pi_{\tilde{w}^{-1}(k)}$.

Let

$$res_P : H^{b_n}(S_G, \mathcal{E}_\mu) \to H^{b_n}(\partial_P S_G, \mathcal{E}_\mu)$$

be the natural restriction of classes to the face $\partial_P S_G$ in the boundary of the Borel–Serre–compactification of S_G. It is obviously Aut($\mathbb{C}$)-equivariant. The image under res_P of a class $[\omega] \in H^{b_n}(S_n, \mathcal{E}_\mu)(\Pi'_f)$, see §1.5.2, is given by the class represented by the constant term along P of the Eisenstein series representing $[\omega]$, cf. [Sch83], Satz 1.10. Hence, recalling the well-defined (i.e., holomorphic at $s = 0$) intertwining operators

$$M(\tau, w) = M(\tau, s, \tilde{w})|_{s=0} : \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}[\tau] \to \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\left[\tau^{\tilde{w}}\right],$$

from [Mœ-Wal95], II.6 (holomorphy at $s = 0$ following from [Mœ-Wal95, IV.1.11]) and the description of the constant term of an Eisenstein series $E_P(h, 0) \in \Pi'$ from II.7, *ibidem*, [Bor80], 2.9–2.13, implies that the image of $res_P([\omega])$ under $I^{b_n}_{\partial_P}$ lies inside the direct sum

$$I^{b_n}_{\partial_P}(res_P([\omega])) \in \bigoplus_{\tilde{w}\in W(A_P)} {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left[\bigotimes_{i=1}^k H^{b_{n_i}}(S_{n_i}, \mathcal{E}_{\mu_w,i})((\Pi_{\tilde{w}^{-1}(i)}\rho_i)_f)\right] \tag{2.2}$$

Here we also used the minimality of the length $\ell(w)$ of the unique Kostant representative $w = w(\tau^{\tilde{w}}, b_n) \in W^P$ giving rise to the coefficients modules $\mathcal{E}_{\mu_w,i}$ with respect to which $\Pi_{\tilde{w}^{-1}(i)}\rho_i$ is cohomological. However, since $id \in W(A_P)$ and since the attached intertwining operator $M(\tau, id) =$

$\mathbf{1}$ is the identity-map, (2.2) implies that the composition $I^{b_n}_{\partial_P} \circ res_P$ induces an isomorphism of $G(\mathbb{A}_f)$-modules

$$r_{\Pi',P} : H^{b_n}(S_n, \mathcal{E}_\mu)(\Pi'_f) \xrightarrow{\sim} {}^a\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}\left[\bigotimes_{i=1}^{k} H^{b_{n_i}}(S_{n_i}, \mathcal{E}_{\mu_w,i})((\Pi_i\rho_i)_f)\right], \tag{2.3}$$

by projecting onto the summand indexed by $\tilde{w} = id$. The following lemma is then obvious by construction.

Lemma 2.4. *The map $r_{\Pi',P}$ is $\mathrm{Aut}(\mathbb{C})$-equivariant, i.e., for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$${}^a\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}\left[\otimes_{i=1}^{k}\tilde{\sigma}^{b_{n_i}}\right] \circ r_{\Pi',P} = r_{^\sigma\Pi',P} \circ \tilde{\sigma}^{b_n}$$

2.2. **A theorem on period relations for isobaric sums.** We are now ready to prove

Theorem 2.5. *Let $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ be an Eisenstein representation of $\mathrm{GL}_n(\mathbb{A}_F)$ as in §1.4.3, which is cohomological with respect to the irreducible algebraic representation $\mathcal{E}_\mu$ of G_∞. Then the bottom-degree Whittaker periods $p(\Pi')$ and $p(\Pi_i\rho_i)$ are all defined, cf. 1.13, and*

(1) *For all $\sigma \in$*$\mathrm{Aut}(\mathbb{C})$

$$\sigma\left(\frac{p(\Pi')\prod_{1\leq i<j\leq k} L^S(1,\Pi_i \times \Pi_j^\vee)^{-1}}{\prod_{1\leq i\leq k} p(\Pi_i\rho_i)}\right) = \frac{p(^\sigma\Pi')\prod_{1\leq i<j\leq k} L^S(1,{}^\sigma\Pi_i \times ({}^\sigma\Pi_j)^\vee)^{-1}}{\prod_{1\leq i\leq k} p(^\sigma(\Pi_i\rho_i))}$$

(2) *Equivalently,*

$$p(\Pi') \sim_{\mathbb{Q}(\Pi')} \prod_{1\leq i\leq k} p(\Pi_i\rho_i) \prod_{1\leq i<j\leq k} L^S(1,\Pi_i \times \Pi_j^\vee)$$

is $\mathrm{Aut}(\mathbb{C})$*-equivariant.*

Proof. We proceed in several steps.
Step 1: We observe that Π' being cohomological implies by [Bor-Wal00], Thm. III.3.3, that $\Pi_i\rho_i$ is cohomological for all $1 \leq i \leq k$. The bottom-degree Whittaker periods $p(\Pi')$ and $p(\Pi_i\rho_i)$ are hence all well-defined by our Cor. 1.13 and depend on our specified choice of embedding Υ_Π, resp. $\Upsilon_{\Pi_i\rho_i}$, see Def. 1.11. Hence, we obtain two isomorphisms of $G(\mathbb{A}_f)$-modules

$$\Delta_G : H^{b_n}(S_n, \mathcal{E}_\mu)(\Pi'_f) \xrightarrow{\sim} W(\Pi'_f)$$

defined as

$$\Delta_G := p(\Pi') \cdot \Upsilon^{-1}_{\Pi'}$$

and

$$\Delta_P : {}^a\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}\left[\bigotimes_{i=1}^{k} H^{b_{n_i}}(S_{n_i}, \mathcal{E}_{\mu_w,i})((\Pi_i\rho_i)_f)\right] \xrightarrow{\sim} {}^a\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}\left[\bigotimes_{i=1}^{k} W((\Pi_i\rho_i)_f)\right]$$

defined as

$$\Delta_P := \prod_{i=1}^{k} p(\Pi_i\rho_i) \cdot {}^a\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}\left[\otimes_{i=1}^{k}\Upsilon^{-1}_{\Pi_i\rho_i}\right]$$

which are $\mathrm{Aut}(\mathbb{C})$-equivariant by definition of the periods.
Take a non-trivial class $[\omega]$ in $H^{b_n}(S_n, \mathcal{E}_\mu)(\Pi'_f)$. By construction, the sheaf-theoretical differential form ω representing it, is of the form

$$\omega = \sum_{\substack{\underline{i}=(i_1,...,i_{b_n})\\ \alpha}} \left(\frac{\partial^{\underline{n}}}{\partial\lambda^{\underline{n}}} E_P(h_{\infty,\underline{i},\alpha} \otimes h_f, \lambda)|_{\lambda=0} \otimes e_\alpha\right)\, dx_{\underline{i}},$$

for appropriate K_∞-finite sections $h_{\infty,\underline{i},\alpha} \otimes h_f \in \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\left[\bigotimes_{i=1}^k \Pi_i\right]$. We may suppose that $\underline{n} = 0$. Indeed, since $E_P(h_{\infty,\underline{i},\alpha} \otimes h_f, \lambda)$ is holomorphic at $\lambda = 0$, see Lem. 1.6, the map

$$H^{b_n}(\mathfrak{g}_\infty, K_\infty, \Pi' \otimes \mathcal{E}_\mu) \to H^{b_n}(S_n, \mathcal{E}_\mu)$$

$$\left[\sum_{\substack{\underline{i}=(i_1,\dots,i_{b_n})\\ \alpha}} X^*_{\underline{i}} \otimes E_P(h_{\infty,\underline{i},\alpha} \otimes h_f, 0) \otimes e_\alpha\right] \mapsto \left[\sum_{\substack{\underline{i}=(i_1,\dots,i_{b_n})\\ \alpha}} \left(E_P(h_{\infty,\underline{i},\alpha} \otimes h_f, 0) \otimes e_\alpha\right)\ dx_{\underline{i}}\right]$$

is injective by [Sch83], Satz 4.11. By construction its image must hence be isotypical to Π'_f and hence by Thm. 1.12 equal to $H^{b_n}(S_n, \mathcal{E}_\mu)(\Pi'_f)$. See also [Bor80], 2.9 or [Spe82], Thm. 1. Given this finer description of $[\omega]$, let us now specify the section h_f: We suppose that it is decomposable, while and at $v \notin S$ we take h_v to be the unique spherical vector in ${}^a\mathrm{Ind}_{P(F_v)}^{G(F_v)}\left[\bigotimes_{i=1}^k (\Pi_i\rho_i)_v\right]$ which is 1 on $id \in G(F_v)$. (Observe that this expression makes sense, once we view the spherical representation ${}^a\mathrm{Ind}_{P(F_v)}^{G(F_v)}\left[\bigotimes_{i=1}^k (\Pi_i\rho_i)_v\right]$ via induction in stages as being induced from local unramified characters (i.e., more precisely, $h_v(id_{G(F_v)})(id_{L_P(F_v)}) = \otimes_{i=1}^n 1$ as an element in the representation space of these characters).) The image of our so-obtained Eisenstein class $[\omega]$ in $H^{b_n}(\mathfrak{g}_\infty, K_\infty, W(\Pi') \otimes \mathcal{E}_\mu)$ under the map described in Prop. 1.10 then equals

$$\left[\sum_{\substack{\underline{i}=(i_1,\dots,i_{b_n})\\ \alpha}} X^*_{\underline{i}} \otimes \left(\prod_{v\in S_\infty} W_{h_{v,\underline{i},\alpha}} \prod_{v\notin S_\infty} W_{h_v}\right) \otimes e_\alpha\right]$$

as shown in [Sha10], Prop. 7.1.3: Here, the $W_{h_{v,\underline{i},\alpha}}$ resp. W_{h_v} are defined as in [Sha10], (7.1.2), i.e., the local Whittaker function in the Whittaker model of ${}^a\mathrm{Ind}_{P(F_v)}^{G(F_v)}\left[\bigotimes_{i=1}^k (\Pi_i\rho_i)_v\right]$, determined by $h_{v,\underline{i},\alpha}$, resp. h_v. Here we use the equality (not only isomorphy!) of the latter space with $W(\Pi'_v)$. Now, choose $h_{v,\underline{i},\alpha}$ such that $\prod_{v\in S_\infty} W_{h_{v,\underline{i},\alpha}} = \xi_{\infty,\underline{i},\alpha}$ as in §1.5.2. Then, finally, $\Delta_G([\omega])$ equals the Whittaker functional

$$\Delta_G([\omega]) : g_f \mapsto p(\Pi') \prod_{v\notin S_\infty} W_{h_v}(g_v)$$

We note that by our choice of h_f we obtain

$$\prod_{v\notin S} W_{h_v}(id) = \prod_{1\le i<j\le k} L^S(1, \Pi_i \times \Pi_j^\vee)^{-1},$$

and choosing h_v at the remaining unspecified places, i.e., $v \in S\backslash S_\infty$ accordingly, we may also suppose that $\prod_{v\in S\backslash S_\infty} W_{h_v}(id) \neq 0$: The first statement is the contents of [Sha10], Thm. 7.1.2, while the second is shown in [Sha10], 3.3.

Having fixed this choice of the vector h_f, let us write ${}^\sigma h_f$ for the corresponding vector of the class $\tilde{\sigma}^{b_n}([\omega]) = \tilde{\sigma}^{b_n}\left(\left[\sum_{\underline{i}=(i_1,\dots,i_{b_n}),\alpha} \left(E_P(h_{\infty,\underline{i},\alpha} \otimes h_f, 0) \otimes e_\alpha\right)\ dx_{\underline{i}}\right]\right)$. Then, Lem. 2.4 shows that this vector ${}^\sigma h_f$ is the non-archimedean component vector of ${}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left[\otimes_{i=1}^k \tilde{\sigma}^{b_{n_i}}\right](r_{\Pi',P}([\omega]))$, and hence we can read of the action of $\sigma \in$Aut$(\mathbb{C})$ on the function h_v directly in terms of the actions on its values, see [Gro-MHar16], Rem. 2.6. In particular, ${}^\sigma h_v(id) = \sigma(h_v(id)) = 1$ at $v \notin S$, so

$$\prod_{v\notin S} W_{{}^\sigma h_v}(id) = \prod_{1\le i<j\le k} L^S(1, \Pi_i^\sigma \times (\Pi_j^\sigma)^\vee)^{-1}$$

where we abbreviated $\Pi_i^\sigma := {}^\sigma(\Pi_i\rho_i)\rho_i^{-1}$ for the cuspidal isobaric summands of ${}^\sigma\Pi'$ as in §1.4.3. Similarly, we obtain $\prod_{v\in S\backslash S_\infty} W_{{}^\sigma h_v}(id)\neq 0$. The Aut($\mathbb{C}$)-equivariance of Δ_G hence finally shows that

$$p({}^\sigma\Pi')\prod_{v\in S\backslash S_\infty} W_{{}^\sigma h_v}(id)\prod_{1\leq i<j\leq k} L^S(1,\Pi_i^\sigma\times(\Pi_j^\sigma)^\vee)^{-1} = \sigma\left(p(\Pi')\prod_{v\in S\backslash S_\infty} W_{h_v}(id)\prod_{1\leq i<j\leq k} L^S(1,\Pi_i\times\Pi_j^\vee)^{-1}\right). \tag{2.6}$$

Step 2: The Whittaker function $W_{h_v}\in W(\Pi'_v)=W({}^a\mathrm{Ind}_{P(F_v)}^{G(F_v)}\left[\bigotimes_{i=1}^k(\Pi_i\rho_i)_v\right])$ from above shall not be confused with the spherical Whittaker function $\xi_{h_v(id)}\in W(\bigotimes_{i=1}^k(\Pi_i\rho_i)_v)$, which is 1 on $id\in L_P(F_v)$ and attached to the local vector $h_v(id)\in\bigotimes_{i=1}^k(\Pi_i\rho_i)_v$. Applying Δ_P to the restriction $r_{\Pi',P}([\omega])$, with $[\omega]$ as above, and invoking the Aut($\mathbb{C}$)-equivariance of Δ_P hence shows that

$$\prod_{1\leq i\leq k} p({}^\sigma(\Pi_i\rho_i))\prod_{v\in S\backslash S_\infty}\xi_{{}^\sigma h_v(id)}(id)=\sigma\left(\prod_{1\leq i\leq k} p(\Pi_i\rho_i)\prod_{v\in S\backslash S_\infty}\xi_{h_v(id)}(id)\right). \tag{2.7}$$

The explicit calculations in [Mah05], §1.4.2–§1.4.4, imply that one may refine one's choice of h_v at $v\in S\backslash S_\infty$, such that

$$\prod_{v\in S\backslash S_\infty}\frac{W_{{}^\sigma h_v}(id)}{\xi_{{}^\sigma h_v(id)}(id)}=\sigma\left(\prod_{v\in S\backslash S_\infty}\frac{W_{h_v}(id)}{\xi_{h_v(id)}(id)}\right).$$

Hence, dividing (2.6) by (2.7) yields

$$\frac{p({}^\sigma\Pi')\prod_{1\leq i<j\leq k}L^S(1,\Pi_i^\sigma\times(\Pi_j^\sigma)^\vee)^{-1}}{\prod_{1\leq i\leq k}p({}^\sigma(\Pi_i\rho_i))}=\sigma\left(\frac{p(\Pi')\prod_{1\leq i<j\leq k}L^S(1,\Pi_i\times\Pi_j^\vee)^{-1}}{\prod_{1\leq i\leq k}p(\Pi_i\rho_i)}\right) \tag{2.8}$$

for all $\sigma\in$Aut($\mathbb{C}$). This proves (1).

Step 3: In order to show (the equivalence of (1) and) (2) in the theorem, observe that

$$\prod_{1\leq i<j\leq k} L^S(1,\Pi_i^\sigma\times(\Pi_j^\sigma)^\vee)=\prod_{1\leq i<j\leq k} L^S(1,\Pi_i\times(\Pi_j)^\vee)$$

for all $\sigma\in$Aut($\mathbb{C}$), for which $\{\Pi_1^\sigma,...,\Pi_k^\sigma\}=\{\Pi_1,...,\Pi_k\}$, because the Π_i are all conjugate self-dual. But this means that ${}^\sigma\Pi'\cong\Pi'$, see §1.4.3, so also the period $p(\Pi')$ is fixed by those σ. As obviously any σ, fixing all $\Pi_i\rho_i$, $1\leq i\leq k$, satisfies the above condition, the relation in (2) holds over the composite of number fields $\prod_{i=1}^k\mathbb{Q}(\Pi_i\rho_i)$. So, finally, in order to reduce the above relation to $\mathbb{Q}(\Pi')$, let $\sigma\in$ Aut($\mathbb{C}$) be any automorphism for which $\{\Pi_1^\sigma,...,\Pi_k^\sigma\}=\{\Pi_1,...,\Pi_k\}$. We need to show that then also

$$\prod_{1\leq j\leq k}p(\Pi_j\rho_j)=\prod_{1\leq i\leq k}p({}^\sigma(\Pi_i\rho_i)). \tag{2.9}$$

To this end, let i, $1\leq i\leq k$, be arbitrary. Then, by our condition on σ, there is a j, $1\leq j\leq k$, such $\Pi_i^\sigma\cong\Pi_j$. This implies that $n_i=n_j$, hence (1.7) shows that ${}^\sigma(\Pi_i\rho_i)\cong\Pi_j\rho_j\cdot\|\cdot\|^b$ for some integer $b\in\mathbb{Z}$. In particular, $p({}^\sigma(\Pi_i\rho_i))=p(\Pi_j\rho_j\cdot\|\cdot\|^b)$ and so furthermore $p({}^\sigma(\Pi_i\rho_i))\sim_{\mathbb{Q}(\Pi_j\rho_j)}p(\Pi_j\rho_j)$, by [Rag-Sha08], Thm. 4.1, observing that $\mathcal{G}(\|\cdot\|_f^b)=1$ and $\mathbb{Q}(\|\cdot\|^b)=\mathbb{Q}$. But since $p(\Pi_j\rho_j)$ is only defined up to non-trivial multiples of elements in $\mathbb{Q}(\Pi_j\rho_j)$, we may hence, without loss of generality, assume that we have

$$p({}^\sigma(\Pi_i\rho_i)) = p(\Pi_j\rho_j)$$

by readjusting $p(\Pi_j\rho_j)$ if necessary. We point out that this modification can be done consistently, i.e., without changing $p({}^\sigma(\Pi_j\rho_j))$ for the same reason: See the last line in the proof of [Rag-Sha08], Def./Prop. 3.3 and recall that $\sigma(\mathbb{Q}(\Pi_j\rho_j)) = \mathbb{Q}({}^\sigma(\Pi_j\rho_j))$. Hence, finally, we see that if $\sigma \in \mathrm{Aut}(\mathbb{C})$ is any automorphism such that $\{\Pi_1^\sigma, ..., \Pi_k^\sigma\} = \{\Pi_1, ..., \Pi_k\}$, then (2.9) holds. This shows the last claim. □

Remark 2.10. The above theorem also holds, when the isobaric summands of Π' are isobaric sums themselves: Let $1 \le \ell \le k$ and let $\bigcup_{1\le j\le \ell}\{i_1, ..., i_j\} = \{1, ..., k\}$ be a partition of $\{1, ..., k\}$ into ℓ disjoint sets. We set $\ell_j := n_{i_1} + ... + n_{i_j}$ and let P' be the standard parabolic subgroup of GL_n with Levi factor equal to $L' \cong \prod_{1\le j\le \ell}\mathrm{GL}_{\ell_j}$. Set $\Pi'_j := \Pi_{i_1} \boxplus ... \boxplus \Pi_{i_j}$ and ρ'_j to be equal to the restriction of $\rho_{P'}$ to $\mathrm{GL}_{\ell_j}(\mathbb{A}_F)$. With only a little more work one can in fact show that

$$p(\Pi') \sim_{\prod_{1\le j\le \ell}\mathbb{Q}(\Pi'_j\rho'_j)} \prod_{1\le j\le \ell} p(\Pi'_j\rho'_j) \prod_{1\le i<j\le \ell} L^S(1, \Pi'_i \times \Pi'^{\vee}_j)$$

is $\mathrm{Aut}(\mathbb{C})$-equivariant.

Let $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ be a cohomological isobaric sum as in §1.4.3 and assume that all cuspidal summands Π_i are conjugate self-dual. Put

$$\Pi_i^{\mathsf{alg}} := \begin{cases} \Pi_i & \text{if } n \equiv n_i \mod 2, \\ \Pi_i \otimes \eta & \text{otherwise.} \end{cases} \tag{2.11}$$

These are unitary, conjugate self-dual, cohomological cuspidal automorphic representations, which can be seen as follows: Π' being cohomological by assumption implies that $\Pi_i\rho_i$ is cohomological for all i. Now, recalling that ρ_i is algebraic if and only if $n \equiv n_i \mod 2$ shows that Π_i is cohomological itself if and only if n and n_i have the same parity, or, otheriwse said, that $\Pi_i \otimes \eta$ is cohomological if and only if n and n_i do not have the same parity. Since η is unitary and conjugate self-dual, this shows that Π_i^{alg} is unitary, conjugate self-dual, cohomological cuspidal for all $1 \le i \le k$. The above theorem now yields the following corollary.

Corollary 2.12. *Let $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ be a cohomological isobaric sum as in §1.4.3 and assume that all cuspidal summands Π_i are conjugate self-dual. Then,*

$$p(\Pi') \sim_{E(\Pi')E(\phi)} \prod_{1\le i\le k} p(\Pi_i^{\mathsf{alg}}) \prod_{1\le i<j\le k} L^S(1, \Pi_i \times \Pi_j^\vee),$$

is $\mathrm{Aut}(\mathbb{C}/F^{Gal})$-equivariant.

Proof. By Thm. 2.5

$$p(\Pi') \sim_{\mathbb{Q}(\Pi')} \prod_{1\le i\le k} p(\Pi_i\rho_i) \prod_{1\le i<j\le k} L^S(1, \Pi_i \times \Pi_j^\vee).$$

One easily checks that (1.7) implies that $\Pi_i\rho_i = \Pi_i^{\mathsf{alg}}\phi^{e_i} \cdot \|\cdot\|^{b_i}$ for some $b_i \in \mathbb{Z}$ and

$$e_i = \begin{cases} 0 & \text{if } n \equiv n_i \mod 2, \\ -1 & \text{otherwise.} \end{cases}$$

Hence, [Rag-Sha08], Thm. 4.1, shows that $p(\Pi_i\rho_i) \sim_{\mathbb{Q}(\Pi_i^{\mathsf{alg}})\mathbb{Q}(\phi^{e_i})} p(\Pi_i^{\mathsf{alg}})\mathcal{G}(\phi_f^{e_i})^{n_i(n_i-1)/2}$ observing that $\mathcal{G}(\|\cdot\|_f^{b_i}) = 1$ and $\mathbb{Q}(\|\cdot\|^{b_i}) = \mathbb{Q}$. At the cost of adjusting $p(\Pi_i^{\mathsf{alg}})$ by an element in $\mathbb{Q}(\Pi_i^{\mathsf{alg}})$, we may hence assume that $p(\Pi_i\rho_i) \sim_{\mathbb{Q}(\phi^{e_i})} p(\Pi_i^{\mathsf{alg}})\mathcal{G}(\phi_f^{e_i})^{n_i(n_i-1)/2}$. In order to finish the proof, it is hence enough to show that $\mathcal{G}(\phi_f^{-1}) \in E(\phi)$. To this end, let $\sigma \in \mathrm{Aut}(\mathbb{C})$. There exists

$t^\sigma \in \hat{\mathbb{Z}}^\times \subseteq \hat{\mathcal{O}}_{F^+}^\times \subset \hat{\mathcal{O}}_F^\times \subset \mathbb{A}_f^\times$ associated to σ given by the cyclotomic character (cf. §3.2 of [Rag-Sha08]). It is easy to verify that (cf., e.g., the proof of Theorem 3.9 of [Gro-MHar16])

$$\frac{\sigma(\mathcal{G}(\phi_f^{-1}))}{\mathcal{G}({}^\sigma\phi_f^{-1})} = \phi_f^{-1}(t^\sigma). \tag{2.13}$$

Since $t^\sigma \in \hat{\mathbb{Z}}$, we know $\phi_f^{-1}(t^\sigma) = \phi\,|^{-1}_{\mathbb{A}_{F^+,f}}(t^\sigma)$. Recall that $\phi = \eta\,\|\cdot\|^{1/2}$ and $\eta\,|_{\mathbb{A}_{F^+}} = \varepsilon$ by definition, so $\phi_f^{-1}(t^\sigma) = \varepsilon_f(t^\sigma)$. This implies that

$$\frac{\sigma(\mathcal{G}(\phi_f^{-1}))}{\mathcal{G}({}^\sigma\phi_f^{-1})} = \frac{\sigma(\mathcal{G}(\varepsilon_f))}{\mathcal{G}({}^\sigma\varepsilon_f)}, \tag{2.14}$$

and so $\mathcal{G}(\phi_f^{-1}) \sim_{\mathbb{Q}(\phi)\mathbb{Q}(\varepsilon)} \mathcal{G}(\varepsilon_f)$. However, ε being quadratic forces $\mathbb{Q}(\varepsilon) = \mathbb{Q}(\{\pm 1\}) = \mathbb{Q}$, so we finally conclude by Lemma 1.21 that $\mathcal{G}(\phi_f^{-1}) \sim_{\mathbb{Q}(\phi)} \mathcal{G}(\varepsilon_f) \sim_{F^{Gal}} 1$ and hence $\mathcal{G}(\phi_f^{-1}) \in E(\phi) = \mathbb{Q}(\phi) \cdot F^{Gal}$. □

3. Asai L-functions of Langlands transfers

In this section, we will calculate the Asai L-functions of conjugate self-dual Eisenstein representations and of a representation automorphically induced from suitable Hecke characters. Having detailed knowledge about these Asai L-functions will turn out to be crucial for the proof of the main theorems.

3.1. **The Asai L-function of an isobaric sum.** Let S^+ be the finite set of places of F^+, containing the restrictions of the places in S. Let $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ be an Eisenstein representation of $\mathrm{GL}_n(\mathbb{A}_F)$ as in 1.4.3 and assume that each Π_i is conjugate self-dual. With these assumptions, unitary conjugate self-dual, cohomological cuspidal automorphic representations Π_i^{alg} have been defined for all $1 \leq i \leq k$ in (2.11). We obtain

Lemma 3.1. *$L^S(s, \Pi_i^{\mathsf{alg}}, \mathrm{As}^{(-1)^{n_i}})$ is holomorphic and non-vanishing at $s = 1$.*

Proof. As all Π_i^{alg} are cohomological and conjugate self-dual unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, $L^S(s, \Pi_i^{\mathsf{alg}}, \mathrm{As}^{(-1)^{n_i}})$ is holomorphic at $s = 1$ by the argument given in [Mok15], Cor. 2.5.9. (See also [Gro-MHar-Lap16], §6.1 for more details.) The non-vanishing follows from [Sha81], Thm. 5.1. □

Remark 3.2. Observing that twisting by η just changes the sign of the Asai-representation, we see that $L^S(s, \Pi_i, \mathrm{As}^{(-1)^n})$, being equal to $L^S(s, \Pi_i^{\mathsf{alg}}, \mathrm{As}^{(-1)^{n_i}})$, is holomorphic and non-vanishing at $s = 1$ for all $1 \leq i \leq k$.

Lemma 3.3. *Let $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ be an Eisenstein representation of $\mathrm{GL}_n(\mathbb{A}_F)$ as in 1.4.3 and assume the each Π_i is conjugate self-dual. Then*

$$L^S(s, \Pi', \mathrm{As}^{(-1)^n}) = \prod_{i=1}^{k} L^S(s, \Pi_i^{\mathsf{alg}}, \mathrm{As}^{(-1)^{n_i}}) \cdot \prod_{1 \leq i < j \leq k} L^S(s, \Pi_i \times \Pi_j^\vee).$$

Proof. Recalling the previous remark, we will show that

$$L^S(s, \Pi', \mathrm{As}^{(-1)^n}) = \prod_{i=1}^{k} L^S(s, \Pi_i, \mathrm{As}^{(-1)^n}) \cdot \prod_{1 \leq i < j \leq k} L^S(s, \Pi_i \times \Pi_j^\vee).$$

We argue locally, distinguishing the two possible types of unramified places $v \notin S^+$. In particular, we will drop the subscript for the local places for all global objects in what follows if this will not cause any confusion. Moreover, let γ be the character from [Gro-MHar16], §2.1.2, i.e.,

$$\gamma = \begin{cases} \mathbf{1} & \text{if } n \equiv 0 \mod 2 \\ \eta & \text{otherwise} \end{cases}$$

Observe that $\gamma|^2_{F^+} = \mathbf{1}$ in any case. Let us first suppose that v is inert, extending to one place $w \notin S$. For every $1 \leq i \leq k$, denote by $\chi^{(i)}_\ell$, $1 \leq \ell \leq n_i$, the unramified characters from which Π_i is fully induced. Then, by induction in stages and applying Lemma 1 of [Gel-Jac-Rog01],

$$\begin{aligned}
& L(s, \Pi'_v, \mathrm{As}^{(-1)^n}) \\
= & \left(\prod_{i,\ell} L(s, \chi^{(i)}_\ell \gamma|_{F^+})\right) \cdot \left(\prod_{\substack{i,j,r,t \\ \chi^{(i)}_r \text{ coming before } \chi^{(j)}_t}} L(s, \chi^{(i)}_r \gamma \cdot \chi^{(j)}_t \gamma)\right) \\
= & \prod_{i=1}^k \left(\prod_{\ell=1}^{n_i} L(s, \chi^{(i)}_\ell \gamma|_{F^+}) \cdot \prod_{1 \leq r < t \leq n_i} L(s, \chi^{(i)}_r \chi^{(i)}_t)\right) \cdot \prod_{1 \leq i < j \leq k} \prod_{\substack{1 \leq r \leq n_i \\ 1 \leq t \leq n_j}} L(s, \chi^{(i)}_r \chi^{(j)}_t) \\
= & \prod_{i=1}^k L(s, \Pi_{i,v}, \mathrm{As}^{(-1)^n}) \cdot \prod_{1 \leq i < j \leq k} L(s, \Pi_{i,w} \times \Pi_{j,w}).
\end{aligned}$$

As v is inert and all Π_j are conjugate self-dual, the latter local Rankin-Selberg L-function is equal to $L(s, \Pi_{i,w} \times \Pi^\vee_{j,w})$ and hence we obtain the desired relation at inert places v. If v is split, extending to a product of two places $w_1, w_2 \notin S$, then $\gamma_{w_1}\gamma_{w_2} = \mathbf{1}$ and the residue fields are of equal cardinality $q_v = q_{w_1} = q_{w_2}$. So we get independently of n,

$$\begin{aligned}
& L(s, \Pi'_v, \mathrm{As}^{(-1)^n}) \\
= & \det(id - (A(\Pi'_{w_1}) \otimes A(\Pi'_{w_2}))q_v^{-s})^{-1} \\
= & \prod_{1 \leq i,j \leq k} \prod_{\substack{1 \leq r \leq n_i \\ 1 \leq t \leq n_j}} (1 - \chi^{(i)}_{r,w_1} \chi^{(j)}_{t,w_2} q_v^{-s})^{-1} \\
= & \prod_{i=1}^k \left(\prod_{\substack{1 \leq r \leq n_i \\ 1 \leq t \leq n_j}} (1 - \chi^{(i)}_{r,w_1} \chi^{(i)}_{t,w_2} q_v^{-s})^{-1}\right) \cdot \prod_{\substack{1 \leq i \neq j \leq k \\ 1 \leq r \leq n_i \\ 1 \leq t \leq n_j}} (1 - \chi^{(i)}_{r,w_1} \chi^{(j)}_{t,w_2} q_v^{-s})^{-1} \\
= & \prod_{i=1}^k L(s, \Pi_{i,v}, \mathrm{As}^{(-1)^n}) \cdot \prod_{\substack{1 \leq i < j \leq k \\ 1 \leq r \leq n_i \\ 1 \leq t \leq n_j}} (1 - \chi^{(i)}_{r,w_1} \chi^{(j)}_{t,w_2} q_v^{-s})^{-1} (1 - \chi^{(i)}_{r,w_2} \chi^{(j)}_{t,w_1} q_v^{-s})^{-1} \\
= & \prod_{i=1}^k L(s, \Pi_{i,v}, \mathrm{As}^{(-1)^n}) \cdot \prod_{\substack{1 \leq i < j \leq k \\ 1 \leq r \leq n_i \\ 1 \leq t \leq n_j}} (1 - \chi^{(i)}_{r,w_1} \overline{\chi}^{(j)}_{t,w_1} q_{w_1}^{-s})^{-1} (1 - \chi^{(i)}_{r,w_2} \overline{\chi}^{(j)}_{t,w_2} q_{w_2}^{-s})^{-1}
\end{aligned}$$

$$= \prod_{i=1}^{k} L(s, \Pi_{i,v}, \mathrm{As}^{(-1)^n}) \cdot \prod_{1\leq i<j\leq k} L(s, \Pi_{i,w_1} \times \overline{\Pi}_{j,w_1}) L(s, \Pi_{i,w_2} \times \overline{\Pi}_{j,w_2}).$$

Since all Π_j are conjugate self-dual, this shows the claim at split places v. □

Corollary 3.4. *Let $\Pi' = \Pi_1 \boxplus ... \boxplus \Pi_k$ be an Eisenstein representation of $\mathrm{GL}_n(\mathbb{A}_F)$ as in 1.4.3 and assume the each Π_i is conjugate self-dual. Then $L^S(s, \Pi', \mathrm{As}^{(-1)^n})$ is holomorphic and non-vanishing at $s = 1$.*

Proof. Follows directly from Lem. 3.3 and 3.1, recalling that by definition all Π_i are pairwise different and unitary. □

3.2. **The Asai L-function of an automorphically induced representation.** Let L be a cyclic extension of F of degree $[L : F] = n \geq 1$ which is still a CM field. We write L^+ for its maximal totally real subfield.

Definition 3.5. Let χ be an algebraic Hecke character of L. We let $\Pi(\chi)$ be the automorphic induction of χ to $\mathrm{GL}_n(\mathbb{A}_F)$ (cf. [Art-Clo89], Chp. 3, Thm. 6.2). We write

$$\boxed{\Pi_\chi := \begin{cases} \Pi(\chi) & \text{if } n = [L : F] \text{ is odd,} \\ \Pi(\chi) \otimes \eta & \text{if } n = [L : F] \text{ is even.} \end{cases}}$$

Then Π_χ is an isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, fully induced from cuspidal automorphic representations, which is algebraic (in the sense of [Clo90], Def. 1.8). Let ϑ be a generator of $\mathrm{Gal}(L/F)$. It induces an automorphism on $\mathbb{A}_L^\times$, denoted by the same letter, and we define χ^ϑ, a Hecke character of L, as the composition $\chi \circ \vartheta$.

Now identify $\mathrm{Gal}(L/F)$ with $\mathrm{Gal}(L^+/F^+)$. If n is even, we define $L^\flat := L^{\vartheta^{\frac{n}{2}} c}$, an index 2 subfield of L. It is also a CM field with maximal totally real subfield $(L^+)^{\vartheta^{\frac{n}{2}}}$. We write ε_{L/L^+} and $\varepsilon_{L/L^\flat}$ for the quadratic Hecke character associated to L/L^+ and $L/L^\flat$ respectively by the class field theory.

Proposition 3.6. *Assume that χ is conjugate self-dual.*
If n is odd then

$$L^S(s, \Pi_\chi, \mathrm{As}^{(-1)^n}) = \prod_{1\leq k\leq \frac{n-1}{2}} L^S(s, \chi \otimes \chi^{\vartheta^k, c}) L^S(s, \varepsilon_{L/L^+}). \tag{3.7}$$

If n is even then

$$L^S(s, \Pi_\chi, \mathrm{As}^{(-1)^n}) = \prod_{1\leq k\leq \frac{n-2}{2}} L^S(s, \chi \otimes \chi^{\vartheta^k, c}) L^S(s, \varepsilon_{L/L^+}) L^S(s, \chi|_{\mathbb{A}_{L^\flat}} \otimes \varepsilon_{L/L^\flat}). \tag{3.8}$$

Proof. Let v be a non-archimedean place of F^+ such that every representation at hand is unramified at v. We now prove that the products of local L-factors at the places over v of both sides of (3.7) and (3.8) are equal. In order to ease our assertions, we simply call these products the "v-parts" of the left hand side, respectively, the right hand side. As a general reference, we refer again to Lemma 1 of [Gel-Jac-Rog01] where the unramified local factors of the Asai L-function have been calculated.

We write q_v for the cardinality of the residue field of F_v^+. We will use similar notations for other finite places of other fields. Let w be a place of F over v. Let $w_1, w_2, \cdots w_m$ be the places of L over w. We know $m \mid n$ and we write l for n/m. We may assume that $\vartheta(w_{i-1}) = w_i$ for any $1 \leq i \leq m$

where we apply the useful extension of notation defined by $w_i := w_{i \bmod m}$ for all $i \in \mathbb{Z}$.

Let ζ be a primitive l-th root of unity. For each i let t_i be the Hecke eigenvalue of χ at w_i, where, similar to above, we wrote $t_i := t_{i \bmod m}$ for $i \in \mathbb{Z}$. Since χ is conjugate self-dual, its Hecke eigenvalue at w_i^c is t_i^{-1} where w_i^c is the complex conjugation of w_i. If $w_i = w_i^c$ then $t_i = \pm 1$. Moreover, since χ is algebraic and conjugate self-dual, χ is trivial on $\mathbb{A}_{L^+}^\times$ (cf. Rem. 1.31). Hence t_i is in fact 1 in this case. For each $1 \leq i \leq m$, we fix a complex l-th root of t_i and denote it by $t_i^{1/l}$. By equation (6.2) in Chapter 3 of [Art-Clo89], we know that the Hecke eigenvalues of Π_χ at w are $t_i^{1/l}\zeta^a$ with $1 \leq i \leq m, 1 \leq a \leq l$, and those at w^c are $t_j^{-1/l}\zeta^b$ with $1 \leq j \leq m, 1 \leq b \leq l$.

Case 1: n odd In this case, both l and m are odd numbers.

(1) When v is split in F, i.e. $v = ww^c$, the v-part of the left hand side of equation (3.7) is equal to

$$\prod_{1\leq i,j\leq m}\prod_{1\leq a\leq l}\prod_{1\leq b\leq l}(1-t_i^{1/l}\zeta^a t_j^{-1/l}\zeta^b q_v^{-s})^{-1} = \prod_{1\leq i,j\leq m}(1-t_it_j^{-1}q_v^{-ls})^{-l}.$$

The v-part of $L(s,\chi\otimes\chi^{\vartheta^k,c})$ is equal to $\prod_{1\leq i\leq m}(1-t_it_{i+k}^{-1}q_{w_i}^{-s})^{-1}(1-t_i^{-1}t_{i+k}q_{w_i}^{-s})^{-1}$. We know $q_{w_i} = q_{w_i^c} = q_v^l$. Hence the v-part of the right hand side of equation (3.7) is equal to:

$$\begin{aligned}
&\left[\prod_{1\leq k\leq \frac{n-1}{2}}\prod_{1\leq i\leq m}(1-t_it_{i+k}^{-1}q_v^{-ls})^{-1}(1-t_i^{-1}t_{i+k}q_v^{-ls})^{-1}\right]\cdot(1-q_v^{-ls})^{-m}\\
=&\prod_{1\leq i\leq m}\left[\left(\prod_{-\frac{n-1}{2}\leq k\leq\frac{n-1}{2},k\neq 0}(1-t_it_{i+k}^{-1}q_v^{-ls})^{-1}\right)\cdot(1-t_it_i^{-1}q_v^{-ls})^{-1}\right]\\
=&\prod_{1\leq i\leq m}\prod_{-\frac{n-1}{2}\leq k\leq\frac{n-1}{2}}(1-t_it_{i+k}^{-1}q_v^{-ls})^{-1}\\
=&\prod_{1\leq i\leq m}\prod_{1\leq j\leq m}(1-t_it_j^{-1}q_v^{-ls})^{-l}.
\end{aligned}$$

(2) Assume now that v is inert in F. Since $\mathrm{Gal}(L/F)$ acts transitively on the set $\{w_i \mid 1 \leq i \leq m\}$ and commutes with complex conjugation, either $w_i^c = w_i$ for all i, or $w_i^c \neq w_i$ for all i. If the latter is true, then for each $1 \leq i \leq m$ there exists $j \neq i$ such that $w_j = w_i^c$. In particular, the set $\{w_i \mid 1 \leq i \leq m\}$ can be divided into disjoint pairs. But this is impossible since m is odd. Therefore we have that $w_i^c = w_i$ and hence the Hecke eigenvalues $t_i = 1$ for all i. Keeping this in mind, one can easily show that the v-parts of both sides of equation (3.7) coincide and are in fact equal to $(1+q_v^{-ls})^{-m}(1-q_v^{-2ls})^{-(m^2l-m)/2}$.

Case 2: n even

(1) When $v = ww^c$ is split, the v-part of the left hand side of equation (3.8) is again equal to $\prod_{1\leq i,j\leq m}(1-t_it_j^{-1}q_v^{-ls})^{-l}$. In order to evaluate the right hand side, observe that the finite places of $L^\flat = L^{\vartheta^{\frac{n}{2}}c}$ over v are $w_iw^c_{\frac{n}{2}+i}$, $1 \leq i \leq m$. Therefor, the Hecke character $\chi|_{L^\flat}$ has

Hecke eigenvalue $t_i t_{\frac{n}{2}+i}^{-1}$ at $w_i w_{\frac{n}{2}+i}^c$. The v-part of the right hand side is hence equal to:

$$\begin{aligned}
&\left[\prod_{1\leq k\leq \frac{n-2}{2}}\prod_{1\leq i\leq m}(1-t_it_{i+k}^{-1}q_v^{-ls})^{-1}(1-t_i^{-1}t_{i+k}q_v^{-ls})^{-1}\right]\cdot(1-q_v^{-ls})^{-m}\\
&\quad\cdot\left[\prod_{1\leq i\leq m}(1-t_it_{\frac{n}{2}+i}^{-1}q^{-ls})^{-1}\right]\\
= &\prod_{1\leq i\leq m}\left[\left(\prod_{-\frac{n-2}{2}\leq k\leq\frac{n}{2},k\neq 0}(1-t_it_{i+k}^{-1}q_v^{-ls})^{-1}\right)\cdot(1-t_it_i^{-1}q_v^{-ls})^{-1}\right]\\
= &\prod_{1\leq i\leq m}\prod_{-\frac{n-2}{2}\leq k\leq\frac{n}{2}}(1-t_it_{i+k}^{-1}q_v^{-ls})^{-1}\\
= &\prod_{1\leq i\leq n}\prod_{1\leq j\leq m}(1-t_it_j^{-1}q_v^{-ls})^{-1}\\
= &\prod_{1\leq i\leq m}\prod_{1\leq j\leq m}(1-t_it_j^{-1}q_v^{-ls})^{-l}.
\end{aligned}$$

(2) When v is inert and $w_i = w_i^c$ for all $1 \leq i \leq m$, we know $t_i = 1$ for all i. So the representation $\Pi_\chi = \Pi(\chi)\otimes\eta$ has Hecke eigenvalues $-\zeta^j$ for $1 \leq j \leq n$.

If l is odd, then m is even. The v-part of the left hand side of equation (3.8) is

$$(1+q_v^{-ls})^{-m}(1-q_v^{-2ls})^{-(m^2l-m)/2}.$$

In this case, the finite places of $L^\flat = L^{\vartheta^{\frac{n}{2}}c}$ over v are $w_iw_{\frac{m}{2}+i}$, $1 \leq i \leq \frac{m}{2}$. The v-part of the right hand side is then equal to:

$$\begin{aligned}
&(1-q_v^{-2ls})^{-m(ml-2)/2}(1+q_v^{-ls})^{-m}(1-q_v^{-2ls})^{-\frac{m}{2}}\\
= &(1+q_v^{-ls})^{-m}(1-q_v^{-2ls})^{-(m^2l-m)/2}.
\end{aligned}$$

Similarly, if l is even then the v-parts of both sides of equation (3.8) are easily seen to be equal to $(1-q_v^{-2ls})^{-m^2l/2}$.

(3) When v is inert and $w_i \neq w_i^c$ for all $1 \leq i \leq m$, there exists t, an integer between 2 and $m-1$, such that $w_1^c = w_{t+1}$. We apply ϑ^t to both sides and get $w_{t+1}^c = w_{2t+1}$. Hence $2t+1 \equiv 1 \mod m$. This implies that $2t = m$. In particular, we know then m is even and $w_i^c = w_{i+\frac{m}{2}}$. The latter implies that $t_{i+\frac{m}{2}} = t_i^{-1}$ for all $1 \leq i \leq m$.

The representation Π_χ has eigenvalues $\{-\zeta^a t_i^{1/l} \mid 1 \leq i \leq m, 1 \leq a \leq l\}$ at v. the v-part of the left hand side of equation (3.8) is:

$$\prod_{1\leq i\leq m}\prod_{1\leq a\leq l}(1+\zeta^at_i^{1/l}q_v^{-s})^{-1}P(q_v^{-2s})^{-1} \tag{3.9}$$

where $P \in \mathbb{C}[X]$ is the unique polynomial such that $P(0) = 1$ and

$$(P(X))^2 = \prod_{1\leq i,j\leq m}\prod_{1\leq a,b\leq m,(i,a)\neq(j,b)}(1-\zeta^at_i^{1/l}\zeta^bt_j^{1/l}X)$$

$$= \frac{\prod_{1\le i,j\le m}\prod_{1\le a,b\le m}(1-\zeta^a t_i^{1/l}\zeta^b t_j^{1/l}X)}{\prod_{1\le i\le m}\prod_{1\le a\le l}(1-\zeta^{2a}t_i^{2/l}X)}$$

$$= \frac{\prod_{1\le i,j\le m}(1-t_it_jX^l)^l}{\prod_{1\le i\le m}\prod_{1\le a\le l}(1-\zeta^{2a}t_i^{2/l}X)}.$$

If l is odd, then

$$(P(X))^2 = \prod_{1\le i\le m}(1-t_i^2X^l)^{l-1}\prod_{1\le i<j\le m}(1-t_it_jX^l)^{2l}.$$

Hence, the v-part of the left hand side of equation (3.8) is:

$$\prod_{1\le i\le m}(1-t_i^2q_v^{-2ls})^{-(l-1)/2}\cdot\prod_{1\le i<j\le m}(1-t_it_jq_v^{-2ls})^{-l}\cdot\prod_{1\le i\le m}(1+t_iq_v^{-ls})^{-1}.$$

Moreover, it is easy to see that $(\vartheta^{\frac{n}{2}}c)w_i = w^c_{i+\frac{m}{2}} = w_i$ for all i. The intersections of w_i, $1\le i\le m$ with the ring of integers in $L^\flat$ are different prime ideals and hence are inert with respect to the extension $L/L^\flat$. The v-part of the right hand side of equation (3.8) is then:

$$\prod_{1\le k\le\frac{n-2}{2}}\prod_{1\le i\le m}(1-t_it_{i+k}^{-1}q_v^{-2sl})^{-1}\cdot(1-q_v^{-2sl})^{-m/2}\cdot\prod_{1\le i\le m}(1+t_iq_v^{-ls})^{-1}.$$

Recall that $t_{i+\frac{m}{2}} = t_i^{-1}$ for all i. We have:

$$\begin{aligned}
&\prod_{1\le k\le\frac{n-2}{2}}\prod_{1\le i\le m}(1-t_it_{i+k}^{-1}q_v^{-2sl})^{-1}\cdot(1-q_v^{-2sl})^{-m/2}\\
=&\prod_{1\le k\le\frac{n-2}{2}}\prod_{1\le i\le m}(1-t_it_{i+k+\frac{m}{2}}q_v^{-2sl})^{-1}\cdot\prod_{1\le i\le m/2}(1-t_it_{\frac{m}{2}+i}q_v^{-2sl})^{-1}\\
=&\prod_{1\le i\le m}\prod_{\frac{m}{2}+1\le k\le\frac{m+n}{2}-1}(1-t_it_{i+k}q_v^{-2sl})^{-1}\cdot\prod_{1\le i\le m/2}(1-t_it_{\frac{m}{2}+i}q_v^{-2sl})^{-1}\\
=&\prod_{1\le i\le m}\Big[\prod_{\frac{m}{2}+1\le k\le m-1}(1-t_it_{i+k}q_v^{-2sl})^{-(l+1)/2}\cdot\prod_{1\le k\le\frac{m}{2}-1}(1-t_it_{i+k}q_v^{-2sl})^{-(l-1)/2}\Big]\cdot\\
&\prod_{1\le i\le m}(1-t_it_{i+\frac{m}{2}}q_v^{-2sl})^{-(l-1)/2}\prod_{1\le i\le m}(1-t_i^2q_v^{-2sl})^{-(l-1)/2}\prod_{1\le i\le m/2}(1-t_it_{\frac{m}{2}+i}q_v^{-2sl})^{-1}\\
=&\prod_{1\le i<j\le m, j-i\ne m/2}(1-t_it_jq_v^{-2sl})^{-l}\prod_{1\le i\le m/2}(1-t_it_{\frac{m}{2}+i}q_v^{-2sl})^{-(l-1)}\cdot\\
&\prod_{1\le i\le m}(1-t_i^2q_v^{-2sl})^{-(l-1)/2}\prod_{1\le i\le m/2}(1-t_it_{\frac{m}{2}+i}q_v^{-2sl})^{-1}\\
=&\prod_{1\le i<j\le m}(1-t_it_jq_v^{-2sl})^{-l}\prod_{1\le i\le m}(1-t_i^2q_v^{-2sl})^{-(l-1)/2}.
\end{aligned}$$

We have deduced that the v-parts of the two sides of equation (3.8) coincide if l is odd.

If l is even, the left hand side of equation (3.8) is equal to

$$\prod_{1\le i<j\le m}(1-t_it_jq_v^{-2sl})^{-l}\prod_{1\le i\le m}(1-t_i^2q_v^{-2sl})^{-l/2}.$$

Moreover, we have $(\vartheta^{\frac{n}{2}}c)w_i=w_i^c=w_{i+\frac{m}{2}}$. Hence $w_iw_{i+\frac{m}{2}}$ for $1\le i\le\frac{m}{2}$ are the places of $L^\flat$ over v. The corresponding right hand side is equal to

$$\begin{aligned}
&\prod_{1\le k\le\frac{n-2}{2}}\prod_{1\le i\le m}(1-t_it_{i+k}^{-1}q_v^{-2sl})^{-1}\cdot(1-q_v^{-2sl})^{-m/2}\cdot\prod_{1\le i\le m/2}(1-t_it_{\frac{m}{2}+i}q_v^{-2ls})^{-1}\\
=&\prod_{1\le i\le m}[\prod_{\frac{m}{2}+1\le k\le m-1}(1-t_it_{i+k}q_v^{-2sl})^{-l/2}\cdot\prod_{1\le k\le\frac{m}{2}-1}(1-t_it_{i+k}q_v^{-2sl})^{-l/2}]\cdot\\
&\prod_{1\le i\le m}(1-t_it_{i+\frac{m}{2}}q_v^{-2sl})^{-(l/2-1)}\prod_{1\le i\le m}(1-t_i^2q_v^{-2sl})^{-l/2}\cdot\prod_{1\le i\le m/2}(1-t_it_{\frac{m}{2}+i}q_v^{-2ls})^{-2}\\
=&\prod_{1\le i<j\le m,j-i\neq m/2}(1-t_it_jq_v^{-2sl})^{-l}\prod_{1\le i\le m/2}(1-t_it_{\frac{m}{2}+i}q_v^{-2ls})^{-(l-2)-2}\prod_{1\le i\le m}(1-t_i^2q_v^{-2sl})^{-l/2}\\
=&\prod_{1\le i<j\le m}(1-t_it_jq_v^{-2sl})^{-l}\prod_{1\le i\le m}(1-t_i^2q_v^{-2sl})^{-l/2}.
\end{aligned}$$

□

4. Period relations with precise powers of $(2\pi i)$

4.1. **Critical characters and CM-periods.** Let χ be a Hecke character of F with infinity-type $z^{a_\iota}\bar{z}^{a_{\bar\iota}}$ at $\iota\in\Sigma$. We say that χ is *critical* if it is algebraic and moreover $a_\iota\neq a_{\bar\iota}$ for all $\iota\in J_F$. This is equivalent to the motive associated to χ having critical points in the sense of Deligne (cf. [Del79]). We remark that 0 and 1 are always critical points in this case.

In the critical case, we can define Φ_χ, a subset of J_F, as follows: An embedding $\iota\in J_F$ is in Φ_χ if and only if $a_\iota<a_{\bar\iota}$. Clearly, Φ_χ is a CM type of F. Given any CM-type Φ, we say that χ is *compatible* with Φ if $\Phi=\Phi_\chi$.

Let χ be an algebraic Hecke character of F and let $\Psi\subset J_F$ be any subset such that $\Psi\cap\bar\Psi=\emptyset$. Attached to (χ,Ψ) one may define a CM Shimura-datum as in section 1.1 of [MHar93], and a number field $E(\chi,\Psi)$ which contains $\mathbb{Q}(\chi)$ and the reflex field of the CM Shimura datum defined by Ψ. Moreover, one may associate a non zero complex number $p_F(\chi,\Psi)$ to this datum, which is well defined modulo $E(\chi,\Psi)^\times$, called a *CM-period*: As CM-periods $p_F(\chi,\Psi)$ will only be a technical ingredient in our arguments, not showing up in the final formulas, we believe that it is justified not to repeat their precise construction here, but refer for the sake if brevity to the appendix of [MHar-Kud91]. We also write $p(\chi,\Psi)$ instead of $p_F(\chi,\Psi)$ if there is no ambiguity concerning the base field F. Slightly abusing our notation, we denote

$$\boxed{E(\chi):=\bigcup_\Psi E(\chi,\Psi).}$$

It contains $\mathbb{Q}(\chi)\cdot F^{Gal}$ (but may in general be bigger than that).

Remark 4.1. The group $\mathrm{Aut}(\mathbb{C})$ acts on the CM Shimura datum. The CM-periods are defined via certain rational structures of cohomological spaces. We may choose the rational structures equivariantly under the action of $\mathrm{Aut}(\mathbb{C})$, and get a family of the CM-periods $\{p({}^\sigma\chi,\Phi_{^\sigma\chi})\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ which only depends on the restriction of σ to $E(\chi)$.

4.2. **Period relations for CM-periods.**

Definition 4.2. Let ϑ be an element in $\mathrm{Aut}(F)$. For $\iota \in J_F$, we define $\iota^\vartheta \in J_F$ as $\iota \circ \vartheta$.

Recall from §3.2 that we may interpret ϑ also as automorphism of $\mathbb{A}_F^\times$ and define $\chi^\vartheta = \chi \circ \vartheta$. Applying Def. 4.2, it has infinity-type $z^{a_{\iota^\vartheta}} \bar{z}^{a_{\bar{\iota}^\vartheta}}$ at ι. In particular, if χ is algebraic (resp. critical) then so is χ^ϑ. In particular, if χ is compatible with a CM type Ψ_χ then χ^ϑ is compatible with the CM type $\Psi_\chi^{\vartheta^{-1}}$.

Proposition 4.3. *Let χ be a critical Hecke character of F. Let Ψ be a subset of J_F such that $\Psi \cap \bar{\Psi} = \emptyset$. Let ϑ be an element in $\mathrm{Aut}(F)$. Then we have:*

$$p(\chi, \Psi) \sim_{E(\chi)} p(\chi^\vartheta, \Psi^{\vartheta^{-1}})$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. Let $T_F := Res_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ be a torus. We define a homomorphism $h_\Psi : Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to T_{F,\mathbb{R}}$ such that for each $\iota \in J_F$, the Hodge structure induced by h_Ψ is of type $(-1, 0)$ if $\iota \in \Psi$, of type $(0, -1)$ if $\iota \in \bar{\Psi}$, and of type $(0, 0)$ otherwise. The pair (T_F, h_Ψ) is then a Shimura datum. The composition with ϑ induces a morphism of Shimura data $h : (T_F, h_\Psi) \to (T_F, h_{\Psi^{\vartheta^{-1}}})$. Now the expected relation between CM-periods follows as in Lemma 1.6 in [MHar93]. We also refer to Proposition 1.2 of [Lin15a] for more details. □

We recall some other properties of CM-periods. The proof is similar to the previous proposition and can be found in Proposition 1.1 of [Lin15a].

Proposition 4.4. *Let L be a CM field containing F, $\iota \in J_L$ and let χ, χ' be critical Hecke characters of F. Let Ψ a subset of J_F such that $\Psi \cap \bar{\Psi} = \emptyset$ and let $\Psi = \Psi_1 \sqcup \Psi_2$ be a partition of Ψ. Then,*

$$\begin{aligned} p(\chi\chi', \Psi) &\sim_{E(\chi_1)E(\chi_2)} p(\chi, \Psi)\, p(\chi', \Psi) \\ p(\chi, \Psi) = p(\chi, \Psi_1 \sqcup \Psi_2) &\sim_{E(\chi)} p(\chi, \Psi_1)\, p(\chi, \Psi_2) \\ p(\chi, \Psi) &\sim_{E(\chi)} p(\bar{\chi}, \bar{\Psi}) \\ p(\chi \circ N_{\mathbb{A}_L/\mathbb{A}_F}, \iota) &\sim_{E(\chi)} p(\chi, \iota|_F) \end{aligned}$$

The first three relations are equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$*, the last one is equivariant under the action of* $\mathrm{Aut}(\mathbb{C}/L^{Gal})$.

We will also need the following lemma (cf. (1.10.9) in [MHar97])

Lemma 4.5. *For any $\iota \in J_F$, we have*

$$p(\|\cdot\|_{\mathbb{A}_F}, \iota) \sim_{\mathbb{Q}} (2\pi i)^{-1} \tag{4.6}$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C})$.

4.3. **A result of Blasius.** The special values of an L-function for a Hecke character over a CM field can be interpreted in terms of CM-periods. The following theorem was proved by Blasius, presented as in [MHar93], Prop. 1.8.1 (and the attached erratum [MHar97], p. 82).

Theorem 4.7. *Let χ be a critical Hecke character of F and recall $\check{\chi} = \chi^{-1,c} = \bar{\chi}^\vee$. For m a critical value of $L(s, \chi)$, we have*

$$L^S(m, \chi) \sim_{E(\chi)} (2\pi i)^{md} p(\check{\chi}, \Phi_\chi)$$

is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

4.4. **Special L-values of automorphically induced representations.**

4.4.1. *Cohomological representations* Π_χ *and* $\Pi_{\chi'}$. Blasius related critical values of Hecke L-functions CM-periods. We now prove two new results of the same form for critical values of Rankin–Selberg and Asai L-functions of automorphically induced representations.

Let L (resp. L') be a cyclic extension over F of degree n (resp. $n-1$) which is still a CM field. Let χ (resp. χ') be a conjugate self-dual algebraic Hecke character of L (resp. L'). We consider L and L' as subfields of $\mathbb{C}$ and denote by $LL' \subset \mathbb{C}$ the compositum of L and L'. We write L^+ (resp. L'^+) for the maximal totally real subfield of L (resp. L'). It is easy to see that $L^+L'^+$ is an index 2 subfield of LL'. Hence LL' is also a CM field.

Let ι be an element inside the CM type Σ of F. We write $\iota_1, \iota_2, \cdots, \iota_n$ (resp. $\iota'_1, \iota'_2, \cdots, \iota'_{n-1}$) for the embeddings of L (resp. L') which extend ι. For each $1 \leq i \leq n$ and $1 \leq j \leq n-1$, write $\iota_{i,j}$ for the unique embedding of LL' which extends ι_i and ι'_j.

We write the infinity-type of χ (resp. χ') at ι_i (resp. ι'_j) as $z^{a_i}\bar{z}^{-a_i}$ (resp. $z^{b_j}\bar{z}^{-b_j}$) with $a_i \in \mathbb{Z}$ (resp. $b_j \in \mathbb{Z}$). By permuting the embeddings we may suppose that the numbers a_i (resp. b_j) are in decreasing order.

We assume moreover that the numbers $\{a_i\}_{1\leq i\leq n}$ (resp. $\{b_j\}_{1\leq j\leq n-1}$) are all different, i.e., the infinity-types of χ and χ' are *regular*. With this extra assumption, [Art-Clo89], Chp. 3, Thm. 6.2 together with [Clo90, Lem. 3.14], imply that both representations Π_χ and $\Pi_{\chi'}$, as defined in §3.2, are unitary conjugate self-dual, cohomological isobaric sums, fully induced from different cuspidal automorphic representations, i.e., serve as Eisenstein representations as in §1.4.3.

4.4.2. *Rationality for the Asai L-function of* Π_χ.

Proposition 4.8. *Let* χ *be a conjugate self-dual algebraic Hecke character of* L *with regular infinity-type. Then,*

$$L^S(1, \Pi_\chi, \mathrm{As}^{(-1)^n}) \sim_{E(\chi)} (2\pi i)^{n(n+1)d/2} \prod_{\iota\in\Sigma} \prod_{1\leq i\leq n} [p(\check{\chi}, \iota_i)^{i-1} p(\check{\chi}, \bar{\iota}_i)^{n-i}] \tag{4.9}$$

equivariant under $\mathrm{Aut}(\mathbb{C}/L^{Gal})$.

Proof. Recall that the left hand side is calculated in Proposition 3.6.

Let ϑ be a generator of $\mathrm{Aut}(L/F)$. For any $1 \leq i \leq n$, there exists $1 \leq s(i) \leq n$ such that $\vartheta\iota_i = \iota_{s(i)}$. Since ϑ is a generator of $\mathrm{Gal}(L/F)$, we know s is of order n in the permutation group $\mathfrak{S}_n$. For any $1 \leq k \leq n-1$, the Hecke character χ^{ϑ^k} has infinity-type $z^{a_{s^k(i)}}\bar{z}^{-a_{s^k(i)}}$ at ι_i for any $1 \leq i \leq n$. Hence the Hecke character $\chi \otimes \chi^{\vartheta^k,c} = \chi \otimes \chi^{\vartheta^k,-1}$ has infinity-type $z^{a_i - a_{s^k(i)}}\bar{z}^{-a_i+a_{s^k(i)}}$ at ι_i for any $1 \leq i \leq n$. Since $s^k(i) \neq i$, $a_i - a_{s^k(i)} \neq 0$, so we know that the Hecke character $\chi \otimes \chi^{\vartheta^k,c}$ is critical. For any $1 \leq k \leq n-1$, we define $\Psi_{\iota,k} := \{\iota_i \mid 1 \leq i \leq n, a_i < a_{s^k(i)}\} = \{\iota_i \mid 1 \leq i \leq n, i > s^k(i)\}$. We define $\Psi_{\bar{\iota},k} := \{\bar{\iota}_i \mid 1 \leq i \leq n, i < s^k(i)\}$ and $\Psi_k := \bigcup_{\iota\in\Sigma} \Psi_{\iota,k} \cup \Psi_{\bar{\iota},k}$ is the CM type of L associated to $\chi \otimes \chi^{\vartheta^k,-1}$. By Blasius's result, Thm. 4.7, and Prop. 4.3 & 4.4, we have:

$$\begin{aligned} L(1, \chi \otimes \chi^{\vartheta^k,c}) &\sim_{E(\chi)} (2\pi i)^{nd} p(\check{\chi} \otimes \check{\chi}^{\vartheta^k,c}, \Psi_k) \\ &\sim_{E(\chi)} (2\pi i)^{nd} p(\check{\chi}, \Psi_k) p(\check{\chi}, \overline{\Psi_k}^{\vartheta^k}). \end{aligned}$$

It is easy to verify that

$$\overline{\Psi_k}^{\vartheta^k} = \bigcup_{\iota\in\Sigma} \left[\{\bar{\iota}_{s^k(i)} \mid 1 \leq i \leq n, i > s^k(i)\} \cup \{\iota_{s^k(i)} \mid 1 \leq i \leq n, i < s^k(i)\} \right]$$

$$= \bigcup_{\iota\in\Sigma}\left[\{\bar{\iota}_i \mid 1\leq i\leq n, i< s^{-k}(i)\}\cup\{\iota_i \mid 1\leq i\leq n, i> s^{-k}(i)\}\right].$$

Hence we deduce that:

$$L(1,\chi\otimes\chi^{\vartheta^k,c})\sim_{E(\chi)} \tag{4.10}$$
$$(2\pi i)^{nd}\prod_{\iota\in\Sigma}[\prod_{1\leq i\leq n, i>s^k(i)} p(\check{\chi},\iota_i)\prod_{1\leq i\leq n, i<s^k(i)} p(\check{\chi},\bar{\iota}_i)\prod_{1\leq i\leq n, i<s^{-k}(i)} p(\check{\chi},\bar{\iota}_i)\prod_{1\leq i\leq n, i>s^{-k}(i)} p(\check{\chi},\iota_i)].$$

We first prove the lemma when n is odd. In this case, we know by Proposition 3.6 that

$$L^S(1,\Pi_\chi,\mathrm{As}^{(-1)^n}) = \prod_{1\leq k\leq\frac{n-1}{2}} L^S(1,\chi\otimes\chi^{\vartheta^k,c})L^S(1,\varepsilon_{L/L^+}). \tag{4.11}$$

Equation (4.10) implies that:

$$\begin{aligned}
&(2\pi i)^{-n(n-1)d/2}\prod_{1\leq k\leq\frac{n-1}{2}} L(1,\chi\otimes\chi^{\vartheta^k,c})\\
\sim_{E(\chi)}&\prod_{1\leq k\leq\frac{n-1}{2}}\prod_{\iota\in\Sigma}[\prod_{1\leq i\leq n, i>s^k(i)} p(\check{\chi},\iota_i)\prod_{1\leq i\leq n, i<s^k(i)} p(\check{\chi},\bar{\iota}_i)\cdot\\
&\prod_{1\leq i\leq n, i<s^{-k}(i)} p(\check{\chi},\bar{\iota}_i)\prod_{1\leq i\leq n, i>s^{-k}(i)} p(\check{\chi},\iota_i)]\\
\sim_{E(\chi)}&\prod_{\iota\in\Sigma}\prod_{1\leq k\leq n-1}[\prod_{1\leq i\leq n, i>s^k(i)} p(\check{\chi},\iota_i)\prod_{1\leq i\leq n, i<s^k(i)} p(\check{\chi},\bar{\iota}_i)]\\
\sim_{E(\chi)}&\prod_{\iota\in\Sigma}\prod_{1\leq i\leq n}[\prod_{1\leq k\leq n-1, i>s^k(i)} p(\check{\chi},\iota_i)\prod_{1\leq k\leq n-1, i<s^k(i)} p(\check{\chi},\bar{\iota}_i)]\\
\sim_{E(\chi)}&\prod_{\iota\in\Sigma}\prod_{1\leq i\leq n}[p(\check{\chi},\iota_i)^{i-1}p(\check{\chi},\bar{\iota}_i)^{n-i}].
\end{aligned}$$

Recall that by equation (1.23) we have $L^S(1,\varepsilon_{L/L^+})\sim_{L^{Gal}}(2\pi i)^{dn}$. We conclude that

$$L^S(1,\Pi_\chi,\mathrm{As}^{(-1)^n})\sim_{E(\chi)}(2\pi i)^{n(n+1)d/2}\prod_{\iota\in\Sigma}\prod_{1\leq i\leq n}[p(\check{\chi},\iota_i)^{i-1}p(\check{\chi},\bar{\iota}_i)^{n-i}]. \tag{4.12}$$

Next, if n is even, again by Proposition 3.6, we have:

$$L^S(1,\Pi_\chi,\mathrm{As}^{(-1)^n}) = \prod_{1\leq k\leq\frac{n-2}{2}} L^S(1,\chi\otimes\chi^{\vartheta^k,c})L^S(1,\varepsilon_{L/L^+})L^S(1,\chi\mid_{\mathbb{A}_{L^\flat}}\otimes\varepsilon_{L/L^\flat}). \tag{4.13}$$

Similar to above, one may deduce from (4.10) by a simple calculation that

$$\prod_{1\leq k\leq\frac{n-2}{2}} L(1,\chi\otimes\chi^{\vartheta^k,c})\sim_{E(\chi)}$$
$$(2\pi i)^{dn(n-2)/2}\prod_{\iota\in\Sigma}\prod_{1\leq i\leq n}[\prod_{1\leq k\leq n-1, k\neq\frac{n}{2}, i>s^k(i)} p(\check{\chi},\iota_i)\prod_{1\leq k\leq n-1, k\neq\frac{n}{2}, i<s^k(i)} p(\check{\chi},\bar{\iota}_i)].$$

Recall that $L^S(1,\varepsilon_{L/L^+})\sim_{L^{Gal}}(2\pi i)^{dn}$. It remains to calculate $L^S(1,\chi\mid_{\mathbb{A}_{L^\flat}}\otimes\varepsilon_{L/L^\flat})$ in (4.13). The complex embeddings of the CM field $L^\flat := L^{\vartheta^{\frac{n}{2}}c}$ are $\iota_i\mid_{L^\flat}$ with $\iota\in\Sigma, 1\leq i\leq n$. We remark that $(\iota_i\mid_{L^\flat})^c = \iota_{s^{n/2}(i)}\mid_{L^\flat}$. The Hecke character $\chi\mid_{\mathbb{A}_{L^\flat}}$ has infinity-type $z^{a_i-a_{s^{n/2}(i)}}\bar{z}^{-a_i+a_{s^{n/2}(i)}}$ at ι_i, and the Hecke character $\varepsilon_{L/L^\flat}$ has trivial infinity-type.

We define $\Psi_\iota^\flat := \{\iota_i |_{L^\flat} | \ a_i < a_{s^{n/2}(i)}, 1 \leq i \leq n\} = \{\iota_i |_{L^\flat} | \ i > s^{n/2}(i), 1 \leq i \leq n\}$. Then the Hecke character $\chi |_{\mathbb{A}_{L^\flat}} \otimes \varepsilon_{L/L^\flat}$ is compatible with the CM type $\bigcup_{\iota\in\Sigma} \Psi_\iota^\flat$.

Using Prop. 4.4 we deduce thereof

$$
\begin{aligned}
& L^S(1, \chi |_{\mathbb{A}_{L^\flat}} \otimes \varepsilon_{L/L^\flat}) \\
\sim_{E(\chi)} \ & (2\pi i)^{dn/2} p(\check{\chi} |_{\mathbb{A}_{L^\flat}} \otimes \check{\eta}_{L/L^\flat}, \bigcup_{\iota\in\Sigma} \Phi_\iota^b) \\
\sim_{E(\chi)} \ & (2\pi i)^{dn/2} \prod_{\iota\in\Sigma} \prod_{1\leq i\leq n, i>s^{n/2}(i)} p(\check{\chi} |_{\mathbb{A}_{L^\flat}} \otimes \check{\eta}_{L/L^\flat}, \iota_i |_{L^\flat}) \\
\sim_{E(\chi)} \ & (2\pi i)^{dn/2} \prod_{\iota\in\Sigma} \prod_{1\leq i\leq n, i>s^{n/2}(i)} p([\check{\chi} |_{\mathbb{A}_{L^\flat}} \otimes \check{\eta}_{L/L^\flat}] \circ N_{\mathbb{A}_L/\mathbb{A}_{L^\flat}}, \iota_i) \\
\sim_{E(\chi)} \ & (2\pi i)^{dn/2} \prod_{\iota\in\Sigma} \prod_{1\leq i\leq n, i>s^{n/2}(i)} p(\check{\chi} \otimes \check{\chi}^{\vartheta^{\frac{n}{2}} c}, \iota_i) \\
\sim_{E(\chi)} \ & (2\pi i)^{dn/2} \prod_{\iota\in\Sigma} [\prod_{1\leq i\leq n, i>s^{n/2}(i)} p(\check{\chi}, \iota_i) \cdot \prod_{1\leq i\leq n, i>s^{n/2}(i)} p(\check{\chi}, \bar{\iota}_{s^{n/2}(i)})] \\
\sim_{E(\chi)} \ & (2\pi i)^{dn/2} \prod_{\iota\in\Sigma} [\prod_{1\leq i\leq n, i>s^{n/2}(i)} p(\check{\chi}, \iota_i) \cdot \prod_{1\leq i\leq n, i<s^{n/2}(i)} p(\check{\chi}, \bar{\iota}_i)].
\end{aligned}
$$

We conclude that when n is even we still have the following relation:

$$
\begin{aligned}
& L^S(1, \Pi_\chi, \mathrm{As}^{(-1)^n}) \\
\sim_{E(\chi)} \ & (2\pi i)^{dn(n+1)/2} \prod_{\iota\in\Sigma} \prod_{1\leq i\leq n} [\prod_{i>s^k(i), 1\leq k\leq n-1} p(\check{\chi}, \iota_i) \prod_{i<s^k(i), 1\leq k\leq n-1} p(\check{\chi}, \bar{\iota}_i)] \\
\sim_{E(\chi)} \ & (2\pi i)^{dn(n+1)/2} \prod_{\iota\in\Sigma} \prod_{1\leq i\leq n} [p(\check{\chi}, \iota_i)^{i-1} p(\check{\chi}, \bar{\iota}_i)^{n-i}].
\end{aligned}
$$

Finally, we remark that all the relations above are equivariant under the action of $\mathrm{Aut}(\mathbb{C}/L^{Gal})$. ◻

The previous lemma and Theorem 1.27 imply immediately the following period relation for cuspidal automorphically induced representations Π_χ:

Corollary 4.14. *Let χ be a conjugate self-dual algebraic Hecke character of L with regular infinity-type. If Π_χ is moreover cuspidal, then*

$$p(\Pi_\chi) \sim_{E(\Pi_\chi)E(\chi)} a(\Pi_{\chi,\infty})^{-1} (2\pi i)^{dn(n+1)/2} \prod_{\iota\in\Sigma} \prod_{1\leq i\leq n} [p(\check{\chi}, \iota_i)^{i-1} p(\check{\chi}, \bar{\iota}_i)^{n-i}]$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/L^{Gal})$.

4.4.3. *Rationality for the Rankin–Selberg L-function of $\Pi_\chi \times \Pi_{\chi'}$.* We obtain

Proposition 4.15. *Let χ (resp. χ') be a conjugate self-dual algebraic Hecke character of L (resp. L') with regular infinity-types. Assume the Π_χ is cuspidal and that $(\Pi_\chi, \Pi_{\chi'})$ satisfies the piano-condition, cf. Hypothesis 1.29. Let $\frac{1}{2} + m \in \mathrm{Crit}(\Pi_\chi \times \Pi_{\chi'})$. Then,*

$$L^S(\tfrac{1}{2} + m, \Pi_\chi \times \Pi_{\chi'}) \sim_{E(\chi)E(\chi')E(\phi)}$$

$$(2\pi i)^{(\frac{1}{2}+m)dn(n-1)} \prod_{\iota\in\Sigma} \left(\prod_{1\leq i\leq n} [p(\check{\chi}, \iota_i)^{i-1} p(\check{\chi}, \overline{\iota_i})^{n-i}] \prod_{1\leq j\leq n-1} [p(\check{\chi}', \iota'_j)^{j-1} p(\check{\chi}', \overline{\iota'_j})^{n-1-j}] \right)$$

equivariant under the action of $\mathrm{Aut}(\mathbb{C}/(LL')^{Gal})$.

Proof. We know that:

$$L^S(\frac{1}{2}+m,\Pi_\chi\times\Pi_{\chi'}) = L^S(\frac{1}{2}+m,(\chi\circ N_{LL'/L})(\chi'\circ N_{LL'/L'})(\eta\circ N_{LL'/F})) = L^S(m,(\chi\circ N_{LL'/L})(\chi'\circ N_{LL'/L'})(\phi\circ N_{LL'/F})) \tag{4.16}$$

Since ϕ as infinity-type $z^1\bar{z}^0$ at each ι, the infinity-type of the Hecke character $\chi^{\#} := (\chi\circ N_{LL'/L})(\chi'\circ N_{LL'/L'})(\phi\circ N_{LL'/F})$ at $\iota_{i,j}$ is $z^{a_i+b_j+1}\bar{z}^{-a_i-b_j}$. The piano-condition implies that:

$$a_1 > -b_{n-1}-\tfrac{1}{2} > a_2 > -b_{n-2}-\tfrac{1}{2} > \cdots > -b_1-\tfrac{1}{2} > a_n.$$

Define $\Phi_\iota := \{\iota_{i,j} \mid a_i+b_j+\frac{1}{2}<0\} = \{\iota_{i,j} \mid i+j\geq n+1\}$ and $\Phi_{\bar\iota} := \{\overline{\iota_{i,j}} \mid i+j\leq n\}$. Then the Hecke character $\chi^{\#}$ is critical with respect to the CM type $\bigcup_{\iota\in\Sigma}\Phi_\iota\cup\Phi_{\bar\iota}$. An easy check shows that $\frac{1}{2}+m$ is critical for $\Pi_\chi\times\Pi_{\chi'}$ if and only if m is critical for $\chi^{\#}$. By Blasius's result, Thm. 4.7, one has

$$L^S(m,\chi^{\#}) \sim_{E(\chi^{\#})} (2\pi i)^{mdn(n-1)} p(\check{\chi}^{\#},\bigcup_{\iota\in\Sigma}\Phi_\iota\cup\Phi_{\bar\iota}).$$

Applying Prop. 4.4 to the CM-period on the right hand side implies that

$$\begin{aligned} p(\chi^{\#},\bigcup_{\iota\in\Sigma}\Phi_\iota\cup\Phi_{\bar\iota}) &\sim_{E(\chi^{\#})} \prod_{\iota\in\Sigma} p(\check{\chi}^{\#},\Phi_\iota\cup\Phi_{\bar\iota}) \\ &\sim_{E(\chi^{\#})} \prod_{\iota\in\Sigma}\prod_{i+j\geq n+1} p(\check{\chi}^{\#},\iota_{i,j}) \prod_{i+j\leq n} p(\check{\chi}^{\#},\overline{\iota_{i,j}}). \end{aligned}$$

Next observe that

$$\begin{aligned} &\prod_{i+j\geq n+1} p(\check{\chi}^{\#},\iota_{i,j}) \\ \sim_{E(\chi^{\#})} &\prod_{i+j\geq n+1} [p((\check{\chi}\circ N_{LL'/L}),\iota_{i,j})p(\check{\chi}'\circ N_{LL'/L'},\iota_{i,j})p((\check{\phi}\circ N_{LL'/F}),\iota_{i,j})] \\ \sim_{E(\chi^{\#})} &\prod_{i+j\geq n+1} [p(\check{\chi},\iota_i)p(\check{\chi}',\iota'_j)p(\check{\phi},\iota)] \\ \sim_{E(\chi^{\#})} &\prod_{1\leq i\leq n} p(\check{\chi},\iota_i)^{i-1}\cdot \prod_{1\leq j\leq n-1} p(\check{\chi}',\iota'_j)^j\cdot p(\check{\phi},\iota)^{n(n-1)/2} \end{aligned}$$

Similarly, we have

$$\prod_{i+j\leq n} p(\check{\chi}^{\#},\overline{\iota_{i,j}}) \sim_{E(\chi^{\#})} \prod_{1\leq i\leq n} p(\check{\chi},\overline{\iota_i})^{n-i}\cdot \prod_{1\leq j\leq n-1} p(\check{\chi}',\overline{\iota'_j})^{n-j}\cdot p(\check{\phi},\bar\iota)^{n(n-1)/2}.$$

Again by Prop. 4.4 and Lem. 4.5, we know that

$$p(\check{\phi},\iota)p(\check{\phi},\bar\iota) \sim_{E(\phi)} p(\check{\phi},\iota)p(\check{\phi}^c,\iota) \sim_{E(\phi)} p(\|\cdot\|^{-1},\iota) \sim_{E(\phi)} 2\pi i.$$

We finally deduce that

$$\begin{aligned} &(2\pi i)^{-(\frac{1}{2}+m)dn(n-1)} L^S(\frac{1}{2}+m,\Pi_\chi\times\Pi_{\chi'}) \\ \sim_{E(\chi)E(\chi')E(\phi)} &\prod_{\iota\in\Sigma}\left(\prod_{1\leq i\leq n}[p(\check{\chi},\iota_i)^{i-1}p(\check{\chi},\overline{\iota_i})^{n-i}] \prod_{1\leq j\leq n-1}[p(\check{\chi}',\iota'_j)^j p(\check{\chi}',\overline{\iota'_j})^{n-j}]\right) \end{aligned}$$

$$\sim_{E(\chi)E(\chi')E(\phi)} \prod_{\iota\in\Sigma}\left(\prod_{1\leq i\leq n}[p(\check{\chi},\iota_i)^{i-1}p(\check{\chi},\overline{\iota_i})^{n-i}]\prod_{1\leq j\leq n-1}[p(\check{\chi}',\iota'_j)^{j-1}p(\check{\chi}',\overline{\iota'_j})^{n-1-j}]\right)$$

where the last equality is due to the fact that χ' is conjugate self-dual, and hence

$$p(\check{\chi}',\iota'_j)p(\check{\chi}',\bar{\iota}'_j)\sim_{E(\chi')}p(\check{\chi}'\otimes\check{\chi}'^c,\iota'_j)\sim_{E(\chi')}p(\mathbf{1},\iota'_j)\sim_{E(\chi')}1.$$

It is easy to see that all relations above are in fact equivariant under $\mathrm{Aut}(\mathbb{C}/(LL')^{Gal})$. □

4.5. **Explicit determination of the archimedean factors $a(\Pi_\infty)$ and $p(m,\Pi_\infty,\Pi'_\infty)$.** Recall the archimedean factors $a(\Pi_\infty)$ and $p(m,\Pi_\infty,\Pi'_\infty)$ from Thm. 1.27 and Thm. 1.30, respectively. Due to a deep theorem of B. Sun, we know that both factors are in fact non-zero. Here, we will determine them explicitly for conjugate self-dual representations Π and Π', revealing them as concrete powers of $(2\pi i)$.

Our main idea of proof is to replace our original representations Π and Π' with particularly simple automorphic representations, *with the same archimedean components*, hence giving rise to the same archimedean factors $a(\Pi_\infty)$ and $p(m,\Pi_\infty,\Pi'_\infty)$. In view of the previous sections, these auxiliary automorphic representations shall be constructed by automorphic induction from suitable Hecke characters, on the one hand, and as isobaric sums of Hecke characters, on the other hand: This approach enables us to use all of our calculations of critical L-values of Ranking-Selberg- and Asai-L-functions.

Taking critical L-values as an anchor, it is clear that we will have to ensure that the L-values entering the proof do not vanish, in order to be able to use the transitivity of our relation "$\sim$", cf. Rem. 1.16. Hence, whenever we are able to manage with a critical value $s_0=\frac{1}{2}+m$, with $m\geq 1$, (i.e., invoking the functional equation, whenever there is a critical L-value apart from the central one $s_0=\frac{1}{2}$) this important condition of non-vanishing is well-known to be satisfied.

Recalling our description of the set of critical points in terms of the infinity-type $\{z^{a_{\iota,i}}\bar{z}^{-a_{\iota,i}}\}_{1\leq i\leq n}$ of a conjugate self-dual representation Π, cf. §1.7.1, and moreover the relation of the infinity-type with the highest weight μ of our coefficient module $\mathcal{E}_\mu$ in cohomology, cf. §1.4.1, one easily sees that we have such a critical point $s_0=\frac{1}{2}+m\geq\frac{3}{2}$ at our disposal, once $\mu_{\iota,j}-\mu_{\iota,j+1}\geq 2$ for all $\iota\in\Sigma$ and $1\leq j\leq n-1$, i.e., if μ lies sufficiently deep inside the open, positive Weyl chamber.

For the (in some sense, rare) case that μ is not sufficiently regular, i.e., is too close to the boundary of the closed, positive Weyl chamber, we will formulate an according non-vanishing hypothesis for each of our auxiliary representations (Hypotheses 4.20 and 4.26) in the proof of the following theorem. We remark, however, that both hypotheses are very well expected to hold in all cases considered, and hence shall rather be considered as a limitation of the current techniques, than as a principal restriction.

Theorem 4.17. *Let Π by a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, which is cohomological with respect to $\mathcal{E}_\mu$. Recall the abstract archimedean factor $a(\Pi_\infty)$ from Thm. 1.27. If μ is sufficiently regular, i.e., $\mu_{\iota,j}-\mu_{\iota,j+1}\geq 2$ for all $\iota\in\Sigma$ and $1\leq j\leq n-1$, or if Hypotheses 4.20 and 4.26 hold, then*

$$a(\Pi_\infty)\sim_{E(\Pi)}(2\pi i)^{dn}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. As pointed out above, we shall prove this theorem by constructing three auxiliary representations, Π_χ, $\Pi_{\chi^\sharp}$ and $\Pi^\flat$, with appropriate archimedean factors.

Construction of Π_χ*:* Since we are only concerned about the infinity-type $\{z^{a_{\iota,i}}\bar{z}^{-a_{\iota,i}}\}_{1\leq i\leq n}$ at $\iota\in\Sigma$ of Π, we first replace Π by a simpler representation with the same infinity-type. We take a CM-field L which is a cyclic extension over F of degree n. We write $\iota_1,\cdots,\iota_n$ for the elements in J_L which extends ι. If n is even, we let $t=\frac{1}{2}$, otherwise we let $t=0$. In any case, $a_{\iota,i}\in\frac{n-1}{2}+\mathbb{Z}$, so $a_{\iota,i}-t\in\mathbb{Z}$. By lemma 5.1 of [Lin15a], there exists an algebraic conjugate self-dual Hecke character χ of L, with infinity-type $z^{a_{\iota,i}-t}\bar{z}^{-a_{\iota,i}+t}$ at ι_i, such that $(\Pi_\chi)_\infty\cong\Pi_\infty$. We recall that if χ satisfies $\chi^\theta\neq\chi$ for any non-trivial $\theta\in\mathrm{Gal}(L/F)$ then Π_χ is cuspidal (cf. Chp. 3, Lem. 6.4 of [Art-Clo89]). Hence, after twisting by an appropriate finite order Hecke character, we may assume that Π_χ is cuspidal.

Construction of $\Pi_{\chi^\sharp}$*:* For each ι, let $c_{\iota,1},c_{\iota,2},\cdots,c_{\iota,n+1}\in(\frac{1}{2}-t)+\mathbb{Z}=\frac{n}{2}+\mathbb{Z}$ such that

$$c_{\iota,1}>-a_{\iota,n}>c_{\iota,2}>\cdots>-a_{\iota,1}>c_{\iota,n+1}.$$

Recalling that $a_{\iota,i}\in\frac{n-1}{2}+\mathbb{Z}$ are all different, such a choice is always possible. We now take another CM field $L^\sharp$ which is a cyclic extension over F of degree $n+1$. Let $\chi^\sharp$ be a conjugate self-dual Hecke character of $L^\sharp$ such that $\chi^\sharp_{\iota_i}(z)=z^{c_{\iota,i}-(\frac{1}{2}-t)}\bar{z}^{-c_{\iota,i}+(\frac{1}{2}-t)}$. At the cost of twisting $\chi^\sharp$ by a Hecke character of finite order, our second auxiliary representation $\Pi_{\chi^\sharp}$, automorphically induced from $\chi^\sharp$ to $\mathrm{GL}_{n+1}(\mathbb{A}_F)$, may again be assumed to be cuspidal. By construction, its infinity-type equals $\{z^{c_{\iota,i}}\bar{z}^{-c_{\iota,i}}\}_{1\leq i\leq n+1}$ at $\iota\in\Sigma$, hence the pair $(\Pi_{\chi^\sharp},\Pi_\chi)$ satisfies the piano-condition, cf. Hypothesis 1.29.

We may hence apply Theorem 1.27 and Thm. 1.30 to $\Pi_{\chi^\sharp}$ and Π_χ, and get that for any critical point $\frac{1}{2}+m\in\mathrm{Crit}(\Pi_{\chi^\sharp}\times\Pi_\chi)$,

$$\frac{L^S(\frac{1}{2}+m,\Pi_{\chi^\sharp}\times\Pi_\chi)}{L^S(1,\Pi_{\chi^\sharp},\mathrm{As}^{(-1)^{n+1}})L^S(1,\Pi_\chi,\mathrm{As}^{(-1)^n})}\sim_{E(\Pi_{\chi^\sharp})E(\Pi_\chi)}\frac{p(m,\Pi_{\chi^\sharp,\infty},\Pi_{\chi,\infty})}{a(\Pi_{\chi^\sharp,\infty})a(\Pi_{\chi,\infty})}\tag{4.18}$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. Here we could remove the Gauß sum $\mathcal{G}(\omega_{\Pi_{\chi,f}})$ by Remark 1.31. On the other hand, Prop. 4.8 & 4.15 imply that the same quotient satisfies the relation

$$\frac{L^S(\frac{1}{2}+m,\Pi_{\chi^\sharp}\times\Pi_\chi)}{L^S(1,\Pi_{\chi^\sharp},\mathrm{As}^{(-1)^{n+1}})L^S(1,\Pi_\chi,\mathrm{As}^{(-1)^n})}\sim_{E(\chi^\sharp)E(\chi)E(\phi)}(2\pi i)^{(\frac{1}{2}+m)dn(n+1)-\frac{1}{2}d(n+1)(n+2)-\frac{1}{2}dn(n+1)}\tag{4.19}$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/(L^\sharp L)^{Gal})$.

If μ is sufficiently regular, i.e., if $\mu_{\iota,j}-\mu_{\iota,j+1}\geq 2$ for all $\iota\in\Sigma$ and $1\leq j\leq n-1$, we may obviously adjust $\chi^\sharp$ such that there exists critical point $\frac{1}{2}+m\in\mathrm{Crit}(\Pi_{\chi^\sharp}\times\Pi_\chi)$ with $m\geq 1$. As $\mathrm{Crit}(\Pi_{\chi^\sharp}\times\Pi_\chi)=\mathrm{Crit}({}^\sigma\Pi_{\chi^\sharp}\times{}^\sigma\Pi_\chi)$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$, there exists then such a critical point for all twists ${}^\sigma\Pi_{\chi^\sharp}\times{}^\sigma\Pi_\chi$ and so, as $\frac{1}{2}+m\geq\frac{3}{2}$, the critical L-value $L^S(\frac{1}{2}+m,{}^\sigma\Pi_{\chi^\sharp}\times{}^\sigma\Pi_\chi)$ is non-zero for all $\sigma\in\mathrm{Aut}(\mathbb{C})$. As a consequence, we can use the transitivity of "$\sim$", cf. Rem. 1.16, if μ is sufficiently regular, and compare (4.18) with (4.19).

If, at the contrary, μ fails to be sufficiently regular, we may always still take $m=0$, but in order to be able to use the transitivity of the relation "$\sim$" (so to compare (4.18) with (4.19)) we have then to assume the validity of

Hypothesis 4.20. *There exists* $\chi^\sharp$ *and* χ *as above such that* $L^S(\frac{1}{2}, {}^\sigma\Pi_{\chi^\sharp} \times {}^\sigma\Pi_\chi) \neq 0$ *for all* $\sigma \in \mathrm{Aut}(\mathbb{C})$.

In any of the two cases, merging (4.18) with (4.19) and applying our Minimizing-Lemma, cf. Lem. 1.19, we finally conclude that

$$\frac{p(m, \Pi_{\chi^\sharp,\infty}, \Pi_{\chi,\infty})}{a(\Pi_{\chi^\sharp,\infty})a(\Pi_{\chi,\infty})} \sim_{E(\Pi_{\chi^\sharp})E(\Pi_\chi)} (2\pi i)^{(\frac{1}{2}+m)dn(n+1)-\frac{1}{2}d(n+1)(n+2)-\frac{1}{2}dn(n+1)} \tag{4.21}$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/(L^\sharp L)^{Gal})$.

Construction of $\Pi^\flat$: We now construct another auxiliary representation of $\mathrm{GL}_n(\mathbb{A}_F)$. For each $1 \leq j \leq n$, let χ_j be a conjugate self-dual Hecke character of F with infinity-type $z^{a_{\iota,j}-t}\bar{z}^{-a_{\iota,j}+t}$ at $\iota \in \Sigma$. We define

$$\Pi^\flat := \begin{cases} \chi_1 \boxplus \ldots \boxplus \chi_n & \text{if } n+1 \text{ is even} \\ (\chi_1\eta) \boxplus \ldots \boxplus (\chi_n\eta) & \text{if } n+1 \text{ is odd} \end{cases}$$

The resulting automorphic $\mathrm{GL}_n(\mathbb{A}_F)$-representation $\Pi^\flat$ is unitary, cohomological and conjugate self-dual, hence comes under the purview of §1.4.3. Moreover, $\Pi^\flat_\infty \cong \Pi_{\chi,\infty}$ and so the pair $(\Pi_{\chi^\sharp}, \Pi^\flat)$ satisfies the piano-condition by construction.

Let m be specified as in our second construction-step above. Since $\mathrm{Crit}(\Pi_{\chi^\sharp} \times \Pi^\flat) = \mathrm{Crit}(\Pi_{\chi^\sharp} \times \Pi_\chi)$, Thm. 1.30 and Rem. 1.31, imply that

$$L^S(\tfrac{1}{2}+m, \Pi_{\chi^\sharp} \times \Pi^\flat) \sim_{\mathbb{Q}(\Pi_{\chi^\sharp})\mathbb{Q}(\Pi^\flat)} p(\Pi_{\chi^\sharp})p(\Pi^\flat)p(m, \Pi_{\chi^\sharp,\infty}, \Pi^\flat_\infty) \tag{4.22}$$

On the other hand, we know that

$$L^S(\tfrac{1}{2}+m, \Pi_{\chi^\sharp} \times \Pi^\flat) = \prod_{1\leq j\leq n} L^S(\tfrac{1}{2}+m, \chi^\sharp \otimes (\chi_j\eta \circ N_{\mathbb{A}_{L^\sharp}/\mathbb{A}_F})) = \prod_{1\leq j\leq n} L^S(m, \chi^\sharp \otimes (\chi_j\phi \circ N_{\mathbb{A}_{L^\sharp}/\mathbb{A}_F})).$$

The Hecke character $\chi^\sharp \otimes (\chi_j\phi \circ N_{\mathbb{A}_{L^\sharp}/\mathbb{A}_F})$ has infinity-type $z^{a_{\iota,j}+c_{\iota,i}+\frac{1}{2}}\bar{z}^{-a_{\iota,j}-c_{\iota,i}+\frac{1}{2}}$ at ι_i. Hence it is compatible, cf. §4.1, with the CM type

$$\bigcup_{\iota\in\Sigma}\{\iota_i \mid i \geq n+2-j\} \cup \{\bar{\iota_i} \mid i \leq n+1-j\}.$$

By Blasius's result, Thm. 4.7, we have:

$$\begin{aligned} & L^S(m, \chi^\sharp \otimes (\chi_j\phi \circ N_{\mathbb{A}_{L^\sharp}/\mathbb{A}_F})) \\ \sim_{E(\chi^\sharp)E(\chi_j)E(\phi)} \; & (2\pi i)^{md(n+1)} \prod_{\iota\in\Sigma}\prod_{i\geq n+2-j} p(\check{\chi}^\sharp, \iota_i)p(\check{\chi}_j, \iota)p(\check{\phi}, \iota) \prod_{i\leq n+1-j} p(\check{\chi}^\sharp, \bar{\iota_i})p(\check{\chi}_j, \bar{\iota})p(\check{\phi}, \bar{\iota}) \end{aligned}$$

Denote $\prod\limits_{1\leq j\leq n} E(\chi_j)$ simply by E'. Then

$$\begin{aligned} (4.23) \qquad & L^S(\tfrac{1}{2}+m, \Pi_{\chi^\sharp} \times \Pi^\flat) \\ = \; & \prod_{1\leq j\leq n} L^S(m, \chi^\sharp \otimes (\chi_j\phi \circ N_{\mathbb{A}_{L^\sharp}/\mathbb{A}_F})) \\ \sim_{E(\chi^\sharp)E'E(\phi)} \; & (2\pi i)^{mdn(n+1)} \prod_{\iota\in\Sigma}[(\prod_{1\leq i\leq n+1} p(\check{\chi}^\sharp, \iota_i)^{i-1}p(\check{\chi}^\sharp, \iota_i)^{n+1-i})(\prod_{1\leq j\leq n} p(\check{\chi}_j, \iota)^{j-1}p(\check{\chi}_j, \bar{\iota})^{n+1-j}) \times \\ & (p(\check{\phi}, \iota)^{n(n+1)/2}p(\check{\phi}, \bar{\iota})^{n(n+1)/2})] \\ \sim_{E(\chi^\sharp)E'E(\phi)} \; & (2\pi i)^{(\frac{1}{2}+m)dn(n+1)} \prod_{\iota\in\Sigma}[(\prod_{1\leq i\leq n+1} p(\check{\chi}^\sharp, \iota_i)^{i-1}p(\check{\chi}^\sharp, \iota_i)^{n+1-i})(\prod_{1\leq j\leq n} p(\check{\chi}_j, \iota)^{j-1}p(\check{\chi}_j, \bar{\iota})^{n+1-j})] \end{aligned}$$

where the last equation is due to the fact that $p(\check{\phi},\iota)p(\check{\phi},\bar{\iota}) \sim_{E(\phi)} 2\pi i$.

On the other hand, by Corollary 4.14 we know that

$$p(\Pi_{\chi^\sharp}) \sim_{E(\Pi_{\chi^\sharp})E(\chi^\sharp)} a(\Pi_{\chi^\sharp,\infty})^{-1}(2\pi i)^{d(n+1)(n+2)/2}\prod_{\iota\in\Sigma}\prod_{1\leq i\leq n+1}[p(\check{\chi}^\sharp,\iota_i)^{i-1}p(\check{\chi}^\sharp,\bar{\iota}_i)^{n+1-i}]. \tag{4.24}$$

Moreover, by Corollary 2.12, we know $p(\Pi^\flat) \sim_{E(\Pi^\flat)E(\phi)} \prod_{1\leq j<k\leq n} L(1,\chi_j\otimes\chi_k^\vee)$. Here we recall that the Whittaker period of an algebraic Hecke character is trivial by construction, cf. §1.5.1.

The Hecke character $\chi_j\otimes\chi_k^\vee = \chi_j\otimes\chi_k^c$ has infinity-type $z^{a_{\iota,j}-a_{\iota,k}}\bar{z}^{-a_{\iota,j}+a_{\iota,k}}$. Since $j<k$, we know $a_{\iota,j}-a_{\iota,k}>0$ and the character $\chi_j\otimes\chi_k^c$ is compatible with $\bar{\Sigma}$. Therefore,

$$\begin{aligned}\prod_{1\leq j<k\leq n} L(1,\chi_j\otimes\chi_k^\vee) &\sim_{E'} (2\pi i)^{dn(n-1)/2}\prod_{1\leq j<k\leq n}\prod_{\iota\in\Sigma} p(\check{\chi}_j\check{\chi}_k{}^c,\bar{\iota})\\ &\sim_{E'} (2\pi i)^{dn(n-1)/2}\prod_{1\leq j<k\leq n}\prod_{\iota\in\Sigma}[p(\check{\chi}_j,\bar{\iota})p(\check{\chi}_k,\iota)]\\ &\sim_{E'} (2\pi i)^{dn(n-1)/2}\prod_{\iota\in\Sigma}\prod_{1\leq j\leq n}[p(\check{\chi}_j,\iota)^{j-1}p(\check{\chi}_j,\bar{\iota})^{n+1-j}]\end{aligned}$$

and hence

$$p(\Pi^\flat) \sim_{E(\Pi^\flat)E'E(\phi)} (2\pi i)^{dn(n-1)/2}\prod_{\iota\in\Sigma}\prod_{1\leq j\leq n}[p(\check{\chi}_j,\iota)^{j-1}p(\check{\chi}_j,\bar{\iota})^{n+1-j}]. \tag{4.25}$$

As above, we assume that either μ is sufficiently regular or, if not, the validity of the following hypothesis

Hypothesis 4.26. *There exists $\chi^\sharp$ and χ_j, $1\leq j\leq n$ as above, such that $L^S(\frac{1}{2},{}^\sigma\Pi_{\chi^\sharp}\times{}^\sigma\Pi^\flat)\neq 0$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$.*

As a consequence, we may again use the transitivity of the relation "$\sim$", see Rem. 1.16. Hence, comparing (4.22), (4.23), (4.24) and (4.25), and invoking our Minimizing-Lemma, Lem. 1.19, we deduce that:

$$\frac{p(m,\Pi_{\chi^\sharp,\infty},\Pi^\flat_\infty)}{a(\Pi_{\chi^\sharp,\infty})} \sim_{E(\Pi_{\chi^\sharp})E(\Pi^\flat)} (2\pi i)^{(\frac{1}{2}+m)dn(n+1)-\frac{1}{2}d(n+1)(n+2)-\frac{1}{2}dn(n-1)}. \tag{4.27}$$

Conclusion: Comparing (4.21) with (4.27) and applying our Minimizing-Lemma once more, we conclude that $a(\Pi_{\chi,\infty}) \sim_{E(\Pi_\chi)} (2\pi i)^{dn}$. Invoking that $\Pi_{\chi,\infty}\cong\Pi_\infty$ and the Minimizing-Lemma, we finally obtain the desired relation $a(\Pi_\infty)\sim_{E(\Pi)}(2\pi i)^{dn}$ for our given cuspidal representation Π.

For the last assertion, observe that the relation for $a(\Pi_\infty)$ is independent of the choice of field extensions $L^\sharp$ and L. Hence, it is in fact equivariant under the union of all groups $\mathrm{Aut}(\mathbb{C}/(L^\sharp L)^{Gal})$, taken over all $L^\sharp$ and L, which are cyclic CM-extensions of F of prescribed degree. By class field theory, this union is $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. □

Remark 4.28. Instead of the regularity-condition on μ, we could have equivalently assumed that there is a conjugate self-dual cuspidal automorphic representation $\Pi^\sharp$ of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$, satisfying the piano-hypothesis when coupled with Π, and $\frac{1}{2}+m\in\mathrm{Crit}(\Pi^\sharp\times\Pi)$ with $m\neq 0$. This assumption, however, just reads far more elaborate than the simple obstruction on the highest weight μ.

Let now Π and Π' be automorphic representations as in Thm. 1.30 and assume that their archimedean components Π_∞ and Π'_∞ are conjugate self-dual. Choose conjugate self-dual Hecke characters χ of L and χ' of L' as in §4.4.1 such that $\Pi_{\chi,\infty}\cong\Pi_\infty$ and $\Pi_{\chi',\infty}\cong\Pi'_\infty$. By re-adjusting the characters, if necessary, we may impose that Π_χ and likewise $\Pi_{\chi'}$ is cuspidal. For later reference we record the following

Hypothesis/Conjecture 4.29. There are characters χ and χ' such that $L^S(\frac{1}{2},{}^\sigma\Pi_\chi\times{}^\sigma\Pi_{\chi'})\neq 0$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$.

The above theorem implies

Corollary 4.30. *Let Π and Π' be cohomological automorphic representations as in Thm. 1.30 and assume that their archimedean components Π_∞ and Π'_∞ are conjugate self-dual. Let $\frac{1}{2}+m\in\mathrm{Crit}(\Pi\times\Pi')$ be a critical point. If $m=0$ we assume Hyp. 4.20 & 4.26 for Π and Π', whenever μ or μ' are not sufficiently regular, and moreover that there exist algebraic conjugate self-dual Hecke characters χ and χ' as in §4.4.1, such that Π_χ and $\Pi_{\chi'}$ are cuspidal representations, satisfying Hyp./Conj. 4.29 and $\Pi_{\chi,\infty}\cong\Pi_\infty$ and $\Pi_{\chi',\infty}\cong\Pi'_\infty$. Then,*

$$p(m,\Pi_\infty,\Pi'_\infty)\sim_{E(\Pi)E(\Pi')}(2\pi i)^{mdn(n-1)-\frac{1}{2}d(n-1)(n-2)}$$

is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. Choose some algebraic conjugate self-dual Hecke characters χ and χ' as in §4.4.1, such that Π_χ and $\Pi_{\chi'}$ are cuspidal with $\Pi_{\chi,\infty}\cong\Pi_\infty$ and $\Pi_{\chi',\infty}\cong\Pi'_\infty$. As pointed out above, this is always possible. Let $\frac{1}{2}+m\in$Crit$(\Pi\times\Pi')=$Crit$(\Pi_\chi\times\Pi_{\chi'})$. If $m\neq 0$, then our description of the set of critical points for Eisenstein representations, which satisfy the piano-condition, given in §1.7.1, tells us that the highest weights μ and μ' of the finite-dimensional coefficient modules $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$, with respect to which Π_∞ and Π'_∞ are cohomological, are sufficiently regular. Hence, Thm. 4.17 holds for Π_χ and $\Pi_{\chi'}$, i.e., we have $a(\Pi_{\chi,\infty})\sim_{E(\Pi_\chi)}(2\pi i)^{dn}$ and $a(\Pi_{\chi',\infty})\sim_{E(\Pi_{\chi'})}(2\pi i)^{d(n-1)}$. Moreover, as $\frac{1}{2}+m\neq\frac{1}{2}$, the critical L-value $L^S(\frac{1}{2}+m,{}^\sigma\Pi_\chi\times{}^\sigma\Pi_{\chi'})$ is non-zero for all $\sigma\in\mathrm{Aut}(\mathbb{C})$. Hence, (4.21) is valid, which yields

$$p(m,\Pi_{\chi,\infty},\Pi_{\chi',\infty})\sim_{E(\Pi_\chi)E(\Pi_{\chi'})}(2\pi i)^{mdn(n-1)-\frac{1}{2}d(n-1)(n-2)}.$$

Invoking our Minimizing-Lemma, cf. Lem. 1.19, shows the claim for $m\neq 0$. If $m=0$, then our additional assumptions imply that one may in fact argue as for $m\neq 0$. This completes the proof. ☐

5. Our four main theorems for special L-values

5.1. **Critical values of Asai L-functions.** Our first main theorem for special values has two major assets: Firstly, it generalizes Thm. 1.27 to isobaric representations $\Pi'=\Pi_1\boxplus...\boxplus\Pi_k$, with an arbitrary number of conjugate self-dual cuspidal summands Π_i. Secondly, we are able to determine the (mysterious) archimedean factor in the resulting relation due to our calculations in §4.4 as a concrete power of $(2\pi i)$.

In what follows, we write μ_i^{alg} for the highest weight of the algebraic representation with respect to which Π_i^{alg}, cf. 2.11, is cohomological.

Theorem 5.1. *Let $\Pi'=\Pi_1\boxplus...\boxplus\Pi_k$ be a cohomological isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ as in 1.4.3, such that each cuspidal automorphic summand Π_i is conjugate self-dual. If μ_i^{alg} is not sufficiently regular, we assume Hyp. 4.20 & 4.26 for Π_i^{alg}. One has*

$$L^S(1,\Pi',\mathrm{As}^{(-1)^n})\ \sim_{E(\Pi')}\ (2\pi i)^{dn}p(\Pi')$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. On the one hand, by Lem. 3.3 we know that

$$L^S(1,\Pi',\mathrm{As}^{(-1)^n}) = \prod_{i=1}^{k} L^S(1,\Pi_i^{\mathsf{alg}},\mathrm{As}^{(-1)^{n_i}}) \cdot \prod_{1\le i<j\le k} L^S(1,\Pi_i\times\Pi_j^\vee).$$

Since Π_i^{alg} is unitary conjugate self-dual, cuspidal and cohomological, we may apply Thm. 1.27 and by our extra assumptions on Π_i^{alg} moreover Thm. 4.17 to get

$$L^S(1,\Pi_i^{\mathsf{alg}},\mathrm{As}^{(-1)^{n_i}}) \sim_{E(\Pi_i^{\mathsf{alg}})} a(\Pi_{i,\infty}^{\mathsf{alg}})p(\Pi_i^{\mathsf{alg}}) \sim_{E(\Pi_i^{\mathsf{alg}})} (2\pi i)^{dn_i} p(\Pi_i^{\mathsf{alg}})$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

On the other hand, by Cor. 2.12, we have

$$p(\Pi') \sim_{E(\Pi')E(\phi)} \prod_{1\le i\le k} p(\Pi_i^{\mathsf{alg}}) \prod_{1\le i<j\le k} L^S(1,\Pi_i\times\Pi_j^\vee)$$

which is also equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. Hence,

$$L^S(1,\Pi',\mathrm{As}^{(-1)^n}) \sim_{\prod_{i=1}^{k} E(\Pi_i^{\mathsf{alg}})E(\Pi')E(\phi)} \prod_{i=1}^{k}(2\pi i)^{dn_i}\cdot p(\Pi') = (2\pi i)^{dn}p(\Pi').$$

We apply the Minimizing-Lemma, cf. Lem. 1.19, in order to shrink the base field of the relation to $E(\Pi')$. This shows the claim. □

5.2. **Critical values of Rankin-Selberg L-functions.** Our second main theorem for special values provides a explicit refinement of Thm. 1.30, revealing the archimedean factor $p(m,\Pi_\infty,\Pi'_\infty)$ – extending Cor. 4.30 – also for non-cuspidal isobaric representations Π' as an explicit power of $(2\pi i)$.

As before, we may choose some appropriate algebraic conjugate self-dual Hecke characters χ and χ' as in §4.4.1, such that $\Pi_{\chi,\infty}\cong\Pi_\infty$ and $\Pi_{\chi',\infty}\cong\Pi'_\infty$. We write $\Pi_{\chi'}=\Pi_{\chi',1}\boxplus\ldots\boxplus\Pi_{\chi',k}$.

Theorem 5.2. *Let Π and Π' be cohomological automorphic representations as in Thm. 1.30 and assume that their archimedean components Π_∞ and Π'_∞ are conjugate self-dual. Let $\frac12+m\in$ $\mathrm{Crit}(\Pi\times\Pi')$ be a critical point. If $m=0$ we assume that there are algebraic conjugate self-dual Hecke characters χ and χ' as in §4.4.1, such that Π_χ is cuspidal, $(\Pi_\chi,\Pi_{\chi'})$ satisfies Hyp./Conj. 4.29 and moreover, that whenever μ or $\mu^{\mathsf{alg}}_{\chi',i}$ is not sufficiently regular, Hyp. 4.20 & 4.26 hold for Π resp. $\Pi^{\mathsf{alg}}_{\chi',i}$. Then,*

$$L^S(\tfrac12+m,\Pi\times\Pi') \ \sim_{E(\Pi)E(\Pi')} (2\pi i)^{mdn(n-1)-\frac12 d(n-1)(n-2)}p(\Pi)\ p(\Pi')\mathcal{G}(\omega_{\Pi'_f})$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. Let χ and χ' be appropriate algebraic conjugate self-dual Hecke characters as in §4.4.1, such that $\Pi_{\chi,\infty}\cong\Pi_\infty$ and $\Pi_{\chi',\infty}\cong\Pi'_\infty$. We may arrange that Π_χ is cuspidal and write $\Pi_{\chi'}=\Pi_{\chi',1}\boxplus\ldots\boxplus\Pi_{\chi',k}$. Let $\frac12+m\in$Crit$(\Pi\times\Pi')=$Crit$(\Pi_\chi\times\Pi_{\chi'})$. If $m\neq0$, then our description of the set of critical points, cf. §1.7.1, implies that the highest weights μ or $\mu^{\mathsf{alg}}_{\chi',i}$ are all sufficiently regular. In particular, the automorphic representations Π_χ and $\Pi_{\chi'}$ then satisfy the assumptions of Thm. 1.30 and Thm. 5.1. Hence, we obtain

$$\frac{L^S(\frac12+m,\Pi_\chi\times\Pi_{\chi'})}{L^S(1,\Pi_\chi,\mathrm{As}^{(-1)^n})L^S(1,\Pi_{\chi'},\mathrm{As}^{(-1)^{n-1}})} \sim_{E(\Pi_\chi)E(\Pi_{\chi'})} \frac{p(m,\Pi_{\chi,\infty},\Pi_{\chi',\infty})}{(2\pi i)^{d(2n-1)}} \tag{5.3}$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. On the other hand, applying Prop. 4.8 and 4.15 shows that the same quotient satisfies the relation

$$\frac{L^S(\frac{1}{2}+m,\Pi_\chi\times\Pi_{\chi'})}{L^S(1,\Pi_\chi,\mathrm{As}^{(-1)^n})L^S(1,\Pi_{\chi'},\mathrm{As}^{(-1)^{n-1}})}\sim_{E(\chi)E(\chi')E(\phi)}(2\pi i)^{(\frac{1}{2}+m)dn(n-1)-dn^2} \tag{5.4}$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/(LL')^{Gal})$. As $\frac{1}{2}+m\neq\frac{1}{2}$ we have $L^S(\frac{1}{2}+m,\Pi_\chi\times\Pi_{\chi'})\neq 0$, so we may combine (5.3) and (5.4) and obtain

$$p(m,\Pi_\infty,\Pi'_\infty)\sim_{E(\Pi)E(\Pi')}(2\pi i)^{mdn(n-1)-\frac{1}{2}d(n-1)(n-2)}$$

by the Minimizing-Lemma, cf. Lem. 1.19. Hence, the result follows for $m\neq 0$ from applying Thm. 1.30 to Π and Π'. If finally $m=0$, then our assumptions on Π and $\Pi^{\mathsf{alg}}_{\chi',i}$ imply that one may argue as for the case $m\neq 0$. This completes the proof. □

5.3. **Quotients of critical L-values.** As a consequence of Thm. 5.1 and Thm. 5.2, we obtain another two rationality-results, both for quotients of critical L-values, see Thm. 5.5 and Thm. 5.6 below. It is one of their advantages that they avoid any reference to bottom-degree Whittaker periods, but express the respective ratio of critical L-values purely in terms of powers of $(2\pi i)$.

Let us point out that the first of these theorems, Thm. 5.5, establishes the main result of [GHar-Rag17] for general CM-fields F, and a general pair of automorphic representations (Π,Π') of $\mathrm{GL}_n(\mathbb{A}_F)\times\mathrm{GL}_{n-1}(\mathbb{A}_F)$ satisfying Thm. 5.6, as compared to the case of totally real fields F^+ and a pair of cuspidal cohomological representations (σ,σ') of $\mathrm{GL}_n(\mathbb{A}_{F^+})\times\mathrm{GL}_{n'}(\mathbb{A}_{F^+})$ considered *ibidem*. While the second theorem, Thm. 5.6, will allow us to prove a version of the refined Gan–Gross–Prasad conjecture for unitary groups in §6 below. It is also closely connected to Deligne's conjecture for motivic L-functions, see Rem. 5.8.

Theorem 5.5. *Let Π and Π' be as in Thm. 5.2 and let $\frac{1}{2}+m,\frac{1}{2}+\ell\in\mathrm{Crit}(\Pi\times\Pi')$ be two critical points. Whenever $L^S(\frac{1}{2}+\ell,\Pi\times\Pi')$ is non-zero (e.g., if $\ell\neq 0$),*

$$\frac{L^S(\frac{1}{2}+m,\Pi\times\Pi')}{L^S(\frac{1}{2}+\ell,\Pi\times\Pi')}\sim_{E(\Pi)E(\Pi')}(2\pi i)^{d(m-\ell)n(n-1)}.$$

and this relation is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. In particular, if $L^S(\frac{3}{2}+m,\Pi\times\Pi')$ is non-zero (e.g., if $m\neq -1$), the quotient of consecutive critical L-values satisfies

$$(2\pi i)^{dn(n-1)}\frac{L^S(\frac{1}{2}+m,\Pi\times\Pi')}{L^S(\frac{3}{2}+m,\Pi\times\Pi')}\in E(\Pi)E(\Pi').$$

Proof. This follows directly from Thm. 5.2. □

This theorem also complements recent achievements of Januszewski, see [Jan16], Thm. A, where an analogously explicit result has been proved (under different assumptions) for pairs of cuspidal representations (π,σ) over totally real fields.

Theorem 5.6. *Let Π be a cohomological conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and let $\Pi'=\Pi_1\boxplus\ldots\boxplus\Pi_k$ be a cohomological isobaric automorphic sum on $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, fully induced from distinct conjugate self-dual cuspidal automorphic representations Π_i. Assume that the highest weight modules $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ of Π and Π' satisfy the piano-hypothesis Hyp. 1.29. Let $\frac{1}{2}+m\in\mathrm{Crit}(\Pi\times\Pi')$ be a critical point. If $m=0$ we assume that there are algebraic conjugate self-dual Hecke characters χ and χ' as in §4.4.1, such that Π_χ is cuspidal, $(\Pi_\chi,\Pi_{\chi'})$ satisfies Hyp./Conj.*

4.29 and moreover, that whenever μ or $\mu^{\mathsf{alg}}_{\chi',i}$ is not sufficiently regular, Hyp. 4.20 & 4.26 hold for Π resp. $\Pi^{\mathsf{alg}}_{\chi',i}$. Then,

$$\frac{L^S(\frac{1}{2}+m,\Pi\times\Pi')}{L^S(1,\Pi,\mathrm{As}^{(-1)^n})\ L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})}\sim_{E(\Pi)E(\Pi')}(2\pi i)^{mdn(n-1)-dn(n+1)/2}.$$

and this relation is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. Let Π and Π' be as stated. By Thm. 5.2, see also Rem. 1.31,

$$L^S(\tfrac{1}{2}+m,\Pi\times\Pi')\ \sim_{E(\Pi)E(\Pi')}(2\pi i)^{mdn(n-1)-\frac{1}{2}d(n-1)(n-2)}p(\Pi)\ p(\Pi').$$

By Thm. 5.1, we have $L^S(1,\Pi,\mathrm{As}^{(-1)^n})\ \sim_{E(\Pi)}\ (2\pi i)^{dn}p(\Pi)$ and $L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})\ \sim_{E(\Pi')}(2\pi i)^{d(n-1)}p(\Pi')$. This shows the claim. □

Remark 5.7. From the proof we can see that the same strategy works as well for certain non-cuspidal Π, for example, if Π is isobaric sum of Hecke characters.

Remark 5.8 (*Relation to Deligne's conjecture*)**.** Due to the absence of our Whittaker periods, it is easiest to interpret Thm. 5.6 from the perspective of Deligne's conjecture on critical values of motivic L-functions. Indeed, in Thm. 5.6, $s_0=\frac{1}{2}+m$ is critical for $L(s,\Pi\times\Pi')$ and $s_0=1$ is critical for $L(s,\Pi,\mathrm{As}^{(-1)^n})\ L(s,\Pi',\mathrm{As}^{(-1)^{n-1}})$ in the sense coined by Deligne, cf. [Del79]. Invoking the conjectural dictionary between automorphic representations Π and Π' and motives, there should hence be irreducible motives $\mathbb{M}$ and $\mathbb{M}'$ over F whose attached Deligne periods capture the transcendental part of the respective L-value. More precisely, we have:

$$\begin{aligned}
&L^S(\tfrac{1}{2}+m,\Pi\times\Pi')=L^S(m+n-1,\mathbb{M}\times\mathbb{M}')=L^S(0,\mathbb{M}\times\mathbb{M}'(m+n-1)),\\
&L^S(1,\Pi,\mathrm{As}^{(-1)^n})=L^S(1,\mathrm{As}^{(-1)^n}(\mathbb{M}))=L^S(0,\mathrm{As}^{(-1)^n}(\mathbb{M})(1)),\\
&L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})=L^S(1,\mathrm{As}^{(-1)^{n-1}}(\mathbb{M}'))=L^S(0,\mathrm{As}^{(-1)^{n-1}}(\mathbb{M}')(1)).
\end{aligned}$$

Moreover, one can show that if (Π,Π') satisfies the piano-hypothesis, then the Deligne periods are related to each other by the formula

$$c^+(\mathbb{M}\times\mathbb{M}'(m+n-1))\sim(2\pi i)^{mdn(n-1)-dn(n+1)/2}c^+(\mathrm{As}^{(-1)^n}(\mathbb{M})(1))c^+(\mathrm{As}^{(-1)^{n-1}}(\mathbb{M}')(1))$$

We refer to §1 of [MHar13], when $F^+=\mathbb{Q}$, and to §2 of [MHar-Lin17] and the forthcoming thesis of P. Lopez for general F^+. As a consequence, Thm. 5.6 is in perfect fit with Deligne's conjecture, [Del79, Conj. 2.8].

One can also compare Thm. 5.1 and Thm. 5.2 with Deligne's conjecture, though the actual presence of Whittaker periods makes it trickier to interpret our formulas motivically. The difficulty relies in the problem to find a motivic analogue of our Whittaker periods: At least when Π and Π' descend to unitary groups of all signatures, one can define so-called *arithmetic automorphic periods* for these representations (cf. [MHar97], [Lin15b]), which in fact have motivic analogues (cf. §4 of [MHar-Lin17]). The final bridge between Whittaker periods and arithmetic automorphic periods is then provided by [Gro-MHar16] and [Lin15b]. We remark that there is an archimedean factor left undecided in the underlying relations. By a strategy, similar to the one presented here, one can show however that this archimedean factor is also equivalent to a power of $2\pi i$. One can then compare the Whittaker periods with the Deligne periods. This shall also be part of the forthcoming thesis of P. Lopez.

6. Our main theorem on the refined GGP-conjecture for unitary groups

6.1. **A short review of the GGP-conjecture.** Broken down to one sentence, the global Gan-Gross-Prasad Conjecture (GGP) asserts that the non-vanishing of the central value $s = \frac{1}{2}$ of an "Rankin-Selberg-type" L-function $L(s, \pi_{\mathcal{V}} \boxtimes \pi_{\mathcal{W}})$ of two tempered cuspidal automorphic representations $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{W}}$ of an isometry group is equivalent to the non-vanishing of a period-integral $\mathcal{P}(\varphi_{\mathcal{V}}, \varphi_{\mathcal{W}})$ (cf. [Gan-Gro-Pra12], Conj. 24.1).

In order to set up notation for later and to put ourselves in medias res, let $\mathbb{E}/\mathbb{F}$ be a field extension of number fields of degree $\dim_{\mathbb{F}} \mathbb{E} \leq 2$ and c the unique automorphism of $\mathbb{E}$ which has $\mathbb{F}$ as fixed points $\mathbb{E}^{c=\mathbf{1}} = \mathbb{F}$ (e.g., $\mathbb{E} = F$ and $\mathbb{F} = F^+$ from §1.1). Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{E}$ and let $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{E}$ be a non-degenerate, c-sesquilinear pairing. The connected component of the identity of the group of isometries with respect to $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is denoted $\mathscr{G}(\mathcal{V})$ and a reductive algebraic group over $\mathbb{F}$ (e.g., $\mathcal{V} = V_n$ and $\mathscr{G}(\mathcal{V}) = H_n$ from §1.2). Not to interfere with low-rank cases, we will have to assume tacitly that $\dim_{\mathbb{E}}(\mathcal{V}) + [\mathbb{E} : \mathbb{F}] \geq 4$ (e.g, that $n \geq 2$ in the terminology of §1.2).

Let $\mathcal{W} \subset \mathcal{V}$ be a non-degenerate subspace of $\mathcal{V}$ of odd codimension $\dim_{\mathbb{E}}(\mathcal{W}^\perp) = 2r + 1$, containing an isotropic subspace $\mathcal{X}$ of dimension $r \geq 0$ (i.e., $\mathcal{W}$ is r-split). (Here, for reasons of precision, we assume that $\mathscr{G}(\mathcal{W})$ is not split if $\dim_{\mathbb{E}}(\mathcal{W}) = 2$.) We define $\mathscr{P} = \mathscr{P}_{\mathfrak{F}}$ to be the parabolic subgroup of $\mathscr{G}(\mathcal{V})$, which stabilizes a fixed complete flag $\mathfrak{F}$ of $r+1$ isotropic $\mathbb{E}$-subspaces in $\mathcal{X}$ and let $\mathscr{G}(\mathcal{W})$ be defined as above, replacing $\mathcal{V}$ by $\mathcal{W}$. Then there are natural inclusions $\mathscr{G}(\mathcal{W}) \hookrightarrow \mathscr{P}_{\mathfrak{F}} \hookrightarrow \mathscr{G}(\mathcal{V})$, where $\mathscr{G}(\mathcal{W})$ embeds into a Levi subgroup of $\mathscr{P}$, whence it acts naturally by conjugation on the unipotent radical $\mathscr{N} = \mathscr{N}_{\mathfrak{F}}$ of $\mathscr{P}$. We set $\mathscr{H} := \mathscr{G}(\mathcal{W}) \rtimes \mathscr{N}$, which is again a natural subgroup of $\mathscr{G}(\mathcal{V})$. For all the above we refer to [Gan-Gro-Pra12], §2 and §12.

In what follows $\mathbb{A} = \mathbb{A}_{\mathbb{F}}$. We chose a generic automorphic character

$$\psi_{\mathfrak{F}} = \otimes_v \psi_{\mathfrak{F},v} : \mathscr{N}_{\mathfrak{F}}(\mathbb{F}) \backslash \mathscr{N}_{\mathfrak{F}}(\mathbb{A}) \to \mathbb{C}^\times,$$

which is invariant under conjugation by $\mathscr{G}(\mathcal{W})(\mathbb{A})$ and define the form

$$\Psi_{\psi_{\mathfrak{F}}}(\varphi)(g) := \int_{\mathscr{N}_{\mathfrak{F}}(\mathbb{F}) \backslash \mathscr{N}_{\mathfrak{F}}(\mathbb{A})} \varphi(n)\ \psi_{\mathfrak{F}}(ng)^{-1}\ dn,$$

for an automorphic form φ of $\mathscr{G}(\mathcal{V})(\mathbb{A})$ and the Tamagawa measure dn of $\mathscr{N}_{\mathfrak{F}}(\mathbb{A})$. Since the domain of integration is compact, the integral converges absolutely. Now, let $\pi_{\mathcal{V}}$ (resp. $\pi_{\mathcal{W}}$) be a cuspidal automorphic representation of $\mathscr{G}(\mathcal{V})(\mathbb{A})$ (resp. $\mathscr{G}(\mathcal{W})(\mathbb{A})$) and $\varphi \in \pi_{\mathcal{V}}$ (resp. $\varphi' \in \pi_{\mathcal{W}}$) be a cusp form. Then the *global period integral*

$$\mathcal{P}(\varphi, \varphi') := \int_{\mathscr{G}(\mathcal{W})(\mathbb{F}) \backslash \mathscr{G}(\mathcal{W})(\mathbb{A})} \Psi_{\psi_{\mathfrak{F}}}(\varphi)(g')\ \varphi'(g')\ dg' \tag{6.1}$$

is absolutely convergent. Again, dg' denotes the Tamagawa measure on $\mathscr{G}(\mathcal{W})(\mathbb{A})$.

Suppose now in addition that $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{W}}$ are tempered at all places and let S be any finite set of places containing all archimedean places and the places where $\pi_{\mathcal{V}}$ or $\pi_{\mathcal{W}}$ ramify. Then the partial L-function $L^S(s, \pi_{\mathcal{V}} \boxtimes \pi_{\mathcal{W}})$ is defined with respect to the local Satake-parameters of $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{W}}$ outside S and the representation

$$R = \begin{cases} \mathrm{St} \otimes \mathrm{St} & \text{if } \mathbb{E} = \mathbb{F} \\ \mathrm{Ind}_{\widehat{\mathscr{G}(\mathcal{V}) \times \mathscr{G}(\mathcal{W})}}^{L(\mathscr{G}(\mathcal{V}) \times \mathscr{G}(\mathcal{W}))}[\mathrm{St} \otimes \mathrm{St}] & \text{if } [\mathbb{E} : \mathbb{F}] = 2 \end{cases}$$

of the L-group ${}^L(\mathscr{G}(\mathcal{V}) \times \mathscr{G}(\mathcal{W}))$. Here, St denotes the standard representation of the respective factor. We have to assume that this L-function allows a meromorphic continuation to whole s-plane.

Then, in the situation at hand, the GGP-conjecture asserts[3]

Conjecture 6.2 ([Gan-Gro-Pra12], Conj. 24.1)**.** *Let $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{W}}$ be tempered cuspidal automorphic representations of $\mathscr{G}(\mathcal{V})(\mathbb{A})$ resp. $\mathscr{G}(\mathcal{W})(\mathbb{A})$, which appear with multiplicity one in the cuspidal spectrum. Then the following statements are equivalent:*

(i) $L^S(\frac{1}{2}, \pi_{\mathcal{V}} \boxtimes \pi_{\mathcal{W}}) \neq 0$ *and* $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathscr{H}(\mathbb{A})}[\pi_{\mathcal{V}} \otimes \pi_{\mathcal{W}}, \psi_{\mathfrak{F}}] = 1$

(ii) $\mathcal{P}(\varphi, \varphi') \neq 0$ *for some cusp forms* $\varphi \in \pi_{\mathcal{V}}$ *and* $\varphi' \in \pi_{\mathcal{W}}$.

6.2. **Refinements of the GGP-conjecture.** In the last couple of years the GGP-Conjecture has undergone a series of increasingly general refinements, which shade a significant amount of new light on the original conjecture of GGP. As it will be of great importance for our major application to know them precisely, we have to recall them shortly. We define L-functions $L^S(s, \pi_{\mathcal{V}}, \mathrm{Ad})$ (resp. $L^S(s, \pi_{\mathcal{W}}, \mathrm{Ad})$) of $\pi_{\mathcal{V}}$ (resp. $\pi_{\mathcal{W}}$) with respect to the Satake parameters and the adjoint representation $R = \mathrm{Ad}$ of the L-group ${}^L\mathscr{G}(\mathcal{V})$ (resp. ${}^L\mathscr{G}(\mathcal{W})$). Again, we shall suppose that these L-functions are meromorphically continuable to all $s \in \mathbb{C}$ and moreover, that they do not vanish at $s = 1$ (Note that in this generality these L-functions don't come under the purview of [Sha81], Thm. 5.1).

Recall now that dg' denotes the Tamagawa measure on $\mathscr{G}(\mathcal{W})$. We choose, once and for all, local Haar measures dg'_v at all places v of $\mathbb{F}$, such that the following holds

(1) $dg' = \prod_v dg'_v$

(2) $vol_{dg'_v}(\mathscr{O}_v) \in \mathbb{Q}$ for all open subsets $\mathscr{O}_v$ of $\mathscr{G}(\mathcal{W})(\mathbb{F}_v)$, if v is non-archimedean

(3) $vol_{dg'_v}(\mathscr{K}_v) = 1$ for a hyperspecial maximal compact subgroup $\mathscr{K}_v$ of $\mathscr{G}(\mathcal{W})(\mathbb{F}_v)$, for $\mathscr{G}(\mathcal{W})$ unramified at v.

Next, pin down a factorization $\pi_{\mathcal{V}} \cong \otimes'_v \pi_{\mathcal{V},v}$ which is compatible with the factorization of global and local inner products, i.e., for the usual L^2-product

$$\langle \varphi_1, \varphi_2 \rangle^{\mathcal{V}}_{\mathbb{A}} = \int_{\mathscr{G}(\mathcal{V})(F)\backslash \mathscr{G}(\mathcal{V})(\mathbb{A})} \varphi_1(g)\ \overline{\varphi_2(g)}\ dg,$$

dg denoting the Tamagawa measure, and the given inner products $\langle \cdot, \cdot \rangle^{\mathcal{V}}_v$ on the Hilbert spaces underlying $\pi_{\mathcal{V},v}$, we have

$$\langle \varphi_1, \varphi_2 \rangle^{\mathcal{V}}_{\mathbb{A}} = \prod_v \langle \varphi_{1,v}, \varphi_{2,v} \rangle^{\mathcal{V}}_v$$

for decomposable data $\varphi_i = \otimes'_v \varphi_{i,v}$, $i = 1, 2$. Likewise, we fix a factorization $\pi_{\mathcal{W}} \cong \otimes'_v \pi_{\mathcal{W},v}$.

Let now v be any non-archimedean place of $\mathbb{F}$. For any pair of (smooth) vectors $\varphi_v, \phi_v \in \pi_{\mathcal{V},v}$, the integral

$$\int_{\mathscr{N}_{\mathfrak{F}}(\mathbb{F}_v)} \langle \pi_{\mathcal{V},v}(n_v)\varphi_v, \phi_v \rangle^{\mathcal{V}}_v\ \psi_{\mathfrak{F},v}(n_v)^{-1}\ dn_v,$$

with dn_v being the self-dual measure, stabilizes at some compact open subgroup $\mathscr{N}_0 \subseteq \mathscr{N}_{\mathfrak{F}}(\mathbb{F}_v)$, i.e., for all compact open subgroups $\mathscr{N}_1 \supseteq \mathscr{N}_0$, integration of $\langle \pi_{\mathcal{V},v}(n_v)\varphi_v, \phi_v \rangle^{\mathcal{V}}_v\ \psi_{\mathfrak{F},v}(n_v)^{-1}$ over $\mathscr{N}_1$ and

[3]Strictly speaking, this is an interpretation of the GGP-conjecture, because it assumes the (expected) holomorphy and non-vanishing for $s > 0$ of the (still partly mysterious) local L-function at the ramified places. Moreover, in view of the focus of this paper, we restricted our attention to sesquilinear forms of sign 1, while the original GGP conjecture allows sign -1 as well. On the other hand, however, GGP deal only with quasisplit groups, a restriction, which we avoided.

$\mathscr{N}_0$ gives rise to the same value, denoted

$$\int^{\mathrm{st}}_{\mathscr{N}_{\mathfrak{F}}(\mathbb{F}_v)} \langle \pi_{\mathcal{V},v}(n_v)\varphi_v, \phi_v\rangle_v^{\mathcal{V}} \; \psi_{\mathfrak{F},v}(n_v)^{-1} \; dn_v.$$

See [Lap-Mao15], §2.1, in particular Prop. 2.3. For our tempered cuspidal automorphic representations $\pi_{\mathcal{V}} \cong \otimes'_v \pi_{\mathcal{V},v}$ and $\pi_{\mathcal{W}} \cong \otimes'_v \pi_{\mathcal{W},v}$ and decomposable cuspidal automorphic forms $\varphi = \otimes'_v \varphi_v$ and $\varphi' = \otimes'_v \varphi'_v$ we may define

(6.3)

$$\alpha_v(\varphi_v, \varphi'_v) := \int_{\mathscr{G}(\mathcal{W})(\mathbb{F}_v)} \left(\int^{\mathrm{st}}_{\mathscr{N}_{\mathfrak{F}}(\mathbb{F}_v)} \langle \pi_{\mathcal{V},v}(g'_v n_v)\varphi_v, \varphi_v\rangle_v^{\mathcal{V}} \; \psi_{\mathfrak{F},v}(n_v)^{-1} \; dn_v \right) \overline{\langle \pi_{\mathcal{W},v}(g'_v)\varphi'_v, \varphi'_v\rangle_v^{\mathcal{W}}} \; dg'_v.$$

By one of the main results in [Liu16], Thm. 2.1, $\alpha_v(\varphi_v, \varphi'_v)$ is absolutely convergent and $\alpha_v(\varphi_v, \varphi'_v) \geq 0$.

If v is archimedean, let $\mathscr{C}_v$ (resp. $\mathscr{C}'_v$) be a maximal compact subgroup of $\mathscr{G}(\mathcal{V})(\mathbb{F}_v)$ (resp. $\mathscr{G}(\mathcal{W})(\mathbb{F}_v)$) and let φ_v (resp. φ'_v) be a $\mathscr{C}_v$-finite (resp. $\mathscr{C}'_v$-finite) function in $\pi_{\mathcal{V},v}$ (resp. $\pi_{\mathcal{W},v}$). For such functions we define $\alpha_v(\varphi_v, \varphi'_v)$ as the Fourier transform of the tempered distribution given by the absolutely convergent integral (cf. [Liu16], Cor. 3.13 and [Sun09], Thm. 1.2)

$$\mathfrak{a}_{\varphi_v,\varphi'_v}(n_v) := \int_{\mathscr{N}_{\mathfrak{F}}(\mathbb{F}_v)_{-\infty} \times \mathscr{G}(\mathcal{W})(\mathbb{F}_v)} \langle \pi_{\mathcal{V},v}(n_v n'_v g'_v)\varphi_v, \varphi_v\rangle_v^{\mathcal{V}} \; \overline{\langle \pi_{\mathcal{W},v}(g'_v)\varphi'_v, \varphi'_v\rangle_v^{\mathcal{W}}} \; dn'_v dg'_v,$$

(where $n_v \in \mathscr{N}_{\mathfrak{F}}(\mathbb{F}_v)$ and $\mathscr{N}_{\mathfrak{F}}(\mathbb{F}_v)_{-\infty}$ denotes the subset of matrices in $\mathscr{N}_{\mathfrak{F}}(\mathbb{F}_v)$, which are 0 at those off-diagonal matrix-entries, on which $\psi_{\mathfrak{F},v}$ is defined, cf. [Liu16], p. 155) evaluated at the generic character $\psi_{\mathfrak{F},v}$,

$$\alpha_v(\varphi_v, \varphi'_v) := \widehat{\mathfrak{a}_{\varphi_v,\varphi'_v}}(\psi_{\mathfrak{F},v}). \tag{6.4}$$

By [Liu16], Thm. 2.1, $\alpha_v(\varphi_v, \varphi'_v) \geq 0$. If $\pi_{\mathcal{V},v}$ is in the discrete series, then $\alpha_v(\varphi_v, \varphi'_v)$ is known to be absolutely convergent, see [Liu16], Prop. 3.15.

Set

$$\Delta_{\mathscr{G}(\mathcal{V})} := \begin{cases} \prod_{i=1}^{n} \zeta_{\mathbb{F}}(2i) & \text{if } \mathbb{E} = \mathbb{F} \text{ and } \dim_{\mathbb{E}} \mathcal{V} = 2n+1 \\ \prod_{i=1}^{n-1} \zeta_{\mathbb{F}}(2i) \cdot L(n, \chi_{\mathcal{V},f}) & \text{if } \mathbb{E} = \mathbb{F} \text{ and } \dim_{\mathbb{E}} \mathcal{V} = 2n \\ \prod_{i=1}^{n} L(i, \epsilon_f^i) & \text{if } [\mathbb{E} : \mathbb{F}] = 2 \text{ (and } \dim_{\mathbb{E}} \mathcal{V} = n) \end{cases}$$

where $\chi_{\mathcal{V}}$ (resp. ϵ) denotes the quadratic Hecke character $\mathbb{F}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ associated with the discriminant of $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ (resp. with the quadratic extension $\mathbb{E} : \mathbb{F}$ by class field theory) in the second (resp. in the last) line.

As a final ingredient, we invoke the theory of global Arthur packets for the square-integrable automorphic spectrum of $\mathscr{G}(\mathcal{V})(\mathbb{A})$ and $\mathscr{G}(\mathcal{W})(\mathbb{A})$. It is expected that $\pi_{\mathcal{V}}$ should be associated with a tempered elliptic Arthur parameter

$$\Psi(\pi_{\mathcal{V}}) : \mathcal{L}_{\mathbb{F}} \to^L \mathscr{G}(\mathcal{V}),$$

uniquely determined by $\pi_{\mathcal{V}}$. We define $\mathcal{S}_{\pi_{\mathcal{V}}} := \mathrm{Cent}_{\widehat{\mathscr{G}(\mathcal{V})}}(\mathrm{Im}\Psi(\pi_{\mathcal{V}}))$ to be the centralizer of the image of $\Psi(\pi_{\mathcal{V}})$ in the Langlands dual group $\widehat{\mathscr{G}(\mathcal{V})}$. Analogously, one obtains $\mathcal{S}_{\pi_{\mathcal{W}}} := \mathrm{Cent}_{\widehat{\mathscr{G}(\mathcal{W})}}(\mathrm{Im}\Psi(\pi_{\mathcal{W}}))$. Both are a elementary 2-abelian groups.

Liu's refinement of the GGP-conjecture now provides a comparison of two adelic pairings, the key-ingredient of this comparison being that one of them is defined *ad hoc* globally (by (6.1)) while the other is only defined *ex post* globally (by forming the product over all places v of the integrals (6.3)). Here is Liu's conjecture[4]

Conjecture 6.5 ([Liu16], Conj. 2.5)**.** *Let $\pi_{\mathcal{V}} \cong \otimes'_v \pi_{\mathcal{V},v}$ (resp. $\pi_{\mathcal{W}} \cong \otimes'_v \pi_{\mathcal{W},v}$) be a tempered cuspidal automorphic representation of $\mathscr{G}(\mathcal{V})(\mathbb{A})$ (resp. $\mathscr{G}(\mathcal{W})(\mathbb{A})$) coming together with a fixed tensor product factorization and appearing with multiplicity one in the cuspidal spectrum. Let S be any finite set of places of $\mathbb{F}$, containing the archimedean ones and such that $\pi_{\mathcal{V}}$, $\pi_{\mathcal{W}}$ and $\psi_{\mathfrak{F}}$ are unramified outside S. Then for all decomposable $\mathscr{C}_\infty$-finite (resp. $\mathscr{C}'_\infty$-finite) smooth functions $\varphi = \otimes'_v \varphi_v \in \pi_{\mathcal{V}}$ resp. $\varphi' = \otimes'_v \varphi'_v \in \pi_{\mathcal{W}}$,*

- *the Fourier transform $\alpha_v(\varphi_v, \varphi'_v)$ is absolutely convergent for all archimedean places v,*
- *$\alpha_v \not\equiv 0$ if and only if $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathscr{H}(\mathbb{F}_v)}[\pi_{\mathcal{V},v} \otimes \pi_{\mathcal{W},v}, \psi_{\mathfrak{F},v}] = 1$ for all v*

and one obtains the identity

$$|\mathcal{P}(\varphi, \varphi')|^2 = \frac{1}{|\mathcal{S}_{\pi_{\mathcal{V}}}| \cdot |\mathcal{S}_{\pi_{\mathcal{W}}}|} \frac{\Delta_{\mathscr{G}(\mathcal{V})}\ L^S(\frac{1}{2}, \pi_{\mathcal{V}} \boxtimes \pi_{\mathcal{W}})}{L^S(1, \pi_{\mathcal{V}}, \mathrm{Ad})\ L^S(1, \pi_{\mathcal{W}}, \mathrm{Ad})} \prod_{v \in S} \alpha_v(\varphi_v, \varphi'_v).$$

Specifying on the data entering Conj. 6.5, one retrieves the older conjectures of Ichino-Ikeda, [Ich-Ike10], Conj. 2.1 and N. Harris, [NHar14], Conj. 1.3:

Conjecture 6.6 (Ichino-Ikeda)**.** *This is Conj. 6.5 with $r = 0$ and $\mathbb{E} = \mathbb{F}$.*

Conjecture 6.7 (N. Harris)**.** *This is Conj. 6.5 with $r = 0$ and $[\mathbb{E} : \mathbb{F}] = 2$.*

6.3. **A major application of Thm. 5.6 - An algebraic version of the refined GGP-conjecture.** It is the goal of this section to prove an algebraic version of certain instances of Liu's refined GGP-conjecture. More precisely, recall the c-hermitian spaces $\mathcal{V} = V_n/F$ and attached unitary groups $\mathscr{G}(\mathcal{V}) = H_n = U(V_n)/F^+$ from §1.2. By an $E(\pi)$-*rational function* $\varphi = \otimes'_v \varphi_v \in \pi$ we mean a function whose π_f-component lies in the fixed $E(\pi)$-structure on π_f, chosen in §1.4.2, while its attached matrix coefficients at the archimedean places define an element of the affine algebra $E(\pi)(H_n)$ of the algebraic group H_n. Likewise for π'. We may now prove

Theorem 6.8. *Let $\pi \cong \otimes'_v \pi_v$ (resp. $\pi' \cong \otimes'_v \pi'_v$) be a cohomological cuspidal automorphic representation of $H_n(\mathbb{A}_{F^+}) = U(V_n)(\mathbb{A}_{F^+})$ (resp. $H_{n-1}(\mathbb{A}_{F^+}) = U(V_{n-1})(\mathbb{A}_{F^+})$), coming together with a fixed tensor product factorization and appearing with multiplicity one in the cuspidal spectrum. Let S be any finite set of places of F^+, containing the archimedean ones and such that π and π' are unramified outside S. We suppose that π and π' are tempered at all $v \in S$. Assume moreover that the quadratic base change $BC(\pi) = \Pi$ is a cohomological cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ as in §1.4.1 and that the quadratic base change $BC(\pi') = \Pi'$ is a cohomological isobaric automorphic representation $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ as in §1.4.3.*

(1) *If $H_{n,\infty}$ and $H_{n-1,\infty}$ are compact and Π and Π' satisfy the conditions of Thm. 5.6, then for all decomposable smooth $E(\pi)$-rational (resp. $E(\pi')$-rational) functions $\varphi = \otimes'_v \varphi_v \in \pi$ (resp. $\varphi' = \otimes'_v \varphi'_v \in \pi'$),*

$$|\mathcal{P}(\varphi, \varphi')|^2 \sim_{E(\pi)E(\pi')} \frac{\Delta_{H_n}\ L^S(\frac{1}{2}, \pi \boxtimes \pi')}{L^S(1, \pi, \mathrm{Ad})\ L^S(1, \pi', \mathrm{Ad})} \prod_{v \in S} \alpha_v(\varphi_v, \varphi'_v) \tag{6.9}$$

[4]Again, our Conj. 6.5 amounts to a "sanded" version of Liu's original conjecture [Liu16], Conj. 2.5: On the one hand, we believe it is more convenient to simply assume that our cusp forms are tempered and appearing with multiplicity one. This emulates Liu's assumption of being *almost locally generic*, but also has the advantage that it is (conjecturally) even less restrictive than his original genericity-supposition and avoids moreover all difficulties arising from questions of convergence of $\alpha_v(\varphi_v, \varphi'_v)$ at non-archimedean places. On the other hand, we have to assume the well-expected local properties of our L-functions at $v \in S$ of being holomorphic and non-zero for $s > 0$.

where $E(\pi)$ or $E(\pi')$ are the number fields defined in §1.4.2.

(2) *If $H_{n,\infty}$ or $H_{n-1,\infty}$ are non-compact, but $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ satisfy the piano-hypothesis, cf. Hypothesis 1.29, then the same conclusion holds trivially for all smooth C_∞- (resp. C'_∞-)finite decomposable functions in π (resp. π').*

Remark 6.10. The factor $q \in E(\pi)E(\pi')$ used in relating the left- and the right-hand-side of (6.9) is independent of the cusp forms φ and φ'. However, as it is obvious by the definition of our relation "$\sim_{E(\pi)E(\pi')}$", see Def. 1.15, our theorem cannot detect whether or not one side in (6.9) is non-zero, but rather compensates this defect: If one side of (6.9) vanishes, then the theorem holds by brute force, multiplying the respective other side by $0 \in E(\pi)E(\pi')$. *The non-trivial assertion of our theorem is hence in fact about the case when both sides of the relation (6.9) do not vanish: Then they are linked by a non-zero number q in the concrete number field $E(\pi)E(\pi')$, q being furthermore independent of φ and φ'.*

Before we prove Thm. 6.8 we state two further important remarks:

Remark 6.11. Before we put our main theorem into relation with the exciting recent literature on Conj. 6.5 (and Conj. 6.7), let us first remark on the various objects in Thm. 6.8, in particular the quantities in (6.9), being well-defined.

Firstly, the results in [Lab11], Cor. 5.3 and [Mor10], Prop. 8.5.3, as jointly refined by Shin, [Shi14], Thm. 1.1, show that quadratic base change BC is well-defined and exists for all unitary groups H_n and H_{n-1} and representations π and π' as above without any further assumptions. Moreover, by the description of its image, it makes sense to specify the properties of the base change lifts $BC(\pi)$ and $BC(\pi')$ as we did in the statement of Thm. 6.8, namely to assume that $BC(\pi)$ is (cohomological) cuspidal (as in §1.4.1) and that $BC(\pi')$ is a (cohomological) isobaric sum of cuspidal automorphic representations (as made precise in §1.4.3).
Secondly, this implies that $L^S(s,\pi\boxtimes\pi') = \prod_{1\leq i\leq k} L^S(s,\Pi\times\Pi_i)$ is holomorphic at $s=\frac{1}{2}$ as well as that the product

$$L^S(s,\pi,\mathrm{Ad})\cdot L^S(s,\pi',\mathrm{Ad}) = L^S(1,\Pi,\mathrm{As}^{(-1)^n})\cdot L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})$$

is holomorphic and non-vanishing at $s=1$, see Cor. 3.4. In particular, the quotient of L-values in our algebraic relation (6.9) makes sense without any assumptions.
Thirdly, we recall that the absolute convergence of $\alpha_v(\varphi_v,\varphi'_v)$ at archimedean v – as demanded by Conj. 6.5 – follows from the fact that a (by assumption) tempered and cohomological representation of a unitary group must be in the discrete series, cf. §1.4.1.
This demonstrates that all objects and quantities in Thm. 6.8 exist and are well-defined.

Remark 6.12 (*A comparison of our theorem on the refined GGP-conjecture with the results of W. Zhang and R. Beuzard-Plessis*)**.** As our algebraicity-result is coarser in its very statement, than Conj. 6.5 resp. Conj. 6.7, we feel that for the reader's sake a careful remark is in order to put our result in a precise relation with the strong results of W. Zhang, [Zha14b] and R. Beuzard-Plessis. Most important for us, Zhang ([Zha14b], Thm. 1.2.(2)) has established the following deep equality

$$|\mathcal{P}(\varphi,\varphi')|^2 = \frac{c_{\pi_\infty,\pi'_\infty}}{4}\frac{\Delta_{H_n}\ L^S(\frac{1}{2},\Pi\times\Pi')}{L^S(1,\Pi,\mathrm{As}^{(-1)^n})\ L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})}\prod_{v\in S}\alpha_v(\varphi_v,\varphi'_v),$$

whenever $\mathscr{G}(\mathcal{V})\times\mathscr{G}(\mathcal{W})$ is compact at every archimedean place v of F^+. Here, $c_{\pi_\infty,\pi'_\infty}$ is a certain constant, only depending on the archimedean components of π and π'.

Zhang's theorem is built on a list of conditions on $\pi_{\mathcal{V}}=\pi$ and $\pi_{\mathcal{W}}=\pi'$ (called **RH(I)** and **RH(II)**,

p. 544). Very recently, Beuzard-Plessis has announced [Beu17] that he had been able to significantly relax Zhang's conditions and to reveal Zhang's constant as a simple (but yet undetermined) sign $c_{\pi_\infty,\pi'_\infty} = \pm 1$. More precisely, he announced to have proved the formula

$$|\mathcal{P}(\varphi,\varphi')|^2 = \frac{\pm 1}{|\mathcal{S}_\pi|\cdot|\mathcal{S}_{\pi'}|}\frac{\Delta_{H_n}\ L^S(\frac{1}{2},\Pi\times\Pi')}{L^S(1,\Pi,\mathrm{As}^{(-1)^n})\ L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}})}\prod_{v\in S}\alpha_v(\varphi_v,\varphi'_v),$$

for cusp forms π and π', which are supercuspidal at one non-archimedean place. This latter assumption, however, is indispensable in the approach taken by Zhang and Beuzard-Plessis due to the current limitations of the Jacquet-Rallis relative trace formulae.

It is important to notice that our result, Thm. 6.8, *avoids any assumption of supercuspidality* of $\pi\otimes\pi'$ at any place: It is the asset of our approach, that we do not rely on any use of the trace formula. In this regard, our result on the refined GGP-conjecture, Thm. 6.8, may be viewed as a complementary theorem to [Zha14b] and its still unwritten refinement by Beuzard-Plessis, *applying to a different (broader) class of cuspidal representations* π and π'.

6.4. **Proof of Thm. 6.8.** Recall $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$, the coefficient modules with respect to which Π, resp. Π' are of non-trivial cohomology. For simplicity, put for each $v\in S_\infty$, $\lambda_v := (\mu_{\iota_v,1},...,\mu_{\iota_v,n})$ (resp. $\lambda'_v := (\mu'_{\iota_v,1},...,\mu'_{\iota_v,n-1})$) and let $\mathcal{F}_\lambda$ (resp. $\mathcal{F}_{\lambda'}$) be the irreducible algebraic representation of $H_{n,\infty}$ (resp. $H_{n-1,\infty}$) given by the highest weight $\lambda := (\lambda_v)_{v\in S_\infty}$ (resp. $\lambda' := (\lambda'_v)_{v\in S_\infty}$) as in §1.3. Then $\mathcal{F}_\lambda$ (resp. $\mathcal{F}_{\lambda'}$) is the highest weight module with respect to which π_∞ (resp. π'_∞) is cohomological, cf. [Lab11], Cor. 5.3.

We assume at first that $H_{n,\infty}$ and $H_{n-1,\infty}$ are compact.

Lemma 6.13. *If $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ do not satisfy the piano-condition, Hyp. 1.29, then $\alpha_v(\varphi_v,\varphi'_v)=0$ for all archimedean places v and functions $\varphi_v\in\pi_v$ and $\varphi'_v\in\pi'_v$.*

Proof. This an easy consequence of the branching law to which the piano hypothesis is equivalent, described, in [Goo-Wal09], Thm. 8.1.1: If $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ do not satisfy the piano-condition, then the branching law says that $\mathrm{Hom}_{H_{n-1}(F_v^+)}[\mathcal{F}_{\lambda_v}\otimes\mathcal{F}_{\lambda'_v},\mathbb{C}]=0$ for all $v\in S_\infty$. Compactness of $H_{n,\infty}$ and $H_{n-1,\infty}$ implies that $\pi_v\cong\mathcal{F}^\vee_{\lambda_v}$ and $\pi'_v\cong\mathcal{F}^\vee_{\lambda'_v}$, so by dualizing also $\mathrm{Hom}_{H_{n-1}(F_v^+)}[\pi_v\otimes\pi'_v,\mathbb{C}]=0$. As $\alpha_v\in\mathrm{Hom}_{H_{n-1}(F_v^+)}[\pi_v\otimes\pi'_v,\mathbb{C}]$, this shows the claim. □

Therefore, if $H_{n,\infty}$ and $H_{n-1,\infty}$ are compact, but $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ do not satisfy the piano-condition, then our main theorem, Thm. 6.8, trivially follows by multiplying the left hand side of relation (6.9) with $q=0$.

Hence, let us now consider the non-trivial case, when $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ do satisfy the piano-condition. Let $\mathcal{A}_{cusp}(H_n,\mathcal{F}_\lambda)$ be the space of automorphic (and hence, by compactness of $H_{n,\infty}$ automatically) cuspidal functions, which transform by $\mathcal{F}^\vee_\lambda$ on the right. Again by compactness of $H_{n,\infty}$, the restriction of functions $\phi\mapsto\phi|_{H_n(\mathbb{A}_f)}$ defines a natural isomorphism

$$R_n:\mathcal{A}_{cusp}(H_n,\mathcal{F}_\lambda)\xrightarrow{\sim}H^0(S_{H_n},\mathcal{F}_\lambda),$$

the right hand side being defined in §1.3.2. Obviously, the analogous construction works for H_{n-1}, defining an isomorphism R_{n-1}. Then, it is proved in [MHar13] that one obtains the following three algebraicity results

Proposition 6.14 ([MHar13], Cor. 2.5.4)**.** *Let $\varphi\in\pi$ and $\varphi'\in\pi'$ be chosen such that they map via R_n (resp. R_{n-1}) into the natural $E(\pi)$- (resp. $E(\pi')$-)structure of $H^0(S_{H_n},\mathcal{F}_\lambda)$ (resp.*

$H^0(S_{H_{n-1}}, \mathcal{F}_{\lambda'})$), defined in §1.3.2. Then

$$|\mathcal{P}(\varphi,\varphi')|^2 \in E(\pi)\cdot E(\pi').$$

Proposition 6.15. *If $v \in S/S_\infty$ and $\varphi_v \in \pi_v$ and $\varphi'_v \in \pi'_v$ are chosen such that the lie in the natural $E(\pi)$- (resp. $E(\pi')$-) structure of π_v (resp. π'_v), induced by the factorization $\pi_f \xrightarrow{\sim} \otimes'_{v\notin S_\infty}\pi_v$ (resp. $\pi'_f \xrightarrow{\sim} \otimes'_{v\notin S_\infty}\pi'_v$), fixed in Thm. 6.8. Then*

$$\alpha_v(\varphi_v,\varphi'_v) \in E(\pi)\cdot E(\pi').$$

Proof. This is [MHar13], Lem. 4.1.5.1 together with the fact that for all $v \in S/S\infty$

$$\frac{L(\frac{1}{2},\Pi_v\times\Pi'_v)}{L(1,\Pi_v,\mathrm{As}^{(-1)^n})\ L(1,\Pi'_v,\mathrm{As}^{(-1)^{n-1}})} \in \mathbb{Q}(\Pi)\mathbb{Q}(\Pi') \subseteq \mathbb{Q}(\pi_f)\mathbb{Q}(\pi'_f) \subset E(\pi)E(\pi'),$$

see [Rag10] Prop. 3.17 and [Gro-MHar-Lap16] §6.4. □

Proposition 6.16 ([MHar13], Cor. 4.1.4.3). *For all $\varphi_\infty = \otimes_{v\in S_\infty}\varphi_v \in \pi_\infty = \otimes_{v\in S_\infty}\pi_v$ and $\varphi'_\infty = \otimes_{v\in S_\infty}\varphi'_v \in \pi'_\infty = \otimes_{v\in S_\infty}\pi'_v$, whose attached matrix coefficients define an element of the affine algebra $E(\pi)(H_n)$ of the algebraic group H_n (resp. $E(\pi')(H_{n-1})$ of H_{n-1}),*

$$\alpha_\infty(\varphi_\infty,\varphi'_\infty) = \prod_{v\in S_\infty}\alpha_v(\varphi_v,\varphi'_v) \in E(\pi)\cdot E(\pi').$$

Consequence. *Thm. 6.8 holds if $H_{n,\infty}$ and $H_{n-1,\infty}$ are compact.*

Proof. Recall that $\Delta_{H_n} = \prod_{j=1}^n L(i,\varepsilon_f^j)$. By (1.22) and (1.23) we know that if $j \geq 1$ is even then $L(j,\varepsilon_f^j) \sim_{F^{Gal}} (2\pi i)^{dj}$, and if $j \geq 1$ is odd then $L(j,\varepsilon_f^j) \sim_{F^{Gal}} (2\pi i)^{dj}$. Hence $\Delta_{H_n} \sim_{F^{Gal}} (2\pi i)^{dn(n+1)/2}$.
Invoking the three propositions, Prop. 6.14, Prop. 6.15 and Prop. 6.16, Thm. 6.8 finally follows from Thm. 5.6. See also Rem. 6.11. □

Now if $H_{n,\infty}$ or $H_{n-1,\infty}$ is non-compact, but $\mathcal{E}_\mu$ and $\mathcal{E}_{\mu'}$ satisfy the piano-condition, then we know by the branching law, [Goo-Wal09], Thm. 8.1.1, that the tempered representation $\pi_\infty \otimes \pi'_\infty$ is distinguished for the pair of compact unitary groups. But by the results in [Beu16a], there is at most one pair of unitary group such that $\pi_\infty\otimes\pi'_\infty$ is distinguished. In particular, the representation $\pi_\infty\otimes\pi'_\infty$ can not be distinguished for the pair $(H_{n-1,\infty}, H_{n,\infty})$. Hence, $\alpha_v(\varphi_v,\varphi'_v) = 0$ for all $v \in S_\infty$ and all $\varphi_v \in \pi_v$, $\varphi'_v \in \pi'_v$ and Thm. 6.8 is trivially true.

Harald Grobner: Fakultät für Mathematik, University of Vienna, Oskar–Morgenstern–Platz 1, A-1090 Vienna, Austria.
E-mail address: harald.grobner@univie.ac.at

Jie Lin: Institut des hautes études scientifiques, 35 Route de Chartres, 91440 Bures-sur-Yvette, France.
E-mail address: linjie@ihes.fr

References

[Art-Clo89] J. Arthur, L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Ann. Math. Studies, **120** (Princeton University Press, 1989) 28, 29, 34, 39
[Beu16a] R. Beuzart-Plessis, *A local trace formula for the local Gan-Gross-Prasad conjecture for unitary groups: The archimedean case*, preprint (2016) 52
[Beu16b] R. Beuzard-Plessis, *Comparison of local spherical characters and the Ichino-Ikeda conjecture for unitary groups*, preprint (2016) 7

[Beu17] R. Beuzard-Plessis, *Factorisations de périodes et formules de Plancherel*, Peccot lecture series, held at the Collège de France, Paris, April, 21 – May, 12 (2017) 51

[Bla97] D. Blasius, *Period relations and critical values of L-functions*, in: *Olga Taussky-Todd, in Memoriam*, eds. M. Aschbacher, D. Blasius and D. Ramakrishnan, Pacific J. Math., Special Issue, **183**, No. 3, (1997), pp. 53–83.

[Bor79] A. Borel, *Automorphic L-functions*, in: Proc. Sympos. Pure Math., Vol. XXXIII, part II, AMS, Providence, R.I., (1979), pp. 27–61 18

[Bor80] A. Borel, *Cohomology and spectrum of an arithmetic group*, in: Proc. of a Conference on Operator Algebras and Group Representations, Neptun, Rumania (1980), Pitman (1983), pp. 28–45 21, 23

[Bor-Cas83] A. Borel, W. Casselman, L^2-Cohomology of locally symmetric manifolds of finite volume, *Duke Math. J.* **50** (1983) 625–647

[Bor-Jac79] A. Borel, H. Jacquet, *Automorphic forms and automorphic representations*, in: Proc. Sympos. Pure Math., Vol. XXXIII, part I, AMS, Providence, R.I., (1979), pp. 189-202

[Bor-Ser73] A. Borel, J.-P. Serre, Corners and arithmetic groups, *Comm. Math. Helvet.* **48** (1973) 436–491 10

[Bor-Wal00] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies **94**, (Princeton Univ. Press, New Jersey, 2000) 21

[Bum97] D. Bump, *Automorphic forms and representations*, Cambridge Univ. Press, (1997). 9, 13, 15, 21, 22 18

[Clo90] L. Clozel, *Motifs et Formes Automorphes: Applications du Principe de Fonctorialité*, in: *Automorphic forms, Shimura varieties, and L-functions*, Vol. I, Perspect. Math., vol. 10, eds. L. Clozel and J. S. Milne, (Ann Arbor, MI, 1988) Academic Press, Boston, MA, 1990, pp. 77–159 11, 14, 28, 34

[Cog-PS-Sha11] J. W. Cogdell, I. I. Piatetski-Shapiro, F. Shahidi, *Functoriality for the Quasisplit Classical Groups*, in: *On Certain L-functions*, Clay Mathematics Proceedings, vol. 13, eds. J. Arthur, J. W. Cogdell, S. Gelbart, D. Goldberg, D. Ramakrishnan, J.-K. Yu, (West Lafayette, IN, 2007) AMS, Providence, RI, 2011, pp. 117–140 12

[Del79] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales, With an appendix by N. Koblitz and A. Ogus* in: Proc. Sympos. Pure Math., Vol. XXXIII, part II, AMS, Providence, R.I., (1979), pp. 313–346 1, 19, 32, 45

[Enr79] T. J. Enright, Relative Lie algebra cohomology and unitary representations of complex Lie groups, *Duke Math. J.* **46** (1979) 513–525 11

[Fra98] J. Franke, Harmonic analysis in weighted L_2-spaces, *Ann. Sci. de l'ENS* **31** (1998) 181–279 16

[Fra-Sch98] J. Franke, J. Schwermer, A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups, *Math. Ann.*, **311** (1998) 765–790 15, 16

[Gan-Gro-Pra12] W. T. Gan, B. H. Gross, D. Prasad, Symplectic local root numbers, central critical L-values and restriction problems in the representation theory of classical groups, *Astérisque* **346** (2012) 111-170 46, 47

[Gan-Ich11] W. T. Gan, A. Ichino, On endoscopy and the refined Gross-Prasad conjecture for (SO_4, SO_5), *J. Inst. Math. Jussieu* **10** (2011) 235–324

[Gel-Jac-Rog01] S. Gelbart, H. Jacquet, J. Rogawski, Generic representations for the unitary group in three variables, *Israel J. Math.* **126** (2001) 173–237 27, 28

[God70] R. Godement, *Notes on Jacquet–Langlands' Theory*, mimeographed notes, I.A.S Princeton, (New Jersey, 1970) 9

[Goo-Wal09] R. Goodman, N. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, **255** (Springer, Dordrecht, 2009) 51, 52

[Gro17a] H. Grobner, *Rationality for isobaric automorphic representations: The general case*, preprint (2017) 3, 4, 11, 13, 17, 19, 20

[Gro17b] H. Grobner, *Rationality results for the exterior and the symmetric square L-function*, preprint (2017) 20

[Gro13] H. Grobner, Residues of Eisenstein series and the automorphic cohomology of reductive groups, *Compos. Math.* **149** (2013) 1061–1090 3, 15, 16, 21

[Gro15] H. Grobner, A note on the arithetmic of residual automorphic representations of reductive groups, *Math. Res. Letters* **22** (2015) 93 – 109 11

[Gro-MHar16] H. Grobner, M. Harris, Whittaker periods, motivic periods, and special values of tensor product L-functions, *J. Inst. Math. Jussieu* **15** (2016) 711–769 3, 4, 5, 19, 23, 26, 27, 45

[Gro-MHar-Lap16] H. Grobner, M. Harris, E. Lapid, Whittaker rational structures and special values of the Asai L-function, *Contemp. Math.* **664** (2016) 119–134 4, 14, 19, 20, 26, 52

[Gro-Rag14] H. Grobner, A. Raghuram, On some arithmetic properties of automorphic forms of GL_m over a division algebra, *Int. J. Number Theory* **10** (2014) 963–1013 11, 16

[Gro-Seb16] H. Grobner, R. Sebastian, *Period relations for cusp forms of* GSp_4 *and general aspects for modular symbols*, perprint 2016 11

[Gro-Pra92] B. H. Gross, D. Prasad, On the decomposition of a representation of SO_n when restricted to SO_{n-1}, *Canad. J. Math.* **44** (1992) 974–1002 6

[GHar83] G. Harder, *General aspects in the theory of modular symbols*, in: *Séminaire de Théorie des Nombres, Paris 1981-82*, ed M.-J. Bertin, Progress in Mathematics **38** (Boston, Basel, Stuttgart 1983), pp. 73–88 3

[GHar-Rag17] G. Harder, A. Raghuram, *Eisenstein cohomology for $GL(N)$ and ratios of critical values of Rankin-Selberg L-functions. With Appendix 1 by Uwe Weselmann and Appendix 2 by Chandrasheel Bhagwat and A. Raghuram*, preprint (2017) 7, 44

[MHar93] M. Harris, L-functions of 2×2 unitary groups and factorization of periods of Hilbert modular forms, *JAMS* **6** (1993) 637–719 32, 33

[MHar13] M. Harris, L-functions and periods of adjoint motives, *Alg. and Number Theory* **7** (2013) 117–155 19, 45, 51, 52

[MHar97] M. Harris, L-functions and periods of polarized regular motives, *J. reine angew. Math.* **483** (1997) 75–161 8, 19, 33, 45

[MHar-Kud91] M. Harris, S. S. Kudla, The central critical value of the triple product L-functions, *Ann. Math.* **133** (1991) 605–672 32

[MHar-Lin17] M. Harris, J. Lin, *Period relations and special values of Rankin–Selberg L-functions*, preprint (2017) 19, 45

[NHar14] R. N. Harris, The refined Gross-Prasad conjecture for unitary groups, *Int. Math. Res. Not.* (2014) 303–389 5, 6, 49

[Ich-Ike10] A. Ichino, T. Ikeda, On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture, *Geom. Funct. Anal.* **19** (2010) 1378–1425 49

[Jac-PS-Sha81] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, Conducteur des représentations du groupe linéaire, *Math. Ann.* **256** (1981) 199–214 14

[Jac-Sha81a] H. Jacquet, J. Shalika, On Euler Products and the Classification of Automorphic Representations I, *Amer. J. Math.* **103** (1981) 499–558

[Jac-Sha81b] H. Jacquet, J. Shalika, On Euler Products and the Classification of Automorphic Representations II, *Amer. J. Math.* **103** (1981) 777–815 16

[Jan16] F. Januszewski, *On Period Relations for Automorphic L-functions I*, preprint (2016) 8, 44

[Lab11] J.-P. Labesse, *Changement de base CM et séries discrètes*, in: *On the Stabilization of the Trace Formula*, Vol. I, eds. L. Clozel, M. Harris, J.-P. Labesse, B.-C. Ngô, International Press, Boston, MA, 2011, pp. 429–470 50, 51

[Lan79] R. P. Langlands, *On the Notion of an Automorphic Representation. A supplement to the preceding paper*, in: Proc. Sympos. Pure Math., Vol. XXXIII, part I, AMS, Providence, R.I., (1979), pp. 203–207 12, 16

[Lap-Mao15] E. Lapid, Z. Mao, A conjecture on Whittaker-Fourier coefficients of cusp forms, *J. Number Th.* **146** (2015) 448–505 48

[Lin15a] J. Lin, Period relations for automorphic induction and applications, *Comptes Rendus Math.* **353** (2015) 95–100, (reference to this source refer to the detailed version, available online at `https://arxiv.org/abs/1511.03517`) 33, 39

[Lin15b] J. Lin, *Special values of automorphic L-functions for $GL_n \times GL_{n'}$ over CM fields, factorization and functoriality of arithmetic automorphic periods*, thèse de docotorat, (2015) 4, 45

[Liu16] Y. Liu, Refined global Gan-Gross-Prasad conjecture for Bessel periods, *J. reine angew. Math.* **717** (2016) 133–194 5, 6, 48, 49

[Mah05] J. Mahnkopf, Cohomology of arithmetic groups, parabolic subgroups and the special values of automorphic L-Functions on $GL(n)$, *J. Inst. Math. Jussieu* **4** (2005) 553–637. 3, 24

[Mah98] J. Mahnkopf, Modular symbols and values of L-functions on GL_3, *J. reine angew. Math.* **497** (1998) pp. 91–112

[Mok15] Chung Pang Mok, *Endoscopic Classification of Representations of Quasi-Split Unitary Groups*, Memoirs of the AMS, **235** (2015) 26

[Mor10] S. Morel, *On the cohomology of certain non-compact Shimura varieties*, Ann. Math. Studies **173** (Princeton Univ. Press, New Jersey, 2010) 50

[Mœ-Wal89] C. Mœglin, J.-L. Waldspurger, Le Spectre Résiduel de $GL(n)$, *Ann. Sci. de l'ENS* **22** (1989) 605–674. 16

[Mœ-Wal95] C. Mœglin, J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Univ. Press (1995) 12, 13, 21

[Paz14] F. Pazuki, Heights and regulators of number fields and elliptic curves, *Publications mathématiques de Besancon* (2014). 18

[Rag10] A. Raghuram, On the special values of certain Rankin-Selberg L-functions and applications to odd symmetric power L-functions of modular forms, *Int. Math. Res. Not.* Vol. (2010) 334–372, doi:10.1093/imrn/rnp127 52

[Rag16] A. Raghuram, Critical values for Rankin-Selberg L-functions for $GL(n) \times GL(n-1)$ and the symmetric cube L-functions for $GL(2)$, *Forum Math.* **28** (2016) 457–489 3, 19

[Rag-Sha08] A. Raghuram, F. Shahidi, On certain period relations for cusp forms on GL_n, *Int. Math. Res. Not.* Vol. (2008), doi:10.1093/imrn/rnn077. 3, 8, 14, 17, 24, 25, 26

[Roh96] J. Rohlfs, Projective limits of locally symmetric spaces and cohomology, *J. Reine Angew. Math.* **479** (1996) pp. 149–182 21

[Sch90] J. Schwermer, *Cohomology of arithmetic groups, automorphic forms and L-functions*, in: *Cohomology of Arithmetic Groups and Automorphic Forms*, eds. J.-P. Labesse, J. Schwermer, Lecture Notes in Math. **1447** (Springer, 1990) pp. 1–29 11, 21

[Sch83] J. Schwermer, *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen*, Lecture Notes in Mathematics, **988** Springer-Verlag (1983) 21, 23

[Sha10] F. Shahidi, *Eisenstein series and Automorphic L-functions*, Colloquium publications **58**, AMS, (2010) 13, 23

[Sha81] F. Shahidi, On certain L-functions, *Amer. J. Math.* **103** (1981) 297–355 26, 47

[Shi14] S. W. Shin, On the cohomological base change for unitary similitude groups, appendix to: W. Goldring, Galois representations associated to holomorphic limits of discrete series I: Unitary Groups, Compos. Math. **150** (2014) 191–228 50

[Sie69] C. L. Siegel, Berechnung von Zetafunktionen an ganzzahligen Stellen. (German) *Nachr. Akad. Wiss. Göttingen Math. Phys. Kl.* **II 10** (1969), 87-102. 18

[Spe82] B. Speh, Induced representations and the cohomology of discrete subgroups, *Duke J.* **49** (1982) 1115–1127 23

[Sun09] B.Sun, Bounding matrix coefficients for smooth vectors of tempered representations, *Proc. Amer. Math. Soc.* **137** (2009) 353–357. 48

[Vog-Zuc84] D. A. Vogan Jr., G. J. Zuckerman, Unitary representations with nonzero cohomology, *Comp. Math.* **53** (1984) 51–90 11

[Wal85a] J.-L. Waldspurger, Quelques propriétés arithmétiques de certaines formes automorphes sur GL(2), *Comp. Math.* **54** (1985) 121–171 11

[Wal85b] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L-automorphes en leur centre de symetrié, *Compos. Math.* **54** (1985) 173–242

[Wei67] A. Weil, *Basic number theory*, Die Grundlehren der mathematischen Wissenschaften, **144**, (Springer-Verlag, New York, 1967) 8, 9

[Zha14a] W. Zhang, Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups, *Ann. of Math.* **180** (2014) 971–1049

[Zha14b] W. Zhang, Automorphic period and the central value of Rankin-Selberg L-function, *JAMS* **27** (2014) 541–612 7, 50, 51