Introduction

In [Har97], M. Harris has defined complex invariants, called automorphic periods, for certain automorphic representations of $GL_n$ over quadratic imaginary field. He proved that critical values of automorphic $L$-functions for $GL_n \times GL_1$ can be interpreted in terms of automorphic periods. His results have been generalized to the case $GL_n \times GL_1$ recently. Moreover, we have formulated a concise expression for general critical values. Our formula is compatible with Deligne’s conjecture (c.f. [Del79]).

Notation and conventions

We fix $\mathbb{Q}$ an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $K \subset \mathbb{Q}$ be a quadratic imaginary field. For $n, n'$ two integers at least 2. Let $\Pi$ (resp. $\Pi'$) be a cuspidal representation of $GL_n(A_K)$ (resp. $GL_{n'}(A_K)$) which is regular, cohomological and conjugate self-dual. For an integer $0 \leq s \leq n$, if $\Pi$ descends by base change to a unitary group over $\mathbb{Q}$ of infinity sign $(n - s, s)$ then $I$ can be realized in the coherent cohomology of the Shimura variety associated to the similitude unitary group. The coherent cohomology has a rational inner product of a rational element in the coherent cohomology of certain Shimura variety associated to unitary groups. We assume that $\Pi$ descends to unitary groups for all infinity signs henceforth. Therefore the automorphic periods can be defined for every $0 \leq s \leq n$. We postulate the similar assumption for $\Pi'$.

Split Index

We write the infinity type of $\Pi$ and $\Pi'$ as $(\delta_1, \delta_2)$, $(\delta_1', \delta_2')$. We denote the length of each part by $m_1$, for arbitrary position for very regular $(\Pi, \Pi')$. We say the pair $(\Pi, \Pi')$ is in good position if $m_1 > 0$ and all $1 \leq j \leq n'$. For an integer $0 \leq a \leq n$, for all $1 \leq i \leq n$ and all $1 \leq j \leq n'$. We split the sequence $(a_1 > a_2 > \cdots > a_{n})$ with the numbers $-b_0 > -b_{-1} > \cdots > -b_{m_1}$. This sequence is split into $n' + 1$ parts. We denote the length of each part by $s(a_1', \Pi', I')$, $s(a_1, \Pi, I)$ and call them the split indices.

An automorphic version of Deligne’s conjecture

The following conjecture is formulated in our work recently. It is already verified in several cases.

Conjecture: Let $\Pi$ and $\Pi'$ be as above. Let $m \in \mathbb{Z} + \frac{m_1}{2}$ be critical for $\Pi \otimes I'$. We predict that:

$L(m, \Pi \otimes I') \sim_{E(ER)} (2\pi i)^{m_1} \prod_{i=1}^{m_1} \Psi(1)^{\nu_i} \prod_{j=1}^{n'} \Psi(1')^{\nu_{j}}$.

Moreover, this relation is equivariant under the action of $Gal(\overline{\mathbb{Q}}/K)$.

Known cases

Definition: We say the pair $(\Pi, \Pi')$ is in good position if $n > n'$ and the numbers $-b_0 > -b_{-1} > \cdots > -b_{m_1}$ are in different gaps between $a_1 > a_2 > \cdots > a_n$. We say $\Pi$ is very regular if $a_{n_1} - a_{n_2} \geq 3$ for all $1 \leq i \leq n - 1$.

Here is a list of known cases for the above conjecture:

Case 1: $n' = 1$ and $m \geq \frac{1}{2}$. It is shown in [Har97].

Case 2: $n > n'$. $\Pi$ very regular. In good position and $m > \frac{1}{2}$ or $m = \frac{1}{2}$ along with a non vanishing condition. When $n' = n - 1$ this is proved in [GH15] and [LIN15]. For general $n'$ this is in the ongoing thesis of the author.

Case 3: arbitrary $n$, $n'$ and arbitrary position for very regular $(\Pi, \Pi')$ but $m = 1$. This is also in the ongoing thesis of the author.

Remark: The above results can be generalized to arbitrary CM field.

Motivic approach and Deligne’s conjecture

Let $M^\#$ be a motive over $\mathbb{Q}$ with coefficients in a number field $E$ of weight $\nu \in \mathbb{Z}$. Recall that its Betti realization $M^\#_B$ and de Rham realization $M^\#_D$ are both finite dimensional vector spaces over $E$ where the former is endowed with a Hodge structure and the latter is endowed with a Hodge filtration.

More precisely, we have a decomposition $M^\#_B \otimes E = \bigoplus_{E} M^\#_{E}$ as $E \otimes \mathbb{C}$-module and a filtration $M^\#_D = \bigoplus_{p \geq 0} M^\#_{D} \otimes E$ as $E$-module. Moreover, there is a comparison isomorphism $\iota_{M^\#} : M^\#_D \otimes E \to M^\#_B \otimes E$ such that $L_{M^\#}(\bigoplus_{E} M^\#_{E}) = M^\#_1 \otimes E$.

The infinity Frobenius acts on $M^\#_D$ and exchanges $M^\#_{E}$ with $M^\#_{E}$. We define $(M^\#_{E})^\circ$ to be the subspace of $M^\#_{E}$ fixed by the infinity Frobenius. For simplicity we assume that $M^\#_{E}$ has no $(\nu/2, \nu/2)$-class and define $F^0(M^\#_{E})$ to be $M^\#_{E}/2$. It is easy to see that the comparison isomorphism induces an isomorphism

$(M^\#_{E})^\circ \otimes E \to M^\#_{E} \otimes E \to (M^\#_{E} \otimes E) / F^0(M^\#_{E}) = (M^\#_{E}/F^0(M^\#_{E})) \otimes E$.

Deligne’s period $c(M^\#_{E})$ is defined to be the determinant of the above isomorphism with respect to any fixed $E$-basis of $(M^\#_{E})^\circ$ and $M^\#_{E}/F^0(M^\#_{E})$. It is well defined up to $E^*$. Deligne has predicted in [Del79] that $L(m, M^\#_{E}) \sim E(2\pi i)^{\mu_{\text{Motive}}(M^\#_{E})} c(M^\#)$ if $m$ is critical for $M^\#_{E}$.

Deligne’s period for automorphic pairs

Let $M$ and $M'$ be two regular motives over $K$ of dimension $n$ and $n'$ with coefficients in $E$ and $E'$ respectively. The motivic periods $Q(M)$ can be defined for $1 \leq i \leq n$ as in [Har13]. It is the ratio of two rational elements respect to two different rational structures in the $i$-th level of the Hodge decomposition. The determinant period $\delta(M)$ is defined as the determinant of the comparison isomorphism $\iota_{E(\Pi)} : M_{D} \otimes E \to M_{B} \otimes E$. It is an analogue as the determinant period in [Del79] where the motives are over $\mathbb{Q}$.

Let $M^\# = (M, M')$ be the restriction of $M \otimes M'$ from $K$ to $\mathbb{Q}$. It is a motive over $\mathbb{Q}$. We may calculate Deligne’s period $c(M^\#)$ explicitly. The right formula should be the inverse of that in Lemma 1.4.1 of [Har13].

An important ingredient of the ongoing thesis of the author is to simplify the expression for $c(M^\#)$ when $M$ and $M'$ are associated to automorphic pairs.

Let us assume that there exists motives $M$ and $M'$ associated to $\Pi$ and $\Pi'$ respectively. For all $0 \leq j \leq n$ we define the motivic periods $Q_j(M) = Q_j(M_1) \cdots Q_j(M_{n'})$ where $Q_j$ is the central character of $\Pi$. We define $Q_j(M^\#)$ for $1 \leq k \leq n'$ similarly.

The motive period $Q_j(M^\#)$ is related to the automorphic period $P_j(\Pi')$. The comparison is done in section 4 of [GH15].

Proposition: If $M^\#$ has no $(p, p)$-class then

$c(M^\#_{E}) \sim_{E} (2\pi i)^{2\nu_{\text{M}}(M^\#_{E})} \prod_{j=0}^{m_1} Q_j(M_{E})^{\nu_{j}}$.

At last, since $L(m, M \otimes I') = L(m + \frac{m_1}{2}, M^\#_{E})$, our conjecture is compatible with Deligne’s conjecture.

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References


