DEFORMATION QUANTIZATION

Maxim Kontsevich

I.H.E.S., 35 Route de Chartres, Bures-sur-Yvette 91440, France; email: maxim@ihes.fr

1. Star-products

Let A be the algebra over $\mathbf{R}$ of functions on C^∞-manifold X. *Star-product* on X is a structure of an associative algebra over $\mathbf{R}[[\hbar]]$ on $A[[\hbar]] := A\widehat{\otimes}_{\mathbf{R}}\mathbf{R}[[\hbar]]$ such that for any $f, g \in A \subset A[[\hbar]]$ the "new " product, denoted by $f \star g$, is given by the formula

$$f \star g = fg + \hbar B_1(f \otimes g) + \hbar^2 B_2(f \otimes g) + \ldots \in A[[\hbar]]$$

where $B_i : A \otimes A \longrightarrow A,\ i \geq 1$ are bidifferential operators on X. Associativity of the star-product

$$(f \star g) \star h = f \star (g \star h) \quad \forall f, g, h \in A[[\hbar]]$$

is a non-trivial quadratic constraint on $(B_i)_{i \in \mathbf{N}}$.

There is an action of infinite-dimensional group $G \subset Aut_{\mathbf{R}[[\hbar]]-mod}(A[[\hbar]])$:

$$G := \{\text{maps } f \mapsto f + \hbar D_1(f) + \hbar^2 D_2(f) + \ldots |\ D_i \text{ are differential operators on } X\}$$

on the set of star-products.

It is easy to see that by G-action one can kill symmetric part of bidifferential operator B_1. Thus, we can assume that B_1 is antisymmetric, $B_1(f \otimes g) = -B_1(g \otimes f)$. It follows from the associativity that B_1 is a bi-derivation, i.e. a bivector field, and also it satisfies the Jacobi identity. The conclusion is that B_1 gives a Poisson structure on X.

Any symplectic manifold (a natural object in classical mechanics) carries non-degenerate Poisson structure and could correspond via term B_1 to a non-commutative algebra (observables in quantum mechanics). It was one of motivations 20 years ago for Bayen, Flato, Fronsdal, Lichnerowitz and Sternheimer to start the study of star-products. A star-product modulo the gauge equivalence (G-action) is called a deformation quantization of manifold X. In 80-ies De Wilde, Lecompte and (later) Fedosov constructed a canonical gauge equivalence class of star-products on all symplectic manifolds. We think nevertheless that the whole line of ideas was based on slightly unnatural assumptions. First of all, the Euler-Lagrange equation in classical mechanics gives a closed 2-form, not a Poisson bracket. In degenerate cases one can not relate 2-forms and bivector fields. Also, there is no intrinsic reason in quantum mechanics to have an associative algebra of observables.

In the next section we will describe all deformation quantizations in geometrical terms. The proof of our main result is based on ideas from string theory. It seems that associative algebras are most closely related with open string theories, not with the quantum mechanics.

2. Classification of star-products

Theorem. *For any manifold X one can canonically identify the set of gauge equivalence classes of star-products on X with the following set:*

$$\{\alpha(\hbar)|\ \alpha(\hbar) = \alpha_1\hbar + \alpha_2\hbar^2 + \ldots \in \Gamma(\wedge^2 T_X)[[\hbar]],\ \ [\alpha(\hbar), \alpha(\hbar)] = 0\}/\widetilde{G}$$

where $[\cdot,\cdot] : \Gamma(\wedge^2 T_X)\otimes\Gamma(\wedge^2 T_X)\longrightarrow\Gamma(\wedge^3 T_X)$ *is the Schouten-Nijenhuis bracket on polyvector fields, and* $\widetilde{G}$ *is the group of formal paths starting at* id_X *in the diffeomorphism group of* X*:*

$$\widetilde{G} := Maps\,((Spec(\mathbf{R}[[\hbar]],0)\longrightarrow(Diff(X), id_X))\quad .$$

We remind that bivector field $\alpha\in\Gamma(\wedge^2 T_X)$ gives a Poisson structure on X iff $[\alpha,\alpha]=0$.

As an immediate corrolary of our theorem we conclude that any Poisson structure $\alpha_1\in\Gamma(\wedge^2 T_X)$ is canonically quantazible. The deformation quantization corresponds to the path $\alpha(\hbar):=\alpha_1\hbar$.

3. Explicit formula for $X=\mathbf{R}^N$

Let $\alpha=\sum\alpha^{ij}(x)\partial_i\wedge\partial_j$ be a Poisson structure on $\mathbf{R}^N$, $\partial_i=\partial/\partial x^i$, $i=1,\dots,N$. First few terms for the star-product corresponding to α are following:

$$f\star g = fg+\hbar\sum_{i,j}\alpha^{ij}\,\partial_i f\,\partial_i g+\frac{\hbar^2}{2}\sum_{i,j,k,l}\alpha^{ij}\,\alpha^{kl}\,\partial_i\partial_k f\,\partial_j\partial_l g+$$

$$+\frac{\hbar^2}{3}\sum_{i,j,k,l}\alpha^{ij}\,\partial_i\alpha^{kl}\,(\partial_j\partial_k f\,\partial_l g-\partial_j\partial_k g\,\partial_l f)+O(\hbar^3)\quad .$$

In the full formula terms are naturally labeled by certain oriented graphs. It is convenient to encode graphs of degree $n\in\mathbf{Z}_{\geq 0}$ (giving terms proportional to $\hbar^n$) by two maps

$$a_1,\, a_2 : \{1,\dots,n\}\longrightarrow\{1,\dots,n+2\}$$

such that for any $k\in\{1,\dots,n\}$ three numbers $k, a_1(k), a_2(k)$ are pairwise distinct.

Graph Γ associated with (a_1,a_2) has $n+2$ enumerated vertices. First n vertices correspond to bivector field α, the $(n+1)$-st vertex corresponds to function f, and the $(n+2)$-nd vertex corresponds to function g. Edges of Γ are oriented. The complete list of edges is

$$\{k\longrightarrow a_1(k),\, k\longrightarrow a_2(k)\,|\, k=1,\dots,n\}\quad .$$

The expression in our formula corresponding to Γ is

$$B_\Gamma(f,g;\alpha) := \sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n\\ 1\leq i_\star,j_\star\leq N}}\prod_{k=1}^{n}\left[\left(\prod_{l_1:a_1(l_1)=k}\partial_{i_{l_1}}\right)\cdot\left(\prod_{l_2:a_2(l_2)=k}\partial_{j_{l_2}}\right)\alpha^{i_k j_k}\right]\cdot$$

$$\cdot\left[\left(\prod_{l_1:a_1(l_1)=n+1}\partial_{i_{l_1}}\right)\cdot\left(\prod_{l_2:a_2(l_2)=n+1}\partial_{j_{l_2}}\right)f\right]\cdot\left[\left(\prod_{l_1:a_1(l_1)=n+2}\partial_{i_{l_1}}\right)\cdot\left(\prod_{l_2:a_2(l_2)=n+2}\partial_{j_{l_2}}\right)g\right]\quad .$$

In short, functions $B_\Gamma(f,g;\alpha)$ are all possible $GL(N,\mathbf{R})$-invariant expressions constructed form partial derivatives of functions f,g and of coefficients of bivector field α by contractions of upper and lower indices, without making an assumption that $[\alpha,\alpha]=0$.

The general formula for the star-product is

$$f\star g=\sum_{n\geq 0}\frac{\hbar^n}{n!}\sum_{\substack{\text{graphs }\Gamma\\ \text{of degree }n}} c_\Gamma\cdot B_\Gamma(f,g;\alpha)$$

where $c_\Gamma\in\mathbf{R}$ are constants defined in the next section. The associativity of star-product follows form certian non-homogeneous quadratic relations between numbers c_Γ.

4. Integral formula for c_Γ

Let $\mathcal{H}=\{z\in\mathbf{C}|\,Im\,z>0\}$ be the standard upper half-plane, $\overline{\mathcal{H}}:=\mathcal{H}\cup\mathbf{R}\subset\mathbf{C}$ be its closure in $\mathbf{C}$. We define a map

$$\phi:\overline{\mathcal{H}}\times\overline{\mathcal{H}}\setminus\text{diagonal}\longrightarrow\mathbf{R}/2\pi\mathbf{Z}$$

by the formula

$$\phi(z,w)=Arg(z-w)-Arg(\overline{z}-w)\ \ .$$

The meaning of this formula is that $\phi(z,w)$ is equal to the angle between lines (z,w) and $(z,+i\infty)$ in the Lobachevsky geometry.

The value of c_Γ is given by the following integral:

$$c_\Gamma=\frac{1}{(8\pi^2)^n}\int\limits_{\substack{(z_1,\dots,z_n)\in\mathcal{H}^n\\ z_i\neq z_j\ \text{for}\ i\neq j}}\bigwedge_{k=1}^{n}\left(d\phi(z_k,z_{a_1(k)})\wedge d\phi(z_k,z_{a_2(k)})\right)$$

where we define z_{n+1},z_{n+2} as points $0,1\in\overline{\mathcal{H}}$ respectively.

The integral from above is absolutely convergent. Probably, all numbers c_Γ are rational, although we cannot prove or disprove this statement at present.

The proof of quadratic relations between numbers c_Γ is essentially an application of the Stockes formula. In order to clarify the combinatorics of the proof, and also to construct star-products on general Poisson manifolds, we have to introduce general notions and constructions from the deformation theory.

5. Deformation theory and quasi-isomorphisms

Let $\mathbf{g}^*$ be a differential $\mathbf{Z}$-graded Lie algebra (DGLA) over field k of characteristic 0. The deformation functor $Def_{\mathbf{g}^*}$ associates with any finite-dimensional Artin algebra $\mathbf{A}$ over k the following set:

$$\{\alpha\in\mathbf{g}^1\otimes m_{\mathbf{A}}|\,d\alpha+\frac{1}{2}[\alpha,\alpha]=0\in\mathbf{g}^2\otimes m_{\mathbf{A}}\}/G_{(\mathbf{A})}$$

where $m_{\mathbf{A}} \subset \mathbf{A}$ is the maximal ideal of $\mathbf{A}$, and group $G_{(\mathbf{A})}$ is the nilpotent group associated with the nilpotent Lie algebra $\mathbf{g}^0 \otimes m_{\mathbf{A}}$. The action of $G_{(\mathbf{A})}$ on the set of solutions of the Maurer-Cartan equation is in infinitesimal form

$$\dot{\alpha} = d\gamma + [\gamma, \alpha], \;\; \gamma \in \mathbf{g}^0 \otimes m_{\mathbf{A}} \;\; .$$

One of most familiar examples is the deformation theory of complex structures on a complex manifold M. In this case $k = \mathbf{C}$, and the DGLA controlling the deformation theory is

$$\mathbf{g}^* = \bigoplus_{k \geq 0} \mathbf{g}^k, \;\; \mathbf{g}^k = \Gamma(M, T^{1,0} \otimes \wedge^k \left((T^{0,1})^{dual}\right)$$

with the differential equal to the usual $\overline{\partial}$-operator and with Lie bracket coming from the usual Lie bracket on vector fields and from the cup-product on differential forms. The set $Def_{\mathbf{g}^*}(\mathbf{A})$ is the set of equivalence classes of flat morphisms of complex analytic spaces $\widetilde{M} \longrightarrow Spec(\mathbf{A})$, endowed with an identification of the special fiber $\widetilde{M} \underset{Spec(\mathbf{C})}{\times} Spec(\mathbf{A})$ with M.

Let $\mathbf{g}_1^*, \mathbf{g}_2^*$ are two DGLAs. We are going to introduce a structure (a *quasi-isomorphism* between $\mathbf{g}_1^*$ and $\mathbf{g}_2^*$) which identifies deformation functors $Def_{\mathbf{g}_1^*}$ and $Def_{\mathbf{g}_2^*}$.

Definition. *An L_∞-morphism $\mathcal{T}$ from $\mathbf{g}_1^*$ to $\mathbf{g}_2^*$ is an homomorphism of differential graded cocommutative coassociative coalgebras*

$$\mathcal{T} : \bigoplus_{k \geq 1} Sym^k(\mathbf{g}_1^*[1]) \longrightarrow \bigoplus_{k \geq 1} Sym^k(\mathbf{g}_2^*[1]) \;\; .$$

In the formula from above symmetric powers are constructed in the tensor category of $\mathbf{Z}$-graded vector spaces (i.e. using the Koszul rule of signs). The graded space $\mathbf{g}^*[1]$ is obtained from $\mathbf{g}^*$ by the shift of degrees by 1:

$$(\mathbf{g}[1])^n := \mathbf{g}^{n+1} \;\; .$$

The differential in the "chain complex" $C_*(\mathbf{g}^*) := \bigoplus_{k \geq 1} Sym^k(\mathbf{g}^*[1])$ of any DGLA $\mathbf{g}^*$ is defined by usual formula using the differential and the Lie bracket in $\mathbf{g}^*$. Geometrically, one can think about coalgbera $C_*(\mathbf{g}^*)$ as of an object encoding an infinite-dimensional formal $\mathbf{Z}$-graded supermanifold. The reason is that the dual space to $C_*(\mathbf{g}^*)$ is the algebra of formal power series. The differential on $C_*(\mathbf{g}^*)$ can be viewed as an odd vector field Q on a supermanifold such that $[Q, Q] = 0$. An L_∞-morphim gives a Q-equivariant map between formal supermanifolds.

One can reformulate the definition of the deformation functor in geometrical terms (i.e. for odd vector field Q). Any L_∞-morphim induces a natural transformation between deformation functors.

Definition. *An L_∞-morphism $\mathcal{T}$ from $\mathbf{g}_1^*$ to $\mathbf{g}_2^*$ is called a quasi-isomorphism iff its component $\mathcal{T}^{(1,1)}$ which maps $\mathbf{g}_1^*[1]$ to $\mathbf{g}_2^*[1]$ is a quasi-isomorphim of complexes.*

Below we state a well-known result in slightly new form:

Theorem. *Any quasi-isomorphism induces an isomorphism between deformation functors.*

6. Formality

Let X be a manifold, A be the algebra of functions on X. We define two DGLAs over $\mathbf{R}$ associated with X. The first algebra $D^*(X)$ is related with the deformation quantization. For each $n \geq -1$ we define $D^n(X)$ by the formula

$$\{\Phi : A^{\otimes(n+1)} \longrightarrow A |\ \Phi(f_0 \otimes f_1 \otimes \ldots \otimes f_n) \text{ is a polydifferential operator in } f_*\} .$$

The differential and the bracket in $D^*(X)$ are given by standard formulas for the differential and the bracket in the Hochschild complex. We define a bilinear operation $(\Phi_1, \Phi_2) \longrightarrow \Phi_1 \circ \Phi_2$ on $D^*(X)$ for $\Phi_1 \in D^m(X)$ and $\Phi_2 \in D^n(X)$

$$(\Phi_1 \circ \Phi_2)(f_0 \otimes \ldots \otimes f_{n+m}) := \sum_{k=0}^{m} \pm \Phi_1(f_1 \otimes \ldots \otimes \Phi_2(f_k \otimes \ldots \otimes f_{k+n}) \otimes \ldots \otimes f_{n+m}) .$$

The Lie bracket in $D^*(X)$ is defined as

$$[\Phi_1, \Phi_2] = \Phi_1 \circ \Phi_2 - (-1)^{mn} \Phi_2 \circ \Phi_1$$

and the differential as

$$d\Phi = [m_X, \Phi]$$

where $m_X \in D^1(X)$ is the product in A: $m_X(f_0 \otimes f_1) = f_0 f_1$.

The second DGLA is denoted by $T^*(X)$. It is simply the cohomology of $D^*(X)$ with respect to the differential in $D^*(X)$. The differential in $T^*(X)$ is defined to be zero. By a version of Hochschild-Kostant-Rosenberg theorem graded components of $T^*(X)$ are spaces of polyvector fields:

$$T^n(X) = \Gamma(X, \wedge^{n+1} T_X), \;\; n \geq -1$$

and the bracket in $T^*(X)$ is the usual Schouten-Nijenhuis bracket.

Theorem. *For any manifold X two DGLAs $D^*(X)$ and $T^*(X)$ are quasi-isomorphic.*

Solutions of the Maurer-Cartan equation in $D^*(X)$ parametrized by $Spec(\mathbf{R}[[\hbar]])$ are exactly star-products on X. Solutions in $T^*(X)$ are Poisson structures. Thus, we get a canonical quantization for arbitrary Poisson structure.

Usually, a differential graded algebra quasi-isomorphic to its cohomology algebra, is called *formal*. For example, the de Rham complex on any Kähler manifold is formal. Our result means that DGLA $D^*(X)$ is formal.

7. Few words about the proof

Firts of all, using a generalization of the construction with graphs as in sections 3,4 we construct an explicit quasi-isomorphism from $T^*(\mathbf{R}^N)$ to $D^*(\mathbf{R}^N)$ for any N. The check of relevant identities uses the Stockes formula on certain compactifications of configurations spaces of $\overline{\mathcal{H}}$, and the following lemma:

Lemma. *Let M be a complex algebraic variety of dimension $d \geq 1$, anf $f_1, \ldots, f_{2d}$ be non-zero rational functions on M. Then the integral*

$$\int_{M(\mathbf{C})} \bigwedge_{k=1}^{2d} d\,\mathrm{Arg}(f_k)$$

is absolutely convergent and equal to zero.

This lemma is used in the study of certain degenerations when several points on $\mathcal{H}$ move close to each other. The main step in the proof of the lemma is the following identity:

$$\bigwedge_{k=1}^{2d} d\,\mathrm{Arg}(f_k) = \bigwedge_{k=1}^{2d} d\,\mathrm{Log}|f_k| \ .$$

The next step is to introduce a Gelfand-Fuks cocyle of the Lie algebra of formal vector fields with coefficients in a module responsible for L_∞-morphisms form $T^*(\mathbf{R}^N)$ to $D^*(\mathbf{R}^N)$. Fortunately, it can be done in essentially the same manner as for an individual L_∞-morphism. Vanishing of some integral over a configuration space of $\mathcal{H}$ guarantees that this cocycle is a relative cocycle with respect to the Lie algebra $gl(N, \mathbf{R}) \subset Vect(\mathbf{R}^n)$. The rest is a generalization of standard constructions of characteristic classes associated with Gelfand-Fuks cohomology.

8. Applications

There many of them. For example, any quadratic Poisson bracket on a finite-dimensional vector space admits a canonical quantization to a graded algebra with quadratic relations. It gives the positive answer to one of questions posed by Drinfeld.

Here is another application (which needs in fact an additional work with graphs and integrals):

Theorem. *Let $\mathbf{g}$ be a Lie algebra in a tensor category $\mathcal{C}$ which is a "finite-dimensional" object of $\mathcal{C}$, i.e. the dual object $\mathbf{g}^{dual}$ exists and $(\mathbf{g}^{dual})^{dual} = \mathbf{g}$. Then the center of the universal enveloping algebra $Z(\mathcal{U}\mathbf{g}) = (\mathcal{U}\mathbf{g})^{\mathbf{g}}$ is isomorphic as an algebra in $\mathcal{C}$ to the algebra $(\oplus_{k\geq 0} Sym^k(\mathbf{g}))^{\mathbf{g}}$ of ad^*-invariant polynomials on $\mathbf{g}^{dual}$.*

In the classical case of the category of vector spaces this fact was proven by Duflo using at certain essential step the classification theory of Lie algebras. In fact, the isomorphism in our theorem is the one predicted by Kirillov and Duflo, and involves a kind of Todd class for elements of finite-dimensional Lie algebras. The analogous statement for Lie superlalgebras was unknown. Now we can say finally that the orbit method has a solid background.

A parallel new theorem in algebraic geometry is

Theorem. *Let M be smooth algebraic variety over a field of characteristic zero. Then the graded algebra $Ext^*_{M\times M}(\mathcal{O}_{diag}, \mathcal{O}_{diag})$ is isomorphic to $\oplus H^*(M, \wedge^* T_X)$.*

Another application is to the Mirror Symmetry, but we will not try to explain it here.

9. Motivations

The vague idea underlying our formula is the following: 1) a Poisson manifold gives after a doubling an odd symplectic manifold with the action of add vector field Q, $[Q, Q] = 0$, 2) few years ago in a joint paper with Alexandrov, Schwarz and Zaboronsky we constructed a Lagrangian for a topological two-dimensional field theory, using as an input an odd symplectic Q-manifold, 3) any topological two-dimensional field theory produces after a coupling with gravity an A_∞-category, a generalization of an associative algebra.

Our formulas seem come from standard Feynman rules in (string) perturbation theory.