NONARCHIMEDEAN KÄHLER GEOMETRY

MAXIM KONTSEVICH AND YURI TSCHINKEL

Abstract. We propose a generalization of basic notions in Kähler geometry to Berkovich's nonarchimedean analytic spaces.

sect:introduction

1. Introduction

The existence of special, canonical metrizations of line bundles is important both in geometry and arithmetic. In arithmetic, one seeks to construct "adelic metrizations", which involve simultaneously archimedean and non-archimedean valuations. We develop basic notions of non-archimedean Kähler geometry, starting from an analogy between harmonic and convex functions. Using a suitable interpretation of Berkovich's analytic spaces we introduce a non-archimedean Monge-Ampére operator and a Mabuchi functional.

Here is a roadmap to the paper. in the first part we recall main definitions and results of Kähler geometry: metrizations of line bundles, Kähler-Einstein metrics, Calabi's conjecture (Yau's theorem) and their analogs in special affine geometry. In the second part we explain our approach to Berkovich's analytic spaces, the correct analogs of complex algebraic varieties. We work systematically with projective limits of "nice models". This allows us to give purely combinatorial definitions for all objects of interest, for example, non-archimedean Monge-Ampére operator and Mabuchi functional. The difficult complex-analytic theorems are reduced to verifications of properties of (locally) convex functions. In the third part we discuss semi-geometry.

Acknowledgments. The second author was partially supported by the NSA. Part of the work was done during visits of the first author to the IAS and the second author to the IHES. We are grateful to both institutions for ideal working conditions.

Notations 1.0.1. We will denote by $X = X_K$ an algebraic variety defined over a field K and by $X(K')$ the set of K'-points of X (for all K'/K).

Date: May 17, 2002.

Part I. Classical positivity

sect:ample

2. Ampleness

Let X be a smooth projective variety over a field K, of dimension n. We denote by

$$A^{\bullet}_{num}(X) = \oplus_{j=0}^{n} A^{j}_{num}(X)$$

the quotient of the Chow ring of X by numerical equivalence. We have canonical isomorphisms

$$A^0_{num}(X) \simeq A^n_{num}(X) \simeq \mathbb{Z}.$$

Graded components of the ring $A^{\bullet}_{num}(X)$ are finitely generated torsion free abelian groups equipped with a non-degenerate intersection pairing

$$\langle \cdot , \cdot \rangle \, : \, A^{j}_{num}(X) \otimes A^{n-j}_{num}(X) \to \mathbb{Z}.$$

For example, if $K = \mathbb{C}$ then, by Hard Lefschetz, $A^{j}_{num}(X)$ is a subgroup of

$$\big(H^{2j}(X(\mathbb{C}), \mathbb{Z})/tors\big) \cap H^{j,j}(X).$$

Denote by

$$\mathrm{Ample}_X \subset \mathrm{NS}(X) \otimes \mathbb{R} = A^1_{num}(X) \otimes \mathbb{R}$$

the nonempty open cone in the Néron-Severi group of X generated by ample divisors. Denote by Nef_X the closure of Ample_X. Obviously, $\mathrm{Nef}_X \cap -\mathrm{Nef}_X = \{0\}$. Call Nef_X the cone of numerically effective divisors. This terminology is justified by

thm:klei

Theorem 2.0.2. (Kleiman criterium) *An element $\alpha \in A^1_{num}(X) \otimes \mathbb{R}$ is numerically effective, i.e., it has the property*

$$\langle \alpha, [Z] \rangle \geq 0$$

for all 1-dimensional subvarieties $Z \subset X$, if and only if $\alpha \in \mathrm{Nef}_X$.

sect:psh

3. Review of Kähler geometry

3.1. **Plurisubharmonic functions.** Let M be a smooth complex analytic manifold of dimension n. Recall that we have a second order differential operator $dd^c = \partial\overline{\partial}/(2i)$ on real-valued differential forms on M.

Definition 3.1.1. *A real current $\alpha \in \mathcal{D}^{j,j}(M)$ is* non-negative *iff for every set of smooth $\mathbb{R}$-valued functions with compact support $f_1, \ldots, f_{n-j} \in C_0^{\infty}(M, \mathbb{R})$ and every non-negative function $f_0 \in C_0^{\infty}(M, \mathbb{R})$), one has*

$$\int_M \alpha \wedge \Big(f_0 \cdot \wedge_{j'=1}^{2(n-j)} dd^c(f_{j'}) \Big) \geq 0.$$

Any closed analytic subset $Z \subset M$ of codimension j gives a non-negative closed (j, j)-current δ_Z (corresponding to the functional on smooth forms given by the integration over Z). The Poincaré pairing between the cohomology class of any closed non-negative (j, j)-current and the product of $(n - j)$ Kähler classes is non-negative.

Definition 3.1.2. *A distribution* $\phi \in \mathcal{D}^0(M, \mathbb{R})$ *is called* plurisubharmonic *iff* $dd^c(\phi)$ *is a non-negative* $(1, 1)$*-current.*

One can show that a plurisubharmonic function (*psh-function* for short) is in fact a measurable function in L^1_{loc}, locally bounded from above and well-defined modulo subsets of measure zero. Every psh-function has a canonical representative which is a locally upper semi-continuous function, with values in $\mathbb{R} \cup \{-\infty\}$, and locally in L^1_{loc} (see []). In what follows we assume that psh-functions *are* the canonical representatives as above. We have the following maximum principle:

lemm:max **Lemma 3.1.3.** *A psh-function is locally constant near its local maximum.*

It follows that every global psh-function on a compact M is constant. Psh-functions which are locally bounded (i.e., essentially locally bounded from *below*) have very mild singularities. In particular, one can extend the cup-product from smooth functions and define currents

$$g \cdot (dd^c(f_1) \wedge \cdots \wedge dd^c(f_j)) \in \mathcal{D}^{j,j}(M)$$

for arbitrary locally bounded psh-functions $f_1, \ldots, f_j$ and a locally bounded measurable function $g \in L^\infty(M)$ (see [2, 12]). Psh-functions which are not locally bounded are much less understood.

The sheaf Psh_M of psh-functions is closed under pointwise maximum, addition and the multiplication by $\mathbb{R}_+^\times$, and it contains the constant sheaf $\mathbb{R}$. The same is true for the sheaf $cPsh_M$ of continuous psh-functions.

For any Stein domain $U \subset M$ the space $cPsh_M(U)$ is closed under uniform convergence on compacts. Moreover, it can be described as the closure (in this topology) of the space of functions of the form

$$f(x) = \max_{i \in I} \left(\lambda_i \log(|\phi_i(x)|^2)\right),$$

where $(\phi_i)_{i \in I}$ is a finite collection of holomorphic functions on U without common zeroes, and $\lambda_i > 0$ are positive real constants.

sect:virt

3.2. Virtual line bundles and metrizations.

A real-valued function f on a complex manifold M is *pluriharmonic* iff it satisfies the following equivalent conditions:

(1) $f \in C^\infty(M)$ and $dd^c(f) = 0$,
(2) $f \in Psh(M)$ and $(-f) \in Psh(M)$,

(3) near every point $x \in M$ the function f can be represented as $\log(|\phi|^2)$, where $\phi \in \mathcal{O}_x^\times$ is the germ of an invertible holomorphic function.

The third description shows that the sheaf of pluriharmonic functions is equivalent to

$$|\mathcal{O}^\times| := \mathcal{O}^\times/\mathbf{U}(1)$$

where $\mathbf{U}(1)$ is the sheaf of locally constant functions of absolute value 1.

Definition 3.2.1. *A* virtual line bundle *on M is a torsor over $|\mathcal{O}^\times|$.*

Any holomorphic line bundle $\mathcal{L}$ on M gives a virtual line bundle $L := |\mathcal{L}|$. The next proposition describes the set of equivalence classes of virtual line bundles on M in the Kähler case.

Proposition 3.2.2. *For compact Kähler M we have*

$$H^1(M, |\mathcal{O}^\times|) = H^{1,1}(M) \cap H^2(M, i\mathbb{R}).$$

Proof. Consider the exact sequence of sheaves

$$0 \longrightarrow i\mathbb{R} \longrightarrow \mathcal{O} \xrightarrow{\exp} |\mathcal{O}^\times| \longrightarrow 0.$$

Hodge theory and the long exact sequence in cohomology now give

$$\begin{array}{rcc} H^1(M, |\mathcal{O}^\times|) & = & \ker[H^2(M, i\mathbb{R}) \to H^2(M, \mathcal{O})] \\ & = & H^{1,1}(M) \cap H^2(M, i\mathbb{R}). \end{array}$$

□

In fact, we get a canonical map

$$\log : H^1(M, |\mathcal{O}^\times|) \to H^2(M, i\mathbb{R}).$$

We define the first Chern class $c_1(L) \in H^2(M, i\mathbb{R})$ as $(\log([L]))/2\pi i$.

The sheaf of abelian groups $|\mathcal{O}^\times|$ acts on the sheaf C_M^∞ of real-valued smooth functions:

$$\phi \mapsto \left(f \mapsto f + \log(|\phi|^2)\right),$$

where $\phi \in (\mathcal{O}^\times/\mathbf{U}(1))(U)$, $f \in C^\infty(U)$ and $U \subset M$ is an open subset. For a virtual line bundle L we denote by $C^\infty_{(L)}$ the twisted by L form of the sheaf C^∞_M.

Definition 3.2.3. *A* smooth metrization *of a virtual line bundle L is a section of $C^\infty_{(L)}$.*

Analogously, using the sheaf $cPsh_M$ we can define cPsh-metrizations etc.

The *Kähler cone* in $H^2(M, i\mathbb{R})$ is an open convex cone consisting of cohomology classes of all Kähler metrics. It can be described as the set of first Chern classes of virtual line bundles admitting strictly plurisubharmonic smooth metrizations. In the case when $M = X(\mathbb{C})$ is projective algebraic and

$H^{2,0}(M) = 0$, the Kähler cone of M coincides with Ample_X. The first Chern class of a virtual line bundle admitting a continuous psh-metrization belongs to the closure of the Kähler cone, but not all classes in this closure are obtained in such a way, in general.

sect:ma

3.3. Monge-Ampére operator and Calabi conjecture.

Definition 3.3.1. *Let $f \in C^\infty(M, \mathbb{R})$ be a smooth real-valued function on a complex n-dimensional manifold M. The* Monge-Ampére charge $MA(f)$ *associated with f is defined by:*

$$MA(f) = (2\pi i)^{-n} (\partial\overline{\partial}(f))^n.$$

Here we identified densities and top-degree forms using the canonical orientation on M.

For a smooth psh-function f the charge $MA(f)$ is non-negative, i.e., it is a measure. The operator MA has a unique extension (by continuity) to an n-linear operator from the sheaf $cPsh_M$ to the sheaf of charges with density in L^1_{loc}.

The charge $MA(f)$ does not change under a shift of the argument

$$f \mapsto f + \log|\phi|,$$

where $\phi \in \mathcal{O}_M^\times(U)$ is an invertible holomorphic function in an open domain $U \subset M$. Thus, the charge MA is well-defined on an arbitrary virtual line bundle with a smooth or cPsh-metrization.

Here is the classical Calabi conjecture proven by S.-T.Yau:

Theorem 3.3.2. *For any compact connected n-dimensional complex manifold M there is a bijection*

$$\omega \mapsto ([\omega], \omega^n)$$

between the space of Kähler metrics on M and the set of pairs

$$\left\{ (\alpha, \mu) | \langle \alpha^n, [M] \rangle = \int_M \mu \right\},$$

where $\alpha \in H^2(M, i\mathbb{R})$ is a Kähler class and $\mu \in \Omega^{2n}(M)$ is a smooth positive density.

In other words, for any given virtual line bundle L with $c_1(L)$ in the Kähler cone the operator MA gives a one-to-one correpondence between the set of smooth strictly positive metrizations (modulo multiplication by $\mathbb{R}_+^\times$) and smooth strictly positive densities on M with total volume equal to

$$\langle c_1(L)^n, [M] \rangle.$$

An optimal weakening of the condition on the metrizations to be *locally* psh, and also of the condition of the Kählerness of $c_1(L)$, seems to be unknown.

sect:mab

3.4. **Mabuchi functional.** For a virtual line bundle L the space of its smooth metrizations $\Gamma(X(\mathbb{C}), C^{\infty}_{(L)})$ is an infinite-dimensional affine space parallel to the space $C^{\infty}(M, \mathbb{R})$. Thus, we can interpret MA as a 1-form on it. An easy calculation shows that this 1-form is closed, i.e., there exists a function on $\Gamma(M, C^{\infty}_{(L)})$ (well-defined up to addition of a constant) whose derivative is MA. This function(al) on metrizations is called the *Mabuchi functional*. It is a polynomial of degree $n+1$. In the case of trivial L it has the form

$$f \mapsto \frac{1}{n+1} \int_M f \cdot (dd^c(f))^n.$$

For Kähler classes $c_1(L)$ the Mabuchi functional is *convex* on the open convex subset

$$U_L \subset \Gamma(M, C^{\infty}_{(L)})$$

of all strictly psh smooth metrizations. Yau's theorem can be reformulated as the statement that the Legendre transform of the Mabuchi functional gives an identification of the quotient $U_L/\mathbb{R}$ with the space of strictly positive smooth densities with total volume equal to $\langle c_1(L)^n, [M]\rangle$.

sect:ricci

3.5. **Ricci curvature and Kähler-Einstein metrics.** The Ricci curvature of a Kähler metric g on an n-dimensional complex manifold M is calculated as follows. The volume element of g coincides with the top-degree of the Kähler form ω. It can be interpreted as a metrization of the canonical bundle $\omega_M := \Omega^n_M$. Ricci curvature is the symmetric 2-tensor associated with the canonical $(1,1)$-form representing $c_1(\omega_M)$. In local coordinates one has

$$Ricci_{i,\bar{j}} = \partial_i \overline{\partial}_j (\log(\det(\omega))).$$

In particular, a Kähler manifold satisfying the Einstein equation

$$Ricci = 0$$

has vanishing $c_1(\omega_M) = 0 \in H^2(M, i\mathbb{R})$. Yau's theorem shows that for compact M with $c_1(\omega_M) = 0$ a Ricci flat metric exists and is unique in each Kähler class. Ricci flat compact Kähler manifolds are called *Calabi-Yau* manifolds.

The volume element of a Ricci flat metric is uniquely determined up to a constant. In the case

$$[\omega_M] = 0 \in \mathrm{Pic}(M) := H^1(M, \mathcal{O}^{\times})$$

there exists a global holomorphic form $\Omega \in \Omega^n(M)$, non-vanishing everywhere, such that the volume element ω^n is equal to $const \times \Omega \wedge \bar{\Omega}$.

When $M = X(\mathbb{C})$, where $X/\mathbb{C}$ is a smooth projective variety of general type, i.e., when $[\omega_X] \in \mathrm{Ample}_X$, there exists a unique strictly psh smooth metrization g of ω_M such that the top degree of the Kähler form associated with g coincides with g. In other words, there exists a unique Kähler metric satysfying the Einstein equation with the negative cosmological constant -1

$$Ricci = -metric.$$

This result is considerably simpler than the Calabi conjecture, and it was proven independently by Aubin and Yau.

sect:spec-aff

3.6. **Analogs in special affine geometry.** Here we describe a less well-known analog of Kähler geometry. Locally it can be modelled on the real affine space $\mathbb{R}^n$. In fact, as we will see, it is closely related to the non-archimedean version of Kähler geometry.

Denote by $\Phi : (\mathbb{C}^\times)^n \to \mathbb{R}^n$ the map

$$(z_1, \ldots, z_n) \mapsto (\log(|z_1|), \ldots, \log(|z_1|)).$$

Lemma 3.6.1. *A function $f \in C^0(U)$ (for open $U \subset \mathbb{R}^n$) is convex iff $\Phi^*(f)$ is plurisubharmonic on $\Phi^{-1}(U)$. Further, f is affine iff $\Phi^*(f)$ is pluriharmonic.*

Unlike the situation in complex geometry here we have no *unbounded* convex functions. An analog of a complex manifold in this geometry is a C^∞-manifold with a smooth affine structure. One can easily define affine versions of virtual line bundles and metrizations. A *Kähler affine* manifold is a manifold with an affine structure and a smooth metric g such that locally in affine coordinates the metric is potential, i.e. there exists a (locally) smooth strictly convex function f such that

$$g_{ij} = \partial_i \partial_j (f).$$

Chang and Yau (see [10]) proved that every compact Kähler affine manifold is diffeomorphic to a torus, and the Kähler affine structure is obtained by a continuous deformation from the standard flat structure on $\mathbb{R}^n/\mathbb{Z}^n$.

In order to define the Monge-Ampére operator one needs additionally a locally constant volume element.

Definition 3.6.2. *Let M be a real n-dimensional C^∞-manifold with a smooth affine structure and a smooth measure μ which is constant in local affine coordinates. For a smooth real-valued function $f \in C^\infty(M, \mathbb{R})$ the* Monge-Ampére charge *$MA(f)$ is given (in local special affine coordinates, $\mu = dx^1 \ldots dx^n$) by the following formula*

$$MA(f) = \det(\partial_i \partial_j (f))\, dx^1 \ldots dx^n.$$

The relation between the complex and the special affine Monge-Ampére operators is as follows. If $f \in C^\infty(\mathbb{R}^n)$ then

$$MA_{\text{special affine}}(f) = \Phi_*(MA_{\mathbb{C}}(\Phi^*(f)).$$

If we rescale the measure μ on M by

$$\mu \mapsto \lambda\mu,, \lambda \in \mathbb{R}_+^\times,$$

then the operator MA will change as

$$MA(f) \mapsto \lambda^{-1} MA(f).$$

For convex f the charge $MA(f)$ is a measure, and the definition of $MA(f)$ extends uniquely (by continuity) to an n-linear operator from the sheaf of convex functions to the sheaf of measures (not necessarily continuous with respect to the Lebesgue measure).

In special affine geometry there are analogs of the Mabuchi functional and of the Calabi conjecture (a theorem). In the concrete example $M = \mathbb{R}^n/\mathbb{Z}^n$ the Calabi conjecture gives the following result.

Theorem 3.6.3. *Let Q be a positive definite quadratic form on $\mathbb{R}^n$. The Monge-Ampére operator gives one-to-one correspondence between the set*

$$\left\{ f \in C^0(\mathbb{R}^n) \,|\, f \text{ is convex, } (f - Q) \text{ is } \mathbb{Z}^n - \text{periodic} \right\}$$

and the set of measures on $\mathbb{R}^n/\mathbb{Z}^n$ with total volume equal to $\det(Q)$.

Part II. Nonarchimedean geometry

In this part K denotes a local field with a discrete nontrivial nonarchimedean valuation

$$\mathrm{val} = \mathrm{val}_K \, : \, K^\times \twoheadrightarrow \mathbb{Z}.$$

satisfying the usual conditions

- $\mathrm{val}(\lambda \cdot \lambda') = \mathrm{val}(\lambda) + \mathrm{val}(\lambda')$;
- $\mathrm{val}(\lambda + \lambda') \leq \max(\mathrm{val}(\lambda), \mathrm{val}(\lambda'))$

for all $\lambda, \lambda' \in K$. The principal examples will be $K = \mathbb{Q}_p$ and $K = \mathbb{C}((t))$. Denote by k the residue field of K and by X a smooth proper variety of dimension n over K.

4. Clemens complexes

4.1. Models.

Definition 4.1.1. *A* model *of X is a scheme of finite type $\mathcal{X}/\mathcal{O}_K$ flat and proper over $\mathcal{O}_K$ together with an identification $\mathcal{X} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\mathcal{O}_K)$ with X over K. Denote the special fiber of $\mathcal{X}$ by*

$$\mathcal{X}^\circ := \mathcal{X} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(k).$$

A model has no nontrivial automorphisms. Thus, the stack of equivalence classes of models is a plain set, which we denote by Mod_X. It carries a natural partial order:

$$\mathcal{X}_1 \geq \mathcal{X}_2$$

if there exists a map $\mathcal{X}_1 \to \mathcal{X}_2$ over $\mathrm{Spec}(\mathcal{O}_K)$. Such a map is automatically unique.

Definition 4.1.2. *A model $\mathcal{X}$ has* normal crossings *iff the scheme $\mathcal{X}$ is regular and the reduced subscheme $\mathcal{X}^\circ_{red}$ is a divisor with normal crossings.*

By the resolution of singularities, in the case $\mathrm{char}(k) = 0$ every model is dominated by a model with normal crossings.

Definition 4.1.3. *A model $\mathcal{X}$ has* simple normal crossings *(*snc model *for short) iff*

- *it has normal crossings;*
- *all irreducible components of $\mathcal{X}^\circ_{red}$ are smooth and*
- *all intersections of irreducible components of $\mathcal{X}^\circ_{red}$ are either empty or irreducible.*

The set of equivalence classes of snc models will be denoted by Mod_X^{snc}.

It is easy to show that starting with any model with normal crossings and applying blow-ups centered at certain self-intersection loci of the special fiber we can get an snc model. In what follows we use snc models only. This choice is dictated by convenience and not by necessity. It seems that in the general case of positive/mixed characteristic one can use the so-called pluri-nodal models (see [7]) - a consequence of de Jong's result. Working with snc models has the advantage that all definitions and calculations can be made very transparent.

sect:clem

4.2. **Clemens complex of an snc model.** Let $\mathcal{X}$ be an snc model and $I = I_{\mathcal{X}}$ the set of irreducible components of $\mathcal{X}^\circ_{red}$. Denote by $D_i \subset \mathcal{X}$ the divisor corresponding to $i \in I$. For any finite non-empty subset $J \subset I$ put

$$D_J := \cap_{j\in J} D_j.$$

By the snc property D_J is either empty or a smooth proper variety over k of dimension $\dim(D_J) = (n - \#J + 1)$. For a divisor $D_i \subset \mathcal{X}^\circ$ we denote by $d_i \in \mathbb{Z}_{>0}$ the order of vanishing of u at D_i, where $u \in K$ is an uniformizing element, $\mathrm{val}_K(u) = 1$. Equivalently, d_i is the multiplicity of D_i in $\mathcal{X}^\circ$.

Definition 4.2.1. *The* Clemens complex $S_{\mathcal{X}}$ *is the finite simplicial subcomplex of the simplex* Δ^I *such that* Δ^J *is a face of* $S_{\mathcal{X}}$ *iff* $D_J \neq \emptyset$.

dfn:big

Definition 4.2.2. *We will say that an snc model* $\mathcal{X}$ *is a* big *model if* $S_{\mathcal{X}}$ *is the closure of the union of* n*-dimensional simplices.*

Clearly, $S_{\mathcal{X}}$ is a nonempty connected CW-complex. We will also consider the cone over $S_{\mathcal{X}}$:

$$C_{\mathcal{X}}(\mathbb{R}) := \left\{ \sum_{i\in I} a_i \langle D_i\rangle \,|\, a_i \geq 0,\ \cap_{i:\, a_i>0} D_i \neq \emptyset \right\} \setminus \{0\} \subset \mathbb{R}^I.$$

Analogously, we can define $C_{\mathcal{X}}(\mathbb{Z})$, $C_{\mathcal{X}}(\mathbb{Q})$, and, more generally, $C_{\mathcal{X}}(\Gamma)$, where Γ is an arbitrary completely ordered abelian group. (The notation $C_{\mathcal{X}}(\Gamma)$ will be justified later.) We identify $S_{\mathcal{X}}$ with the following subset of $C_{\mathcal{X}}(\mathbb{R})$:

$$\left\{ \sum_{i\in I} a_i \langle D_i\rangle \in C_{\mathcal{X}}(\mathbb{R}) \,|\, \sum_i a_i d_i = 1 \right\}.$$

Obviously, we can also desribe $S_{\mathcal{X}}$ as a quotient of $C_{\mathcal{X}}(\mathbb{R})$:

$$S_{\mathcal{X}} = C_{\mathcal{X}}(\mathbb{R})/\mathbb{R}_+^\times.$$

This identification gives a bijection between $\mathbb{R}$-valued functions on $S_{\mathcal{X}}$ and homogeneous of degree $+1$ functions on $C_{\mathcal{X}}(\mathbb{R})$. For a function $f \in C^0(S_{\mathcal{X}})$ we denote by f^{lin} corresponding homogeneous function on $C_{\mathcal{X}}(\mathbb{R})$.

sect:simple

4.3. **Simple blow-ups.** Let $\mathcal{X}$ be an snc model, $J \subset I_{\mathcal{X}}$ a non-empty subset and $Y \subset D_J$ a smooth irreducible variety of dimension $< n$. Let us assume that Y intersects transversally (in D_J) all subvarieties $D_{J'}$ of D_J (for $J' \supset J$) and that all intersections $Y \cap D_J$ are either empty or irreducible. It is obvious that the blow-up $\mathcal{X}' := \mathrm{Bl}_Y(\mathcal{X})$ of $\mathcal{X}$ with center at Y is again an snc model.

Definition 4.3.1. *For a pair of snc models $\mathcal{X}' \geq \mathcal{X}$ as above we say that $\mathcal{X}'$ is obtained from $\mathcal{X}$ by a* simple blow-up. *If $Y = D_J$ we say that we have a simple blow-up of the* first type. *Otherwise (when $dim(Y) < dim(D_J)$), we have a simple blow-up of the* second type.

Let us describe the behavior of $S_{\mathcal{X}}$ under simple blow-ups. To the set of vertices we add a new vertex corresponding to the divisor $\widetilde{Y}$ obtained from Y:

$$I_{\mathcal{X}'} = I_{\mathcal{X}} \sqcup \{new\}, \;\; D_{new} := \widetilde{Y}.$$

The degree of the new divisor is (for both the first and the second type)

$$d_{new} := \sum_{i \in J} d_j.$$

For blow-ups of the first type we have automatically $\#J > 1$. Here is the list of faces of $S_{\mathcal{X}'}$:

- I' for $I' \in \mathrm{Faces}(S_{\mathcal{X}})$, $I' \not\subset J$;
- $I' \sqcup \{new\}$ for $I' \in \mathrm{Faces}(S_{\mathcal{X}})$, $I' \neq J$, $I' \cup J \in \mathrm{Faces}(S_{\mathcal{X}})$;
- the vertex $\{new\}$.

For blow-ups of the second type the list of faces of $S_{\mathcal{X}'}$ is

- I' for $I' \in \mathrm{Faces}(S_{\mathcal{X}})$;
- $I' \sqcup \{new\}$ for $I' \in \mathrm{Faces}(S_{\mathcal{X}})$, $I' \supset J$, $Y \cup D_{I'} \neq \emptyset$;
- the vertex $\{new\}$.

Conjecture 4.3.2. (Weak factorization) For any two snc models $\mathcal{X}$, $\mathcal{X}'$ there exists a finite alternating sequence of simple blow-ups

$$\mathcal{X} < \mathcal{X}_1 > \mathcal{X}_2 < \cdots < \mathcal{X}_{2m+1} > \mathcal{X}'$$

Presumably, in $\mathrm{char}(k) = 0$ case it follows from results of Abramovich ét al.

Corollary 4.3.3. *Simple homotopy type of $S_{\mathcal{X}}$ does not depend on the choice of an snc model $\mathcal{X}$.*

A stronger conjecture would be:

conj:strong

Conjecture 4.3.4. (Strong factorization) For any two snc models $\mathcal{X}$, $\mathcal{X}'$ there exists a finite sequence of simple blow-ups

$$\mathcal{X} < \mathcal{X}_1 < \mathcal{X}_2 < \cdots < \mathcal{X}_m > \mathcal{X}_{m+1} > \cdots > \mathcal{X}_{m+l} > \mathcal{X}'$$

This would mean that if we generate a partial order $>_s$ on Mod_X^{snc} by simple blow-ups then in $(\mathrm{Mod}_X^{snc}, >_s)$ there exists an upper bound for an arbitrary finite set.

sect:simple1

4.4. **Simple blow-ups of the first type and Farey fractions.** For a simple blow-up of the first type $\mathcal{X}' > \mathcal{X}$ we define a map

$$p_{\mathcal{X}',\mathcal{X}} \,:\, C_{\mathcal{X}'}(\mathbb{R}) \to C_{\mathcal{X}}(\mathbb{R}),$$

equivariant with respect to the $\mathbb{R}_+^\times$-action. Namely, in notations of Section 4.3, we define $p_{\mathcal{X}',\mathcal{X}}$ as the restriction of the linear map

$$\mathbb{R}^{I_{\mathcal{X}'}} \to \mathbb{R}^{I_{\mathcal{X}}}$$

defined on generators as

$$\langle D_i \rangle \mapsto \langle D_i \rangle \text{ for any } i \in I_{\mathcal{X}}, \;\; \langle D_{new} \rangle \mapsto \sum_{j \in J} \langle D_j \rangle.$$

One can easily check that $p_{\mathcal{X}',\mathcal{X}}$ is a homeomorphism and that it identifies the sets $C_{\mathcal{X}'}(\mathbb{Z})$ and $C_{\mathcal{X}}(\mathbb{Z})$.

We will use the same notation $p_{\mathcal{X}',\mathcal{X}}$ for the induced homeomorphism between $S_{\mathcal{X}'}$ and $S_{\mathcal{X}}$. Thus, performing a simple blow-up of the first type we get a subdivision of the cell decomposition of the Clemens complex.

thml:small

Theorem 4.4.1. *For every snc model $\mathcal{X}$ and for every real constant $\epsilon > 0$ there exists a finite sequence of simple blow-ups such that all cells of the resulting decomposition of $S_{\mathcal{X}}$ have diameter $< \epsilon$ (in the standard Euclidean metric on $S_{\mathcal{X}} \subset \Delta^{I_{\mathcal{X}}}$). A closed subset $Z \subset S_{\mathcal{X}}$ is a finite union of cells for the cell decomposition obtained from a finite sequence of simple blow-ups of the first type iff Z is a finite union of closed simplexes with rational vertices.*

Proof. It is enough to prove the result in the case when $S_{\mathcal{X}}$ is a simplex. The question is reduced to the study of decompositions by simplicial cones of $\mathbb{R}_+^I$ (for a finite set I). In the first non-trivial case $\#I = 2$ we get the usual Farey fractions as slopes of faces of decompositions of $\mathbb{R}_+^2$ obtained inductively after iterations of simple blow-ups of the first type. Thus, we get all rational directions, and both statements of the theorem are obvious. In the general case one proceeds with an induction on dimension. $\square$

sect:berk

5. Berkovich spectra

sect:bas

5.1. **Basic notions.** Let R/K be a ring. The underlying set of the Berkovich spectrum $(\mathrm{Spec}(R/K))^{an}$ can be defined in two ways:

Definition 5.1.1. *(Valuations) A point x of $(\mathrm{Spec}(R/K))^{an}$ is an additive valuation*

$$\mathrm{val}_x : R \to \mathbb{R} \sqcup \{-\infty\}$$

extending val_K, i.e. it is a map satisfying the conditions

- $\mathrm{val}_x(r + r') \leq \max(\mathrm{val}_x(r), \mathrm{val}_x(r'))$;
- $\mathrm{val}_x(rr') = \mathrm{val}_x(r) + \mathrm{val}_x(r')$;
- $\mathrm{val}_x(\lambda) = \mathrm{val}_K(\lambda)$

for all $r, r' \in R$ and all $\lambda \in K$.

Definition 5.1.2. *(Evaluation maps) A point x of $(\mathrm{Spec}(R/K))^{an}$ is an equivalence class of homomorphisms of K-algebras*

$$\mathrm{eval}_x : R \to K_x$$

where $K_x \supset K$ is a complete field equipped with a valuation (with values in a subset of $\mathbb{R}$) extending the valuation val_K, such that K_x is generated by the closure of the image of eval_x.

In order to pass from the first description of $(\mathrm{Spec}(R/K))^{an}$ to the second, starting with a valuation val_x one defines the field K_x as the completion of the field of fractions of R/I_x, where $I_x = (\mathrm{val}_x)^{-1}(\{-\infty\})$.

Definition 5.1.3. *The topology on $(\mathrm{Spec}(R/K))^{an}$ is the weakest topology such that for all $r \in R$ the map*

$$\begin{array}{ccc} (\mathrm{Spec}(R/K))^{\mathrm{an}} & \to & \mathbb{R}_{\geq 0}, \\ x & \mapsto & \mathrm{val}_x(r) \end{array}$$

is continuous.

A neighborhood $U = U_x \subset X^{an}$ of a point x is described as follows: there exist finite sets of functions

$$(f_i)_{i \in I},\ (g_j)_{j \in J}$$

and numbers

$$\beta_i^+, \beta_i^-, \gamma_j \in \mathbb{R}_{>0}$$

such that

- $|f_i(x)| > 0,\ |g_j(x)| = 0$;
- $\beta_i^- < |f_i(x')| < \beta_i^+,\ |g_j(x')| < \gamma_j$

for all $i \in I, j \in J$ and $x' \in U$. In this language, germs of functions $s_U \in \mathcal{O}_{X^{an},x}$ are given as series

$$s_U = \sum_{n_I \in \mathbb{Z}^I, m_J \in \mathbb{N}^J} c_{I,J} f_I^{n_I} g_J^{m_J}$$

with constants $c_{I,J}$ (subject to some convergence conditions).

We have a similar description for Zariski open subsets of $\mathrm{Spec}(R)$ and we can glue schemes. We get a functor

$$\begin{array}{ccc} (\mathrm{Sch}/K) & \to & (K-\text{analytic spaces}) \\ X & \mapsto & (X^{an}, \mathcal{O}_{X^{an}}). \end{array}$$

We will denote by $\mathrm{Coh}(X^{an})$ the category of coherent sheaves on X^{an}.

Define the *cone* over X^{an} as

$$C_{X^{an}}(\mathbb{R}) := X^{an} \times \mathbb{R}_{>0}.$$

We interpret a point $\mathbf{x} = (x, \lambda)$ of $C_{X^{an}}(\mathbb{R})$ as a K_x-point of X, where $K_x \supset K$ is a complete field with the $\mathbb{R}$-valued valuation

$$\mathrm{val}_{\mathbf{x}} := \lambda \, \mathrm{val}_x,$$

whose restriction to K is proportional to val_K. The set of points $\mathbf{x} \in C_{X^{an}}(\mathbb{R})$ such that the valuation $\mathrm{val}_{\mathbf{x}}$ is $\mathbb{Z}$-valued is denoted by $C_{X^{an}}(\mathbb{Z})$.

prope

5.2. **Properties.**

Proposition 5.2.1. *The space X^{an}*

- *is a locally compact Hausdorff space;*
- *has the homotopy type of a finite CW-complex;*
- *is contractible if X has good reduction with irreducible special fiber.*

Proof. Let K'/K be a finite Galois extension with ring of integers $\mathcal{O}_{K'}$. We denote by $\mathcal{X}'$ a stable reduction model for $X_{K'}$ over $\mathcal{O}_{K'}$ (it exists over some finite extension): we assume that the special fiber

$$\mathcal{X}'_0 = \cup_{\alpha \in \mathcal{A}} D'_\alpha$$

is a (strict) normal crossing divisor with smooth D'_α. The homotopy type of X^{an} is the geometric realization of the poset

$$\bigsqcup_{K', \mathcal{X}', j} \pi_0(D'_{\alpha_1} \cap \ldots \cap D'_{\alpha_j}) / \mathrm{Gal}(K'/K).$$

$\square$

Remark 5.2.2. R. Huber considered maps $|\cdot|_x : R \to \mathcal{I}$ into ordered groups $\mathcal{I}$. In such a setup the associated analytic spaces need not be Hausdorff (see [15]).

Example 5.2.3. Let $X = \mathbb{A}^1 = \operatorname{Spec}(K[x])$ be the affine line. The analytic space X^{an} contains, among others, points of the following types:

- $X(K) \hookrightarrow X^{an}$;
- $X(\bar{K})/\operatorname{Gal}(\bar{K}/K) \hookrightarrow X^{an}$;
- for $r \in \mathbb{R}$ define
$$|\sum_{j=0}^{d} c_j z^j|_r := \max_j(|c_j|\, r^j).$$
This gives an embedding $\mathbb{R}_{\geq 0} \hookrightarrow X^{an}$.

We see that X^{an} contains, in a sense, both p-adic and real points.

If $K = k((t))$, where k is an algebraically closed field of characteristic 0, an explicit description of $X^{an} = (\mathbb{A}^1)^{an}$ is as follows: As a set, X^{an} consists of equivalence classes of pairs $\{(a, \mu)\}$, where
$$\mu \in \mathbb{R} \cup \{+\infty\} = (-\infty, +\infty]$$
and a is a map
$$a \,:\, \mathbb{Q} \cap (-\infty, \mu) \to k$$
such that for all $\mu' < \mu$ one has
$$\#\{\lambda \,|\, \lambda < \mu' \textit{ and } a(\lambda) \neq 0\} < +\infty.$$
In other words, the support of the function a is an increasing sequence (finite or infinite)
$$\lambda_1 < \lambda_2 < \cdots <$$
of rational numbers, such that, in case it is infinite, its accumulation point is μ. The equivalence is induced by the action of
$$\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, k^*) = \operatorname{Gal}(\bar{K}/K),$$
$$\phi \,:\, a(\lambda) \mapsto a(\lambda) \cdot \phi(\lambda \bmod \mathbb{Z}).$$
We shall write such an expression as a formal power series
$$x^{an} = \sum a(\lambda_i) t^{\lambda_i} + O(t^{\mu}).$$
Clearly, substituting x^{an} into a polynomial $P \in k((t))[x]$ we get a series of the same type. The exponent at the lowest (nonzero) term of this series defines the valuation on P. We leave it to the reader to define the topology on X^{an} in this language. In this description
$$X(\bar{K})/\operatorname{Gal}(\bar{K}/K)$$

corresponds to Puiseux series with $\mu = +\infty$ and $\mathrm{Supp}(a) \subset \frac{1}{N}\mathbb{Z}$, for some $N \in \mathbb{N}$. The topological completion of $X(\bar{K})$ corresponds to to series with $\mu = +\infty$.

Theorem 5.2.4. *Let X be a smooth projective variety over K and $\mathcal{X}$ its snc model over $\mathcal{O}_K$. Then*

- *(GAGA) one has an equivalence of categories* $\mathrm{Coh}(X) = \mathrm{Coh}(X^{an})$*;*
- *étale cohomology;*
- *X^{an} admits a retraction onto the Clemens complex $S_{\mathcal{X}}$.*

We define the subset

$$\mathcal{P}_{\mathrm{abs}}(X) \subset X^{an}$$

of "divisorial" points of X^{an} as follows. As above, let K'/K be a finite (Galois) extension, $\mathcal{O}_{K'}$ its ring of integers and $\mathcal{X}'$ a model of $X_{K'}$ over $\mathcal{O}_{K'}$. Denote by $\mathrm{Irr}(\mathcal{X}')$ the set of all irreducible components of the special fiber $\mathcal{X}^{0\prime}$ of $\mathcal{X}'$. We put

$$\mathcal{P}(X, K') := \varinjlim_{\mathcal{X}'} \mathrm{Irr}(\mathcal{X}')$$

We have an embedding

$$\begin{array}{ccc} \mathcal{P}(X, K') & \hookrightarrow & X^{an} \\ D & \mapsto & (r \in R \mapsto \mathrm{val}_D(r)) \,. \end{array}$$

There is a natural action of $\mathrm{Gal}(K'/K)$ on $\mathcal{P}(X, K')$ and

$$\mathcal{P}(X, K')/\mathrm{Gal}(K'/K) \hookrightarrow X^{\mathrm{an}}.$$

We define

$$\mathcal{P}_{\mathrm{abs}}(X) := \varinjlim_{K'/K} \mathcal{P}(X, K')/\mathrm{Gal}(K'/K).$$

Remark 5.2.5. The set $\mathcal{P}_{\mathrm{abs}}(X)$ is a dense subset in X^{an}, countable if $K = \mathbb{Q}_p$.

Example 5.2.6. For $X = \mathbb{P}^1$ the set $\mathcal{P}(X, K')$ coincides with the set of vertices of the Bruhat-Tits tree.

sect:cl-cones

5.3. **Clemens cones and valuations.** Let $\mathcal{X}$ be a snc model. We define a map

$$i_{\mathcal{X}}^{\mathbb{Z}} : C_{\mathcal{X}}(\mathbb{Z}) \to C_{X^{an}}(\mathbb{Z})$$

as follows: For $J \subset I_{\mathcal{X}}$ such that $D_J \neq \emptyset$ consider a point $x \in C_{\mathcal{X}}(\mathbb{Z})$

$$x = \sum_{i \in J \subset I_{\mathcal{X}}} a_i \langle D_i \rangle, \;\; a_i > 0 \; \forall i \in J$$

and an affine Zariski open subset $U \subset \mathcal{X}$ containing the generic point of D_J. The algebra $\mathcal{O}(U)$ contains non-trivial principal ideals $\mathcal{I}_i$ corresponding to divisors D_i, $i \in J$. We define valuation v_x of $\mathcal{O}(U)$ as follows:

$$v_x(r) = \sup\left\{\sum a_i n_i \mid r \in \prod \mathcal{I}_i^{n_i},\ (n_i)_{i\in J} \in \mathbb{Z}_{\geq 0}^J\right\}$$

We define $i_X^{\mathbb{Z}}(x)$ to be the image of the point $v_x \in (\mathrm{Spec}(\mathcal{O}(U)/K))^{\mathrm{an}}$ in X^{an}. It is easy to check that the element $i_X^{\mathbb{Z}}(x)$ does not depend on the choice of the open subset U.

prop:contiu **Proposition 5.3.1.** *There exists a unique continuous $\mathbb{R}_+^{\times}$-equivariant map*

$$i_{\mathcal{X}}^{\mathbb{R}} : C_{\mathcal{X}}(\mathbb{R}) \to C_{X^{an}}(\mathbb{R})$$

extending $i_{\mathcal{X}}^{\mathbb{Z}}$. The map $i_{\mathcal{X}}^{\mathbb{R}}$ is an embedding.

Proof. □

We denote by

$$i_{\mathcal{X}} : S_{\mathcal{X}} \hookrightarrow X^{an}$$

the induced embedding.

For simple blow-ups $\mathcal{X}' \geq \mathcal{X}$ we define an injection

$$i_{\mathcal{X},\mathcal{X}'}^{\mathbb{Z}} : C_{\mathcal{X}}(\mathbb{Z}) \hookrightarrow C_{\mathcal{X}'}(\mathbb{Z})$$

by the explicit formulas (in the notations of Section 4.3) :

- first type:
- second type

We extend this definition to

$$i_{\mathcal{X},\mathcal{X}'} : S_{\mathcal{X}} \hookrightarrow S_{\mathcal{X}'}$$

and

$$i_{\mathcal{X},\mathcal{X}'}^{\mathbb{R}} : C_{\mathcal{X}}(\mathbb{R}) \hookrightarrow C_{\mathcal{X}'}(\mathbb{R}).$$

lemm:523 **Lemma 5.3.2.** *For a simple blow-up $\mathcal{X}' \geq \mathcal{X}$ we have*

$$i_{\mathcal{X},\mathcal{X}'} \circ i_X = i_{X'}.$$

Proof. □

Definition 5.3.3. *The points of $C_{X^{an}}(\mathbb{R})$ lying in the union of images of $C_{\mathcal{X}}(\mathbb{Z})$ over all snc models are called* divisorial points.

sect:cones-paths

5.4. **Clemens cones and paths.** For a model $\mathcal{X}$ we can interpret elements of $C_{X^{an}}(\mathbb{Z})$ as *paths* in $\mathcal{X}$, i.e. equivalence classes of maps

$$\phi : \mathrm{Spec}(\mathcal{O}_L) \to \mathcal{X},$$

where $\mathcal{O}_L$ is the ring of integers in a field L with discrete valuation in $\mathbb{Z}$, such that the image of ϕ does not lie in $\mathcal{X}$. We define the map

$$p_{\mathcal{X}}^{\mathbb{Z}} \, : \, C_{X^{an}}(\mathbb{Z}) \to C_{\mathcal{X}}(\mathbb{Z})$$

as

$$p_{\mathcal{X}}^{\mathbb{Z}}([\phi]) := \sum_i a_i \langle D_i \rangle,$$

where $a_i \in \mathbb{Z}_{\geq 0}$ is the multiplicity of the intersection of the path ϕ with the divisor D_i, $i \in I_{\mathcal{X}}$.

Proposition 5.4.1. *The map $p_{\mathcal{X}}^{\mathbb{Z}}$ extends uniquely to a continuous $\mathbb{R}_+^{\times}$-equivariant map $p_{\mathcal{X}}^{\mathbb{R}} \, : \, C_{X^{an}}(\mathbb{R}) \to C_{\mathcal{X}}(\mathbb{R})$. The map $p_{\mathcal{X}}^{\mathbb{R}}$ is a surjection.*

We denote by $p_{\mathcal{X}} \, : \, S_{\mathcal{X}} \to X^{\mathrm{an}}$ the map induced by $p_{\mathcal{X}}^{\mathbb{R}}$.

Proof. ???? □

For any dominating map of models $\mathcal{X}' \geq \mathcal{X}$ we define a projection

$$p_{\mathcal{X}',\mathcal{X}}^{\mathbb{Z}} \, : \, C_{\mathcal{X}'}(\mathbb{Z}) \hookrightarrow C_{\mathcal{X}}(\mathbb{Z})$$

by the formula

$$\sum a_i \langle D_i \rangle \mapsto \sum a_i c_{ij} \langle D_j \rangle$$

where $i \in I_{\mathcal{X}}$, $j \in I_{\mathcal{X}'}$ and $c_{ij} \in \mathbb{Z}_{\geq 0}$ are the multiplicities of the full preimage of D_i in $\mathcal{X}'$. For simple blow-ups the projection $p_{\mathcal{X}',\mathcal{X}}$ is a simplicial map with contractible fibers and formula simplifies to:

- first type:
- second type:

lemm:ddd

Lemma 5.4.2. *For any dominating map of models $\mathcal{X}' \to \mathcal{X}$ we have*

$$p_{\mathcal{X}}^{\mathbb{Z}} = p_{\mathcal{X}',\mathcal{X}}^{\mathbb{Z}} \circ p_{\mathcal{X}'}^{\mathbb{Z}}.$$

Proof. □

Corollary 5.4.3. *For dominating maps $\mathcal{X}'' \geq \mathcal{X}' \geq \mathcal{X}$ we have*

$$p_{\mathcal{X}'',\mathcal{X}}^{\mathbb{Z}} = p_{\mathcal{X}',\mathcal{X}}^{\mathbb{Z}} \circ p_{\mathcal{X}'',\mathcal{X}'}^{\mathbb{Z}}.$$

Remark 5.4.4. For any dominating map of models $\mathcal{X}' \geq \mathcal{X}$ the fibers of $p^{\mathbb{Z}}_{\mathcal{X}',\mathcal{X}}$ are contractible (assuming 4.3.4). Namely, there exists a model $X'' \geq X'$ such that both morphisms $\mathcal{X}''{\rightarrow}\mathcal{X}'$ and $\mathcal{X}''{\rightarrow}\mathcal{X}$ are compositions of simple blow-ups. Thus fibers of the corresponding maps between polyhedra are contractible. Now the statement follows from a standard fact in topology of CW-complexes.

5.5. **Berkovich spaces as projective limits.**

Theorem 5.5.1. *For any algebraic X the analytic space X^{an} is a projective limit over the partially ordered set of snc models $\mathcal{X}$ of Clemens polytopes $S_{\mathcal{X}}$. The connecting maps are $p_{\mathcal{X}',\mathcal{X}}$.*

Proof. □

Remark 5.5.2. Assuming the Strong Factorization Conjecture 4.3.4 we can restrict to the suborder of the set of snc models generated by simple blow-ups.

prop:xy

Proposition 5.5.3. *Let $X{\rightarrow}Y$ be a morphism of smooth projective algebraic varieties and $X^{an}{\rightarrow}Y^{an}$ the induced morphism of Berkovich spaces. Then for any model $\mathcal{Y}$ of Y there exists a model $\mathcal{Y}'$ (obtained from $\mathcal{Y}$ by blow-ups of the first type) and a model $\mathcal{X}$ of X together with a simplicial map $S_{\mathcal{X}}{\rightarrow}S_{\mathcal{Y}'}$ such that the diagram*

$$\begin{array}{ccc} X^{an} & \rightarrow & Y^{an} \\ p_{\mathcal{X}} \downarrow & & \downarrow p_{\mathcal{Y}'} \\ S_{\mathcal{X}} & \rightarrow & S_{\mathcal{Y}'} \end{array}$$

commutes.

Proof. □

sect:log

5.6. **Log-geometry.**

The decription of the analytic space X^{an} as a projective limit of polytopes can be extended to arbitrary smooth, not necessary proper, schemes. Namely, any smooth X over K has a smooth compactification $\bar{X}$ such that $Z := \bar{X} \setminus X$ is a divisor with strict normal crossings. Let $\bar{\mathcal{X}}$ be a model of $\bar{X}$ such that the special fiber together with the closure of Z in $\bar{\mathcal{X}}$ form an *snc*-divisor $\mathcal{D}$ (we will call this an *snc*-model of X). The divisors $\mathcal{D}$ has a decomposition

$$\mathcal{D} = \cup_{i \in I} \mathcal{D}_i$$

of irreducible components, where $I = I^{vert} \sqcup I^{hor}$, with $I^{vert} = Irr(\bar{\mathcal{X}}^{\circ})$ and I^{hor} labels irreducible components of Z, considered as divisors over a nonalgebraically closed field K. Similarly to Section 4.2 we introduce the cone

$$C^{log}_{\bar{\mathcal{X}}}(\mathbb{R}).$$

We have a map

$$\mathrm{mult} \,:\, C^{log}_{\bar{\mathcal{X}}}(\mathbb{R}) \to \mathbb{R}_{\geq 0}.$$

$$\mathrm{mult}(\left\{\sum_{i\in I} a_i\langle D_i\rangle) := \sum_{i\in I^{vert}} a_i d_i\right\},$$

where d_i are the multipliplicity of D_i in $\bar{\mathcal{X}}^{\circ}$. The map mult is proper iff X is proper. We denote by

$$S^{log}_{\bar{\mathcal{X}}} := \mathrm{mult}^{-1}(1)$$

the logarithmic Clemens polytope (not necessarily compact). Obviously, it is a finite union of products of simplices and positive octants.

exam:t *Example* 5.6.1. Let $X = T = \mathbb{G}_m^n$ be a torus and $\bar{X}$ a smooth toric compactification defined by a fan Σ on $\mathrm{Hom}(\mathbb{G}_m, T)$. It has a natural model $\bar{\mathcal{X}}$. The associated space $S^{log}_{\bar{X}}$ is canonically isomorphic to the vector space $\mathrm{Hom}(\mathbb{G}_m, T) \otimes \mathbb{R}$ endowed with a cone decomposition induced by Σ.

exam:mgn *Example* 5.6.2. Our theory has a straightforward generalization to Deligne-Mumford stacks. Let $X = M_{g,n}$ be the moduli stack of curves of genus g with n marked points and $\bar{X}$ its Deligne-Knutson-Mumford compactification. It has an obvious model $\bar{\mathcal{X}}$ over an arbitrary local ring (since it is defined over $\mathrm{Spec}(\mathbb{Z})$). We claim that the associated logarithmic Clemens polytope is canonically isomorphic to the moduli space of *stable metrized graphs.*

More precisely, this moduli space (stack) is the set of equivalence classes, with fixed g, n, of the following objects. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a nonempty connected graph (a CW-complex of dimension ≤ 1), where Γ_0 is the set of vertices and Γ_1 the set of edges. Let

$$genus \,:\, \Gamma_0 \to \mathbb{Z}_{\geq 0}$$

be a map such that

$$g = 1 - \chi(\Gamma) + \sum_{v\in\Gamma_0} genus(v)$$

(where χ is the Euler characteristic of Γ). Let

$$point \,:\, [1, ..., n] \to \Gamma_0$$

and

$$length\,:\,\Gamma_1 \to \mathbb{R}_{>0}$$

be arbitrary maps. We call the collection $(\Gamma, genus, point, length)$ a *stable metrized graph* if for every $v \in \Gamma_0$ one has

$$2 - 2g - \#\{i = 1, ..., n \,|\, point(i) = v\} - \deg(v) < 0.$$

Similarly to Sections 5.3 and **??** one can define the maps

$$i^{\log}_{\bar{\mathcal{X}}}\,:\,S^{log}_{\bar{X}} \to X^{an}$$

and

$$p^{log}_{\bar{\mathcal{X}}}\,:\,X^{an} \to S^{log}_{\bar{X}}.$$

The analogs of Lemma 5.3.2 and Lemma 5.4.2 hold.

Theorem 5.6.3. *The space X^{an} is the projective limit over the partially ordered set of logarithmic snc-models of Clemens polytopes $S^{log}_{\bar{\mathcal{X}}}$. The connecting maps $p_{\bar{\mathcal{X}}'\bar{\mathcal{X}}}$ are proper and piecewise-linear.*

sect:am

5.7. **Amoebas.** Let $Z \subset X$ be a subvariety of dimension $m < n = \dim X$ and let $\mathcal{X}$ be a model of X. We call the image of Z^{an} under the map

$$p_{\mathcal{X}}\,:\,X^{an} \to S_{\mathcal{X}}.$$

the *amoeba* of Z associated with the model $\mathcal{X}$. We will show that it is a closed piecewise-linear subset of $S_{\mathcal{X}}$ of (local) codimension at most $n - m$ at every point.

Remark 5.7.1. This generalizes the amoebas of Gelfand, Kapranov and Zelevinski [13] and Mikhalkin [17]. Namely, for $X = \mathbb{G}^n_m$ and any toric compactification $\bar{X}$ the projection

$$p_{\bar{\mathcal{X}}}\,:\,X^{an} \to S^{log}_{\bar{\mathcal{X}}} = \mathbb{R}^n$$

is given (on $\bar{K}$-points) by

$$p_{\bar{\mathcal{X}}}(x_1, ..., x_n) = (\log(|x_1|), ..., \log(|x_n|).$$

The (classical) amoeba is the image of $Z(\mathbb{C})$ under the analogous map in the archimedean case $K = \mathbb{C}$.

prop:am

Proposition 5.7.2.

sect:pls

6. Plurisubharmonic functions on X^{an}

In this section X denotes a smooth proper variety of dimension n over K.

6.1. **Some classes of functions.** We will say that an $\mathbb{R}$-valued function on X^{an} is *of finite type* iff it is a pullback of a function on $S_{\mathcal{X}}(\mathbb{R})$ for some snc model $\mathcal{X}$. We define the sheaf $Fun_{lft,X}$ (or simply Fun_{lft}) of functions of locally finite type as the sheaf of functions whose germs are germs of functions of finite type. The compactness of X^{an} implies that global sections of Fun_{lft} are exactly functions of finite type. Below we always assume that all sheaves of functions and global functions are subsheaves of Fun_{lft}.

Remark 6.1.1. The restriction to functions of finite type simplifies many analytical problems in the comparison with the archimedean (complex) case. Presumably, one should develop a more general theory including certain limits of functions of finite type.

Definition 6.1.2. *Let $\mathcal{X}$ be an snc model. A continuous function $f \in C^0(S_{\mathcal{X}})$ is called* simple *if its extension f^{lin} (to a homogeneous of degree $+1$ function on $C_{\mathcal{X}}(\mathbb{R})$) is linear on each face of the cone $C_{\mathcal{X}}(\mathbb{R})$.*

Obviously, the set of simple functions is canonically identified with $\mathbb{R}^{I_{\mathcal{X}}}$, i.e., every simple function is uniquely determined by values of f^{lin} on points $\langle D_i \rangle \in C_{\mathcal{X}}(\mathbb{Z})$. Also, it is easy to see that for any simple blow-up $\mathcal{X}_1 < \mathcal{X}_2$ the pullback under the projection $\mathrm{pr}_{\mathcal{X}_2,\mathcal{X}_1}$ of a simple function on $S_{\mathcal{X}_1}$ is a simple function on $S_{\mathcal{X}_2}$.

A function on $S_{\mathcal{X}}$ is called simple iff it is simple in the above sense.

Definition 6.1.3. *A* piecewise linear *(PL for short) function on X^{an} is a pullback of a simple function from $S_{\mathcal{X}}$ (for some model $\mathcal{X}$).*

We define the subsheaf $\mathbb{R}PL_X$ of Fun_{lft} as the sheaf of functions whose germs at every point are germs of a PL function. We define the sheaf $\mathbb{Z}PL_X$ as the subsheaf of $\mathbb{R}PL_X$ consisting of functions f such that the corresponding homogeneous functions f^{lin} take *integer* values at integer points $C_{X^{\mathrm{an}}}(\mathbb{Z}) \subset C_{X^{\mathrm{an}}}(\mathbb{R})$.

Proposition 6.1.4. *The sheaf $\mathbb{R}PL_X$ is soft. The space of global sections $\Gamma(X^{an}, \mathbb{R}PL_X)$ coincides with the space of PL functions. The subsheaf $\mathbb{Z}PL_X$ is closed under pointwise addition, subtraction and maximum.*

Proof. The first statement follows from the fact that using blow-ups of the first type we can make all faces of the simplicial subdivision of $S_{\mathcal{X}}$ arbitrarily small (see **??**). The second statement follows immediately from the compactness of X^{an}. The third statement also follows from **??**. Namely, if f, g are two (global) integer-valued PL functions then there exists a finite sequence of blow-ups of the first type such that the functions $f+g, f-g, \max(f,g)$ are linear on all faces of the resulting subdivision. □

For a given model $\mathcal{X}$ we denote by $\mathbb{R}PL_{\mathcal{X}}$ and $\mathbb{Z}PL_{\mathcal{X}}$ the soft sheaves on $S_{\mathcal{X}}$ which are the obvious cousins of the corresponding sheaves on X^{an}.

6.2. **Convex simple functions.**

Definition 6.2.1. *Let $\mathcal{X}$ be an snc model and $J \subset I$ a non-empty subset such that $D_J \neq \emptyset$. A simple function $f \in C^0(S_{\mathcal{X}})$ is* convex *along the face Δ^J iff the following linear combination of divisors in D_J*

$$\sum_{i \in I_{\mathcal{X}}} f^{lin}(\langle D_i \rangle) \times [D_i]_{|D_J} \in NS(D_J) \otimes \mathbb{R}$$

is nef, i.e., if it belongs to the closure of the ample cone. A function f is linear *along Δ^J iff the above linear combination vanishes.*

Obviously, a simple function f is linear along Δ^J iff both functions f and $-f$ are convex along Δ^J.

Lemma 6.2.2. *If a simple function is convex (resp. linear) along Δ^J then it is also convex (resp. linear) along $\Delta^{J'}$ for any $J' \supset J$ such that $D_{J'} \neq \emptyset$.*

Proof. Follows immediately from the fact that the property of being nef is preserved under restrictions to submanifolds. □

This lemma shows that for any simple function f the union $ConvLoc(f)$ of simplicial faces in S_X along which f is convex, is an open subset of $S_{\mathcal{X}}$. We call the set $ConvLoc(f)$ the *convexity locus* of f.

Theorem 6.2.3. *For every simple blow-up $\mathcal{X}_1 < \mathcal{X}_2$ and every simple function f on $S_{\mathcal{X}_1}$ the pullback of $ConvLoc(f)$ coincides with the convexity locus of the pullback of f.*

Proof. We should study separately ?????? □

This theorem implies that the convexity locus is well-defined for any PL function on X^{an}. As before, we define the sheaf $Conv_X^{\mathbb{R}PL}$ of locally convex piecewise linear functions X^{an}, and also its integer version $Conv_X^{\mathbb{Z}PL}$. Analogously, we define sheaves (of abelian groups) $Lin_X^{\mathbb{R}}$ and $Lin_X^{\mathbb{Z}}$ of locally linear functions. For an snc model $\mathcal{X}$ we also have the corresponding sheaves on $S_{\mathcal{X}}$

$$Conv_{\mathcal{X}}^{\mathbb{R}PL}, Conv_{\mathcal{X}}^{\mathbb{Z}PL}, Lin_{\mathcal{X}}^{\mathbb{R}}, Lin_{\mathcal{X}}^{\mathbb{Z}}.$$

All sheaves defined above are soft.

Theorem 6.2.4. *The sheaf $Conv_X^{\mathbb{Z}PL}$ is closed under addition and pointwise maximum.*

Proof. Let f, g be two germs of convex $\mathbb{Z}PL$ functions. We can reduce the problem of convexity of $f+g$ and of $\max(f,g)$ to a finite level, and moreover, we can assume that all functions $f, g, f+g, \max(f,g)$ are pullbacks from simple functions on the same model $\mathcal{X}$. Now the property of convexity for the sum is obvious. For maximum, - ???? □

The relation between the notion of convexity introduced above and the usual notion is explained by the following result:

Theorem 6.2.5. *Let $\mathcal{X}$ be a model and $\Delta^J \subset S_{\mathcal{X}}$ an n-dimensional simplex (where $n = \dim X$). A germ of a rational piecewise linear function $f \in (C^0_{S_{\mathcal{X}}})_x$ for $x \in Int(\Delta^J)$ is a germ of a convex function (in the usual sense) in the interior of the simplex iff the germ of the pullback $p^*_{\mathcal{X}}(f)$ at x is convex on X^{an}.*

Proof. First of all, germs of integral linear functions on Δ^J are linear in our sense. Thus they are also convex in our sense. The reason is that D_J is a point. Any germ of a convex piecewise linear function is the maximum of germs of linear functions. The use of closedness of the sheaf $Conv_X^{\mathbb{Z}PL}$ under taking maxima finishes the proof that usual convex function are convex in out sense. For the proof of the converse ???? □

6.3. **Plurisubharmonic functions: first definition.**

Theorem 6.3.1. *Let $U \subset X$ be a Zariski open subscheme and $\phi \in \mathcal{O}^\times(U)$ be an invertible function. Then* $\log|\phi|$ *is a section of $Lin_X^{\mathbb{Z}}$ on U^{an}.*

Proof. ?????? □

Thus, we see that a germ f of a real-valued function on X^{an} which can be represented as

$$f(x) = \max_{i \in I} \left(\lambda_i \log(|\phi_i(x)|^2) \right),$$

where $(\phi_i)_{i \in I}$ is a finite collection of analytic functions on U without common zeroes, and $\lambda_i > 0$ are positive *rational* constants, is a germ of a convex simple function.

We denote by $Conv_X$ the sheaf of convex functions on X^{an}. Sections of this sheaf are anlogous to continuous psh functions in the complex case.

In Section **??** we will give a characterization of convex functions, similar to the Definition **??**.

??????

6.4. **Quasi-curves.** Here we define a class of piecewise-linear graphs embedded in $\mathcal{X}$ for an arbitrary snc model $\mathcal{X}$.

Definition 6.4.1. *A* metrized *graph is a graph (i.e. a* 1*-dimensional CW complex) G endowed with parametrization of each open* 1*-cell*

$$e \in Edges(G) := \{\textit{1-cells of } G\}$$

by an open interval $par_e : (0, l) \simeq e$ *defined modulo involution* $x \mapsto l - x$ *of* $(0, l)$*. The positive number* $l = l(e)$ *is called the* length *of edge* e*.*

Any metrized graph carries a natural metric. Also, for any metrized graph G and any collection $(S_e)_{e\in Edges(G)}$ of finite subsets of edges of G one can define a subdivision of G (adding points in $\sqcup_e S_e$ as new vertices), which is isometric to G.

Definition 6.4.2. *Two metrized graphs* G_1 *and* G_2 *are* equivalent *iff there exist subdivisions* G'_1 *and* G'_2 *isomorphic as metrized graphs.*

It is clear how to define the notion of a *piecewise-linear* (or PL for short) map from a finite metrized graph to a finite simplicial complex.

One can define in an obvious way the sheaf of sets $MetGraphs_{\mathcal{X}}$ on $\mathcal{X}$ consisting of equivalence classes of locally finite PL-embedded metrized graphs in $\mathcal{X}$.

Definition 6.4.3. *Let* G *be a finite metrized graph,* $x \in G$ *a point and* f *a real-valued continuous function on* G*. We say that* f *is* convex at *the point* $x \in G$ *iff*

- *if* $x \in e$ *is an interior point of an edge, then* $par_e^*(f)$ *is convex in a neighborhood of* $par_e^{-1}(x)$ *in* $(0, l(e))$*,*
- *if* $x \in G$ *is a vertex, then* f *is convex near* x *on each edge adjacent to* x*, and the sum of derivatives at* x *of* f *along edges adjacent to* x *is non-negative.*

In this definition we used the fact that any continuos convex function f on a *semi-closed* interval $[0, l) \subset \mathbb{R}$ has a well-defined derivative $f'(0) \in \mathbb{R} \sqcup \{-\infty\}$. We denote by $Conv_G$ the sheaf of convex functions on G.

Lemma 6.4.4. *The sheaf* $Conv_G$ *does not change after a subdivision of* G*.*

Proof. It follows from the obvious gluing property of convex functions on $\mathbb{R}$: a function f on $(-a, b) \subset \mathbb{R}$, $a, b > 0$ is convex iff its restrictions to $(-a, 0]$ and $[0, b)$ are convex, and if the sum of the left and the right derivative at 0 is non-negative. □

This lemma shows that $Conv_G$ is the same for equivalent metrized graphs.

Lemma 6.4.5. *If* G *is a finite graph then the sheaf* $Conv_G$ *is closed under convergence on compacts.*

The proof is easy and is left to the reader as an exercise.

For any algebraic curve (possibly singular and not reduced) $C \in X$ and for any snc model $\mathcal{X}$ we will show that the image of C

$$pr_{\mathcal{X}}(C^{\rm an}) \subset S_{\mathcal{X}}$$

is a global section of $MetGraphs_{\mathcal{X}}$.

??????????

Definition 6.4.6. *The sheaf $QCurves_{\mathcal{X}}$ of* quasi-curves *is the subsheaf of $MetGraphs_{\mathcal{X}}$ consisting of sections whose germs at every point are germs of graphs associated with algebraic curves in X.*

Definition 6.4.7. *A germ f of a continuous function at $x \in \mathcal{X}$ is* convex *at x iff its restriction to any germ $G \in (QCurves_{\mathcal{X}})_x$ is convex at x.*

Proposition 6.4.8. *A germ f_x of a function of finite type on X^{an} at at a point x is called* convex *iff there exist an snc model $\mathcal{X}$, a neighborhood V containing $pr_{\mathcal{X}}(x)$ and a continuous function $f \in C^0(V)$ such that*

- *the germ of $pr_{\mathcal{X}}^*(f)$ at x is f_x and*
- *f is a uniform limit of elements of $Conv_{\mathcal{X}}^{\mathbb{R}PL}(V)$.*

Proposition 6.4.9. *(Maximum Principle) A germ of a convex function at any local maximum is locally constant.*

Proof. It is enough to check for finite metrized graphs. ??? □

sect:lb

6.5. **Line bundles and their metrizations.** We claim that sheaf $Lin_X^{\mathbb{R}}$ is an analog of the sheaf $|O^\times|$ in the archimedean situation.

Theorem 6.5.1. *For any x $inX^{\rm an}$ and any germ f of a section of $Lin_{\mathcal{X}}^{\mathbb{R}}$ there exist a neighborhood U of x, a sequence of sections $\phi_i \in \Gamma(U, \mathcal{O}^\times)$ and sequence $\lambda_i \in \mathbb{R}$ of real numbers such that f is the uniform limit of $\lambda_i \log(|\phi_i|)$.*

Proof. ???

□

Also, for any model $\mathcal{X}$ the sheaf $Lin_{\mathcal{X}}^{\mathbb{R}}$ is closed under convergence on compacts. Thus, the sheaf $Lin_{\mathcal{X}}^{\mathbb{R}}$ is a satisfactory analog of $|\mathcal{O}^\times|$, at least when one restricts to functions of finite type.

Definition 6.5.2. A virtual line bundle *on $\mathcal{X}^{an}$ is a torsor over the sheaf $Lin_X^{\mathbb{R}}$.*

Using the morphism

$$\log|\cdot| \,:\, \mathcal{O} \to Lin_X^{\mathbb{R}}$$

we associate a virtual line bundle with an arbitrary line bundle on X.

Analogously, for any model $\mathcal{X}$ we define a virtual line bundle on $\mathcal{X}$ as a torsor over $Lin_{\mathcal{X}}^{\mathbb{R}}$.

Definition 6.5.3. *TO CORRECT !!! A K-analytic metrization of L is a homogeneous of degree 1 function*

$$\varphi : \hat{L}^{an} \to \mathbb{R}_{\geq 0}.$$

A metrization is convex (resp. simple, continuous etc) if φ is such.

Remark 6.5.4. An K-analytic metrization on L^{an} defines compatible metrizations of $L_{K'}$ on $X_{K'}$ for all finite extensions K'/K.

Conjecture 6.5.5 (Kodaira)**.** A line bundle L on X is nef iff there exists a PSH-metrization of L^{an} on X^{an}.

IT IS NOT TRUE IN COMPLEX CASE. IN OUR SITUATION IT SEEMS TO BE EASIER TO PROVE USING COMPACTNESS.

Theorem 6.5.6. *For any given virtual line bundle $\mathcal{L}$ on a model $\mathcal{X}$ the quotient space of the space of convex metrizations of $\mathcal{L}$ modulo additive costants, is compact.*

Proof. We will use the following lemma

Lemma 6.5.7. *The statement of the theorem holds if one replaces $\mathcal{X}$ by a finite connected metrized graph.*

???

This lemma implies that for any two points x, y in $\mathcal{X}$ belonging to a connected pseudo-curve in $\mathcal{X}$ the difference (???) between metrizations of $\mathcal{L}$ at x and y is uniformly bounded over all convex metrizations of $\mathcal{L}$. Then we should use the existence of a large family of pseudo-curves on $\mathcal{X}$. ???? □

7. Non-archimedean Calabi conjecture

7.1. Non-archimedean Monge-Ampére-operator.

Definition 7.1.1. *For a simple function f on an snc model $\mathcal{X}$ we define its* Monge-Ampére charge *$MA(f)$ on $S_{\mathcal{X}}$ as follows:*

$$MA(f) = \sum_{i \in I_{\mathcal{X}}} w_i \times \delta_i,$$

where δ_i is the singleton measure at point

$$\mathrm{pr}_{C_{\mathcal{X}} \twoheadrightarrow S_{\mathcal{X}}}(\langle D_i \rangle)$$

and the weight $w_i \in \mathbb{Z}$ is given by

$$w_i := d_i \sum_{i_1,\dots,i_n \in I_{\mathcal{X}}} \#([D_i] \cdot [D_{i_1}] \cdot \dots \cdot [D_{i_n}]) \times \prod_{\alpha=1}^{n} f(\langle D_{i_\alpha} \rangle).$$

Theorem 7.1.2. *The non-linear operator MA has the following properties:*

(1) *$MA(f)$ is a homogeneous of degree n polynomial in f (with values in charges on $S_{\mathcal{X}}$);*
(2) *the integral of $MA(f)$ vanishes, $\int_{S_{\mathcal{X}}} MA(f) = 0$;*
(3) *the weight of $MA(f)$ at $\mathrm{pr}_{C_{\mathcal{X}} \twoheadrightarrow S_{\mathcal{X}}}(\langle D_i \rangle)$ does not change under shifts of the argument $f \mapsto f + \ell$, where ℓ is a simple function which is* linear *at the point $\langle D_i \rangle$;*
(4) *the weight of $MA(f)$ at $\mathrm{pr}_{C_{\mathcal{X}} \twoheadrightarrow S_{\mathcal{X}}}(\langle D_i \rangle)$ is non-negative if f is convex at the point $\langle D_i \rangle$;*
(5) *if $\mathcal{X}' > \mathcal{X}$ is a simple blow-up then*

$$(i_{\mathcal{X},\mathcal{X}'})_*(MA(f)) = MA(p^*_{\mathcal{X}',\mathcal{X}}(f)) \ .$$

Proof. The first statement is obvious from the definition of the operator MA. For the proof of the second statement we write

$$\int_{S_{\mathcal{X}}} MA(f) = \sum_i d_i w_i = \sum_{i,i_1,\dots,i_n} d_i \#(D_i \cdot D_{i_1} \cdot \dots \cdot D_{i_n}) \times \prod_{\alpha=1}^{n} f(\langle D_{i_\alpha} \rangle) =$$

$$= \# \left(\left(\sum_i d_i D_i \right) \cdot \left(\sum_i f(\langle D_i \rangle) D_i \right)^n \right) = 0.$$

To justify the last identity notice that the divisor $\sum_i d_i D_i = \mathcal{X}^\circ$ is the pullback of the closed point $\mathrm{Spec}(k)$ in $\mathrm{Spec}(\mathcal{O}_K)$ and thus it is movable "outside" of itself. Therefore, its intersection with an arbitrary cycle supported at the special fiber $\mathcal{X}^\circ$ vanishes.

For the third statement notice that simple function f is linear at $[D_i]$ iff .... □

Theorem 7.1.3. *The Monge-Ampére -operator $MA(f)$ extends uniquely to a continuous functional from the sheaf of convex functions of finite type to the sheaf of measures.*

Proof. ???? □

Definition 7.1.4. *The* Mabuchi functional *Mab on the space of simple convex metrization of a virtual line budle is ????*

sect:cc

7.2. Calabi conjecture.

thm:ma **Theorem 7.2.1.** *Let $\mathcal{X}$ be an snc model of a smooth projective algebraic variety X and L an ample line bundle on X such that the associated virtual line bundle $\mathcal{L}$ on the analytic space X^{an} is equivalent to the pullback $p_{\mathcal{X}}^*(\mathcal{L}_{\mathcal{X}})$, where $\mathcal{L}_{\mathcal{X}}$ is a virtual line bundle on $\mathcal{X}$. Then the Monge-Ampére -operator on $\mathcal{X}$ establishes a bijection between the set of convex metrizations of $\mathcal{L}_{\mathcal{X}}$ (up to constant factors) and the set of measures on $\mathcal{X}$ with total volume equal to $\langle c_1(L)^{\dim X}, [X]\rangle$.*

Plan of the proof:

- First we want to prove that the image of the Monge-Ampére -operator contains all measures on $S_{\mathcal{X}}$ with total volume as above. By convexity of the space of metrizations and of the Monge-Ampére -operator, the image is convex. Thus it is enough to show that the image contains singleton measures supported at x, for every point $x \in S_{\mathcal{X}}$.
- For every point $x \in S_{\mathcal{X}}$ and every metrization ϕ_x of the fiber L_x of L in x we define a canonical convex metrization $\|\cdot\|_{\phi_x}$ of $\mathcal{L}_{\mathcal{X}}$ as the maximum over the set of all convex metrizations of $\mathcal{L}_{\mathcal{X}}$ such that the metrization ϕ restricted to L_x is $\leq \|\cdot\|_{\phi_x}$.
- We claim that the Monge-Ampére -measure μ_x associated with $\|\cdot\|_{\phi_x}$ is supported at x. Thus it is a singleton measure.
- Proof by contradiction: suppose that there exists a closed subset $Z \subset S_{\mathcal{X}}$, with $x \notin Z$, such that the integral of μ_x over Z is strictly positive. Then we prove that, for large n, there exist a section $s_0 \in \Gamma(X, L^{\otimes n})$ and a point $z \in Z$ such that

$$\|s_0\|_{\phi_x}(z) < 1 \text{ and } \|s_0\|_{\phi_x}(z) > 1.$$

- This section s_0 allows us to modify the norm:

$$\|s\|_{new}(y) := \min(\|s\|_{\phi_x}(y), |s^n(y)/s_0(y)|^{1/n})$$

  for all all (local) sections s over $y \in S_{\mathcal{X}}$, where $s_0(y) \neq 0$.
- The norm $\|\cdot\|_{new}$ is strictly larger than the old norm $\|\cdot\|_{\phi_x}$ and also satisfies the upper bound at x. This contradicts the maximality of $\|\cdot\|_{\phi_x}$.
- In order to prove the existence of the section s_0 we study the asymptotic distribution of possible values of norms of sections of $L^{\otimes n}$, for large n. We introduce a continuous functional $\mathcal{M}$ on the space of all continuous metrizations of $\mathcal{L}_{\mathcal{X}}$, defined up to an additive constant. To fix the

ambiguity we choose any metrization ϕ_0. Define

$$\mathcal{M}(\phi) := \lim_{n\to\infty} n^{-(\dim X+1)} \cdot \log(\mathrm{vol}_n(\phi)),$$

where $\mathrm{vol}_n(\phi)$ is the volume of the ball

$$\{s \in\in \Gamma(X, L^{\otimes n}) \,|\, \|s\|_\phi(x) \leq 1,\, \forall x \in S_{\mathcal{X}}\}$$

and the Lebesgues measure on $\Gamma(X, L^{\otimes n})$ is normalized by the condition that $\mathrm{vol}_n(\phi_0) = 1$.

- The functional is well defined (the limit exists). We define unconditionally two analogous functionals using lim sup and lim inf. We prove that the restrictions of these functionals to piecewise linear metrizations coincide, that both are monotone and that the multiplication of the metrization by a positive constant leads to an additive shift by the logarithm of this constant. We can uniformly approximate every metrization by a piecewise linear metrization. This implies the coincidence of upper and lower limits.
- Let ϕ be convex and f an arbitrary continuous real-valued function. Then there exists a constant $c > 0$, depending on L and $\mathcal{X}$ only, such that

$$\mathcal{M}(\phi\cdot e^f) = \mathcal{M}(\phi) + \int_{S_{\mathcal{X}}} f\cdot MA(\phi) + \lambda\cdot max_{x\in S_{\mathcal{X}}} f(x)^2,$$

  where $|\lambda| \leq c$. This lemma implies the existence of s_0.
- We need to write an explicit formula for this functional in terms of certain dual polyhedra.
- Uniqueness: if $\mathcal{M}(\phi_1) = \mathcal{M}(\phi_2)$ for two convex metrizations ϕ_1 and ϕ_2 then they are proportional.

ex:p11 *Example* 7.2.2. We explain in detail the case of $X = \mathbb{P}^1$. Let $K = k((t))$ and $\mathcal{X}$ be the blow-up of $\mathbb{P}^1 \times \mathrm{Spec}(k[[t]])$ at the point $(\infty, 0)$. The associtated polytope can be identified with the interval $[0, 1]$. Let $u = \log(\phi)$, where ϕ is a metrization of $\mathcal{O}(1)$ on X, compatible with $\mathcal{X}$. In the standard trivialization of $\mathcal{O}(1)$ over $\mathbb{P}^1 \setminus \{\infty\}$ the function u is just a continuous function on $[0, 1]$. Convex metrizations ϕ correspond to u such that

- u is convex,
- the right derivative of u at 0 is ≥ 0;
- the left derivative of u at 1 is ≤ 1.

We have

$$\mathcal{M}(u) = \int_0^1 \min_{x\in[0,1]} (u(x) - xy)dy.$$

Proof. □

8. Real cohomology

9. Motivic integration

9.1. **Tamagawa measures.** Let X be a smooth n-dimensional projective variety over a local field K and that $\mathcal{X}$ an snc model of X over $\mathcal{O}_K$. Assume that $\Omega^n_{\mathcal{X}}$ is trivial and fix $\Omega \in H^0(\mathcal{X}, \Omega_{\mathcal{X}})$. Introduce local (analytic) coordinates around $D_j \in \mathcal{X}^\circ$ such that D_j is given by $x_j = 0$ and write (locally)

$$\Omega = x_j^{a_j} f,$$

where f is an invertible function. Put

$$a := \min(a_j),$$

call the divisors D_j with $a_j = a$ *essential* and denote by $S^{\mathrm{e}}_{\mathcal{X}}$ the Clemens polytope corresponding to the essential divisors in $\mathcal{X}^\circ$.

Proposition 9.1.1. *The essential Clemens polytope $S^{\mathrm{e}}_{\mathcal{X}}$ is independent of the choice of $\mathcal{X}$.*

Proof. We use the notations of Section 5.2. There is a map

$$\mathrm{vol}(\Omega_{\mathcal{X}}, \cdot) : \mathcal{P}_{\mathrm{abs}}(X) \to \mathbb{Q},$$

defined as follows: For a finite extension K'/K and a $D \in \mathcal{P}(X, K')$ we put

$$\mathrm{vol}(\Omega_{\mathcal{X}}, D) := \frac{1 + (\text{order of } \Omega_{\mathcal{X}\otimes K'} \text{ at } D)}{\mathbf{e}},$$

where $\mathbf{e}$ is the ramification index of K'/K. We claim that $S^{\mathrm{e}}_{\mathcal{X}}$ is the closure in X^{an} of all those $D \in \mathcal{P}_{\mathrm{abs}}(X)$ where $\mathrm{vol}(\Omega_{\mathcal{X}}, D)$ achieves its minimal value. This shows the claimed independence. □

9.2. **Skeleta.**

Conjecture 9.2.1. Assume that ω_X is nef and that ω_X^{an} is equipped with a convex metrization φ. Then

- the associated measure μ is supported on a finite connected *CW*-complex;
- the homotopy type of this complex does not depend on φ;
- this complex is homotopy equivalent to X^{an};
- if X is a Calabi-Yau variety then the canonical measure μ is supported on the "canonical" skeleton.

Theorem 9.2.2 (Construction)**.** *Let K be a local field and X a smooth algebraic variety over K. Assume that ω_X is nef. Then ω_X^{an} has a canonical metrization.*

Proof. The canonical metrization is the maximal PSH-metrization such that μ has no poles. $\square$

Thus, for every finite extension K'/K we have a canonical measure $|\Omega_{K'}|$ on $X(K')$. Moreover,

$$\int_{X(K')} |\Omega_{K'}| = c\mathbf{e}^{n_1} q^{\mathbf{f}n_2}(1 + o(1))$$

where $n_1, n_2 \in \mathbb{N}$ and $\mathbf{e}$ (resp. $\mathbf{f}$) is the ramification index (resp. residual degree). We have $0 \le n_1 \le \dim X$. We will say that X has maximal degeneration if $n_1 = \dim X$.

9.3. **Calabi-Yau varieties.**

9.4. **Varieties of general type.** Let $\{\varphi\}$ be a constructible metrization of ω_X. For any $x \in \mathcal{D}_{\mathrm{abs}}(X)$ we define $\mathrm{val}_x(\varphi) \in \mathbb{Q}$ as follows:

10. Constructions

Let K'/K be a finite (maximally ramified) extension with $[K' : K] \le \infty$.

Let $\mathcal{L}$ be a very ample line bundle on X and

$$f_{\mathcal{L}} : X \hookrightarrow \mathbb{P}(V)$$

the corresponding embedding. A choice of an $\mathcal{O}_F$ sublattice $V_o \subset V$ defines a metrization $\|\cdot\|$ of $\mathcal{L}$ (up to a constant multiple?). Such a metric will be called an algebraic Kähler metric. It defines:

- a metrization $\|\cdot\|_{\mathcal{L},\omega_X}$ of the canonical line bundle ω_X of X;
- a measure on $X(K')$.

By the standard construction, we obtain metrizations $\|\cdot\|_{\mathcal{L}^n}$ of $\mathcal{L}^{\otimes n}$ for all $n \in \mathbb{N}$.

The canonical class on a Calabi-Yau variety carries a canonical metrization $\|\cdot\|^{\mathrm{can}}_{\omega_X}$ (use the isomorphism $\mathcal{O}_X = \omega_X$).

Conjecture 10.0.1. Let X be a Calabi-Yau variety and $\mathcal{L}$ a very ample line bundle on X. Then

$$\lim_{n\to\infty} \|\cdot\|_{\mathcal{L}^n,\omega_X} \to \|\cdot\|^{\mathrm{can}}_{\omega_X}.$$

Definition 10.0.2. *Let X be a Calabi-Yau variety and $\mathcal{L}$ a very ample line bundle on X. We define the canonical metric $\|\cdot\|^{\mathrm{can}}_{\mathcal{L}}$ on $\mathcal{L}$ by*

$$\|\cdot\|^{\mathrm{can}}_{\mathcal{L}} := \lim_{n\to\infty} \|\cdot\|^{1/n}_{\mathcal{L}^n}.$$

11. Examples

Part III. Common grounds

12. Idempotent algebra

sectidemp

12.1. Idempotent semifields.

Definition 12.1.1. *A* semifield *is a set F together with operations* $(+, \times, :)$ *and an element $1_F \in F$ satisfying the following axioms:*

- $(F, \times, :, 1)$ *is an abelian group (in the multiplicative notation)*
- $(F, +)$ *is an abelian semigroup (in the additive notation)*
- *the multiplication is distributive with respect to addition,*

$$(a + a') \times a'' = (a \times a'') + (a' \times a'')$$

for all a, a', a''.

The basic example of a semifield is $\mathbb{Q}_{>0}$ with the usual arithmetic operations. An important difference between semifields and usual fields is that the category of semifields is a category of algebras over a set-theoretic operad. In particular, there are *free* semifields, and the category of semifields admits limits and colimits.

Definition 12.1.2. *An* idempotent semifield *is a semifield F such that the addition is idempotent:*

$$a + a = a, \ \ \forall a \in F.$$

We define a partial order in any idempotent semifield by declaring that $a \leq a'$ iff $a + a' = a'$. Any nonempty finite subset $S \subset F$ has a minimal upper bound $\sup(S)$ and the addition can be calculated via the partial order as

$$a + a' = \sup\left(\{a\} \cup \{a'\}\right).$$

The simplest idempotent semifield is the set $\mathbb{Z}$ with multiplication/division equal to the usual addition/subtraction, and the addition equal to the usual maximum. More generally, any completely ordered abelian group carries a canonical structure of a semifield. We will consider only idempotent semifields and often use the logarithmic notations for the operations, i.e. $(\max, +, -)$ and 0 instead of $(+, \times, :)$ and 1.

Theorem 12.1.3. *For $N \in \mathbb{Z}_{\geq 0}$ let ISF_N be the set of all those homogeneous of degree $+1$ continuous piecewise-linear real-valued functions on $\mathbb{R}^N$ which take integer values at integer points. Endow ISF_N with point-wise operations* $(\max, +, -)$. *Then ISF_N is a free idempotent semifield whose N generators are the standard coordinate functions $x_1, \ldots, x_N$ on $\mathbb{R}^N$).*

If F is a quotient of ISF_N by a finitely generated ideal then $\mathrm{Hom}(F,\mathbb{R})$ is a closed conical subset of $\mathbb{R}^N = \mathrm{Hom}(ISF_N,\mathbb{R})$ which is a finite union of rational simplicial cones. Conversely, any such conical subset corresponds to a finitely generated ideal in ISF_N. The set $\mathrm{Hom}(F,\mathbb{Z})$ is the set of integer points in the conical set $\mathrm{Hom}(F,\mathbb{R})$.

We introduce formally the category of $\mathbb{Z}$-PL *cones* as the opposite category to the category of idempotent semifields. The $\mathbb{Z}$-PL cone corresponding to an idempotent semifield F is denoted by $\mathrm{Spec}(F)$. Conversely, if C is a $\mathbb{Z}$-PL cone then the corresponding semifield is denoted as $\mathcal{O}(C)$. For a $\mathbb{Z}$-PL cone C and an idempotent semifield F denote by $C(F)$ the set of homomorphisms $\mathrm{Hom}(\mathcal{O}(C),F)$ and call it the set of F-points of C. Naturally, these notations are designed to make apparent the analogy with the usual algebraic geometry. The following simple result eliminates the need for gluing:

thm:glue

Theorem 12.1.4. *The functor*

$$C \mapsto (C(\mathbb{R}) \setminus \{0\})/\mathbb{R}_+^\times$$

from the category of $\mathbb{Z}$-PL *cones to the category of sets preserves all finite limits and colimits.*

The "true" algebraic geometry will start in the next subsection.

sect:idemp

12.2. **Idempotent semirings.**

Definition 12.2.1. *A* semiring *is a set R endowed with operations* $(+,\times)$ *and elements* $0_R \in R$, $1_R \in R$ *satisfying the following axioms:*

- $(R,\times,1_R)$ *is an abelian monoid (in the multiplicative notation);*
- $(R,+,0_R)$ *is an abelian monoid (in the additive notation);*
- *multiplication is distributive with respect to addition,*
$$(r+r')\times r'' = (r\times r'') + (r'\times r'');$$
-
$$0_R \times r = r$$
for all $r, r', r'' \in R$.

Notice that in our definitions a semifield is *not* a semiring because it has no zero. Nevertheless, after adding to a semifield an additional element $-\infty$ (or 0 in the multiplicative notations) we obtain a semiring.

Definition 12.2.2. *An* idempotent semiring *is a semiring R with an idempotent addition.*

The basic example of an idempotent semiring is $\mathbb{Z}\cup\{-\infty\}$. Usually we will use the logarithmic notations $\max, +, 0, -\infty$ for idempotent semirings.

The free idempotent semiring R_N with N generators is ????????????

12.3. **Idempotent semischemes.**

sect:unive

12.4. Universal valuations. Let R be a commutative ring and V an idempotent semiring. In the next definition we will write the operations in V in the multiplicative form.

Definition 12.4.1. *A* multiplicative valuation *of R with values in V is a map* val $: R \to V$ *such that*

- $\mathrm{val}(0_R) = 0_V,\ \mathrm{val}(1_R) = 1_V$
- $\mathrm{val}(rr') = \mathrm{val}(r)\,\mathrm{val}(r')$
- $\mathrm{val}(r + r') \leq \mathrm{val}(r) + \mathrm{val}(r')$ *$(= \max(val(r), \mathrm{val}(r')))$*

Notice that the last condition can rewritten as

$$\mathrm{val}(r + r') + \mathrm{val}(r) + \mathrm{val}(r') = \mathrm{val}(r) + \mathrm{val}(r').$$

It is easy to see that there exists a universal idempotent semiring $U(R)$ and a valuation uval $: R \to U(R)$ such that for any idempotent semiring V any valuation of R with values in V is induced from uval by a unique homomorphism $U(R) \to V$. The semiring $U(R)$ is huge; it can be described explicitly as the idempotent semiring of all finitely generated additive submonoids in $(R, +, 1_R)$.

12.5. **Semischeme structure on the Berkovich spectrum.**

References

abramovich [1] D. Abramovich, K. Karu, K. Matsuki, J. Wlodarczyk, *Torification and Factorization of Birational Maps*, `alg-geom 0004135`.

bedford-taylor [2] E. Bedford, B. A. Taylor, *Plurisubharmonic functions with logarithmic singularities*, Ann. Inst. Fourier (Grenoble) 38, (1988), no. 4, 133–171.

bedford-taylor2 [3] E. Bedford, B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Mathematica 149, (1982), 1–41.

berk1 [4] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, 33, AMS, (1990)

berk2 [5] V. G. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. No. 78, (1993), 5–161.

berk3 [6] V. G. Berkovich, *p-adic analytic spaces*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), 141–151.

berk4 [7] V. G. Berkovich, *On the comparison theorem for étale cohomology of non-Archimedean analytic spaces*, Israel J. Math. 92, (1995), no. 1-3, 45–59.

berk5 [8] V. G. Berkovich, *Smooth p-adic analytic spaces are locally contractible*, Invent. Math. 137 (1999), no. 1, 1–84.

bgs [9] S. Bloch, H. Gillet, C. Soulé, *Non-Archimedean Arakelov theory*, J. Alg. Geom. 4 (1995), no. 3, 427–485.

chang-yau [10] S. Cheng, S-T. Yau, *The real Monge-Ampére equation and affine flat structures*, Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980), 339–370, Science Press, Beijing, 1982.

demailly1 [11] J.-P. Demailly, *Courants positifs et théorie de l'intersection*, Gaz. Math. 53 (1992), 131-158.

demailly2 [12] J.-P.Demailly, *Monge-Ampére -operators, Lelong numbers and intersection theory*, in Complex Analysis and Geometry, Univ. Ser. Math., (1993), 115–193.

gkz [13] I. M. Gelfand, M. Kapranov, A. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

gillet-soule [14] H. Gillet, C. Soulé, *Direct images in non-Archimedean Arakelov theory* Ann. Inst. Fourier (Grenoble) 50 (2000), no. 2, 363–399.

huber [15] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, Friedr. Vieweg, Braunschweig, (1996).

huber2 [16]

m [17] G. Mikhalkin, *Real algebraic curves, the moment map and amoebas*, Ann. of Math. (2) **151** (2000), no. 1, 309–326.

yau [18]

I.H.E.S., Le Bois-Marie, 35, route de Chartres, F-91440, Bures-sur-Yvette, FRANCE

E-mail address: maxim@ihes.fr

Department of Mathematics, Princeton University, Princeton, NJ 08544-1000, U.S.A.

E-mail address: ytschink@math.princeton.edu