Periods

M. Kontsevich

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1 Numbers and their arithmetic properties

NATURAL NUMBERS are

$$\mathbf{N} = \{1, 2, 3, \ldots\}$$

Adding zero and negative numbers, we get INTEGERS:

$$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Adding indecomposable fractions, we get RATIONAL NUMBERS:

$$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z},\ q \in \mathbf{N},\ g.c.d.(p,q) = 1\}$$

Taking limits of sequences of rational numbers, we get REAL NUMBERS. Finally, it is very natural to extend the class of real numbers adding formally symbol "i" whose square is (-1). After that we get COMPLEX NUMBERS:

$$\mathbf{C} = \{x + i \cdot y |\ x, y \in \mathbf{R}\}$$

There are many remarkable advantages coming from the introduction of complex numbers, one of them is the Main Theorem of Algebra (C. F. Gauss):

Any polynomial equation

$$a_0 + a_1 x + \ldots + a_n x^n = 0$$

with complex coefficients has a solution $x \in \mathbf{C}$.

In particular, we can consider the set of all $x \in \mathbf{C}$ such that x satisfies an algebraic equation with *rational* coeffients. In this way we obtain the set of ALGEBRAIC NUMBERS, denoted by $\overline{\mathbf{Q}} \subset \mathbf{C}$. The simplest irrational real algebraic number is $\sqrt{2} = 1.4142135\ldots$ (Pythagoras). Triginometric functions of any rational angle are algebraic numbers:

$$\sin(60^\circ) = \frac{\sqrt{3}}{2},\ \tan(18^\circ) = \sqrt{1 - \frac{2}{\sqrt{5}}},\ \ldots$$

It is an ancient tradition to describe numbers according to their position in the hierarchy

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \overline{\mathbf{Q}} \subset \mathbf{C} \supset \mathbf{R}$$

Numbers which are not algebraic are called TRANSCEDENTAL. There is a huge difference in size between algebraic and transcedental numbers. The set $\overline{\mathbf{Q}}$ of algebraic numbers is *countable* and the set of transcedental numbers is *uncountable*. It means that one can not really describe a "generic" transcedental number using a finite amount of words. A transcedental number contains an infinite information. Also, if we meet a number for which there is no apparent reason to be algebraic, then it is most natural to assume that this number is transcedental.

2 Periods

I am going to define a "new" class of numbers lying between $\overline{\mathbf{Q}}$ and $\mathbf{C}$, called PERIODS, which is still countable and seems to be the next most important class in the hierarchy of numbers according to their arithmetic properties. Periods appear surprisingly often in various formulas in mathematics.

The simplest nonalgebraic period is the number π, the area of the unit circle:

$$\pi = 3.1415926\ldots$$

It is ubiquitous. For example, the volume of 3-dimensional unit ball is $\frac{4}{3}\pi$. Also π appears in formulas for volumes of higher-dimensional balls, spheres, cones, cylinders, ellipsoids etc. Number π is transcedental (F. Lindemann, 1882).

There are two other famous numbers which got special notations:

$$e = 2.7182818\ldots = \sum_{n\geq 0} \frac{1}{n!} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$\text{Euler constant } \gamma = 0.5772156\ldots = \lim_{n\to\infty} \left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log(n)\right)$$

These two numbers (conjecturally) are not periods. It is known only that e is transcedental.

Here is an elementary definition of a period:

Definition 1 *A period is the value of an absolutely convergent multi-variable integral of a rational function with algebraic coefficients, over a domain in* $\mathbf{R}^n$ *given by polynomial inequalities with algebraic coefficients.*

In this definition one can replace the rational function by an algebraic function. For example, number π has following integral representations:

$$\pi = \int\int_{\{x^2+y^2\leq 1\}} dx\,dy$$

$$\pi = 2\int_{-1}^{1} \sqrt{1-x^2}dx$$

$$\pi = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

and also, after the multiplication by the algebraic number $2i$ we get a contour integral

$$2\pi i = \oint \frac{dz}{z}$$

in the complex plane around point $z = 0$.

It is not enough to have number π in addition to algebraic numbers. For example, the perimeter of an ellips with radii a and b is the *elliptic integral*

$$2\int_{-b}^{b} \sqrt{1 + \frac{a^2x^2}{b^4 - b^2x^2}}\, dx$$

and it can not be expressed algebraically using π.

It can be said without much overstretching, that a large part of algebraic geometry is in a hidden form a study of integrals of rational functions of several variables.

I am going to propose the following principle for mathematical practice: *whenever you meet a new number, and decided (or convinced yourself) that it is transcedental, try to figure out whether it is a period!*

3 Further examples of periods

First of all, all algebraic numbers are periods, and logarithms of algebraic numbers are periods, e.g.

$$\log(2) = \int_1^2 \frac{dx}{x}$$

Also, periods form an algebra, i.e. the sum and the product of two periods is a period.

Many infinite sums of elementary expressions are periods. For example,

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{= 3^3} + \ldots = 1.2020569\ldots$$

has the following representation as an integral:

$$\zeta(3) = \int\int\int_{0<x<y<z} \frac{dx\, dy\, dz}{= (1-x)yz}$$

Analogously, all values of the Riemann zeta function

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$$

at integers $s \geq 2$ are periods (and also rational multiples of π^s for even s). Some integrals of transcedental functions are also periods,

$$\int_0^1 \frac{x}{\log \frac{1}{1-x}} dx = \log(2)$$

Values of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

at rational points $t \in \mathbf{Q}$ are closely related to periods:

$$(\Gamma(p/q))^q \in Periods, \;\; p, q \in \mathbf{N}$$

For example, $\Gamma(1/2) = \sqrt{\pi}$. In general, it seems to be no general rule explaining why certain infnite sums or integrals of transcedental functions are periods. Each time one has to invent a new trick to prove that a given transcedental expression is a period.

4 Identities between periods

The main problem for periods with respect to algebraic numbers, is that *a priori* there are many ways to write a complex number as an integral, and it is not clear how to check when two periods given by explicit integrals are equal or different.

For algebraic numbers there is an algorithmic procedure deciding whether two numbers are equal or different. The idea is that two *different* solutions of a given algebraic equation with integer coefficients can not be too close to each other, there exists an explicit bound on the distance between two different roots. Thus, calculating algebraic numbers with sufficient precision we can decide whether two of them are equal or not.

For integrals there are several transformation rules, i.e. ways to prove identities between integrals. I will illustrate these rules in the case of integrals of functions in one variable.

1) Additivity in the integrand and in the domain of integration.

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

2) Change of variables. For example, for one-dimensional integrals, if $y = f(x)$ is an invertible change of variables, then

$$\int_{f(a)}^{f(b)} F(y)dy = \int_a^b F(f(x))f'(x)dx$$

3) Newton-Leibniz formula.

$$\int_a^b f'(x)dx = f(b) - f(a)$$

In the case of multi-dimensional integrals one puts the Jacobian of an invertible change of coordinates in the rule 2), and replaces Newton-Leibniz formula by Stokes formula in the rule 3).

Centuries of experience lead to the following

Conjecture 1 *If a period has two integral representations, then one can pass from one formula to another using only rules 1),2),3) in which all functions and domains of integration are algebraic with coefficients in* $\overline{\mathbf{Q}}$.

In other words, we do not expect any miraculous coincidence of two integrals of algebraic functions which will be not possible to prove using three simple rules. The conjecture from above is one of central conjectures about algebraic independence and transcedental numbers.

5 L-functions and Beilinson conjectures

Let $P_1, \ldots, P_n \in \mathbf{Z}[x_1, \ldots, x_m]$ be a collection of polynomials with integral coefficients in several variables. The system of equations

$$\big(P_1(x_1, \ldots, x_m) = 0, \ldots, P_n(x_1, \ldots, x_m) = 0\big)$$

makes sense in any field, in particularly in any finite field. It is well known that for any natural number $q = p^r$ which is a power of a prime, there exists a unique up to an isomorphism finite field $\mathbf{F}_q$ with exactly q elements, an extension of the field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ of residues modulo prime number p.

Denote by $N_q (= N_q(P_1, \ldots, P_n))$ where $q = p^r$ is a power of a prime number, the number of solutions of the system of equations $(P_i = 0)_{1 \le i \le n}$ in $\mathbf{F}_q$. The L-function of the scheme $X/\mathbf{Z}$ given by the system of equation $(P_i = 0)_{1 \le i \le n}$, is defined as

$$L_X(s) := \exp\left(\sum_{p:\, prime} \sum_{r \ge 1} \left(-N_{p^r} \frac{p^{-rs}}{r}\right)\right)$$

for real $s \gg 0$. Nobody knows *why* such an expression should be written and *what* does it mean. The whole history of number theory showed that the most deep mysteries there (and probably in mathematics in

general) are related with L-functions.

The L-function of the empty system of equations in 0 variables (when $N_q = 1$ for all q) is exactly the Riemann zeta-function

$$\zeta(s) = \sum_{n \ge 1} \frac{1}{n^s} = \prod_{p:\, prime} \left(1 - \frac{1}{p^s}\right) = \exp\left(-\sum_{p:\, prime} \sum_{r \ge 1} \frac{p^{-rs}}{r}\right)$$

It is conjectured that L-function of any system of equations has an analytic continuation to a meromorphic function on the whole complex plane $\mathbf{C}$. Also, all zeroes and poles of any L-function should lie on vertical lines $s \in \frac{1}{2}\mathbf{Z} + i \cdot \mathbf{R}$ (generalized Riemann conjecture).

Values of the Riemann zeta-function at integers $s > 1$ are in fact periods, e.g. we showed already the integral formula for $\zeta(3)$. Also, values of ζ at integers $s \le 0$ are rational numbers (e.g. $\zeta(0) = -1/2,\ \zeta(-1) = -1/12$, etc), and at $s = 1$ function $\zeta(s)$ has a simple pole with residue 1.

A. Beilinson (and partially P. Deligne) proposed at the beginning of 80-ies a vast generalization of previous conjectures in number theory concerning values of L-functions. Beilinson gave formula for the first non-zero coeficient of any L-function at $s = s_0 \in \mathbf{Z}$. It was not clear *a priori* that his formula gives a period, but later A. Scholl proposed a reformulation in which it was explicitly expressed as a period.

Conjecture 2 *For any L-function in number theory, for any $n \in \mathbf{Z}$, the leading coefficent c of the Laurent expansion of $L(s)$ at $s = n$*

$$L(s) = c \cdot (s-n)^k + o((s-n)^k), \;\; c \in \mathbf{C} \setminus \{0\}, \;\; k \in \mathbf{Z}$$

is a period.

6 L-functions of modular forms

The next most studied case after the Riemann zeta function (and its generalizations, such as Dedekind zeta-functions and Artin L-functions) is the case of L-functions of automorphic forms for congruence subgroups of the group $SL(2, \mathbf{Z})$. Recall that the modular form of weight $k \in \mathbf{Z}$ and level $N \in \mathbf{N}$ is a holomorphic function on the upper-half plane $\{\tau \mid \Im(\tau) > 0\}$:

$$f(\tau) = \sum_{n \geq 0} a(n) q^n, \;\; q := \exp(2\pi i \tau)$$

satisfying the transformation property

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for $a, b, c, d \in \mathbf{Z}$, $ad - bc = 1$, $a \equiv 1, c \equiv 0, d \equiv 1 (mod\ N)$. The L-function associated with modular form f is

$$L_f(s) := \sum_{n \geq 1} \frac{a(n)}{n^s}$$

I will not remind here the definition of automorphic forms, only mention that the L-function of any elliptic curve defined over $\mathbf{Q}$ is the L-function of an automorphic form of weight 2.

Modular forms can be associated with integral lattices. If

$$Q(x_1, \ldots, x_d) = \sum_{1 \leq i,j, \leq d} a_{ij} x_i = x_j \in \mathbf{Z}[x_1, \ldots, x_d]$$

is a positive definite quadratic form with integer coefficients, then the so called Epstein zeta function

$$Z_Q(s) := \sum_{(x_1, \ldots, x_d) \in \mathbf{Z}^d \setminus \{0\}} Q(x_1, \ldots, x_d)^{-s}$$

is convergent for $\Re(s) > d/2$, and has an analytic continuation to a meromorphic function on $\mathbf{C}$. It is known that if number d, the rank of form Q, is *even*, then $Z_Q(s)$

is a finite linear combination with algebraic coefficents of L-functions associated with modular forms of weight $d/2$. Thus, Beilinson conjectures imply that $Z(s)$ for all integer $s > d/2$ are periods. In fact, it is an established result:

Theorem 1 *If $f = \sum_n a(n)q^n$ is a modular form of weight k, then values $L_f(s)$ at integers $s > k/2$ are periods.*

It seems that this basic fact is unknown to many leading specialists in number theory! This theorem was proven by Beilinson for the case of forms of weight $k = 2$, and was announced already many years ago by A. Scholl for general $k \geq 2$. The proof uses a well-known trick (Rankin method) relating $L_f(s)$ with certain integral of a transcedental function which does not look as a period. Nevertheless, using cohomological interpretations, one can show that this integral is a period.

Corollary 1 *For positive definite integral quadratic form Q in $d \in 2\mathbf{Z}$ variables, the value of the Epstein zeta function $Z_Q(s)$ at any integer $s > d/2$ is a period.*

It is not clear to me whether the analogous statement holds for quadratic forms in *odd* number of variables.

I would like to mention a much more elementary result:

Theorem 2 . *If $f = \sum_n a(n)q^n$ is a modular form with algebraic coefficients, then values of f at $\tau \in \mathbf{Q} + i \cdot \sqrt{\mathbf{Q}^\times}$ (at Heegner points), are periods.*

7 Birch and Swinnerton-Dyer conjecture

This conjecture is a predecessor of Beilinson conjectures, and it was discovered in computer experiments in 60-ies. The Birch and Swinnerton-Dyer conjecture gives a formula for the order of zero and of the leading Taylor coefficient at $s = 1$ of L-function of any automorphic form of weight 2 with integral coefficients. By famous Taniyama-Weil conjecture (essentially proven now by A. Wiles) these L-functions are exactly L-functions of all elliptic curves defined over $\mathbf{Q}$:

$$y^2 = x^3 + ax + b, \ \ a, b \in \mathbf{Q}$$

Rational points on an elliptic curve form a finitely generated abelian group. Let $p_1, \ldots, p_r$ be generators (modulo torsion) of $E(\mathbf{Q})$, $r = rank(E(\mathbf{Q}) \otimes \mathbf{Q})$. The leading term in the Taylor expansion of $L_E(s)$ at $s = 1$ is

$$(s-1)^r \cdot \det((p_i, p_j)_{1 \leq i,j \leq r}) \cdot \int_{E(\mathbf{R})} \frac{dx}{y} \cdot \text{rational number}$$

where $(p_i, p_j) \in \mathbf{R}$ is so called height pairing between points p_i and p_j. There are some standard formulas for the height pairing in terms of logarithms of a theta-function associated with elliptic curve E. What many people missed is that there is another formula for the height pairing in terms of periods. This formula looks as

$$(p_i, p_j) = \text{period} - \frac{\text{another period}}{\int_{E(\mathbf{R})} \frac{dx}{y}}$$

Periods in this formula are integrals over arcs in $E(\mathbf{R})$ of certain 1-forms with poles. It is not a polynomial in periods, but remarkably the total product in the BSD conjecture is again a polynomial of periods (thus it is itself a period), being a determinant of certain $(r+1)\times(r+1)$ matrix whose entries are periods (S. Bloch). The way to write the formula in the BSD conjecture was somehow misleading.

Now the BSD conjecture can be checked in two cases, when the order of zero of $L_E(s)$ at $s=1$ is either 0 or 1 (D. Gross and D. Zagier). Moreover, it is easy to see that the value $L_E(1)$ is essentially by definition a period (covering the case $r=0$), and that in the case of the vanishing of odd order at $s=1$, the first derivative $L'_E(1)$ can be reduced to a period using a trick (D. Goldfield). Thus, in cases $r=0,1$ the BSD conjecture reduces to an identity between to periods, which in principle should be provable by elementary rules of calculus (Conjecture 1). The proof of Gross and Zagier does not go along these lines, it uses many transcedental functions in intermediate calculations.

In the case $r\geq 2$ (e.g. for the elliptic curve $y^2=x^3+73=x$), there is *no* example in which we can prove the identity predicted by the BSD conjecture. The problem is that the natural integral representation for higher derivatives

$$\left(\frac{\partial}{\partial s}\right)^r L(s)_{|s=1}$$

can not be reduced using all tricks known today to an integral of an algebraic function. It is a great challenge for number theorists to pass the barrier $r\leq 1$.

8 Colmez conjecture

In general, one does not expect any interesting number-theoretic property for subleading coeffcients in the Laurent expansion of $L(s)$ at integer point $s=s_0\in\mathbf{Z}$. Still, there are remarakble exceptions. For example,

$$\zeta(s)=-\frac{1}{2}+\log\left(\frac{1}{\sqrt{2\pi}}\right)\cdot s+O(s^2),\ \ s\to 0$$

or, in a more suggestive form,

$$\log(\zeta(s))=\log(-1/2)+s\cdot\log(2\pi)+O(s^2)$$

Recently P. Colmez proposed a conjecture generalizing this identity.

Conjecture 3 *Let*

$$\rho: Gal(\overline{\mathbf{Q}}/\mathbf{Q})\to GL(n,\overline{\mathbf{Q}})$$

be a representation of the absolute Galois group, such that

$$\rho(\text{complex conjugation})=-\mathbf{1}_{n\times n}$$

Then the logarithmic derivative of the Artin L-function $L_\rho(s)$ at $s=0$ (and also at $s=1$ by the functional equation) is a finite linear combination with coefficients in $\overline{\mathbf{Q}}$ of logarithms of periods of abelian varieties with complex multiplication.

Periods P whose logarithms appear in Colmez conjecture are exactly such that for sufficiently large integral power of π the number

$$\frac{\pi^k}{P}$$

is again a period.

If $K_2 \supset K_1$ are two algebraic number fields in $\overline{\mathbf{Q}}$ such that

$$K_1 \otimes \mathbf{R} \simeq \mathbf{R}^n, \;\; K_2 \otimes \mathbf{R} \simeq \mathbf{C}^n$$

then the ratio of Dedekind zeta-functions $\zeta_{K_2}(s)/\zeta_{K_1}(s)$ is an L-function of the type considered in Colmez conjecture.

Colmez himself proved his conjecture in the case of abelian representations (when all fields entering the game are cyclotomic fields). In essence, it reduces to well-known identities between values of Gamma function at rational points and periods. It seems that today nobody has ideas how to prove the identity predicted by Colmez conjecture in any case of a nonabelian representation. As far as I know, there were no even numerical checks (using computers) in the nonabelian case as well.