On Malliavin measures, SLE and CFT

Maxim Kontsevich
IHES, 35 Route de Chartres,
91440 Bures-sur-Yvette, France

Yuri Suhov
Statistical Laboratory, DPMMS/CMS,
University of Cambridge, Wilberforce Road
Cambridge, CB3 0WB, UK

Abstract. This paper is motivated by emerging connections between the conformal field theory (CFT) on the one hand and stochastic Löwner evolution (SLE) processes and measures that play the rôle of the Haar measures for the diffeomorphism group of a circle, on the other hand. We attempt to build a framework for widely spread beliefs that SLE-processes would provide a picture of phase separation in a small massive perturbation of the CFT.

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1 Introduction

The main object of study in this paper are certain natural determinant bundles, on spaces of simple Jordan curves in surfaces. We consider two classes of such curves: (i) loops in an open surface Σ and (ii) intervals in a surface Σ with a non-empty boundary $\partial\Sigma$, joining two distinct points $x, y \in \partial\Sigma$. In case (i), we put forward, in chapter 2 of the paper, a conjecture of existence and uniqueness (up to a positive scalar factor) of a one-parameter family of (locally) conformally covariant assignments

$$\Sigma \mapsto \boldsymbol{\lambda}_\Sigma. \tag{1.1}$$

Here $\boldsymbol{\lambda}_\Sigma$ is a measure on the space of loops in Σ (called a Malliavin measure), with values in a given power of a determinant bundle. In case (ii), our aim is to prove a theorem of existence of a one-parameter family of (locally) conformally covariant assignments

$$(\Sigma, x, y) \mapsto \boldsymbol{\lambda}_{\Sigma,x,y}. \tag{1.2}$$

Here $\boldsymbol{\lambda}_{\Sigma,x,y}$ is a measure on the space of intervals in Σ joining distinct points $x \in \partial\Sigma$ and $y \in \partial\Sigma$ (called an SLE-measure), with values in a given product of determinant bundles. To this end, in chapter 3 of the paper we develop some useful geometric techniques. Next, in section 4 we state and the aforementioned existence theorem. The proof is carried in sections 4 and 5 and is based on the so-called restriction covariance property introduced and verified for (scalar) probability measures on intervals, in a special situation where the surface Σ (with a boundary) is a closed disk, $x, y \in \partial\Sigma$ are the endpoints of a diameter, and the measure is generated by the (chordal) SLE_κ process, with

$$0 < \kappa \leq 4, \tag{1.3}$$

Condition (1.4) is necessary and sufficient for an SLE_κ-process to generate simple Jordan curves; a similar condition is introduced in the conjecture in case (i).

In the concluding chapter 6 we discuss possible applications of Malliavin and SLE-measures to the problem of describing probability distributions on phase-separating curves (domain walls) in two-dimensional Gibbs random fields just below the critical temperatures.

The paper contains 6 chapters numbered from 1 to 6. Chapters 2–6 are divided into sections numbered by 2.1, 2.2, and so on. Most of the sections contain subsections labeled by triple numbers: 2.2.1, 2.2.2, and so on. Throughout the paper, symbol $\square$ marks the end of a proof. Symbol $\blacksquare$ is used to mark the end of a definition or a remark.

The results of this paper have been announced in [K2].

2 Malliavin measures

2.1 The space of simple loops

Throughout this paper all surfaces are supposed to be open, paracompact and endowed with a conformal structure. Correspondingly, an embedding of a surface into another surface will mean a conformal embedding. In chapters 2 and 3 the surfaces are assumed open, whereas in chapters 4 and 5 they should have a non-empty boundary. A basic examples of an (oriented) compact surface repeatedly mentioned below is a Riemann sphere; other examples are a torus (also oriented) or a Klein bottle (non-oriented). Basic examples of non-compact surfaces repeatedly mentioned throughout the paper are an open disk, an open annulus and a punctured plane (oriented), or an open Möbius strip (non-oriented).

Speaking about a metric on a surface Σ, we always have in mind a Riemannian metric compatible with the conformal structure.

A standard way of producing a compact surface from a non-compact one is to pass to a Schottki double (or briefly a double). Suppose that Σ is a non-compact surface of finite topological type (i.e. with finite Betti numbers). Then there exists a unique oriented compact surface Σ_{double}, with an orientation-reversing conformal involution σ and an embedding $\eta : \Sigma \hookrightarrow \Sigma_{\text{double}}$, such that

1. $\sigma[\eta(\Sigma)] \cap \eta(\Sigma) = \emptyset$,
2. the complement $\Sigma_{\text{double}} \setminus (\sigma[\eta(\Sigma)] \cup \eta(\Sigma))$ is a disjoint union of finitely many isolated points and closed loops.

Given a surface Σ, we denote by $Comp(\Sigma)$ the space of compact subsets of Σ equipped with the standard topology. $Comp(\Sigma)$ is a locally compact Hausdorff space.

By definition, a simple closed Jordan loop on Σ (or, shortly, a loop) is a compact subset of Σ homeomorphic to S^1. The space of loops on Σ is denoted by $Loop(\Sigma)$; it is a Borel subset of $Comp(\Sigma)$, but not closed and not locally compact.[1] An embedding of surfaces $\beta : \Sigma_1 \hookrightarrow \Sigma_2$ gives rise to an open embedding of corresponding spaces of loops $\beta_* : Loop(\Sigma_1) \hookrightarrow Loop(\Sigma_2)$.

An important special case is where surface Σ is an annulus A. In this case we denote by $Loop^1(A)$ the component of $Loop(A)$ consisting of single-winding loops $\mathcal{L} \subset A$. For a general oriented surface Σ and a loop $\mathcal{L} \in Loop(\Sigma)$ there exists a fundamental system of neighborhoods of $\mathcal{L}$ in $Loop(\Sigma)$ formed by the images of $Loop^1(A)$ under embeddings $A \hookrightarrow \Sigma$ of A in Σ. In the case of a non-oriented surface Σ, a similar role is played by a Möbius strip M.

As was said earlier, our goal in this paper is to study measures on spaces of loops (and intervals) which are not locally compact. In this situation, a natural analog of a sigma-finite measure on a non locally compact space $\mathfrak{X}$ is a *locally finite measure* $\boldsymbol{\mu}$, with the property that every point $x \in \mathfrak{X}$ has a neighborhood $\mathfrak{U}$ of finite measure: $\boldsymbol{\mu}(\mathfrak{U}) < \infty$. More generally, if Λ is a continuous oriented real line bundle on $\mathfrak{X}$, then one can speak of locally finite measures with values in Λ. For any local trivialisation s of Λ around point $x \in \mathfrak{X}$ (i.e., positive section of the dual bundle Λ^*), every such Λ-valued measure $\boldsymbol{\mu}$ gives an ordinary locally finite measure $\boldsymbol{\mu}_s$ on a neighborhood of x. Further, for every two local trivialisations $\mathfrak{s}$ and $\mathfrak{s}'$, the Radon–Nikodym derivative $\dfrac{\mathrm{d}\boldsymbol{\mu}_{\mathfrak{s}}}{\mathrm{d}\boldsymbol{\mu}_{\mathfrak{s}'}}$ is a continuous strictly positive function equal to $\mathfrak{s}/\mathfrak{s}'$ in a neighborhood of x.

[1] In [K2] it was wrongly stated that $Loop(\Sigma)$ is locally compact.

In what follows, speaking of a measure with values in a continuous oriented real line bundle, we always mean a locally finite measure. The same agreement is applied to scalar measures.

Note that space *Loop* (Σ) depends only on the *topological* structure on surface Σ. However, the continuous oriented real line bundle $\left|\mathrm{Det}\right|_\Sigma$ on *Loop* (Σ) which is introduced below depends non-trivially on the choice of the *conformal* structure.

2.2 Determinant lines

2.2.1. Liouville action. Let Σ be a compact surface. Given a pair of metrics, g_1 and g_2, on Σ (compatible with the conformal structure), we define the Liouville action $\mathrm{S}_{\mathrm{Liouv}}(g_1, g_2) \in \mathbb{R}$ by

$$\begin{aligned} \mathrm{S}_{\mathrm{Liouv}}(g_1, g_2) &= \frac{1}{48\pi \mathrm{i}} \int_\Sigma \mathrm{L}_{\mathrm{Liouv}}(g_1, g_2) \\ &:= \frac{1}{48\pi \mathrm{i}} \int_\Sigma (\varphi_1 - \varphi_2) \partial \overline{\partial} (\varphi_1 + \varphi_2). \end{aligned} \tag{2.1}$$

Here we use the representation $g_i = \exp\,(\varphi_i)|\mathrm{d}z|^2$ where z is an arbitrary local complex coordinate on Σ. Equation (2.1) gives a natural definition, as the density $\mathrm{L}_{\mathrm{Liouv}}(g_1, g_2)$ does not depend on the choice of coordinate z; it also shows a 'local character' of the Liouville action, where the integrand depends on the values of functions φ_1, φ_2 and their derivatives at a single point.

The main property of Liouville action is the following well-known cocycle identity:

Lemma 2.1.

$$\mathrm{S}_{\mathrm{Liouv}}(g_1, g_3) = \mathrm{S}_{\mathrm{Liouv}}(g_1, g_2) + \mathrm{S}_{\mathrm{Liouv}}(g_2, g_3)\,. \tag{2.2}$$

Proof: Obviously, $\mathrm{S}_{\mathrm{Liouv}}(g_1, g_2)$ is antisymmetric in g_1, g_2. A straightforward calculation shows that

$$\mathrm{L}_{\mathrm{Liouv}}(g_1, g_2) + \mathrm{L}_{\mathrm{Liouv}}(g_2, g_3) + \mathrm{L}_{\mathrm{Liouv}}(g_3, g_1) = \mathrm{d}\alpha(g_1, g_2, g_3). \tag{2.3}$$

Here

$$\alpha(g_1, g_2, g_3) = \frac{-1}{6} \sum_{1 \le i,j,k \le 3} \epsilon^{ijk} \log \frac{g_i}{g_j} \left(\partial - \overline{\partial}\right) \log \frac{g_j}{g_k}\,, \tag{2.4}$$

where ϵ^{ijk} is the standard fully antisymmetric tensor. $\square$

Form $\alpha(g_1, g_2, g_3)$ introduced in (2.4) also satisfies a useful cocycle identity:

Lemma 2.2. *For any collection of four metrics $(g_i)_{1\le i\le 4}$ on Σ, the following equation holds:*

$$\sum_{i=1}^{4}(-1)^i\alpha(g_1,\ldots,\hat{g}_i,\ldots,g_4) = 0. \tag{2.5}$$

Proof : It is easy to see that if we write $g_i = \exp\,(\phi_i)|\mathrm{d}z|^2$ in local coordinate z, then

$$\alpha(g_1,g_2,g_3) = \frac{-1}{2}\sum_{1\le i,j,k\le 3}\epsilon^{ijk}\phi_i\left(\partial-\overline{\partial}\right)\phi_j\ .$$

Each term in this formula depends only on *two* functions ϕ_i. It is easy to see that it leads to the assertion of Lemma 2.2. $\square$

In what follows we repeatedly use the following straightforward assertion

Lemma 2.3. *The Liouville density* $\mathrm{L}_{\mathrm{Liouv}}(g_1, g_2)$ *vanishes at points where both metrics g_1, g_2 are flat.*

Proof: The curvature of metric $\exp\,(\phi)|\mathrm{d}z|^2$ equals $(-2)\exp\,(-\phi)\dfrac{\partial^2\phi}{\partial z\partial\overline{z}}$. This implies the statement of Lemma 2.3. $\square$

Remark 2.1. In the physical literature (see, e.g., the contributions by K. Gawedzki and E. D'Hoker in [20]), the Liouville action (with the cosmological constant zero) is written as a functional of $\sigma\in\mathrm{C}^\infty(\Sigma)$, depending on a background metric g:

$$\mathrm{S}_{\mathrm{Liouv},g}(\sigma) = \frac{1}{12\pi}\int_\Sigma\left(\frac{1}{2}\left|\mathrm{grad}_g\sigma\right|^2 + R_g\cdot\sigma\right)\mathrm{area}_g. \tag{2.6}$$

However, one can check that the following identity holds:

$$\mathrm{S}_{\mathrm{Liouv},g}(\sigma) = -\mathrm{S}_{\mathrm{Liouv}}(g, e^{2\sigma}g), \tag{2.7}$$

establishing the connection between the two forms of the action. Thus our choice of the local density in Eqn (2.1) differs from that in (2.6) by a total derivative; an advantage being the property stated in Lemma 2.3. ■

2.2.2. Determinant lines for compact surfaces. For a compact surface Σ, we define the determinant line $|\det|_\Sigma$, an oriented one-dimensional vector space over $\mathbb{R}$, as follows. Any smooth metric g on Σ compatible with conformal structure gives a positive point (a base vector) in $|\det|_\Sigma$, denoted by $[g]$. For two such metrics, g_1, g_2, the ratio of corresponding vectors is given by

$$[g_2]/[g_1] := \exp\ \left[\mathrm{S}_{\mathrm{Liouv}}(g_1, g_2)\right]. \tag{2.8}$$

Cocycle identity (2.2) ensures that $|\det|_\Sigma$ is correctly defined. Obviously, for any finite collection $(\Sigma_i)_{i=1,n}$ of compact surfaces we have a canonical isomorphism

$$|\det|_{\sqcup_{i=1}^n \Sigma_i} \simeq \bigotimes_{i=1}^n |\det|_{\Sigma_i}\,. \tag{2.9}$$

For any real c we define the c-th tensor power $(|\det|_\Sigma)^{\otimes c}$ using the homomorphism

$$\lambda \mapsto \lambda^c,\ \ \lambda \in \mathbb{R}^\times_{>0}. \tag{2.10}$$

If Σ is a connected orientable compact surface of genus zero, then there is a canonical vector $v_\Sigma \in |\det|_\Sigma$. Namely, let us choose a conformal isomorphism between Σ and $\mathbb{C}P^1$. Then the round metric on $\mathbb{C}P^1$ (the standard metric on the unit sphere $S^2 \subset \mathbb{R}^3$) gives rise to an element v_Σ of $|\det|_\Sigma$. Vector v_Σ does not depend on the choice of the aforementioned conformal isomorphism because there is no non-trivial homomorphism from the group $\mathbb{Z}_2 \ltimes PSL(2,\mathbb{C})$ of conformal automorphisms of $\mathbb{C}P^1$ to $\mathbb{R}^\times_{>0}$.

2.2.3. Determinant lines for non-compact surfaces. For a non-compact surface Σ of finite type, we define the oriented line $|\det|_\Sigma$ as

$$\left(|\det|_{\Sigma_{\mathrm{double}}}\right)^{\otimes(1/2)}\ . \tag{2.11}$$

We say that Σ is puncture-free if the complement $\Sigma_{\mathrm{double}} \setminus (\sigma[i(\Sigma)] \cup i(\Sigma))$ does not contain isolated points. For a puncture-free surface Σ, we can define line $|\det|_\Sigma$ in an alternative way. Namely, we say metric g is well-behaving at infinity if there exists a relatively compact open subset $U \subset \Sigma$ such that the set $\Sigma \setminus U$ with metric g is isometric to a finite disjoint union $\sqcup_i (0, \epsilon_i] \times S^1$ of semi-open flat cylinders $(0, \epsilon_i] \times S^1$, where ϵ_i is a positive number, and

S^1 is the standard circle of length 2π. Such a metric compatible with the conformal structure exists iff Σ is puncture-free.

The Liouville action $\mathrm{S}_{\mathrm{Liouv}}(g_1, g_2)$ is then alternatively defined as in Eqn (2.1), for any two metrics g_1, g_2 well-behaving at infinity. Moreover, metrics g_1 and g_2 well-behaving at infinity can be extended to σ-invariant metrics $\widetilde{g}_1$ and $\widetilde{g}_2$ on Σ_{double}, with the property that $\mathrm{S}_{\mathrm{Liouv}}(g_1, g_2) = \dfrac{1}{2}\mathrm{S}_{\mathrm{Liouv}}(\widetilde{g}_1, \widetilde{g}_2)$. Therefore, the alternative definition of $|\det|_\Sigma$ coincides with the original one.

2.2.4. Canonical vectors. Recall, for a compact orientable surface Σ of genus 0, we defined a canonical vector v_Σ in $|\det|_\Sigma$ by using a conformal isomorphism of σ and S^2. Now, if Σ is a non-compact puncture-free conformal surface homeomorphic to an open disk, we define a canonical element v_Σ in $|\det|_\Sigma$ by using the fact that Σ_{double} is homeomorphic to a sphere.

Next, if Σ is a puncture-free conformal surface homeomorphic to an annulus or to a Möbius strip, then we define a canonical element v_Σ in $|\det|_\Sigma$ by using the unique flat metric on Σ well-behaving at infinity. Moreover, any multiple of this metric gives the same vector v_Σ, as can be seen from Eqn (2.7). Later on (see Lemma 2.4 below), it will be convenient to use a special *normalised* metric g_Σ^{norm}, in the case where Σ is an annulus. Metric g_Σ^{norm} is simply a mutliple of the flat metric, specified by the condition that the height of Σ is equal to 1. This metric can also be specified as follows. Consider a harmonic function h on Σ which tends to 0 at one component of the boundary of Σ and to 1 at the other component (such h is defined uniquely up to the involution $h \mapsto 1 - h$). The normalised metric is given by

$$g_\Sigma^{\mathrm{can}} = \left| 2\frac{\partial h}{\partial z} \right|^2 |\mathrm{d}z|^2 \tag{2.12}$$

in any local coordinate z.

2.2.5. Neutral collections. In this subsection, we describe a construction of special vectors in tensor products of determinant lines for several compact surfaces with conformal structures. Informally, we will deal with two-dimensional non-Hausdorff "manifolds" where some points are non-separable. A particular example is where such a "manifold" is the Cartesian product $N = S^1 \times T^1$ of a circle S^1 and a "train track" T^1, the quotient of $\mathbb{R}^1 \times \{1, 2\}$ by the equivalence relation

$$(x, 1) \sim (x, 2), \quad x < 0$$

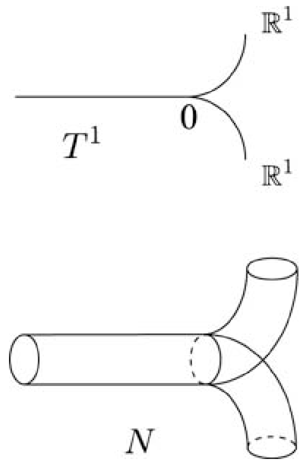

Figure 1: Train track T^1 and its product with S^1

In general, we define an admissible (non-Hausdorff) surface Σ^{nH} as a topological space with countable base, such that any point $x \in \Sigma^{\mathrm{nH}}$ has a neighbourhood homeomorphic to an open disk and such that there exists a neighbourhood of the set of non-separable points homeomorphic to a disjoint union of finitely many copies of space M defined above. Clearly, such surfaces can be endowed with smooth or even conformal structures. We will assume that inseparable points form smooth curves on Σ^{nH}.

Further, consider a finite collection $\Sigma_1, \ldots, \Sigma_n$ of compact surfaces. Assume also that for all $i = 1, \ldots, n$ a 'weight' $m_i \in \mathbb{Z}$ is given. Next, let us fix an admissible surface Σ^{nH} with a conformal structure, and an n-tuple of embeddings $\phi_i : \ \Sigma_i \to \Sigma^{\mathrm{nH}}$.

We call the collection

$$\mathfrak{C} = \big(\{(\Sigma_i, \phi_i, m_i)\}; \Sigma^{\mathrm{nH}}\big)$$

neutral if for any separable point $x \in \Sigma^{\mathrm{nH}}$, the sum of weights of surfaces Σ_i whose images $\phi_i(\Sigma_i)$ contain x equals 0. Given a neutral collection $\mathfrak{C}$, we define a vector $v_{\mathfrak{C}} \in \otimes_i \left(|\det|_{\Sigma_i}\right)^{\otimes m_i}$ by

$$v_{\mathfrak{C}} = \bigotimes_i \left([\phi_i^* g]\right)^{\otimes m_i}, \tag{2.13}$$

where $\phi_i^* g$ is the pullback image of a metric g on Σ^{nH} compatible with the given conformal structure. It follows from the locality of Liouville action that this definition does not depend on the choice of g.[2]

A working example of a neutral collection is where we take the non-Hausdorff surface

$$\mathbb{R} \times \{1,2\} / (x,1) \sim (x,2) \text{ for } x \in (0,1)\,,$$

mulitply it by the unit circle S^1 and 'compactify' the product by adding four 'caps' (closed disks) to four ends; this gives an admissible surface which we will denote by S^{nH}. There are four distinct embeddings of sphere $\mathbb{C}P^1$ in S^{nH}; we deem them ϕ_i, $i = 1,2,3,4$, and denote by S_1, S_2, S_3 and $S_4 \simeq S^2$ the images $\phi_1(\mathbb{C}P^1)$, $\phi_2(\mathbb{C}P^1)$, $\phi_3(\mathbb{C}P^1)$, and $\phi_4(\mathbb{C}P^1)$. Then take any pair of embeddings covering together all four end caps and assign to them multiplicities $+1$. The remaining pair of embeddings gets multiplicities -1. We denote these basic multiplicities by μ_i, $i = 1,2,3,4$. This gives a neutral collection which we will denote by $\mathfrak{F}$.

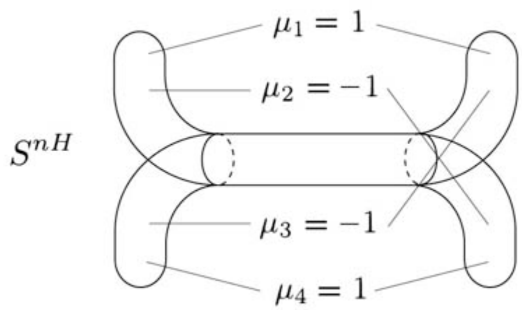

Figure 2: Non-Hausdorff surface S^{nH}

Other working examples are a neutral collection $\mathfrak{E}$ of eight spheres in subsection 2.5.1 and a neutral collection $\mathfrak{S}$ of six spheres in subsection 4.2.3.

A useful fact is as follows. Suppose we have a neutral collection $\mathfrak{C} = \big(\{(\Sigma_i, \phi_i, m_i)\}; \Sigma^{\mathrm{nH}}\big)$, and a continuous map $\psi:\ \Sigma^{\mathrm{nH}} \to \Sigma'^{\mathrm{nH}}$ where Σ'^{nH} is another admissible (non-Hausdorff) surface with a conformal structure, and ψ is locally a conformal homeomorphism. Then the compositions $\phi_i' = \psi \circ \phi_i$

[2]More generally, one can allow maps ϕ_i to be only immersions (local homeomorphisms). In the definition of neutrality one should count each weight w_i with the multiplicity equal to the number of points in $\phi_i^{-1}(x)$.

give a new neutral collection $\mathfrak{C}' = \Big(\{(\Sigma_i, \phi'_i, m_i)\}; \Sigma'^{\,\mathrm{nH}}\Big)$, and vectors $v_{\mathfrak{C}}$ and $v_{\mathfrak{C}'}$ coincide. This follows from the fact that we can choose a metric on Σ^{nH} which is a pullback image of the chosen metric on $\Sigma'^{\,\mathrm{nH}}$.

Informally, it means that we can "move" sets of nonseparable points in a zip-like fashion.

2.3 Determinant bundles on loops

2.3.1. Determinant line for an individual loop. Suppose we are given a surface Σ and a loop $\mathcal{L} \in Loop(\Sigma)$. Next, let $D \subset \Sigma$ be a puncture-free domain that is a surface of a finite topological type containing $\mathcal{L}$ and contractible to $\mathcal{L}$). We define the oriented line $|\det|_{\mathcal{L},\Sigma}$ as the quotient:

$$|\det|_{\mathcal{L},\Sigma} = \frac{|\det|_D}{|\det|_{D\setminus\mathcal{L}}} :\simeq |\det|_D \otimes \Big(|\det|_{D\setminus\mathcal{L}}\Big)^{\otimes(-1)}. \tag{2.14}$$

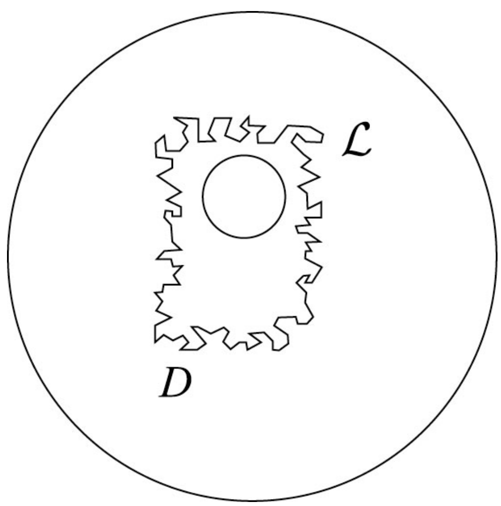

Figure 3: Loop $\mathcal{L}$ in an annulus domain D

To make this definition independent of D, we construct for every pair of domains $D_1, D_2 \supset \mathcal{L}$, of the same kind as above, an isomorphism

$$i_{D_1,D_2} : |\det|_{D_1} \Big/ |\det|_{D_1\setminus\mathcal{L}} \to |\det|_{D_2} \Big/ |\det|_{D_2\setminus\mathcal{L}} \tag{2.15}$$

satisfying the corresponding cocycle identity

$$i_{D_2,D_3} \circ i_{D_1,D_2} = i_{D_1,D_3}.$$

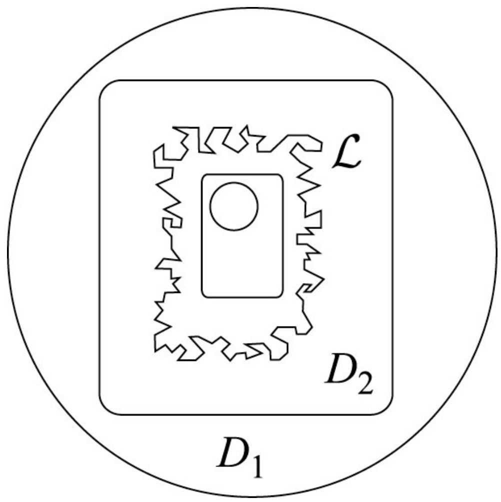

Figure 4: Loop in two domains $D_2 \subset D_1$

The construction of isomorphism i_{D_1,D_2} is as follows. First, assume that D_2 is relatively compact in D_1, and the boundary ∂D_2 consists of two real analytic loops. Next, choose a metric $g_{1\backslash 2}$ on $D_1 \backslash \overline{D}_2$ well-behaving at infinity. Then $\exists$ metrics g_1 on D_1 and $g_{1,\mathcal{L}}$ on $D_1 \setminus \mathcal{L}$ well-behaving at infinity, such that their restrictions on $D_1 \setminus \overline{D}_2$ coincide with $g_{1\backslash 2}$. Further, define metrics g_2 on D_2 and $g_{2,\mathcal{L}}$ on $D_2 \setminus \mathcal{L}$ as the restrictions of g_1 and $g_{1,\mathcal{L}}$, respectively. The isomorphism i_{D_1,D_2} is then determined by

$$i_{D_1,D_2}\left(\frac{[g_1]}{[g_{1,\mathcal{L}}]}\right) = \frac{[g_2]}{[g_{2,\mathcal{L}}]}\,. \tag{2.16}$$

In general, we choose a domain $D_3 \subset D_1 \cap D_2$ which is relatively compact in $D_1 \cap D_2$ and with boundary ∂D_3 consisting of two real analytic loops. Then define i_{D_1,D_2} by

$$i_{D_1,D_2} = i^{-1}_{D_2,D_3} \circ i_{D_1,D_3}\,, \tag{2.17}$$

to guarantee the cocycle identity. The independence of the choice of D_3 follows from the locality of the Liuoville action.

Remark 2.2. It is instructive to give an alternative definition of isomorphism i_{D_1,D_2}, by using the construction of vector $v_{\mathfrak{C}}$ for a particular neutral collection $\mathfrak{C}$ described as follows. Consider four compact surfaces

$$\Sigma_1 = (D_1)_{\text{double}},\ \Sigma_2 = (D_1 \setminus \mathcal{L})_{\text{double}},\ \Sigma_3 = (D_2)_{\text{double}},\ \Sigma_4 = (D_2 \setminus \mathcal{L})_{\text{double}}\,. \tag{2.18}$$

Assign to them multiplicities $(m_1, m_2, m_3, m_4) = (-1, +1, +1, -1)$. Next, fix two relatively compact open neighbourhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of loop $\mathcal{L}$, with smooth boundaries, such that $D_1 \cap D_2 \supset \bar{\mathcal{U}}_1$ and $\mathcal{U}_1 \supset \bar{\mathcal{U}}_2$.

As a non-Hausdorff 'manifold' Σ^{nH}, we take the union $\Sigma_1 \sqcup_{\mathcal{U}} \Sigma_4$ where the set $\mathcal{U}$ (along which Σ_1 and Σ_4 are identified) is the formed by the pullback images of $\mathcal{U}_1 \setminus \bar{\mathcal{U}}_2$. It is easy to see that Σ_2 and Σ_3 are naturally embedded in $\Sigma_1 \sqcup_{\mathcal{U}} \Sigma_4$; this yields the neutral collection $\mathfrak{C}$ under consideration. Then i_{D_1,D_2} is given by multiplication by the vector

$$(v_{\mathfrak{C}})^{\otimes 1/2} \in \left(|\det|_{D_1}\right)^{\otimes(-1)} \otimes |\det|_{D_1\setminus\mathcal{L}} \otimes |\det|_{D_2} \otimes \Big(|\det|_{D_2\setminus\mathcal{L}}\Big)^{\otimes(-1)}\,. \quad \blacksquare \tag{2.19}$$

2.3.2. Topology on the determinant bundle. Our goal in this subsection is to define a continuous real line bundle $\left|\mathrm{Det}\right|_{\Sigma}$ on the space $\mathit{Loop}(\Sigma)$ whose fiber at each point $\mathcal{L} \in \mathit{Loop}(\Sigma)$ is canonically identified with $|\det|_{\mathcal{L},\Sigma}$. Here we will assume for simplicity that Σ is orientable near $\mathcal{L}$; the non-orientable case follows by passing to the double cover.

To start with, assume that surface Σ is a puncture-free annulus A. Recall (see subsection 2.2.4) that in this case we have the canonical vector $v_A \in |\det|_A$. Hence we have a canonical vector $v_{\mathcal{L},A} \in |\det|_{\mathcal{L},A}$ for an arbitrary non-contractible loop $\mathcal{L} \in \mathit{Loop}^1(A)$, namely:

$$v_{\mathcal{L},A} = \frac{v_A}{v_{A_1} \otimes v_{A_2}}\,. \tag{2.20}$$

Here A_1 and A_2 are two annuli forming the connected components of $A \setminus \mathcal{L}$. This yields a trivialisation of the bundle $\left|\mathrm{Det}\right|_A$ on the space $\mathit{Loop}^1(A)$. We then define the continuous structure on $\left|\mathrm{Det}\right|_A$ by declaring that the map $L \mapsto (v_{\mathcal{L},A})$ is continuous.

Next, this construction is extended to the case of a loop $\mathcal{L}$ on a general surface Σ orientable near $\mathcal{L}$. Here, we use the fact that there exists

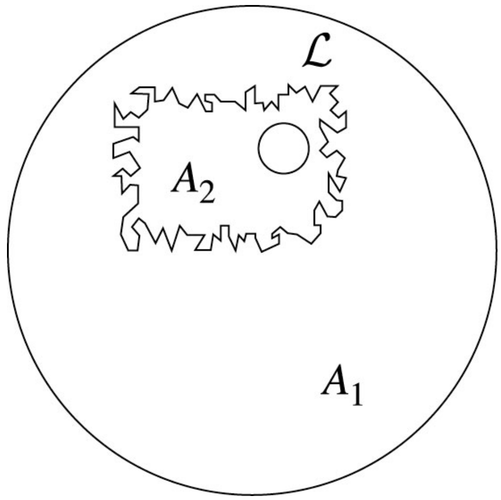

Figure 5: Loop $\mathcal{L}$ and two annuli A_1, A_2, connected components of $A \setminus \mathcal{L}$

a fundamental system of neighborhoods of $\mathcal{L} \in \mathit{Loop}(\Sigma)$ consisting of sets $i_*[\mathit{Loop}^1(A)]$ where $i : A \hookrightarrow \Sigma$ is an embedding of an annulus A in Σ. To justify the correctness of the definition, we have to check that for any two annuli, $A \subset \Sigma$ and $A' \subset \Sigma$, such that $\mathcal{L} \in \mathit{Loop}^1(A) \cap \mathit{Loop}^1(A')$, the ratio $v_{\mathcal{L},A}/v_{\mathcal{L},A'}$ is a continuous function in a neighborhood of $\mathcal{L}$. In order to calculate this ratio, we have to introduce certain interpolations between six flat metrics: the normalised metrics on annuli A and A', and the normalised metrics on the connected components of $A \setminus \mathcal{L}$ and $A' \setminus \mathcal{L}$. The continuity of the ratio follows from Lemma 2.3 and the following assertion.

Lemma 2.4. *For any annulus A, the normalised metrics (see Eqn (2.12)) on both connected components of $A \setminus \mathcal{L}$ depend continuously on $\mathcal{L} \in \mathit{Loop}(A)$ on compacts in $A \setminus \mathcal{L}$.*

Proof: The harmonic function h used in the definition of the normalised metric coincides with the probability of hitting a component of the boundary ∂A in the Brownian motion. Hence it depends continuously on the boundary curve. The expression for the normalised metric includes the first derivative of h which can be replaced by a suitable contour integral because h is harmonic. $\square$

Remark 2.3. The concept of line $|\det|_{\mathcal{L},\Sigma}$ associated with loop $\mathcal{L} \in \mathit{Loop}(\Sigma)$ seems novel and is central for this paper. It is easy to see that the restriction of $\left|\mathrm{Det}\right|_{\Sigma}$ to the subspace of $Loop(\Sigma)$ formed by sufficiently smooth curves (e.g., curves of class C^2) is canonically trivialised. In a sense, one can interpret a non-smooth loop $\mathcal{L}$ as an infinitesimally tiny open subset of Σ, and $|\det|_{\Sigma}$ can be seen as an analog of the determinant line for such an 'open surface'. ■

2.4 The covariance property and the main conjecture

Given an embedding $\xi:\ \Sigma \hookrightarrow \Sigma'$, we have an associated open embedding

$$\xi_* :\ Loop\,(\Sigma) \hookrightarrow Loop\,(\Sigma').$$

Further, it generates the canonical isomorphism of line bundles

$$\xi_{\det} :\ (\xi_*)^*\left|\mathrm{Det}\right|_{\Sigma'} \to \left|\mathrm{Det}\right|_{\Sigma}.$$

(We can use here any annulus A containing a given loop $\mathcal{L} \in Loop(\Sigma)$.

Definition 2.1. Fix a real number c and assume that for every surface Σ we are given a measure $\boldsymbol{\lambda}_\Sigma$ on $Loop\,(\Sigma)$ with values in $\left(\left|\mathrm{Det}\right|_{\Sigma}\right)^{\otimes c}$. We say that the assignment $\Sigma \mapsto \boldsymbol{\lambda}_\Sigma$ is *locally conformally covariant*, with parameter c (briefly: c-LCC, or, simply, LCC) if for any embedding $\xi :\ \Sigma \hookrightarrow \Sigma'$ we have

$$\xi^*(\boldsymbol{\lambda}_{\Sigma'}) = \boldsymbol{\lambda}_\Sigma, \tag{2.21}$$

where we use the obvious identification of the line bundles via isomorphism ξ. ■

Conjecture 1. *For any* $\mathrm{c} \in (-\infty, 1]$, *there exists a unique (up to a positive constant factor) non-zero* c*-LCC assignment* $\Sigma \mapsto \boldsymbol{\lambda}_\Sigma$.

The bound $\mathrm{c} \le 1$ is motivated by properties of the family of random SLE_κ-processes (see chapter 4). A well-known fact is that a trajectory of an SLE_κ-process remains 'simple' (dividing a unit disk or a half-plane into two domains) for $\kappa \in (0, 4]$ which implies the above bound on c.

We will call measures $\boldsymbol{\lambda}_\Sigma$ figuring in Conjecture 1 *Malliavin measures.* The relation between Conjecture 1 and a series of papers by Malliavin and his followers initiated in [M] and [AM] is discussed in subsection 2.5.2.

Observe that if Conjecture 1 has been verified when Σ is an arbitrary annulus $A = A_{r_1,r_2} = \{z \in \mathbb{C} :\ r_1 < |z| < r_2\}$, $0 < r_1 < r_2 < +\infty$, and $\boldsymbol{\lambda}_A$ is a measure on the space $Loop^1(A)$ of single-winding loops in A and ξ is an embedding $A_{r_1,r_2} \to A_{r'_1,r'_2}$, then it will be verified in full generality, for all orientable surfaces Σ.

Similarly, to establish Conjecture 1 for non-orientable surfaces, it is enough to check the conjecture when Σ is an arbitrary Möbius strip M, $\boldsymbol{\lambda}_M$ is a measure on $Loop^1(M)$ and ξ is an embedding $M \to M'$. In our view, the first step in proving Conjecture 1 would be a construction of such measures $\boldsymbol{\lambda}_A$ and $\boldsymbol{\lambda}_M$.

Next, in the orientable case, in section 2.5 we provide a further reduction, which we believe is equivalent to the initial problem of constructing an LCC assignment $\Sigma \mapsto \boldsymbol{\lambda}_\Sigma$. It will be stated in terms of scalar measures on the space of single-winding loops on a punctured plane.

Parameter c is interpreted as the *central charge* (in the corresponding conformal field theory; see section 6.2).

In the case $c = 0$, Conjecture 1 was recently established (in the case of an orientable surface Σ) by Werner [W4]. Unfortunately, the method in [W4] works (for both existence and uniqueness) specifically for $c = 0$; it seems that an extension to other values of c requires new ideas.

In chapter 4 we will define a space of intervals $Int_{x,y}(\Sigma)$, an analog of space $Loop(\Sigma)$ for a surface Σ with a boundary, and natural determinant line bundles on $Int_{x,y}(\Sigma)$. The main result of chapter 4 is the verification that an SLE_κ process, with $0 < \kappa \le 4$, gives rise to an LCC assignment $(\Sigma, x, y) \mapsto \boldsymbol{\lambda}_{\Sigma,x,y}$. Here $\boldsymbol{\lambda}_{\Sigma,x,y}$ is a measure on $Int_{x,y}(\Sigma)$ with values in a tensor product of the aforementioned determinant line bundles. This will extend the LCC property that was previously established in [W4] for $\mathrm{SLE}_{4/3}$ by direct methods.

2.5 Reduction to $\mathbb{C}^*$

2.5.1. Restriction covariance property for measures on $Loop^1(\mathbb{C}^*)$. Fix $c \in (-\infty, 1]$. Under an additional assumption of strong local finiteness (see below), we will reduce the problem of constructing an LCC assignment $\Sigma \mapsto \boldsymbol{\lambda}_\Sigma$ to a simpler problem (in the orientable case), of constructing a scalar measure on the set $Loop^1(\mathbb{C}^*)$ of single-winding loops in $\mathbb{C}^*$ satisfying a restriction covariance property. Here, and below,

$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

is a punctured plane. First, if Σ is a sphere $\mathbb{C}P^1$, we have a scalar-valued measure

$$\boldsymbol{\nu}^{\mathrm{c}}_{\mathbb{C}P^1} = \boldsymbol{\lambda}_{\mathbb{C}P^1} \otimes \left(\frac{v_{\mathbb{C}P^1}}{v_{D_L} \otimes v_{D_R}} \right)^{\otimes(-\mathrm{c})} \tag{2.22}$$

on $Loop(\mathbb{C}P^1)$, where $D_{\mathrm{L}}\big(= D_{\mathrm{L},\mathcal{L}}\big) \subset \mathbb{C}P^1$ and $D_{\mathrm{R}}\big(= D_{\mathrm{R},\mathcal{L}}\big) \subset \mathbb{C}P^1$ are two open disks, to the left and to the right of $\mathcal{L} \in Loop(\mathbb{C}P^1)$, respectively. Measure $\boldsymbol{\nu}^{\mathrm{c}}_{\mathbb{C}P^1}$ is invariant under the action of $PSL(2,\mathbb{C})$ on $Loop(\mathbb{C}P^1)$.

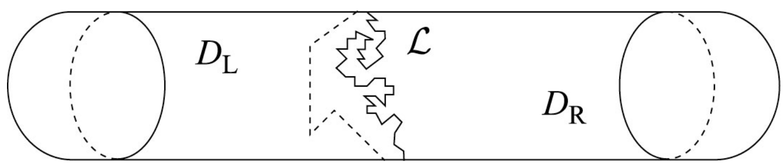

Figure 6: Loop $\mathcal{L}$ on the complement to two caps D_L, D_R

Next, denote by $\boldsymbol{\nu}^{\mathrm{c}}$ the restriction of measure $\boldsymbol{\nu}^{\mathrm{c}}_{\mathbb{C}P^1}$ to $Loop^1$ where

$$Loop^1 := Loop^1(\mathbb{C}^*) \tag{2.23}$$

is an open subset in $Loop(\mathbb{C}P^1)$. Measure $\boldsymbol{\nu}^{\mathrm{c}}$ is invariant under dilations $z \mapsto tz$, $z \in \mathbb{C}^*$, for any fixed $t \in \mathbb{C}^*$.

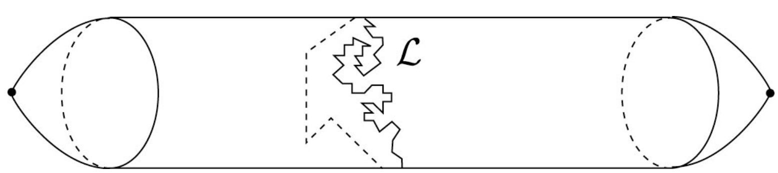

Figure 7: A single-winding loop on $\mathbb{C}^*$

In what follows, we assume that measure $\boldsymbol{\nu}^{\mathrm{c}}$ satisfies the following

Strong local finiteness condition. For any annulus

$$A_{r_1,r_2} = \{z \in \mathbb{C}^* : \ \ r_1 < |z| < r_2\},$$

the set $Loop^1(A_{r_1,r_2}) \subset Loop^1$ of simple loops in A_{r_1,r_2} has a finite $\boldsymbol{\nu}^{\mathrm{c}}$-measure:

$$\boldsymbol{\nu}^{\mathrm{c}}\left(Loop^1(A_{r_1,r_2})\right) < \infty, \quad 0 < r_1 < r_2 < \infty. \tag{2.24}$$

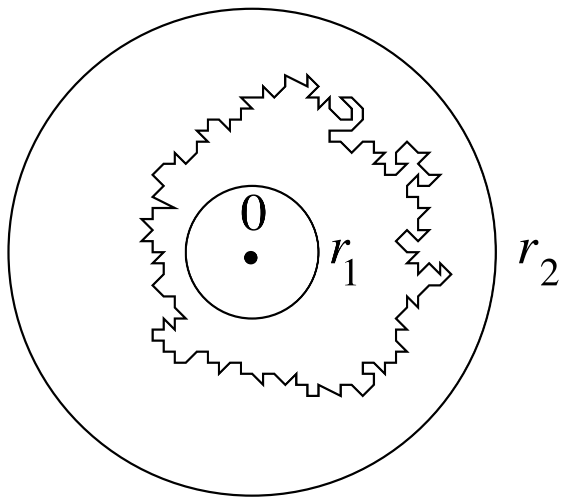

Figure 8: Loop in the annulus A_{r_1,r_2}

Observe that the condition of local finiteness of $\boldsymbol{\nu}^{\mathrm{c}}$ implies only that the volume of the above set is finite when $|r_2/r_1 - 1| < \delta$ for some $\delta > 0$. It is not clear whether the strong local finiteness condition would hold $\forall\, \mathrm{c} \in (-\infty, 1]$. However, we will assume that this property holds true. (It holds for $\mathrm{c} = 0$; see [W4].)

Let $A \subset \mathbb{C}^*$ be a relatively compact annulus and α be an embedding $A \hookrightarrow \mathbb{C}^*$. Assume that both annuli A and $\alpha(A)$ surround the origin. Then α induces an open embedding

$$\alpha_* : \; Loop^1(A) \hookrightarrow Loop^1(\alpha(A)) \,. \tag{2.25}$$

Given A and α as above, there is defined a positive continuous function, $q_\alpha^{\det}(\mathcal{L})$, $\mathcal{L} \in Loop^1$. In terms of this function we will state a condition on a scalar measure $\boldsymbol{\nu}$ on $Loop^1$ called restriction covariance and guaranteeing that $\boldsymbol{\nu}$ obtained from an LCC assignment. In fact, the assignment will be reconstructed from a scalar measure $\boldsymbol{\nu}$ satisfying the restriction covariance condition.

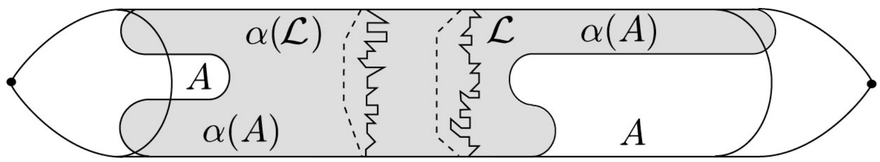

Figure 9: Loop in an annulus, and their images under the embedding α

To define function $q_\alpha^{\det}(\mathcal{L})$, we construct a neutral collection $\mathfrak{E}_{\alpha,\mathcal{L}}$ associated with loop $\mathcal{L} \in \mathit{Loop}^1$ (more precisely, with the corresponding loop in $\mathbb{C}P^1$ which we denote by the same symbol $\mathcal{L}$). Collection $\mathfrak{E}_{\alpha,\mathcal{L}}$ consists of eight spheres S_i, $1 \leq i \leq 8$. Spheres Σ_1, Σ_2, Σ_3, and Σ_4 in the collection are the doubles of four open disks D_1, D_2, D_3, and D_4, respectively. In turn, the disks are identified as follows:

$$D_1 = D_{\mathrm{L},\mathcal{L}},\ D_2 = D_{\mathrm{R},\mathcal{L}},\ D_3 = D_{\mathrm{L},\alpha(\mathcal{L})},\ D_4 = D_{\mathrm{R},\alpha(\mathcal{L})}.$$

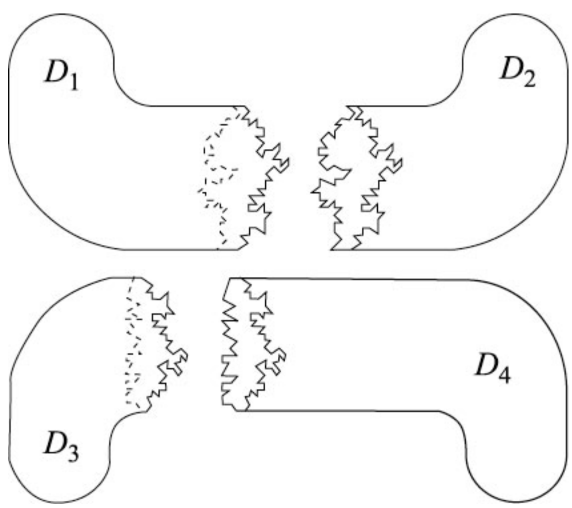

Figure 10: Four discs $D_1, \ldots, D_4$ in $\mathbb{C}P^1$

Geometrically, disks $D_{\mathrm{L},\mathcal{L}}$ and $D_{\mathrm{R},\mathcal{L}}$ are domains in $\mathbb{C}P^1$ to the left and to the right of loop $\mathcal{L}$, respectively, and disks $D_{\mathrm{L},\mathcal{L}}$ and $D_{\mathrm{R},\mathcal{L}}$ are domains in $\mathbb{C}P^1$ to the left and to the right of loop $\alpha(\mathcal{L})$, respectively.

Further, spheres Σ_5 and Σ_6 in the collection constitute the double of sphere $S_{\text{in}} = \overline{D_1 \cup D_2}$, and spheres Σ_7 and Σ_8 the doubles of sphere $S_{\text{fin}} = \overline{D_3 \cup D_4}$, correspondingly. (Subscript in stands for initial and fin for final.) Formally,

$$S_{\text{in}} = \mathbb{C}P^1, \quad S_{\text{fin}} = \mathbb{C}P^1.$$

The non-Hausdorf surface S^{nH} for collection $\mathfrak{E}_{\alpha,\mathcal{L}}$ is formed by glueing the above eight spheres $S_1, \ldots, S_8$. These spheres are glued all along the domains that are the pullback to the double covering of the union of two thin strips on the left and on the right of loop $\mathcal{L}$ in $\mathbb{C}^*$. In the figure below, each sphere S_i is identified by a pair of half-spherical caps which have value i among the pair of indeces attached to them. (So, spheres S_1, S_2, S_3, and S_4 'live' on both levels while S_5, S_6, S_7 and S_8 on a single level. Spheres S_6 and S_5 are drown horizontal.)

The weights μ_i are: $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = -1$, $\mu_4 = -1$, $\mu_5 = -1$, $\mu_6 = -1$, $\mu_7 = 1$ and $\mu_8 = 1$.

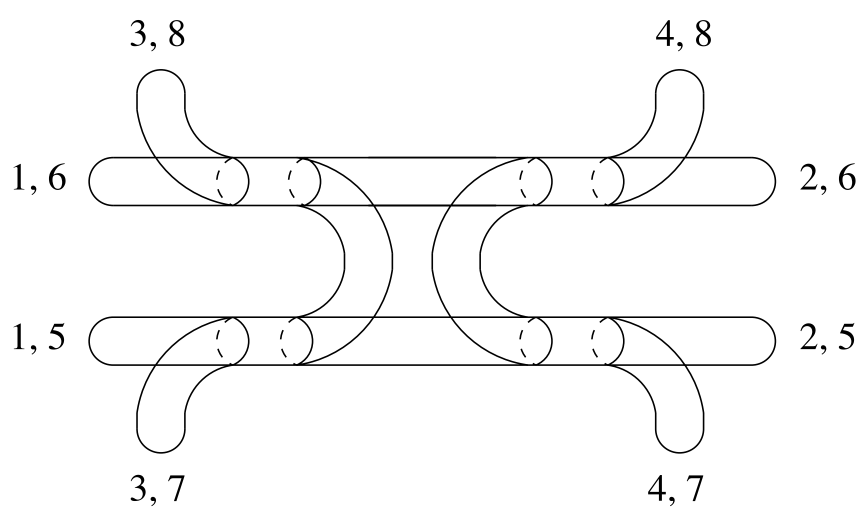

Figure 11: Neutral collection of 8 spheres

Value $q_\alpha^{\det}(\mathcal{L})$ is then defined as follows:

$$q_\alpha^{\det}(\mathcal{L}) = \left(v_{\mathfrak{E}_{\alpha,\mathcal{L}}} \Big/ \bigotimes_{k=1}^{8} v_{S_k}^{\otimes \mu_k} \right)^{1/2}. \tag{2.26}$$

Definition 2.2. We say that a scalar measure $\boldsymbol{\nu}$ on $Loop^1$ is c-*restriction covariant* (c-RC, or briefly, RC) if, for each relatively compact annulus $A \subset \mathbb{C}^*$ and an embedding $\alpha: \ A \hookrightarrow \mathbb{C}^*$, the pullback $\alpha^*\big(\boldsymbol{\nu}\big|_{\alpha_*(Loop^1(A))}\big)$ of the restriction $\boldsymbol{\nu}\big|_{\alpha_*(Loop^1(A))}$ of measure $\boldsymbol{\nu}$ to the image $\alpha_*(Loop^1)$ (which is an open subset in $Loop^1$) is absolutely continuous with respect to $\boldsymbol{\nu}$ and has the Radon-Nikodym derivative

$$\frac{\mathrm{d}\big[\alpha^*(\boldsymbol{\nu})\big]\big|_{\alpha_*(Loop^1(A)}}{\mathrm{d}\boldsymbol{\nu}}(\mathcal{L}) = \big(q_\alpha^{\det}(\mathcal{L})\big)^{\mathrm{c}}, \ \ \mathcal{L} \in Loop^1(A). \quad \blacksquare \tag{2.27}$$

As follows from definitions, if $\Sigma \mapsto \boldsymbol{\lambda}_\Sigma$ is an c-LCC assignment, then scalar measure $\boldsymbol{\nu}^{\mathrm{c}}$ on $Loop^1$ is c-RC. Moreover, any c-RC measure ν on $Loop^1$ gives rise to a unique c-LCC assignment. In fact, it suffices to define measures $\boldsymbol{\lambda}_\Sigma$ when Σ is an arbitrary annulus (and restrict the measures on $Loop^1(\Sigma)$). Further, an annulus can be embedded in $\mathbb{C}^*$. Hence, the RC property of ν is necessary and sufficient for constructing an LCC assignment.

2.5.2. Infinitesimal restriction covariance property for measures on $Loop^1(\mathbb{C}^*)$**.** Perhaps a simpler task is to check the RC property in an infinitesimal form, where embedding α is close to identity. To this end, observe that the Lie algebra (over $\mathbb{R}$)

$$\mathfrak{v} = \mathbb{C}[z, z^{-1}]\frac{\partial}{\partial z} \tag{2.28}$$

acts on $Loop^1$. The basis of algebra $\mathfrak{v}$ consists of elements

$$L_n = -z^{n+1}\frac{\partial}{\partial z} \ \text{ and } \ L_n' = \mathrm{i}z^{n+1}\frac{\partial}{\partial z}, \ \ n \in \mathbb{Z}. \tag{2.29}$$

Formally speaking, the infinitesimal RC property is that

$$\mathrm{div}_{\boldsymbol{\nu}^{\mathrm{c}}} L_n = \mathrm{c}P_n, \ \ \mathrm{div}_{\boldsymbol{\nu}^{\mathrm{c}}} L_n' = \mathrm{c}P_n', \ \ n \in \mathbb{Z},$$

where P_n, P'_n are certain explicit functions on *Loop*^{1} related to so-called Neretin polynomials, and $\mathrm{div}_{\boldsymbol{\nu}^c}$ stands for the divergence relative to measure $\boldsymbol{\nu}^c$ (see [AM]). In reality, it is enough to check this property when $|n| \leq 2$, because algebra $\mathfrak{v}$ is generated by L_n and L'_n with $n = -2, -1, 0, 1, 2$.

It looks plausible that the property of restriction covariance can be deduced from that of infinitesimal restriction covariance. However, in this paper we do not offer a formal proof of this fact. We consider this as an interesting open question.

Explicit formulas for P_n and P'_n for $n = -2, -1, 0, 1, 2$, are given below. First,

$$P_n = P'_n = 0, \quad n = -1, 0, 1, \tag{2.30}$$

which follows from invariance of measure $\boldsymbol{\nu}^c$ under the action of $PSL(2, \mathbb{C})$. Next,

$$P_{-2}(\mathcal{L}) = \frac{1}{12} \mathrm{Re}\, \mathcal{S}_{\phi_{\mathrm{L},\mathcal{L}}}(0), \quad P'_{-2}(\mathcal{L}) = \frac{1}{12} \mathrm{Im}\, \mathcal{S}_{\phi_{\mathrm{L},\mathcal{L}}}(0). \tag{2.31}$$

Here, $\mathcal{S}_f$ stands for the Schwartzian derivative of function f:

$$\mathcal{S}_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3(f''(z))^2}{2(f'(z))^2}\,. \tag{2.32}$$

Next, $\phi_{\mathrm{L},\mathcal{L}} : \ U \to \mathbb{C}$ is an embedding of the open unit disk $U = \{t \in \mathbb{C} : |t| < 1\}$, with the image $\phi_{\mathrm{L},\mathcal{L}}(U) = D_{\mathrm{L},\mathcal{L}}$, normalised so that $\phi_{\mathrm{L},\mathcal{L}}(0) = 0$. For P_2 and P'_2, the formulas are similar to (2.31); the only change is that one uses embedding $\phi_{\mathrm{R},\mathcal{L}} : \ U \to \mathbb{C}$ with the image $\phi_{\mathrm{R},\mathcal{L}}(U) = \varpi[D_{\mathrm{R},\mathcal{L}}]$ where $\varpi : \ \mathbb{C}P^1 \to \mathbb{C}P^1$ is the involution $z \mapsto 1/z$.

The justification of formula (2.31) will not be given here: we refer the reader to section 3.3 where a similar argument is used in a slightly different situation.

We will consider two coordinates on *Loop*^{1}:

$$\{A, a_1, a_2, \ldots\} \ \text{ and } \ \{B, b_1, b_2, \ldots\}.$$

Here, the components are as follows:

A, B are real positive numbers, and a_k, b_k complex numbers, $k \geq 1$,

identified from the representations

$$\begin{array}{l} \phi_{\mathrm{L},\mathcal{L}}(t) = A(t + a_1 t^2 + a_2 t^3 + \ldots), \\ \phi_{\mathrm{R},\mathcal{L}}(t) = B(t + b_1 t^2 + b_2 t^3 + \ldots), \end{array} \quad t \in U. \tag{2.33}$$

One can show that the inequality

$$0 < AB \leq 1$$

holds, with equality only when the loop $\mathcal{L} \in Loop^1$ is a circle $\{z \in \mathbb{C}^* : \ |z| = r\}$, for some $r \in (0, +\infty)$. In fact, we may assume that $AB < 1$, as equality $AB = 1$ holds on a set of $\boldsymbol{\nu}^{\mathrm{c}}$-measure 0.

Definition 2.3. It is convenient to introduce a set $\mathbf{B}$ of functions $F : \ Loop^1 \to \mathbb{C}$ written as finite sums over quadruples of multi-indeces (I, I', J, J'):

$$F(\mathcal{L}) = \sum_{I,I'J,J'} f_{I,I',J,J'}(A,B) a^I \overline{a}^{I'} b^J \overline{b}^{J'}. \tag{2.34}$$

Here $a^I = a_1^{i_1} a_2^{i_2} \dots$ is a monomial in a_1, a_2, $\dots$ associated with an integer-valued multi-index $I = (i_1, i_2, \dots)$, of a finite total degree ($|I| = |i_1| + |i_2| + \dots < +\infty$). Similarly, $\overline{a}^{I'}$, b^J and $\overline{b}^{J'}$ are monomials in the corresponding variables associated with finite-degree multi-indeces $I' = (i'_1, i'_2, \dots)$, $J = (j_1, j_2, \dots)$ and $J' = (j'_1, j'_2, \dots)$. Further, $f_{I,I',J,J'}$ is a function with compact support on $\{(A, B) \in \mathbb{R}^2 : \quad A, B > 0, \ AB < 1\}$. Then $\mathbf{B}$ is a non-unital commutative $*$-algebra, separating points of $Loop^1$. Thus, measure $\boldsymbol{\nu}^{\mathrm{c}}$ is uniquely determined by its integrals for functions from $\mathbf{B}$ (generalised moments). ■

Remark 2.4. The Bieberbach conjecture established by L. De Branges implies that, $\forall \ \mathcal{L} \in Loop^1$,

$$|a_k|, |b_k| \leq k + 1, \quad k \geq 1. \quad ■$$

Note that the Lie algebra $\mathbb{C}[z, z^{-1}] \dfrac{\partial}{\partial z}$ and the operators of multiplication by P_n, P'_n, $n \in \mathbb{Z}$, preserve $\mathbf{B}$.

Remark 2.5. Coordinates a_1, a_2, $\dots$ were used in paper [AM], in an attempt to identify a 'natural' measure on the quotient space $Loop^1/\mathbb{R}$. In our context, it is not enough to use a single coordinate, say $(A, a_1, a_2, \dots)$. The reason is that for a non-zero function $f_{I,I'} \in C_0^\infty(0,1)$, the integral

$$\int_{Loop^1} f_{I,I'}(A) a^I \overline{a}^{I'} \boldsymbol{\nu}^{\mathrm{c}}$$

diverges. Hence, there is no obvious algebra of functions in variables A, a_1, a_2, ... for which the generalised moments are finite. Therefore, the second coordinate $(B, b_1, b_2, \ldots)$ is needed. (An indication of this fact can be found in [AMT].) ■

Definition 2.4. We say that a strongly locally finite measure $\boldsymbol{\nu}$ on $Loop^1$ is *infinitesimally* c*-restriction covariant* (c-IRC, or briefly, IRC) if, $\forall\ F \in \mathbf{B}$ and $\forall\ n \in \mathbb{Z}$,

$$\int\limits_{Loop^1} \big[(L_n + \mathrm{c}P_n)F\big]\ \boldsymbol{\nu} = 0 \tag{2.35}$$

and

$$\int\limits_{Loop^1} \big[(L'_n + \mathrm{c}P'_n)F\big]\ \boldsymbol{\nu} = 0. \quad ■ \tag{2.36}$$

In fact, as we mentioned earlier, in order to check that $\boldsymbol{\nu}$ is c-IRC, it suffices to verify the above equations for $|n| \leq 2$.

We note that the equations for $n \leq 0$ coincide with conditions (2.3.3) from paper [AM]; the case $n = 1$ was considered in article [AMT].

It also possible to consider a larger space $AHull^1$ formed by 'annular hulls' (called 'bubbles' in [LW]). An annular hull is a closed compact in $\mathbb{C}^*$ homotopically equivalent to S^1 and separating 0 from ∞ on $\mathbb{C}P^1$. Coordinates $(A, a_1, a_2, \ldots)$ and $(B, b_1, b_2, \ldots)$, and thus algebra $\mathbf{B}$, can be extended to $AHull^1$. In turn, it allows us to define the IRC property for a measure on $AHull^1$. The domain

$$\{A \geq A_0,\ \ B \geq B_0\}$$

is a compact in $AHull^1$, in the topology generated jointly by the pair of coordinates $(A, a_1, a_2, \ldots)$ and $(B, b_1, b_2, \ldots)$. Then measures on $AHull^1$ are identified with positive functionals on $\mathbf{B}$.

Remark 2.6. A (Borel) measure $\boldsymbol{\pi}$ on $AHull^1$ invariant under the action of $\mathbb{R}_+^\times$ gives rise to a countable collection of distributions (generalised functions) $M_{I,I',J,J'}$ on $(0,1)$ (more precisely, on the test-function space $C_0^\infty(0,1)$), labeled by quadruples of integer-valued multi-indeces I, I', J, J' of finite total degree. Namely,

$$\int\limits_{AHull^1} f_{I,I',J,J'}(A,B)a^I\overline{a}^{I'}b^J\overline{b}^{J'}\boldsymbol{\pi} = \int\limits_0^{+\infty}\int\limits_0^{+\infty} f(A,B)M_{I,I',J,J'}(AB)\frac{\mathrm{d}A\times\mathrm{d}B}{AB}. \tag{2.37}$$

The fact that $\boldsymbol{\pi}$ is IRC gives rise to a countable system of differential equations involving distributions $M_{I,I',J,J'}$. One can show that any solution to this system of differential equations can be uniquely reconstructed from distribution $M_{\underline{0}} := M_{0,0,0,0}$. The latter can be arbitrary, provided that it satisfies a countable system of inequalities, depending on c (which follow from non-negativity of measure $\boldsymbol{\pi}$). In particular, $M_{\underline{0}}$ is a (non-negative) measure on $(0,1)$. ■

In relation to measure $M_{\underline{0}}$, we put forward the following comment.

Remark 2.7. It is plausible that the measures $M_{\underline{0}}$ associated with IRC measures on $AHull^1$ form an infinite-dimensional cone, with a continuum of extremal rays. We expect that $\forall\ r \in (0,1)$, there is a 'canonical' extremal measure $M_{\underline{0}}^{(r)}$, unique up to a scalar factor, and the associated IRC measure $\boldsymbol{\pi}^{(r)}$ on $AHull^1$ admits the following description. Consider the measure on the Cartesian product $Loop^1 \times Loop^1$ which is the product $\boldsymbol{\nu}^{\mathrm{c}} \times \boldsymbol{\nu}^{\mathrm{c}}$ of two copies of the (hypothetic) c-IRC measure $\boldsymbol{\nu}^{\mathrm{c}}$. Consider the restriction of $\boldsymbol{\nu}^{\mathrm{c}} \times \boldsymbol{\nu}^{\mathrm{c}}$ on the open subset $(Loop^1)^{\times 2}_{\mathrm{dis}} \subset Loop^1 \times Loop^1$ consisting of pairs of disjoint loops. With each pair of disjoint loops there is associated an annular hull which is the set bounded by these loops. The conformal parameter of this annular hull generates a map $\Upsilon\colon (Loop^1)^{\times 2}_{\mathrm{dis}} \to (0,1)$. We conjecture that, $\forall\ r \in (0,1)$, $\boldsymbol{\pi}^{(r)}$ is the measure, on the pullback image of $\Upsilon^{-1}r$, induced by the above restriction $\left(\boldsymbol{\nu}^{\mathrm{c}} \times \boldsymbol{\nu}^{\mathrm{c}}\right)\big|_{(Loop^1)^{\times 2}_{\mathrm{dis}}}$.

Finally, we conjecture that for $r = 1$, the limiting measure $\lim\limits_{r\to 1} \boldsymbol{\pi}^{(r)}$ is supported by $Loop^1$ and coincides with $\boldsymbol{\nu}^{\mathrm{c}}$. ■

There are two open problems related to IRC measures on $AHull^1$.

1. *Write explicitly the system of inequalities upon measure $M_{\underline{0}}$ associated with an IRC measure $\boldsymbol{\pi}$ on $AHull^1$.*
2. *Calculate, in a closed form, measure $M_{\underline{0}}$ associated with a (hypothetic) IRC measure $\boldsymbol{\nu}^{\mathrm{c}}$ on $Loop^1$.*

We expect that the latter measure $M_{\underline{0}}$ has a real analytic density relative to Lebesgue's measure on $(0,1)$, and the Radon-Nikodym derivative $\dfrac{\mathrm{d}M_{\underline{0}}}{\mathrm{d}\log\ r}$ is a kind of indefinite θ-series, presumably related to Kac' character formulas for representations of the Virasoro algebra.

3 Properties of determinant lines

In this chapter we prove some useful results relating the determinant lines of various surfaces. These results (Propositions 1 and 2) will be used in chapter 5. In a sense, the results of this chapter are not new and have been known to specialists in a somewhat different form.

3.1 A preliminary: metrics with pole singularities.

In this subsection we spell out some general concepts needed in the context of subsequent parts of the paper. Assume that Σ is a compact surface and $\mathcal{D} = \sum_{i=1}^{n} k_i p_i$ is a divisor on Σ, i.e. a formal linear combination of distinct points $p_i \in \Sigma$ with integral weights $k_i \in \mathbb{Z}$. We define a metric on Σ with singularities given by $\mathcal{D}$ as a metric g on non-compact surface $\Sigma \setminus \{p_1, \ldots, p_n\}$ such that near each point p_i there exists a local holomorphic coordinate z_i in which metric g has form

$$g = |z_i^{k_i}\mathrm{d}z|^2 \,.$$

We claim that such a metric defines a positive vector $[g]$ in the tensor product

$$\big|\det\big|_\Sigma \otimes \left(\bigotimes_{i=1}^{n} \big|\det T_{p_i}\Sigma\big|^{\otimes k_i/24} \right) . \tag{3.1}$$

Here and below, $\det T_p\Sigma$ stands for the wedge square $\wedge^2 T_p\Sigma$. Next, given a one-dimensional real vector space V, we denote by $|V|$ the oriented one-dimensional real vector space associated with the homomorphism

$$GL(1,\mathbb{R}) \to \mathbb{R}^\times_{>0}, \quad x \in GL(1,\mathbb{R}) \mapsto |x|. \tag{3.2}$$

In order to define $[g]$, it suffices to define the ratio

$$[g]/[g_0] \in \bigotimes_{i=1}^{n} \big|\det T_{p_i}\Sigma\big|^{\otimes k_i/24}, \tag{3.3}$$

for any non-singular metric g_0 on Σ. Furthermore, we can assume that g_0 is flat near each point p_i. In this case we set

$$[g]/[g_0] := \exp\left[\frac{1}{48\pi \mathrm{i}} \int\limits_{\Sigma\setminus\{p_1,\ldots,p_n\}} \mathrm{L}_{\mathrm{Liouv}}(g_0,g)\right] \bigotimes_{i=1}^{n} [g_0]_{p_i}^{\otimes k_i/24}. \tag{3.4}$$

Here $[g_0]_p \in \left|\det T_p\Sigma\right|$ is the inverse to the natural volume element on $\det T_p\Sigma$ generated by metric g_0. Notice that the integral in the above expression is absolutely convergent as the density $\mathrm{L}_{\mathrm{Liouv}}(g_0,g)$ vanishes near points p_i (because both metrics g_0 and g are flat there).

The consistency of the above definition is guaranteed by Eqn (2.3) and the following general lemma that is valid for any surface Σ.

Lemma 3.1. *Let Σ be a surface with a marked point p and z, w_1, w_2 be local coordinates near point p, vanishing at p. Let k be an integer. Consider the* 1*-form α defined in Eqn* (2.4)*. Then the integral, over a small, anticlockwise oriented, circle around p, of the closed* 1*-form*

$$\alpha\Big(|z^k \mathrm{d}z|^2, |\mathrm{d}w_1|^2, |\mathrm{d}w_2|^2\Big)$$

is equal to $2\pi \mathrm{i} k \log|(\mathrm{d}w_1/\mathrm{d}w_2)(p)|^2$.

Proof : Observe that for any three flat metrics g_1, g_2 and g_3 on Σ, the form $\alpha(g_1,g_2,g_3)$ is closed. Furthermore, after rescaling one of the metrics as $g_i \to bg_i$, $b>0$, the above integral increases by the amount $\log\ b$ times the difference of the rotation numbers of the two other metrics. Next, let us consider the integral of $\alpha(g_1,g_2,g_3)$ over the unit circle in coordinate $\tilde{z} := tz$, for real $t \to +\infty$, where

$$g_1 = |z^k \mathrm{d}z|^2/t^{2(k+1)} = |\tilde{z}^k \mathrm{d}\tilde{z}|^2,$$

and

$$g_2 = |\mathrm{d}w_1|^2/|(\mathrm{d}w_1/\mathrm{d}\tilde{z})(p)|^2,\ g_3 = |\mathrm{d}w_2|^2/|(\mathrm{d}w_2/\mathrm{d}\tilde{z})(p)|^2\ .$$

This integral tends to zero as $t\to\infty$ because g_2 and g_3 become close to $|\mathrm{d}\tilde{z}|^2$, and form $\alpha(g_1,g_2,g_3)$ is antisymmetric in indices $1,2,3$. By the above remark on rescaling, the difference of the integral in the statement of Lemma 3.1 and the integral of $\alpha(g_1,g_2,g_3)$ is equal to $2\pi \mathrm{i} k \log\left|\frac{\mathrm{d}w_1}{\mathrm{d}w_2}(p)\right|^2$. The assertion of Lemma 3.1 then follows. $\square$

Later on, we will also need

Lemma 3.2. *Let Σ be a surface with a marked point p and z_1, z_2, w be local coordinates near point p, vanishing at p and such that $\dfrac{\mathrm{d}z_1}{\mathrm{d}z_2}(p) = 1$. Given an integer k, the integral, over a small circle around p, of the* 1*-form*

$$\alpha\Big(|z_1^k \mathrm{d}z_1|^2, |z_2^k \mathrm{d}z_2|^2, |\mathrm{d}w|^2\Big)$$

equals zero.

The proof of Lemma 3.2 is similar to that of Lemma 3.1, and we omit it.

3.2 The canonical vector for the special four-sphere neutral collection

The central result of section 3.2 is a formula (see Eqn (3.8)) for the ratio between the canonical vector $v_{\mathfrak{F}}$ and the product of canonical vectors $(v_{S_i})^{\otimes m_i}$. Here and throughout the rest of the paper, $\mathfrak{F}$ stands for the neutral collection $\big(\{(S_i, \phi_i, \mu_i)\}_{i=1}^4; S^{\mathrm{nH}}\big)$ consisting of four spheres introduced in subsection 2.2.5.

3.2.1. A formula for the canonical vector for general metrics Assume that the common part of spheres S_1, S_2, S_3 and S_4 contains a closed cylinder C. Moreover, we assume that

$$\begin{aligned} S_1 = S_{11} = S_{1,L} \cup C \cup S_{1,R}, \quad S_2 = S_{12} = S_{1,L} \cup C \cup S_{2,R}, \\ S_3 = S_{21} = S_{2,L} \cup C \cup S_{1,R}, \quad S_4 = S_{22} = S_{2,L} \cup C \cup S_{2,R}, \end{aligned} \tag{3.5}$$

where S_{iL}, S_{iR}, $i = 1,2,3,4$, are half-spheres whose boundary circle is identified with the corresponding boundary circle of C (left or right, respectively).

From now on we will use the pair of lower indices (ij), $1 \le i,j \le 2$, instead of a single index i, $1 \le i \le 4$. The weights will be

$$\mu_{11} = +1, \;\; \mu_{12} = -1, \;\; \mu_{21} = -1, \;\; \mu_{22} = +1. \tag{3.6}$$

Suppose that g_{ij} are metrics on surfaces S_{ij}, $1 \le i,j \le 2$. Lemma 3.3 below gives an expression for the logarithm of the ratio $v_{\mathfrak{F}} \Big/ \Big(\bigotimes_{1 \le i,j \le 2} [g_{ij}]^{\otimes \mu_{ij}} \Big)$:

Lemma 3.3.

$$
\begin{aligned}
&\log\left[v_{\mathfrak{F}}\Big/\left(\bigotimes_{1\le i,j\le 2}[g_{ij}]^{\otimes\mu_{ij}}\right)\right]\\
&=\frac{1}{48\pi\mathrm{i}}\left[\int\limits_{S_{1,L}}\mathrm{L}_{\mathrm{Liouv}}(g_{11},g_{12})+\int\limits_{S_{2,L}}\mathrm{L}_{\mathrm{Liouv}}(g_{22},g_{21})\right.\\
&+\int\limits_{S_{1,R}\cup C}\mathrm{L}_{\mathrm{Liouv}}(g_{11},g_{21})+\int\limits_{S_{2,R}\cup C}\mathrm{L}_{\mathrm{Liouv}}(g_{22},g_{12})\\
&\left.+\int\limits_{L}\alpha(g_{11},g_{12},g_{21})-\int\limits_{L}\alpha(g_{22},g_{12},g_{21})\right].
\end{aligned}
\tag{3.7}
$$

Here L is the left boundary circle of cylinder C endowed with the standard orientation on ∂C.

Proof: First, assume that all metrics g_{ij} are restrictions of a metric on the non-Hausdorff surface associated with collection $\mathfrak{F}$. In this case, the LHS in (3.7) vanishes. On the other hand, every term in the sum in the RHS also vanishes. Hence, in this case Eqn (3.7) holds.

Thus, we should check that, after the change of metric g_{ij} for some (i,j), both the LHS and the RHS of (3.7) increase by same amount. This follows directly from Lemmas 2.1 and 2.2 and the Stokes formula. $\square$

3.2.2. The residue formula. Now let us apply results from section 3.1 to the special neutral four-sphere collection $\mathfrak{F}$. Choose points $p_{1,L}, p_{2,L}$ on pieces $S_{1,L}$ and $S_{2,L}$ respectively, and fix a holomorphic parametrisation z_{ij} of each surface S_{ij} by $\mathbb{C}P^1$ such that $z_{ij}(p_{i,L})=\infty$. Then $|\mathrm{d}z_{ij}|^2$ is a metric with singularities on S_{ij} at divisor $-2p_{i,L}$. Combining the results from section 3.1 with the explicit formula for form α in Eqn (2.4), we obtain the following assertion for the logarithm of the ratio $v_{\mathfrak{F}}\Big/\left(\bigotimes_{1\le i,j\le 2} v_{S_{ij}}^{\otimes\mu_{ij}}\right)$:

Lemma 3.4. *In the above notation, the following formula holds true:*

$$
\log\left[v_{\mathfrak{F}}\Big/\left(\bigotimes_{1\le i,j\le 2} v_{S_{ij}}^{\otimes\mu_{ij}}\right)\right]=\frac{-1}{24\pi}\mathrm{Im}\int\limits_{L}\log\left(\frac{\mathrm{d}z_{11}}{\mathrm{d}z_{22}}\right)\mathrm{d}\log\left(\frac{\mathrm{d}z_{12}}{\mathrm{d}z_{21}}\right). \tag{3.8}
$$

Proof: Without loss of generality, assume that for $i = 1, 2$,

$$\frac{\mathrm{d}z_{i1}^{-1}}{\mathrm{d}z_{i2}^{-1}}(p_{i,L}) = 1; \tag{3.9}$$

this can be achieved by rescaling coordinates z_{ij}. Set $\widetilde{g}_{ij} = |\mathrm{d}z_{ij}|^2$, and denote by g_{ij} the round metric on σ_{ij} determined by the stereographic projection in coordinate z_{ij}. The LHS in (3.8) is equal by definition to

$$\log\left[v_{\mathfrak{F}} \Big/ \left(\bigotimes_{1\leq i,j\leq 2}[g_{ij}]^{\otimes\mu_{ij}}\right)\right].$$

Owing to Lemma 3.3, this expression coincides with a certain sum of integrals over pieces of Σ and over contour $\mathcal{L}$. For singular metrics $\widetilde{g}_{ij}$, the expression

$$\log\left[v_{\mathfrak{F}} \Big/ \left(\bigotimes_{1\leq i,j\leq 2}[\widetilde{g}_{ij}]^{\otimes\mu_{ij}}\right)\right]$$

also makes sense, because terms taking values in $\left|\wedge^2 T_{p_{i,L}} S_{i,L}\right|$ vanish. Further, for metrics $\widetilde{g}_{ij}$, the RHS in (3.7) is well-defined.

Next, we claim that the assertion of Lemma 3.3 remains valid for metrics $\widetilde{g}_{ij}$. The reason is as follows. Take the difference of the LHSs in (3.7) for metrics g_{ij} and $\widetilde{g}_{ij}$. It is equal to

$$\frac{-1}{48\pi\mathrm{i}}\sum_{i,j=1,2}\mu_{ij}\int\limits_{S_{ij}}\mathrm{L}_{\mathrm{Liouv}}\left(g_{ij}, \widetilde{g}_{ij}\right). \tag{3.10}$$

On the other hand, the difference of the RHSs in (3.7) for metrics g_{ij} and $\widetilde{g}_{ij}$ coincides with (3.8) modulo possible boundary terms around points $p_{i,L}$. This is because the proof of Eqn (3.7) for smooth metrics is based on combination of Eqn (2.4) and Lemma 2.2; hence it works for singular metrics, too.

Near each point $p_{i,L}$ we have four metrics, two smooth and two singular. The integral of 1-form α over a small circle surrounding $p_{i,L}$ vanishes for any choice of three of them, by virtue of Lemmas 3.1 and 3.2. Therefore, we have

$$\log\left[v_{\mathfrak{F}} \Big/ \left(\bigotimes_{1\leq i,j\leq 2}[\widetilde{g}_{ij}]^{\otimes\mu_{ij}}\right)\right] = \frac{1}{48\pi\mathrm{i}}\int_L\Big[\alpha\big(\widetilde{g}_{11},\widetilde{g}_{12},\widetilde{g}_{21}\big) - \alpha\big(\widetilde{g}_{22},\widetilde{g}_{12},\widetilde{g}_{21}\big)\Big]. \tag{3.11}$$

The assertion of Lemma 3.4 then follows, as the expression in (3.11) coincides with the RHS of (3.8) by a straightforward calculation. □.

3.3 A variation formula for the special neutral collection

3.3.1. Schiffer variation. Let Σ be a surface with a conformal structure and z be a local holomorphic coordinate on Σ defined in a neighborhood U_p of point $p \in \Sigma$, such that $z(p) = 0$. We associate with the triple (Σ, p, z) the germ of a one-parameter family $(\Sigma_t)_{0 \le t < \epsilon}$ of new surfaces with conformal structures such that Σ_0 is canonically identified with Σ. Moreover, on each Σ_t for $t \neq 0$ there will be an open part identified conformally with $\Sigma \setminus U_p$.

Namely, we define Σ_t for $t \in [0, \epsilon)$ as the result of glueing of

$$\Sigma \setminus \{p' \in U : \ |z(p')| \le \delta_1\}$$

with the disk $\{w \in \mathbb{C} : \ |w| \le \delta_2\}$, by the correspondence

$$z(p') = \sqrt{w^2 + t}\ . \tag{3.12}$$

Here ϵ^2/δ_1, δ_1/δ_2 and δ_2 are small enough:

$$0 \ll \epsilon^2 \ll \delta_1 \ll \delta_2 \ll 1.$$

Family $(\Sigma_t)_{0 \le t < \epsilon}$ is called the *Schiffer variation* (of the complex structure on Σ). Informally, this construction describes the following modification of the surface. We cut a segment

$$\{p' : \ z(p') \in [-\sqrt{t}, \sqrt{t}] \subset \mathbb{R}\} \tag{3.13}$$

from our surface. The resulting surface has the boundary which consists of two copies of interval $[-\sqrt{t}, \sqrt{t}]$. Then the boundary is glued with itself in a different manner. More precisely, we glue together the sides of the two cuts with the same number $i = 1, 2, 3, 4$; see the figure below.

Transformation $w \mapsto \sqrt{w^2 + t}$ is the exponential map (at time t) of the meromorphic vector field

$$\dot{w} = \frac{1}{2w} \tag{3.14}$$

in a certain domain in $\mathbb{C}$.

Let us assume that $\Sigma = \Sigma_0$ is a sphere. Let $x : \Sigma \to \mathbb{C}P^1$ be a holomorphic parametrisation of Σ such that $x(p) = 0$ and $(\mathrm{d}z/dx)(p) = 1$. Denote by $q \in$

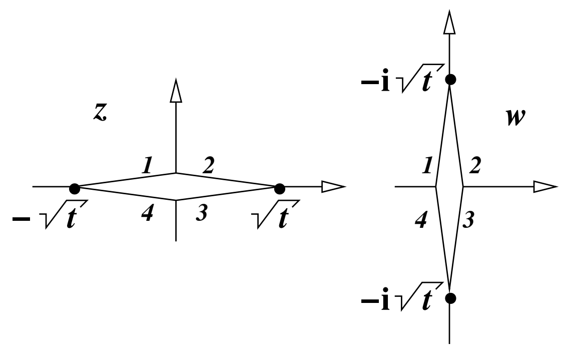

Figure 12: Coordinate planes z and w with cuts

Σ the point corresponding to $\infty \in \mathbb{C}P^1$ in coordinate x. There exists a unique family $x_t : \Sigma_t \to \mathbb{C}P^1$ of holomorphic parametrisations of Σ_t, depending smoothly on t outside of U_p and such that

$$x_t(q) = \infty, \ \left(\frac{d(1/x_t)}{d(1/x)}\right)(q) = 1, \ \left(\frac{d}{d(1/x)}\right)^2 (1/x_t)(q) = 0 \ .$$

In other words, near point q we have $x_t = x + O(x^{-1})$.

Lemma 3.5. *On $\Sigma \setminus U_p$, one has:*

$$\frac{\partial x_t}{\partial t}_{|t=0} = -\frac{1}{2x} \, . \tag{3.15}$$

Proof: First, observe that

$$x_t = x + \frac{c_{-1}(t)}{x} + \frac{c_{-2}(t)}{x^2} + \dots \, .$$

Hence $\left.\dfrac{\partial x_t}{\partial t}\right|_{t=0}$ is a Laurent series in x consisting of strictly negative powers of x. For small t function $w = \sqrt{z^2 - t}$ is a convergent series in non-negative powers of x_t:

$$\sqrt{z^2 - t} = \sum_{i \geq 0} a_i(t) x_t^i =: f_t(x_t) \ .$$

Expanding this identity in t up to t^1 we get

$$z - \frac{t}{2z} + O(t^2) = f_0(x) + t\left(\left.\frac{\partial f_t}{\partial t}\right|_{t=0}(x) + f_0'(x)\left.\frac{\partial x_t}{\partial t}\right|_{t=0}\right) + O(t^2) \ .$$

Comparing coefficients in front of t^1 we conclude that $\left.\frac{\partial x_t}{\partial t}\right|_{t=0}$ is the negative power part of the series

$$-\frac{1}{2f_0(x)f_0'(x)} = -\frac{1}{2x} + O(x)\,. \quad \square$$

Remark 3.1. The Schiffer variation corresponds, up to a scalar factor, to the action of the generator $L_2 = -z^{-1}\mathrm{d}/\mathrm{d}z$ (see (2.29)), in the so-called Virasoro uniformisation of moduli spaces. Cf. [BS], [K1].

3.3.2. Connection with the Schwarzian derivative. In this subsection we continue to work with special neutral four-sphere collection $\mathfrak{F}$. Such a collection gives rise to a real number

$$\rho_{\mathfrak{F}} := \log\left[v_{\mathfrak{F}} \Big/ \Big(\bigotimes_{1\le i,j\le 2} v_{S_{ij}}^{\otimes \mu_{ij}}\Big)\right].$$

Let us assume that a point $p \in S_{1,L}$ is given, together with a germ of local coordinate z at p. Then we can perform Schiffer variations of surfaces S_1 and S_2 and obtain a one-parameter family of neutral collections $\mathfrak{F}_t$. Our goal here is to calculate the first derivative $\left.\frac{\partial \rho_{\mathfrak{F}_t}}{\partial t}\right|_{t=0}$.

It follows easily from the definitions that the expression in question coincides with the limit, as $t \to 0$, of the value $\frac{1}{t}\rho_{\widetilde{\mathfrak{F}}_t}$. Here $\widetilde{\mathfrak{F}}_t$ is a 'perturbed' neutral four-sphere collection consisting of S_1, S_2, $S_{1,t}$ and $S_{2,t}$, with multiplicities $(-1,+1,+1,-1)$.

Let us choose parametrisations $x_1, x_2, x_{1,t}, x_{2,t}$ of spheres S_1, S_2, $S_{1,t}$ and $S_{2,t}$ by $\mathbb{C}P^1$ such that $x_1(p) = x_2(p) = 0$ and $x_{1,t} = x_1 + O(1/x_1)$ at $x_1 = \infty$, and a similar condition for $x_{2,t}$. Moreover, we can assume that $x_1 = z + O(z^2)$, $x_2 = z + O(z^2)$ near p. From subsection 3.3.1, we know that $x_{i,t} = x_i - \frac{t}{2x_i} + O(t^2)$.

The application of the residue formula (3.8) from subsection 3.2.2 yields the following integral

$$\frac{-1}{24\pi}\mathrm{Im}\int\limits_L \log\left[\frac{\mathrm{d}x_1^{-1}}{\mathrm{d}\left(x_2-\dfrac{t}{2x_2}+O(t^2)\right)^{-1}}\right] \times \mathrm{d}\log\left[\frac{\mathrm{d}x_2^{-1}}{\mathrm{d}\left(x_1-\dfrac{t}{2x_1}+O(t^2)\right)^{-1}}\right]. \tag{3.16}$$

A straightforward calculation then shows that the above expression is equal to

$$\frac{t}{12}\mathrm{Re}\,\mathcal{S}_f(0)+O(t^2), \tag{3.17}$$

where function f is defined by $f(x_1)=x_2$ and its Schwarzian derivative $\mathcal{S}_f$ is given by the standard formula

$$\mathcal{S}_f=\frac{f'''}{f'}-\frac{3(f'')^2}{2(f')^2}. \tag{3.18}$$

Thus, we have proved the following

Proposition 1. *In the above notation,*

$$\left.\frac{\partial\rho_{\mathfrak{F}_t}}{\partial t}\right|_{t=0}=\frac{1}{12}\mathrm{Re}\,\mathcal{S}_f(0). \tag{3.19}$$

3.4 The limit formula for degenerating neutral collections

Let Σ be a compact surface with two marked points p_1, p_2, and $(\Sigma_t)_{t\in[t_0,+\infty)}$ be a one-parameter family of compact surfaces which approach in a certain sense the singular surface $\Sigma_\infty := \Sigma/\{p_1=p_2\}$, the result of identification of points p_1 and p_2 on Σ. More precisely, we assume that for each t an open part $U_t\subset\Sigma_t$ is identified with an open domain $U_t'\subset\Sigma\setminus\{p_1,p_2\}$, and U_t is the complement to a closed cylinder in Σ_t, U_t' is the complement to the union of two small closed $\epsilon(t)$-neighborhoods of points p_1 and p_2 in a certain metric on Σ, such that $\epsilon(t) \to 0$ as $t \to +\infty$.

First, we will define a determinant line $\big|\mathrm{Det}\big|_{\Sigma_\infty}$ and the notion of convergence of points $v_t \in \big|\mathrm{Det}\big|_{\Sigma_t}$ to a point in $\big|\mathrm{Det}\big|_{\Sigma_\infty}$ as $t \to +\infty$. Namely, we define an admissible metric g on Σ_∞ as a singular metric on Σ with divisor $-(p_1 + p_2)$.

By definition, there will be a vector $[g] \in \big|\mathrm{Det}\big|_{\Sigma_\infty}$ for every admissible metric g. For any two admissible metrics g_1, g_2 we define the ratio of corresponding vectors by the same formula as in the non-singular case:

$$[g_2]/[g_1] := \exp\ \left[\mathrm{S}_{\mathrm{Liouv}}(g_1, g_2)\right] \tag{3.20}$$

where

$$\mathrm{S}_{\mathrm{Liouv}}(g_1, g_2) := \frac{1}{48\pi\mathrm{i}} \int\limits_{\Sigma\setminus\{p_1,p_2\}} \mathrm{L}_{\mathrm{Liouv}}(g_1, g_2)\,. \tag{3.21}$$

The cocycle identity for singular metrics

$$\mathrm{S}_{\mathrm{Liouv}}(g_1, g_3) = \mathrm{S}_{\mathrm{Liouv}}(g_1, g_2) + \mathrm{S}_{\mathrm{Liouv}}(g_2, g_3) \tag{3.22}$$

again follows from (2.3); the argument here is similar to the one used in Lemmas 3.1 and 3.2.

Next, from results in section 3.1 it follows that there is a canonical isomorphism

$$i_{\Sigma_\infty} : \big|\mathrm{Det}\big|_{\Sigma} \otimes \big|\det T_{p_1}\Sigma\big|^{\otimes 1/24} \otimes \big|\det T_{p_2}\Sigma\big|^{\otimes 1/24} \to \big|\mathrm{Det}\big|_{\Sigma_\infty}. \tag{3.23}$$

Further, we say that a family of metrics $(g_t)_{t\in[t_0,+\infty)}$ on surfaces Σ_t is convergent to an admissible metric g_∞ on Σ_∞ if the following holds. There exists a pair of closed geodesics L_1, L_2, in metric g_∞, surrounding points p_1, p_2, such that, on the cylinders $C_t \subset \Sigma_t$ bounded by curves L_1, L_2, metric g_t is flat, and both curves L_1, L_2 are geodesics of length 2π in metric g_t.We can also assume that metrics g_t converge uniformly to g_∞ on the part of Σ lying outside to punctured disks bounded by L_1 and L_2. Indeed, such families of metrics exist because of the following result:

Lemma 3.6. *Given $s \in [0, 1)$, set*

$$A_s = \{z \in \mathbb{C} :\ \ s < |z| \leq 1\}.$$

Assume that a positive function $r(t)$ is given, where $t > 0$, such that $r(t) \to 0$ as $t \to \infty$. Let ϕ_t be a holomorphic embedding $A_{r(t)} \to A_0$ mapping the boundary circle $|z| = 1$ to itself. Denote by g_t the pullback by ϕ_t of the flat metric $\left|\mathrm{d}z/z\right|^2$. Then, as $t \to \infty$, the metrics g_t converge, uniformly in the C^∞ topology on compacts in the punctured disk A_0 =, to metric $\left|\mathrm{d}z/z\right|^2$.

Proof: The main part of the proof of Lemma 3.6 is the following fact [SS]. Given $s \in (0,1)$, consider an embedding ϕ: $A_s \to A_0$ such that $|\phi(z)| = 1$ for $|z| = 1$. Then, as $s \to 0$, the image $\phi(A_s)$ contains the annulus $\{z \in \mathbb{C} : \ (4 + o(s))s < |z| < 1\}$. The assertion of Lemma 3.6 is then deduced by means of a straightforward argument using the potential theory. □

Next, assume that $(g_t)_{t\in[t_0,+\infty)}$ and $(g'_t)_{t\in[t_0,+\infty)}$ are two families of metrics converging, respectively, to admissible metrics g_∞ and g'_∞ on Σ_∞. Then we have that

$$\lim_{t\to\infty} [g_t]/[g'_t] = [g_\infty]/[g'_\infty] \,. \tag{3.24}$$

It allows us to define a topology near $+\infty$, on the line bundle over $[t_0, +\infty]$ with fibers $\left|\mathrm{Det}\right|_{\Sigma_t}$.

Further, we are going to introduce a map

$$dist : [t_0, +\infty) \to \left|\det T_{p_1}\Sigma\right| \otimes \left|\det T_{p_2}\Sigma\right| \tag{3.25}$$

defined up to a mutliplication by a positive function $f(t)$ such that $\lim\limits_{t\to\infty} f(t) = 1$. Let us choose two local coordinates z_1 and z_2 near points p_1, p_2. These coordinates give an identification of each line $\left|\det T_{p_i}\Sigma\right|$, $i = 1, 2$, with $\mathbb{R}$. Hence, to define map $dist$, it suffices to fix a real-valued function of t. We choose this function to be equal to the conformal parameter of the cylinder on Σ_t bounded by circles $|z_1| = 1$ and $|z_2| = 1$. Here, the conformal parameter of a cylinder C is a number $t \in (0,1)$ such that C is conformally equivalent to $\{z \in \mathbb{C} : \quad t < |z| < 1\}$. The fact that map $dist$ is defined up to a mutliplication by a positive function $f(t)$ such that $\lim_{t\to\infty} f(t) = 1$, for different choices of pairs of local coordinates z_1, z_2, follows easily from arguments similar to those used earlier in this subsection.

Now assume that Σ is a disjoint union of two spheres, and that points p_1, p_2 belong to different components. Then each surface Σ_t, $t \in [t_0, \infty)$, is a sphere. Therefore, we have a canonical vector $v_{\Sigma_t} \in \left|\mathrm{Det}\right|_{\Sigma_t}$, and also a canonical vector $v_{\Sigma_\infty} \in \left|\mathrm{Det}\right|_{\Sigma_\infty} \otimes \left|\det T_{p_1}\Sigma\right|^{\otimes 1/24} \otimes \left|\det T_{p_2}\Sigma\right|^{\otimes 1/24}$ (the tensor product of the canonical vectors of two connected components). Our goal in this subsection is to understand the behavior, when $t \to \infty$, of vectors $v_{\Sigma_t} \in \left|\mathrm{Det}\right|_{\Sigma_t}$ in relation to $v_{\Sigma_\infty} \in \left|\mathrm{Det}\right|_{\Sigma_\infty}$.

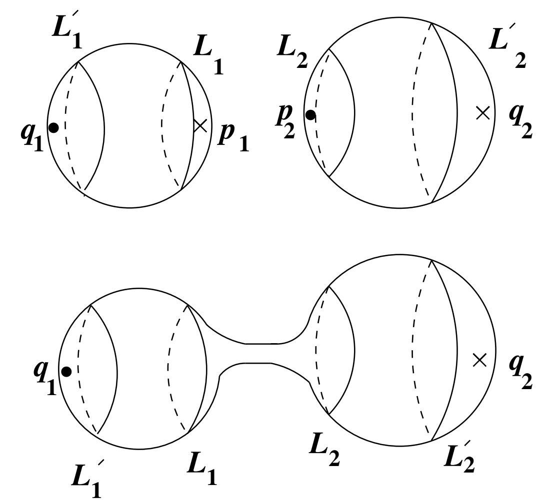

Figure 13: Two surfaces, disjoint, and with a small connecting tube

Proposition 2. *In the case where Σ is a disjoint union of two spheres and points p_1, p_2 belong to different components as above, one has*

$$\lim_{t\to\infty} \left[v_{\Sigma_\infty} \otimes dist(t)^{\otimes -1/24}\right] / v_{\Sigma_t} = 1\,. \tag{3.26}$$

Here we use an identification of lines $\left|\mathrm{Det}\right|_{\Sigma_t}$ with $\left|\mathrm{Det}\right|_{\Sigma_\infty}$ compatible with the topology at $t = \infty$ introduced above.

Proof: It is convenient here to use singular metrics with two simple poles. We choose two points q_1, q_2 on two corresponding components of $\Sigma \setminus \{p_1, p_2\}$. Surface $\Sigma \setminus \{p_1, p_2, q_1, q_2\}$ is represented as a disjoint union of two copies of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Thus, we have on $\Sigma\setminus\{p_1, p_2, q_1, q_2\}$ a unique flat metric g_Σ with singularity at divisor $-(p_1 + p_2 + q_1 + q_2)$. Similarly, on surface Σ_t, $t \geq t_0$, we have a unique flat metric g_{Σ_t} with singularity at $-(q_1 + q_2)$. Also let us choose positive elements $d_i \in \left|\det T_{q_i}\Sigma\right|$.

Owing to results in section 3.1, metric g_Σ, together with pair d_1, d_2, gives a vector $\delta_\infty \in \left|\mathrm{Det}\right|_{\Sigma_\infty}$. Also for any $t \in [t_0, +\infty)$ metric g_{Σ_t} together with pair d_1, d_2 yields a vector $\delta_t \in \left|\mathrm{Det}\right|_{\Sigma_t}$. It follows from the above definitions that

$$\lim_{t\to\infty} \delta_t = \delta_\infty \ .$$

Now choose positive elements $d_i' \in \left|\det T_{p_i}\Sigma\right|$. They can be represented as closed geodesics L_i, $i = 1, 2$, in metric g_Σ. For large t circles L_i are close to geodesics in metric g_{Σ_t}.

It is easy to see that function $dist(t)$ is equal, asymptotically as $t \to +\infty$, to the conformal parameter of the cylinder on Σ_t bounded by circles L_1 and L_2, in the trivialisation of real line $\left|\det T_{p_1}\Sigma\right| \otimes \left|\det T_{p_2}\Sigma\right|$ given by $d_1' \otimes d_2'$. Finally, we should compare our vectors with the canonical vectors in the determinant line of spheres Σ, Σ_1 and Σ_2. This can be done using the following straightforward fact which we give without proof:

Lemma 3.7. *Let d_0 be a vector from $\left|\det T_0\mathbb{C}P^1\right|$ and d_∞ be a vector from $\left|\det T_\infty\mathbb{C}P^1\right|$. Then*

$$v_{\mathbb{C}P^1}\big/(d_0 \otimes d_\infty) = \mathrm{const} \cdot h^{1/24}_{C_{d_0,d_\infty}}. \tag{3.27}$$

Here $h_{C_{d_0,d_\infty}}$ is the conformal parameter of the cylinder C_{d_0,d_∞} bounded by two circles corresponding to d_0 and d_∞ and const > 0 *is an absolute constant.*

Let L_i',$i = 1, 2$, denote circles (in metric g_Σ) surrounding points q_1, q_2, corresponding to vectors d_1, d_2. The assertion of Proposition 2 can be restated as follows:

Lemma 3.8. *Let h_1 and h_2 be conformal parameters of cylinders $C_{L_1,L_1'}$ and $C_{L_2,L_2'}$ bounded by pairs of circles (L_1, L_1') and (L_2, L_2') respectively. Let $h_{\mathrm{in}}(t)$ be the conformal parameter of cylinder C_{L_1,L_2} in Σ_t bounded by (L_1, L_2), and $h_{\mathrm{out}}(t)$ the conformal paramater of cylinder $C_{L_1',L_2'}$ in Σ_t bounded by (L_1', L_2'). Then one has*

$$\lim_{t\to\infty} h_{out}(t)/h_{in}(t) = h_1 h_2\,. \tag{3.28}$$

Proof of Lemma 3.8: Let g_{int} be the unique flat metric with geodesic boundaries of length 2π on the cylinder C_{L_1,L_2}. Let us glue two flat cylinders with conformal parameters h_1 and h_2 to the ends of C_{L_1,L_2}. We obtain a flat

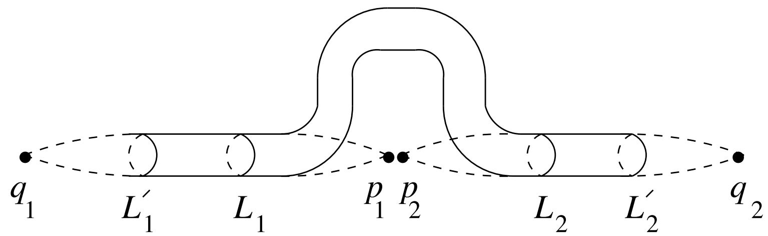

Figure 14: Another picture of glued surfaces and of tube C_{L_1,L_2}

metric on a cylinder C' embedded into Σ_t such that the geodesic boundaries of C' are close to to lines L_1', L_2'. This follows from Lemma 3.6 and the reflection principle. The conformal parameter of C' will be close to that of cylinder $C_{L_1',L_2'}$, owing to monotonicity of the conformal parameter with respect to embeddings of annuli. By construction, the conformal parameter of C' is equal to $h_1 h_2 h_{\text{in}}(t)$.

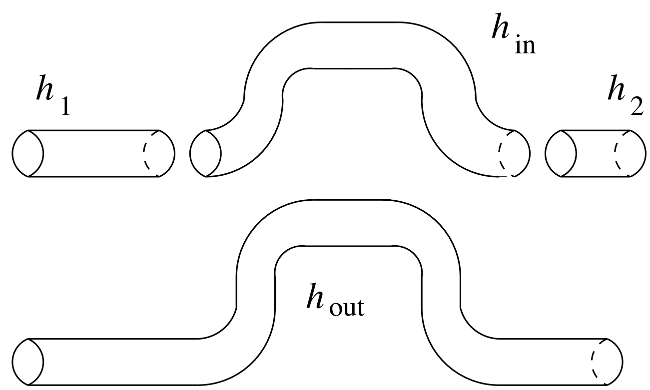

Figure 15: Various tubes and their conformal parameters

This completes the proof of Lemma 3.8 and that of Proposition 2. $\square$

4 The SLE-measures, I

4.1 Spaces of intervals and associated line bundles

In chapters 4 and 5 we work with a surface Σ with a non-empty boundary $\partial\Sigma \subset \Sigma$, and with a conformal structure that is smooth everywhere including $\partial\Sigma$, and a pair of distinct points $x, y \in \partial\Sigma$. Note that it is not meant that Σ should be necessarily closed; a working example of a surface with a boundary is a semi-open rectangle $(a,b)\times[c,d]$ where $a < b$ and $c < d$ are real numbers. Here, the boundary $\partial\Sigma$ is $(a,b) \times \{c,d\}$. The above conditions on surface Σ and points x, y are assumed in this and the following chapter without stressing them every time again.

We define an (oriented) *interval* $\mathcal{I}$ in Σ with endpoints x and y as an equivalence class of homeomorphic embeddings of the unit segment

$$\iota : \; [0,1] \hookrightarrow \Sigma, \;\; \text{with } \iota(0) = x, \; \iota(1) = y \text{ and } \iota((0,1)) \subset \Sigma \setminus \partial\Sigma, \tag{4.1}$$

modulo the action of the group of orientation-preserving homeomorphisms $[0,1] \to [0,1]$ acting by re-parametrisations.

The space of intervals in Σ with endpoints x and y is denoted by $Int_{x,y}(\Sigma)$ and is endowed with the topology induced from $Comp(\Sigma)$. Like $Loop(\Sigma)$, space $Int_{x,y}(\Sigma)$ is not closed in $Comp(\Sigma)$ and not locally compact.

Assume that $\mathcal{I} \in Int_{x,y}(\Sigma)$ is an interval. First, we introduce line $|\det|_{\mathcal{I},\Sigma}$. Suppose we are given an open subset $U \subset \Sigma$ containing $\mathcal{I}$ and such that U as a surface is of finite type. (A surface with boundary is called of finite type iff it has finite Betti numbers and its boundary has finitely many components. An example is the union of an open disk $U = \{z \in \mathbb{C} : \; |z| < 1\}$ with a finite number of open disjoint arcs lying in the circle $\{z \in \mathbb{C} : \; |z| = 1\}$). Following (2.14), we set:

$$|\det|_{\mathcal{I},\Sigma} = \frac{|\det|_{U\setminus\partial\Sigma}}{|\det|_{U\setminus(\mathcal{I}\cup\partial\Sigma)}} :\simeq |\det|_{U\setminus\partial\Sigma} \otimes \Big(|\det|_{U\setminus(\mathcal{I}\cup\partial\Sigma)}\Big)^{\otimes(-1)}. \tag{4.2}$$

The identification of lines defined as above for different subsets $U \supset \mathcal{I}$ is given in the same manner as for the case of loops; cf (2.15)–(2.17). Next, we define the continuous line bundle $\big|\mathrm{Det}\big|_{\Sigma,x,y}$ on the space of intervals $Int_{x,y}(\Sigma)$, similarly to the analogous bundle for loops.

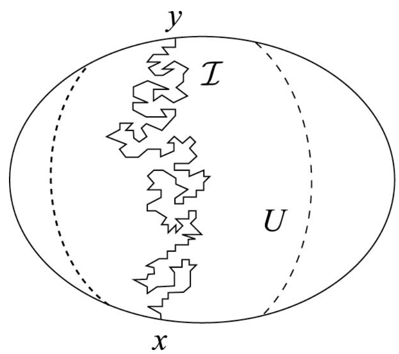

Figure 16: An interval and its open neighborhood

Remark 4.1. We would like to warn the reader of a possible caveat. Namely, one may try to define a determinant line bundle on $Int_{x,y}(\Sigma)$ using the following observation. On surface

$$\Sigma' := (\Sigma \setminus \partial\Sigma)_{\mathrm{double}}$$

we have involution σ that exchanges the copies of Σ. Obviously, any interval $\mathcal{I} \in Int_{x,y}(\Sigma)$ gives a loop $\mathcal{I}'$ on Σ' invariant under involution σ. An alternative approach to the definition of the determinant line of $\mathcal{I}$ would be as $|\det|^{\otimes 1/2}_{\mathcal{I}',\Sigma'}$. This line is *not* isomorphic to our $|\det|_{\mathcal{I},\Sigma}$, the ratio is certain line bundle on $Int_{x,y}(\Sigma)$ depending on $\mathcal{I}$ is only via the germ of $\mathcal{I}$ near its endpoints. ■

We will also need another *trivial* line bundle $\left|\mathrm{Tan}\right|_{\Sigma,x,y}$ on $Int_{x,y}(\Sigma)$. The fiber of $\left|\mathrm{Tan}\right|_{\Sigma,x,y}$ at any point $\mathcal{I} \in Int_{x,y}(\Sigma)$ is the product

$$|T_x\partial\Sigma| \otimes |T_y\partial\Sigma|. \tag{4.3}$$

Here we use the notation $|V|$, where V is a non-oriented one-dimensional real vector space, introduced in subsection 3.1.2.

Definition 4.1. Fix real numbers c and h and assume that for every surface Σ and pair of points $x, y \in \partial\Sigma$ we are given a measure $\boldsymbol{\lambda}_{\Sigma,x,y}$ on $Int_{x,y}(\Sigma)$ with values in

$$\left|\mathrm{Tan}\right|^{\otimes(-\mathrm{h})}_{\Sigma,x,y} \otimes \left|\mathrm{Det}\right|^{\otimes \mathrm{c}}_{\Sigma,x,y}. \tag{4.4}$$

We say that the (measure-valued) assignment $(\Sigma, x, y) \mapsto \boldsymbol{\lambda}_{\Sigma,x,y}$ is (c, h)-LCC (or briefly, LCC) if for any embedding $\xi : \Sigma \hookrightarrow \Sigma'$ we have

$$\xi^*\left(\boldsymbol{\lambda}_{\Sigma',\xi(x),\xi(y)}\right) = \boldsymbol{\lambda}_{\Sigma,x,y}, \tag{4.5}$$

where we again use the obvious identification of the line bundles, associated with ξ. ■

Now consider a family of values $\mathrm{c}(\theta)$ and $\mathrm{h}(\theta)$ parametrised by $\theta \in (0,1]$:

$$\mathrm{c} = (3-2\theta)\left(3-\frac{2}{\theta}\right), \quad \mathrm{h} = \frac{3/\theta - 2}{4}. \tag{4.6}$$

Note that the correspondence between θ and c and between θ and h is one-to-one, the range for $\mathrm{c}(\theta)$ is $(-\infty, 1]$ and the range for $\mathrm{h}(\theta)$ is $[1/4, +\infty)$.

Theorem 1. *For any $0 < \theta \leq 1$ there exists a non-zero (c, h)-LCC assignment $(\Sigma, x, y) \mapsto \boldsymbol{\lambda}_{\Sigma,x,y}$. Here $\mathrm{c} = \mathrm{c}(\theta)$, and $\mathrm{h} = \mathrm{h}(\theta)$ are given by* (4.6).

We also put forward

Conjecture 2. *For any $0 < \theta \leq 1$, the LCC assignment in Theorem 1 is unique, up to a scalar factor.*

Sections 4.2–5.2 aim at the proof of Theorem 1. In fact, we will prove that an LCC assignment is generated by the chordal SLE_κ processes (see section 4.3), with $\kappa \in (0, 4]$. The key property here is that paths produced by the process SLE_κ are simple Jordan curves precisely for $\kappa \in (0, 4]$. The relation between κ and θ is straightforward: $\kappa = 4\theta$. As to uniqueness, it can be verified for $\mathrm{c} = 0$; see Remark 5.2 in section 5.3.

Measures $\boldsymbol{\lambda}_{\Sigma,x,y}$ will be called the *SLE measures*, in analogy with the Malliavin measures.

In what follows, we use relation (4.6) between θ and pair (c, h) without specifying it every time again.

Exponents c and h come from highest vectors in level 2 degenerate Virasoro modules, see section 6.3.

4.2 Reduction to $\overline{\mathbb{H}}$

4.2.1. Space $Int_{0,\infty}$. The first (obvious) step of the proof of Theorem 1 is that it suffices to construct measures $\boldsymbol{\lambda}_{\Sigma,x,y}$ on $Int_{x,y}(\Sigma)$ in the case where Σ is a semi-open rectangle $R_\epsilon = (-\epsilon, \epsilon) \times [0,1]$, with the boundary $\partial R_\epsilon = (-\epsilon, \epsilon) \times \{0, 1\}$, where $x = (0;0)$, and $y = (0;1)$, such that property (4.5) holds for all embeddings $\xi : R_\epsilon \hookrightarrow R_\epsilon$, with $\xi(x) = x$, $\xi(y) = y$.

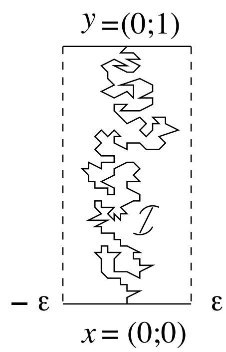

Figure 17: An interval in a rectangle

Next, a semi-open rectangle R_ϵ can be embedded in a closed disk identified with the compactified upper half-plane $\overline{\mathbb{H}}$, so that x is taken to 0 and y to ∞. Formally:

$$\overline{\mathbb{H}} = \mathbb{H} \sqcup \mathbb{R}P^1 \tag{4.7}$$

where $\mathbb{H}$ is the open upper half-plane and $\mathbb{R}P^1 = \partial\overline{\mathbb{H}}$ is the extended real line

$$\mathbb{H} = \{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}, \ \mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}. \tag{4.8}$$

Note that with every such embedding we have $R_\epsilon \subset \overline{R_\epsilon} = \overline{\mathbb{H}}$. Thus, we can associate with R_ϵ a natural isomorphism

$$Int_{x,y}(R_\epsilon) \simeq Int_{0,\infty}(\overline{\mathbb{H}}),$$

and the corresponding identification of line bundles

$$\big|\mathrm{Det}\big|_{R_\epsilon,x,y} \simeq \big|\mathrm{Det}\big|_{\overline{\mathbb{H}},0,\infty}, \ \ \big|\mathrm{Tan}\big|_{R_\epsilon,x,y} \simeq \big|\mathrm{Tan}\big|_{\overline{\mathbb{H}},0,\infty}.$$

The reason is that bundles $\big|\mathrm{Det}\big|_{\Sigma,x,y}$ and $\big|\mathrm{Tan}\big|_{\Sigma,x,y}$ (for a general surface Σ) do not change if we modify $\partial\Sigma$ without changing neighbourhoods $U_x, U_y \subset \partial\Sigma$ of points x and y in $\partial\Sigma$ and the interior $\Sigma \setminus \partial\Sigma$.

Therefore, the assertion of Theorem 1 follows if, $\forall\, \theta \in (0,1]$, we construct a measure $\boldsymbol{\lambda}_{\overline{\mathbb{H}},0,\infty}$ on

$$Int_{0,\infty} := Int_{0,\infty}(\overline{\mathbb{H}}), \tag{4.9}$$

with values in the bundle

$$\left(\big|\mathrm{Tan}\big|_{0,\infty,\overline{\mathbb{H}}}\right)^{\otimes(-\mathrm{h})} \otimes \left(\big|\mathrm{Det}\big|_{\overline{\mathbb{H}}}\right)^{\otimes \mathrm{c}}. \tag{4.10}$$

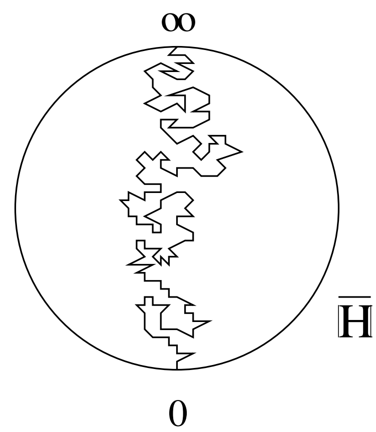

Figure 18: Interval in $\overline{\mathbb{H}}$ connecting 0 and ∞

such that the property (4.5) holds for any continuous map $\xi:\ \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ such that $\xi(0) = 0$, $\xi(\infty) = \infty$, and the restriction $\xi\big|_{\mathbb{H}\cup U_0 \cup U_\infty}$ is a holomorphic embedding, for some open neighbourhoods $U_0, U_\infty \subset \mathbb{C}P^1$ of points 0, ∞ in $\mathbb{C}P^1$:

$$\xi^*(\boldsymbol{\lambda}_{\overline{\mathbb{H}},0,\infty}) = \boldsymbol{\lambda}_{\overline{\mathbb{H}},0,\infty}.$$

4.2.2. Trivialisations of line bundles on $Int_{0,\infty}$. Group $\mathbb{R}^\times_{>0} = Aut(\overline{\mathbb{H}}, 0, \infty)$ acts by dilations on $\overline{\mathbb{H}}$ and hence on $Int_{0,\infty}$ and line bundles $\big|\mathrm{Tan}\big|_{\overline{\mathbb{H}},0,\infty}$ and $\big|\mathrm{Det}\big|_{\overline{\mathbb{H}},o,\infty}$. We construct a $\mathbb{R}^\times_{>0}$-equivariant trivialisation of both these bundles. By the definition of determinant line, we have a canonical isomorphism

$$|\det|_{\mathcal{I},\overline{\mathbb{H}}} \simeq |\det|_{\mathbb{H}} \Big/ |\det|_{\mathbb{H}\backslash\mathcal{I}} \quad .$$

Observe that for any interval $\mathcal{I} \in Int_{0,\infty}(\overline{\mathbb{H}})$ the complement $\mathbb{H}\backslash\mathcal{I}$ is isomorphic to the disjoint union $(\mathbb{H}\setminus\mathcal{I})^{\mathrm{left}} \sqcup (\mathbb{H}\setminus\mathcal{I})^{\mathrm{right}}$ of two copies of an open disk. Therefore the ratio of canonical vectors gives a trivialisation

$$v^{\det}_{\mathcal{I}} := v_{\mathbb{H}} \otimes \left(v_{(\mathbb{H}\backslash\mathcal{I})^{\mathrm{left}}} \otimes v_{(\mathbb{H}\backslash\mathcal{I})^{\mathrm{right}}}\right)^{\otimes(-1)}$$

of bundle $\big|\mathrm{Det}\big|_{\overline{\mathbb{H}}}$, obviously invariant under $\mathbb{R}^\times_{>0}$ action.

Next, the tensor product of the unit tangent vector at $x = 0$ to $\mathbb{C}P^1$ and its image under inversion at $y = \infty$ is a vector

$$v^{\tan} \in \big|\mathrm{Tan}\big|_{0,\infty,\overline{\mathbb{H}}}$$

invariant under the action of $\mathbb{R}^{\times}_{>0}$. Further, any $\mathbb{R}^{\times}_{>0}$-invariant measure $\boldsymbol{\lambda}_{\overline{\mathbb{H}},0,\infty}$ on $Int_{0,\infty}$ with values in bundle (4.10) gives an ordinary (scalar) $\mathbb{R}^{\times}_{>0}$-invariant measure $\boldsymbol{\nu}^{\mathrm{c,h}}$ on $Int_{0,\infty}$, after division by the following section of this line bundle:

$$\mathcal{I} \in Int\,(\overline{\mathbb{H}}) \mapsto (v^{\tan})^{\otimes(-\mathrm{h})} \otimes (v^{\det}_{\mathcal{I}})^{\otimes \mathrm{c}}.$$

The last measure should satisfy a certain condition, called the restriction covariance property and discussed below.

4.2.3. Restriction covariance property for measures on $Int_{0,\infty}$. Let $\alpha:\ \mathbb{H} \hookrightarrow \mathbb{H}$ be an embedding of the open half-plane into itself, which extends by continuity to a continuous map $\overline{\mathbb{H}} \hookrightarrow \overline{\mathbb{H}}$, denoted again by α, such that $\alpha(0) = 0$, $\alpha(\infty) = \infty$, and α can be continued to a holomorphic map to $\mathbb{C}P^1$ near points 0 and ∞. Then α induces an open embedding

$$\alpha_*:\ Int_{0,\infty} \hookrightarrow Int_{0,\infty}\,. \tag{4.11}$$

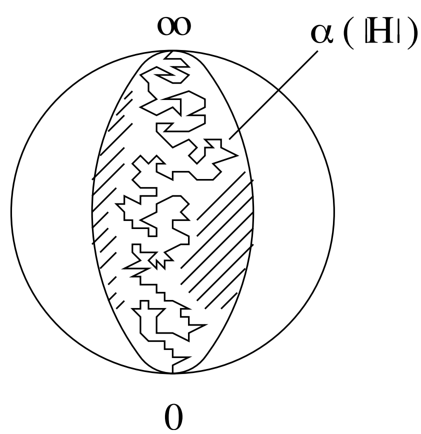

Figure 19: Interval in $\alpha(\mathbb{H}) \subset \mathbb{H}$

Given α as above, there are defined a positive constant, $q^{\tan}_{\alpha}$, and a positive continuous function, $q^{\det}_{\alpha}(\mathcal{I})$, $\mathcal{I} \in Int_{0,\infty}$. Constant $q^{\tan}_{\alpha}$ is given by the product

$$q^{\tan}_{\alpha} = q^{\tan}_{\alpha,0}\, q^{\tan}_{\alpha,\infty}. \tag{4.12}$$

Here numbers $q^{\tan}_{\alpha,0}$, $q^{\tan}_{\alpha,\infty} > 0$ are determined from the Taylor expansions at 0 and ∞:

$$\alpha(z) = q^{\tan}_{\alpha,0}\, z + O(z^2), \quad z \to 0; \quad \frac{1}{\alpha(z)} = q^{\tan}_{\alpha,\infty}\, \frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty. \tag{4.13}$$

Next, function $q^{\det}_{\alpha}(\mathcal{I})$ on $Int_{0,\infty}$ is defined as follows. Given $\mathcal{I} \in Int_{0,\infty}$ and map α, we construct a neutral collection $\mathfrak{S}_{\alpha,\mathcal{I}}$ consisting of six spheres Σ_1, Σ_2, Σ_3, Σ_4, Σ_5 and Σ_6 that are the doubles of six open disks D_1, D_2, D_3, D_4, D_5 and D_6, correspondingly. Namely, these disks will be

$$\alpha(\mathbb{H}), \alpha((\mathbb{H} \setminus \mathcal{I})_{\mathrm{L}}), \alpha((\mathbb{H} \setminus \mathcal{I})_{\mathrm{R}}), \mathbb{H}, (\mathbb{H} \setminus \alpha(\mathcal{I}))_{\mathrm{L}}, (\mathbb{H} \setminus \alpha(\mathcal{I}))_{\mathrm{R}} \tag{4.14}$$

taken with weights $\mu_1 = +1$, $\mu_2 = -1$, $\mu_3 = -1$, $\mu_4 = -1$, $\mu_5 = +1$ and $\mu_6 = +1$. As before, subscripts L stand for left and R for right.

The non-Hausdorff surface S^{nH} containing these six spheres is the union of the doubles of disks $D_1, \ldots, D_6$ glued all along the domain that is the pullback to the double covering of the union of two thin strips on the left and on the right of $\mathcal{I} \subset \overline{\mathbb{H}}$ (see Figure 20).

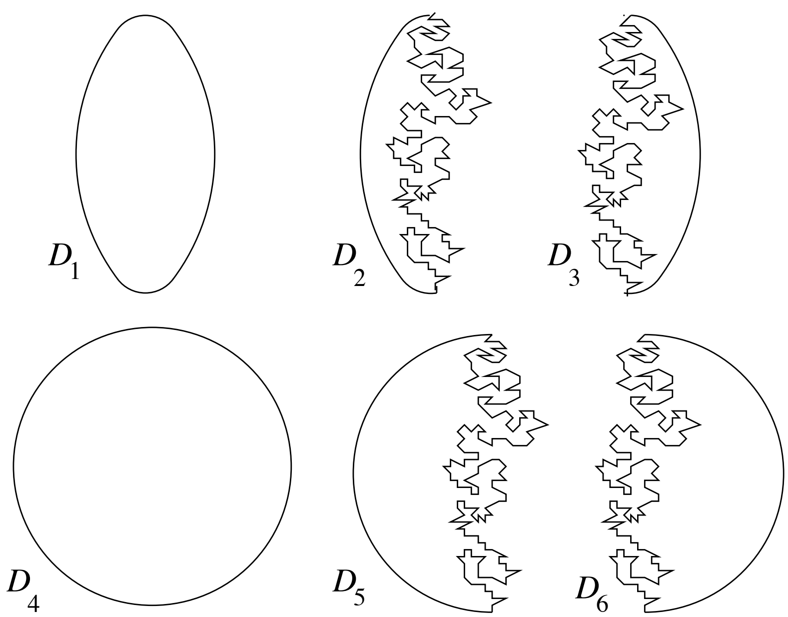

Figure 20: Six disks $D_1, \ldots, D_6$

Schematically, one can draw surface S^{nH} for collection $\mathfrak{S}_{\alpha,\mathcal{I}}$ as drawn on Figure 21.

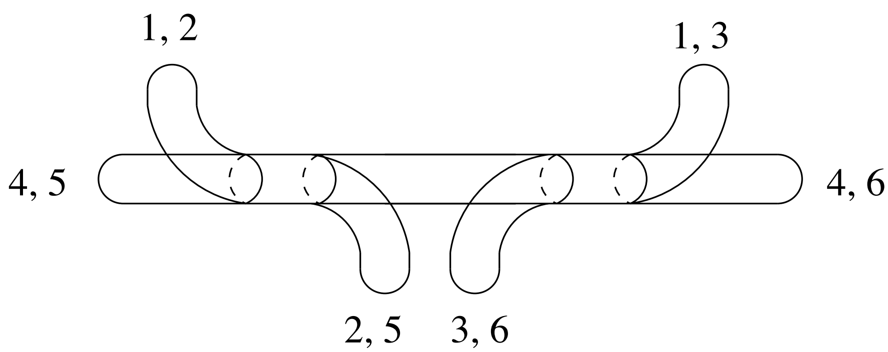

Figure 21: Non-Hausdorff surface S^{nH}

Here sphere S_i connects two half-spherical caps which have value i among the pair of indeces attached to them. (So, sphere S_4 is the horizontal one.)

Value $q_\alpha^{\det}(\mathcal{I})$ is then defined as follows:

$$q_\alpha^{\det}(\mathcal{I}) = \left(v_{\mathfrak{S}_{\alpha,\mathcal{I}}} \Big/ \bigotimes_{k=1}^{6} v_{S_k}^{\otimes \mu_k} \right)^{1/2}. \tag{4.15}$$

Definition 4.2. We call a (scalar) measure $\boldsymbol{\nu}$ on $Int_{0,\infty}$ (c,h)-*restriction covariant*, or briefly, restriction covariant (RC) if, for any embedding $\alpha : \mathbb{H} \hookrightarrow \mathbb{H}$ as above, the pullback $\alpha^*\big(\boldsymbol{\nu}\big|_{\alpha_*(Int_{0,\infty})}\big)$ of the restriction $\boldsymbol{\nu}\big|_{\alpha_*(Int_{0,\infty})}$ of measure $\boldsymbol{\nu}$ to the image $\alpha_*(Int_{0,\infty})$ (which is an open subset in $Int_{0,\infty}$) is absolutely continuous with respect to $\boldsymbol{\nu}$ and has the Radon-Nikodym derivative

$$\frac{\mathrm{d}\big[\alpha^*(\boldsymbol{\nu})\big]\big|_{\alpha_*(Int_{0,\infty})}}{\mathrm{d}\boldsymbol{\nu}}(\mathcal{I}) = \big(q_\alpha^{\tan}\big)^{\mathrm{h}} \big(q_\alpha^{\det}(\mathcal{I})\big)^{\mathrm{c}}, \ \ \mathcal{I} \in Int_{0,\infty}. \quad \blacksquare \tag{4.16}$$

By definition, measure $\boldsymbol{\nu}^{\mathrm{c,h}}$ identified in subsection 4.2.2 is (c,h)-RC. Summarising the arguments produced in section 4.2, we obtain the following lemma

Lemma 4.1. *There is a one-to-one correspondence between LCC assignments* $(\Sigma, x, y) \mapsto \boldsymbol{\lambda}_{\Sigma,x,y}$ *and scalar RC measures* $\boldsymbol{\nu}^{\mathrm{c,h}}$ *on* $Int_{0,\infty}$ *invariant under the antiholomorphic involution* $\sigma_{\overline{\mathbb{H}}} : z \mapsto -\overline{z}$.

Invariance of $\boldsymbol{\nu}^{\mathrm{c,h}}$ under $\sigma_{\overline{\mathbb{H}}}$ (also valid by definition of this measure in subsection 4.2.2) is needed here for independence of $\boldsymbol{\lambda}_{\Sigma,x,y}$ of the orientation of Σ near interval $\mathcal{I} \subset \Sigma$.

Note that the assertion of Lemma 4.1 remains correct regardless of condition $\theta \in (0,1]$. However, we need this condition in the course of constructing and RC measure $\boldsymbol{\nu}^{\mathrm{c,h}}$.

4.3 A reminder on SLE processes

4.3.1. The space of hulls and the canonical time parametrisation. Here we follow works [Sc1], [LSW] and their sequel, where a one-parameter family of random processes SLE_κ, $0 < \kappa < +\infty$ was introduced and investigated in great detail. For recent reviews of progress in this direction, see [Sc2], [W2], [W5] and the bibliography therein.

Define a *hull* as a closed subset $\mathcal{K} \subset \mathbb{H}$ with a contractible complement $\mathbb{H} \setminus \mathcal{K}$ and such that ∞ does not lie in the closure of $\mathcal{K}$ in $\overline{\mathbb{H}}$.

Remark 4.2. Our definition of a hull slightly differs from the standard one, see the aforementioned references. In the standard definition, a hull is the closure of a hull in our sense, in $\overline{\mathbb{H}}$. The advantage of our definition is that there is a canonical one-to-one correspondence between hulls and certain holomorphic mappings, see below. ■

For every hull $\mathcal{K}$ there exists a unique uniformisation of its complement $\mathbb{H} \setminus \mathcal{K}$. It is a bijection

$$\gamma_{\mathcal{K}} : \mathbb{H} \setminus \mathcal{K} \simeq \mathbb{H}, \tag{4.17}$$

admitting a holomorphic extension to a neighborhood of point $\infty \in \mathbb{C}P^1$ (which, for simplicity, we denote by the same symbol $\gamma_{\mathcal{K}}$) such that

$$\gamma_{\mathcal{K}}(\bar{z}) = \overline{\gamma_{\mathcal{K}}(z)},\ \gamma_{\mathcal{K}}(\infty) = \infty,\ \gamma_{\mathcal{K}}(z) = z + o(1) \text{ as } |z| \to \infty,\ \ z \in \mathbb{H}.$$

The space *Hull* of hulls is endowed with the following Hausdorff separable topology (and the associated Borel structure). A sequence of hulls $\mathcal{K}_n$ is convergent to $\mathcal{K}$ iff (i) all Taylor coefficients of $1/\gamma_{\mathcal{K}_n}(z)$ at $z = \infty$ converge to those of $1/\gamma_{\mathcal{K}}(z)$, and (ii) $\exists$ a neighbourhood U_∞ of point ∞ such that $\mathcal{K} \cap U_\infty = \emptyset \ \forall\ n$.

We introduce a continuous function $Time : Hull \to [0, +\infty)$ by

$$Time(\mathcal{K}) = 2\gamma_{\mathcal{K}}^{(-1)} \tag{4.18}$$

where $\gamma_{\mathcal{K}}^{(-1)}$ is the first non-trivial coefficient[3] of the Taylor expansion of $\gamma_{\mathcal{K}}$ at $z=\infty$ (i.e., the coefficient in front of $1/z$):

$$\gamma_{\mathcal{K}}(z) = z + \frac{\gamma_{\mathcal{K}}^{(-1)}}{z} + \ldots . \tag{4.19}$$

The inequality $\gamma_{\mathcal{K}}^{(-1)} \geq 0$ (in fact, $\gamma_{\emptyset}^{(-1)} = 0$ and $\gamma_{\mathcal{K}}^{(-1)} > 0$ for $\mathcal{K} \neq \emptyset$) is well known in the theory of conformal embeddings. See, e.g., [W2]. Function $Time$ defines a foliation of space $Hull$ into its level sets $Time^{-1}(t)$, which we repeatedly use below.

For a given real-valued continuous function $w = (w_s)_{s\geq 0}$, taking $s \in [0, +\infty)$ to $w_s \in \mathbb{R}$, with $w_0 = 0$, there exists a unique solution $g_t(z)(= g_t(z; (w_s)))$ of the Loewner equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - w_t}, \quad t > 0, \quad z \in \mathbb{H}, \tag{4.20}$$

with the initial condition

$$g_0(z) = z, \quad z \in \mathbb{H}. \tag{4.21}$$

This solution determines a family of hulls $\mathcal{K}_t \big(= \mathcal{K}_t\big((w_s)_{s\geq 0}\big)\big)$, with $\mathcal{K}_0 = \emptyset$, via the identification

$$g_t(z) = \gamma_{\mathcal{K}_t}(z), \quad z \in \mathbb{H}. \tag{4.22}$$

In what follows, we repeatedly use identification (4.22), without stressing it every time again. It follows immediately from the Loewner equation that

$$Time(\mathcal{K}_t) = t. \tag{4.23}$$

Furthermore, with respect to the above topology on $Hull$, for any given real-valued continuous function (w_s) such that $w_0 = 0$, the solution $g_t(z)$ of the Loewner equation determines a continuous path $(\mathcal{K}_t)_{t\geq 0}$ in $Hull$, with $\mathcal{K}_0 = \emptyset$. We will call $(\mathcal{K}_t)_{t\geq 0}$ a path (or a trajectory) driven by $w = (w_s)_{s\geq 0}$. On the other hand, w is called a driving function (for path $(\mathcal{K}_t)$).

The *chordal process* SLE_κ is the (Borel) probability measure on continuous paths $(\mathcal{K}_t)_{t\geq 0}$ in $Hull$, with $\mathcal{K}_0 = \emptyset$, generated by the standard Brownian motion $(B_s)_{s\geq 0}$ with diffusion coefficient $\kappa > 0$, by means of the above

[3] In [W4], number $Time(\mathcal{K})/2 = \gamma_{\mathcal{K}}^{(-1)}$ is called the *capacity* of hull $\mathcal{K}$ (from infinity).

construction (i.e., via the random function $g_t(z,(B_s))$ emerging via (4.20)–(4.22). We denote this probability measure by $\boldsymbol{\mu}^\kappa$. In short, SLE_κ is a random path $(\boldsymbol{\mathcal{K}}_t)_{t\geq 0}$ in $Hull$ driven by Brownian motion (B_s) with diffusion coefficient $\kappa > 0$: $\boldsymbol{\mathcal{K}}_t = \boldsymbol{\mathcal{K}}_t\big((B_s)\big)$. The scaling property of the Brownian motion implies the scale covariance of process SLE_κ. Namely, $\forall\ \lambda > 0$ the dilation of time $t \mapsto \lambda t$ corresponds to the dilation of the hull $\mathcal{K}_t \mapsto \sqrt{\lambda}\mathcal{K}_t$:

$$(\boldsymbol{\mathcal{K}}_{\lambda t}) \sim (\sqrt{\lambda}\boldsymbol{\mathcal{K}}_t). \tag{4.24}$$

Formally, it means that two probability measures obtained from $\boldsymbol{\mu}^\kappa$ by the above dilations, coincide.

It is convenient to slightly generalise the above set-up and introduce a Borel subset $\widehat{Hull}$ of the Cartesian product $Hull \times \mathbb{R}$ whose points are pairs $(\mathcal{K}, x)$, or, equivalently, $(\gamma_{\mathcal{K}}, x)$, such that

$$\text{either } \ (\mathcal{K}, x) = (\emptyset, 0) \ \text{ or } \ \overline{\mathcal{K}} \cap \partial\overline{\mathbb{H}} = \{0\} \text{ and } \gamma_{\mathcal{K}}^{-1}(x) \in \partial\overline{\mathcal{K}}. \tag{4.25}$$

Here and below, $\gamma_{\mathcal{K}}^{-1}(x)$ stands for the embedding $\mathbb{H} \hookrightarrow \mathbb{H}\setminus\mathcal{K}$, inverse to $\gamma_{\mathcal{K}}$.

The reason for introducing $\widehat{Hull}$ is that if we start the SLE_κ process at a point from $\widehat{Hull}$, it stays in $\widehat{Hull}$. More precisely, given $(\mathcal{K}, x) \in \widehat{Hull}$, for any real-valued continuous function $w = (w_s)_{s\geq 0}$ with $w_0 = x$, we can define a path $\big(\mathcal{K}_t, w_t\big)_{t\geq 0}$ in $\widehat{Hull}$, with $\mathcal{K}_0 = \mathcal{K}$. Namely, we set $\gamma_{\mathcal{K}_t}(z) = g_t(z, \mathcal{K})$ where $g_t(z, \mathcal{K})$ satisfies Loewner equation (4.20) driven by (w_s), with the initial condition

$$g_0(z, \mathcal{K}) = \gamma_{\mathcal{K}}(z), \quad z \in \mathbb{H} \setminus \mathcal{K}, \tag{4.26}$$

instead of (4.21). We again call $\big(\mathcal{K}_t, w_t\big)$ a path driven by (w_s), and starting from $(\mathcal{K}, x)$. From the above definitions (and independence of increments in Brownian motion) it follows that SLE_κ generates a time-homogeneous Markov process on $\widehat{Hull}$. Namely, the process starting from point $(\mathcal{K}, x)$ is represented by a random path $(\boldsymbol{\mathcal{K}}_t, B_t + x)$ driven by the shifted Brownian motion $(B_s + x)_{s\geq 0}$.

We will call the above Markov process on $\widehat{Hull}$ an *extended* SLE_κ process. Correspondingly, $\widehat{Hull}$ is called the *extended phase space* of the extended SLE_κ process.

We will also denote by $Time$ the pullback of the time function from $Hull$ to $\widehat{Hull}$.

The infinitesimal generator of process SLE_κ in coordinate $(\gamma_\mathcal{K}, x)$ on $\widehat{Hull}$ is given by

$$\frac{\kappa}{2}\left(\frac{\partial}{\partial x}\right)^2 + \frac{2}{\gamma_\mathcal{K} - x}\,\frac{\delta}{\delta\gamma_\mathcal{K}} \tag{4.27}$$

where vector field $\frac{2}{\gamma_\mathcal{K} - x}\frac{\delta}{\delta\gamma_\mathcal{K}}$ is defined by

$$\begin{cases} \dot{x} = 0, \\ \dot{\gamma_\mathcal{K}} = \dfrac{2}{\gamma_\mathcal{K} - x}. \end{cases} \tag{4.28}$$

Definition 4.3. There is a convenient algebra $\mathbf{A}$ of measurable functions on $\widehat{Hull}$ (separating all points) consisting of polynomials in w and all non-trivial Taylor coefficients $\gamma_\mathcal{K}^{(-1)}$, $\gamma_\mathcal{K}^{(-2)}$, ... We endow $\mathbf{A}$ with a graduation by associating weights

$$\mathrm{weight}(x) = 1,\ \ \mathrm{weight}(\gamma_\mathcal{K}^{(-n)}) = n \text{ for } n \geq 1$$

to its generators. Algebra $\mathbf{A}$ has a natural exhaustive increasing filtration by finite-dimensional linear subspaces $\mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}$, where $\mathbf{A}_n$ consists of linear combinations of monomials of weight $\leq n$. $\blacksquare$

It is easy to see that the generator of the extended SLE_κ process preserves finite-dimensional spaces $\mathbf{A}_n$, hence the action of the evolution operator on $\mathbf{A}$ is well-defined.

4.3.2. Hulls and intervals for $\kappa \leq 4$. From now on we assume that $0 < \kappa \leq 4$. The reason is that, as was shown in [RS], if $\kappa \in (0, 4]$ (and only if this condition holds), then with $\boldsymbol{\mu}^\kappa$-probability 1 the path $(\mathcal{K}_t)$ of the SLE_κ process satisfies the following property. Sets $\mathcal{K}_t \cup \{0\}$, $t > 0$, are intervals embedded in $\overline{\mathbb{H}}$ and increasing with t: $\mathcal{K}_{t_1} \subset \mathcal{K}_{t_2}$ for $0 < t_1 < t_2$. Next, the 'tip' of the interval $\mathcal{K}_t$ approaches point ∞, in the limit $t \to +\infty$, By continuity, process SLE_κ, with $0 < \kappa \leq 4$, gives rise to a probability measure on $Int_{0,\infty}$ which we denote by $\boldsymbol{\mu}^\kappa_\infty$. Like before, we can associate this probability measure with a random interval $\boldsymbol{\mathcal{I}}$ in $Int_{0,\infty}$. Scaling covariance of SLE_κ (see (4.24)) implies a similar property of $\boldsymbol{\mathcal{I}}$ in $Int_{0,\infty}$.

To analyse properties of probability measure $\boldsymbol{\mu}^\kappa_\infty$ on $Int_{0,\infty}$, it is convenient to introduce the space $SInt$ of finite *semi-intervals* (in $\overline{\mathbb{H}}$). A finite semi-interval is denoted by $\mathcal{J}$ and is defined an equivalence class of homeomorphic embeddings of the unit segment

$$\iota: \ [0,1] \hookrightarrow \overline{\mathbb{H}}, \ \text{ with } \iota(0) = 0 \text{ and } \iota((0,1]) \subset \mathbb{H}, \tag{4.29}$$

modulo the action of the group of orientation-preserving homeomorphisms $[0,1] \to [0,1]$ preserving point 0. Obviously, a finite semi-interval is a particular case of a hull; viewed in this way, $SInt$ is a Borel subset in $Comp(\overline{\mathbb{H}})$, and we consider it as a topological space, with the induced topology. However, $SInt$ is not closed and not locally compact.

Note that $SInt$ can also be naturally identified with a subspace of $\widehat{Hull}$. The reason is that the real number x giving the second entry of the coordinate $(\mathcal{K}, x)$ in $\widehat{Hull}$ can be uniquely determined from the first entry, $\mathcal{K}$ (which is, in general, a hull, but under condition $0 < \theta \leq 1$, a semi-interval). In fact, if $\mathcal{J} \in SInt$ is a semi-interval and $\mathcal{J} = \iota([0,1])$, then

$$x(= x(\mathcal{J})) = \gamma_{\iota(0,1]}(\iota(1)). \tag{4.30}$$

At the same time, the union $\{0\} \sqcup SInt$ can be treated as the path space of the SLE$_\kappa$ process. More precisely, semi-intervals $\mathcal{J} \in \sqcup SInt$ can be parametrised by means of function $Time$ and will then represent 'stopped trajectories' SLE$_\kappa$. This picture can be extended to intervals $\mathcal{I} \in Int_{0,\infty}$: points of such an interval will be parametrised by $[0, +\infty]$. Furthermore, for any $\mathcal{I} \in Int_{0,\infty}$ of the form $\mathcal{I} = \iota([0,1])$, the map $[0,1] \to [0,+\infty]$ given by

$$\tau \mapsto Time\ (\iota([0,\tau])) \tag{4.31}$$

is a homeomorphism. The function $t(\tau) = Time(\iota([0,\tau]))$ provides a convenient *canonical* parametrisation of the interval $\mathcal{I}$ by $[0, +\infty]$. As a result, we associate with $\boldsymbol{\mu}_\infty^\kappa$ a family of a probability measures $\boldsymbol{\mu}_t^\kappa$ on the level set $Time^{-1}(t) \subset SInt \subset \widehat{Hull}$, where

$$Time^{-1}(t) = \{\mathcal{K}: \ \ Time(\mathcal{K}) = t\}, \ \ 0 < t < \infty. \tag{4.32}$$

5 The SLE-measures, II

5.1 The restriction martingale

Let $\alpha: \mathbb{H} \hookrightarrow \mathbb{H}$ be an embedding, as in subsection 4.2.3, such that $q^{\tan}_{\alpha,\infty} = 1$. We associate with α an open embedding

$$\alpha_*: \widehat{Hull} \hookrightarrow \widehat{Hull}$$

in the following fashion: $\forall\ (\widetilde{\mathcal{K}}, \widetilde{x}) \in \widehat{Hull}$,

$$\alpha_*(\widetilde{\mathcal{K}}, \widetilde{x}) = (\mathcal{K}, x). \tag{5.1}$$

Here $\mathcal{K}$ is the closure $\overline{\alpha(\widetilde{\mathcal{K}})}$ of the image of $\widetilde{\mathcal{K}}$ under α in $\mathbb{H}$.

Next, in order to determine $x \in \mathbb{R}$, we introduce the (partially defined holomorphic) mapping $h(= h_{\widetilde{\mathcal{K}},\mathcal{K}})$: $\mathbb{H} \hookrightarrow \mathbb{H}$, by

$$h = \gamma_{\widetilde{\mathcal{K}}} \circ \alpha^{-1} \circ \gamma_{\mathcal{K}}^{-1}. \tag{5.2}$$

It is easy to see that both h and the inverse mapping $h^{-1} = \gamma_{\mathcal{K}} \circ \alpha \circ \gamma_{\widetilde{\mathcal{K}}}^{-1}$ can be extended continuously to an invertible real analytic map (with strictly positive derivative) in a neighbourhood of $\widetilde{x} \in \mathbb{R}$, $\forall\ (\widetilde{\mathcal{K}}, \widetilde{x}) \in \widehat{Hull}$. We then define x in (5.1) by

$$x = h^{-1}(\widetilde{x}). \tag{5.3}$$

Therefore, we obtain two coordinate systems, $(\widetilde{\mathcal{K}}, \widetilde{x})$ and $(\mathcal{K}, x)$, on $\widehat{Hull}$, related by (5.1). In what follows we will treat h as a function on $\widehat{Hull}$ with values in (partially defined) holomorphic mappings $\mathbb{H} \hookrightarrow \mathbb{H}$.

Embedding α_* generates (by restriction) a similar embedding $SInt \hookrightarrow SInt$.

Let us introduce a new random process $\mathrm{SLE}_{\kappa,\alpha}$ whose phase space is the same space $\{0\} \sqcup SInt \subset \widehat{Hull}$ as for the original process SLE_κ. The time function $Time^\alpha$ for $\mathrm{SLE}_{\kappa,\alpha}$ is equal to $Time \circ \alpha_*$. We then introduce process $\mathrm{SLE}_{\kappa,\alpha}$ as the result of the time redefinition (from $Time$ to $Time^\alpha$) of process SLE_κ.

So, $\forall\ t \in (0, +\infty)$ we have two probability measures on the level set $Time^{-1}(t)$ (see (4.32)). The first measure, $\boldsymbol{\mu}_t\big(= \boldsymbol{\mu}_t^\kappa\big)$, is generated by the process SLE_κ. The second measure, $\boldsymbol{\mu}_{t,\alpha}\big(= \boldsymbol{\mu}_{t,\alpha}^\kappa\big)$, is the pushforward of the measure generated by $\mathrm{SLE}_{\kappa,\alpha}$ under map α_*.

We associate with pair $(\alpha, \mathcal{K})$, where $\mathcal{K} = \overline{\alpha(\widetilde{\mathcal{K}})}$ for some $\widetilde{\mathcal{K}} \in Hull$, the neutral collection $\mathfrak{F}_{\alpha,\mathcal{K}}$ consisting of four spheres S_1, S_2, S_3, S_4, identified as the doubles of four open disks

$$\alpha(\mathbb{H}),\ \ \alpha(\mathbb{H}) \setminus \mathcal{K},\ \ \mathbb{H},\ \ \mathbb{H} \setminus \mathcal{K} \tag{5.4}$$

taken with weights $+1, -1, -1, +1$. The glueing of these spheres is defined similarly to that in subsection 2.5.1.

On $\widehat{Hull}$ we consider the function $r_\alpha^{\rm det}$ which is defined, in coordinate $(\mathcal{K}, x)$, by

$$r_\alpha^{\rm det}(\mathcal{K}) = \left(v_{\mathfrak{F}_{\alpha,\mathcal{K}}} \Big/ \bigotimes_{k=1}^{4} v_{S_k}^{\otimes w_k} \right)^{1/2}, \tag{5.5}$$

and depends on hull $\mathcal{K}$ but not on x. The relation of this function to function $q_\alpha^{\rm det}$ defined in (4.15) on $Int_{0,\infty}$ will be explained in section 5.2 (see Eqn (5.24)).

Next, set:

$$r(\mathcal{K}, x) = (h'(x)\alpha'(0))^{\rm h} \left[r_\alpha^{\rm det}(\mathcal{K}) \right]^{\rm c}, \tag{5.6}$$

where $(\mathcal{K}, x) \in \alpha_*(\widehat{Hull})$, and parameters $\rm h, c$ are defined by (4.6) with $\theta := \kappa/4$.

Theorem 2. *$\forall\, t > 0$, measure $\boldsymbol{\mu}_{t,\alpha}$ is absolutely continuous with respect to $\boldsymbol{\mu}_t$. The Radon-Nikodym derivative*

$$r_t := \frac{{\rm d}\boldsymbol{\mu}_{t,\alpha}}{{\rm d}\boldsymbol{\mu}_t}$$

coincides with the restriction to $Time^{-1}(t)$ of function r defined in Eqn (5.6). Moreover, the extension of function $\alpha_* r$ by 0 outside the image $\alpha_*(\widehat{Hull})$ gives a martingale for process SLE_κ.

Proof: The proof of Theorem 2 is based on Propositions 3 and 4 below. Here we perform a series of formal calculations with second order differential operators on $\widehat{Hull}$ related to the generators of processes SLE_κ and $\mathrm{SLE}_\kappa^\alpha$. These can be converted into assertions about processes in the same way as in [W2], [W3]. (The fact that the SLE_κ-process is specified by its generator on $\widehat{Hull}$ is helpful here.)

Consider a positive function H on $\widehat{Hull}$ given, in coordinate $(\mathcal{K}, x)$, by

$$H(\mathcal{K}, x) = h'(x) = \frac{\partial \widetilde{x}}{\partial x}. \tag{5.7}$$

Further, for a function F on $\widehat{Hull}$ (like r, r^{-1}, H^2, and so on), we denote by the same symbol F the operator of multiplication by F.

We claim that

Proposition 3. *The following operator identity holds true:*

$$r^{-1} \circ \left(\frac{\kappa}{2} \left(\frac{\partial}{\partial x} \right)^2 + \frac{2}{\gamma_{\mathcal{K}} - x} \frac{\delta}{\delta \gamma_{\mathcal{K}}} \right) \circ r = H^2 \circ \left(\frac{\kappa}{2} \left(\frac{\partial}{\partial \widetilde{x}} \right)^2 + \frac{2}{\gamma_{\widetilde{\mathcal{K}}} - \widetilde{x}} \frac{\delta}{\delta \gamma_{\widetilde{\mathcal{K}}}} \right). \tag{5.8}$$

Proposition 3 guarantees that r determines a positive local martingale, and hence a semi-martingale, for SLE_κ.

The RHS of (5.8) gives the generator of $\mathrm{SLE}_\kappa^\alpha$, as follows from Proposition 4:

Proposition 4. *In the above notation, one has the following functional identity on* $\widehat{Hull}$*:*

$$\left(\frac{2}{\gamma_{\widetilde{\mathcal{K}}} - \widetilde{x}} \frac{\delta}{\delta \gamma_{\widetilde{\mathcal{K}}}} \right) (Time^\alpha) = \frac{1}{H^2}. \tag{5.9}$$

The assertion of Theorem 2 then follows from Propositions 3 and 4, by applying Girsanov's formula and the fact that $r(\widetilde{\mathcal{K}}, x) \to r(\emptyset, 0) = 1$ as pair $(\widetilde{\mathcal{K}}, x)$ approaches $(\emptyset, 0)$ in the topology on $\widehat{Hull}$.

Proof of Proposition 3. The first summand in the LHS of (5.8) is the following operator:

$$r^{-1} \circ \left[\frac{\kappa}{2} \left(\frac{\partial}{\partial x} \right)^2 \right] \circ r = \frac{\kappa}{2} \circ H^{-\mathrm{h}} \circ \left(\frac{\partial}{\partial x} \right)^2 \circ H^{\mathrm{h}}. \tag{5.10}$$

The reason is that the other factors figuring in the formula for r (see (5.6)) do not depend on x. In what follows we will denote by H' and H'' the result of application of operators $\dfrac{\partial}{\partial x}$ and $\left(\dfrac{\partial}{\partial x} \right)^2$ to function H. Then for the RHS of (5.10) we have the formula

$$\frac{\kappa}{2} \circ H^{-\mathrm{h}} \circ \left(\frac{\partial}{\partial x} \right)^2 \circ H^{\mathrm{h}} = 2\theta \left[\left(\frac{\partial}{\partial x} \right)^2 + 2\mathrm{h} \left(\frac{H'}{H} \right) \circ \frac{\partial}{\partial x} + \mathrm{h}(\mathrm{h}-1) \left(\frac{H'}{H} \right)^2 + \mathrm{h} \left(\frac{H''}{H} \right) \right]. \tag{5.11}$$

For the second summand in the LHS of (5.8) we have the following operator representation

$$r^{-1} \circ \left(\frac{2}{\gamma_{\mathcal{K}} - x} \frac{\delta}{\delta \gamma_{\mathcal{K}}} \right) \circ r = \frac{2}{\gamma_{\mathcal{K}} - x} \frac{\delta}{\delta \gamma_{\mathcal{K}}} + \left[\frac{2}{\gamma_{\mathcal{K}} - x} \frac{\delta}{\delta \gamma_{\mathcal{K}}} \right] (\log\ r). \qquad (5.12)$$

Our next goal is to calculate the zero degree term in the RHS of (5.12). To this end, we consider the vector field

$$V = \frac{2}{\gamma_{\mathcal{K}} - x} \frac{\delta}{\delta \gamma_{\mathcal{K}}}. \qquad (5.13)$$

and calculate the action of field V on the function

$$(\log\ r)(\mathcal{K}, x) = \mathrm{h} \log\ \alpha'(0) + \mathrm{h} \log\ h'(x) + \mathrm{c} \log\ r_\alpha^{\det}(\mathcal{K}). \qquad (5.14)$$

Observe that the first summand in the RHS of (5.14) is constant; hence we can discard it in future calculations.

Lemma 5.1. *Vector field V acts on function $h = h_{\mathcal{K}}$ as*

$$\dot{h}(z) = h'(x)^2 \frac{2}{h(z) - h(x)} - h'(z) \frac{2}{z - x}. \qquad (5.15)$$

Proof of Lemma 5.1. We have the following identity (cf (5.2)):

$$h \circ \gamma_{\mathcal{K}} = \gamma_{\mathcal{K}} \circ \alpha^{-1}.$$

Next, applying field V, we obtain the identity

$$\dot{h} \circ \gamma_{\mathcal{K}} + \frac{2}{\gamma_{\mathcal{K}} - x} \cdot h' \circ \gamma_{\mathcal{K}} = \dot{\gamma_{\widetilde{\mathcal{K}}}} \circ \alpha^{-1}.$$

It is clear, geometrically, that $\dot{\gamma_{\widetilde{\mathcal{K}}}}$ is proportional to $\dfrac{2}{\gamma_{\widetilde{\mathcal{K}}} - \widetilde{x}}$, with $\widetilde{x} = h(x)$, as we perform a Schiffer variation here. Hence, we obtain, for given $(\mathcal{K}, x)$, that

$$\dot{h}(z) + \frac{2}{z - x} h'(z) = \mathrm{const} \frac{2}{h(z) - h(x)}.$$

The proportionality coefficient is equal to $h'(x)^2$ as follows from the condition that $h(z)$ is non-singular at $z = x$. This completes the proof of Lemma 5.1. □

Lemma 5.2. *Vector field V acts on function* $\log\ H$*:* $(\mathcal{K}, x) \mapsto \log\ h'(x)$ *as follows:*

$$V(\log\ H) = -\frac{4}{3}\frac{H''}{H} + \frac{1}{2}\left(\frac{H'}{H}\right)^2.$$

Proof of Lemma 5.2. Expand h near point x:

$$h(z) = a_0 + a_1(z-x) + a_2(z-x)^2 + a_3(z-x)^3 + \ldots$$

where coefficients a_i are functions on $\widehat{Hull}$:

$$a_0 = \widetilde{x},\ a_1 = H,\ a_2 = \frac{H'}{2},\ a_3 = \frac{H''}{6}, \ldots.$$

By Lemma 5.1, we have:

$$\begin{aligned} \dot{h}(z) &= \frac{2h'(x)^2}{h(z)-h(x)} - \frac{2h'(z)}{z-x} \\ &= \frac{2a_1^2}{a_1(z-x) + a_2(z-x)^2 + a_3(z-x)^3 + \ldots} \\ &\qquad - \frac{2[a_1 + 2a_2(z-x) + 3a_3(z-x)^2 + \ldots]}{z-x}. \end{aligned} \tag{5.16}$$

The coefficient at $(z-x)$ in the RHS equals

$$\dot{H} = -2a_3 + \frac{2a_2^2}{a_1} - 6a_3 = -\frac{4}{3}H'' + \frac{1}{2}\frac{(H')^2}{H}.$$

This completes the proof of Lemma 5.2. □

Lemma 5.3. *Vector field V acts on function* $\log\ q_\alpha^{\det}$ *as follows:*

$$\left[V(\log\ r_\alpha^{\det})\right](\mathcal{K}, x) = -\frac{1}{6}\mathcal{S}_h(x) = -\frac{1}{6}\left(\frac{H''}{H} - \frac{3(H')^2}{2H^2}\right). \tag{5.17}$$

Proof of Lemma 5.3: follows immediately from Proposition 1 in subsection 3.3.2. □

We now can calculate the LHS of (5.8):

$$
\begin{aligned}
&r^{-1} \circ \left(\frac{\kappa}{2} \left(\frac{\partial}{\partial x} \right)^2 + \frac{2}{\gamma_{\mathcal{K}} - x} \frac{\delta}{\delta \gamma_{\mathcal{K}}} \right) \circ r \\
&= 2\theta \left(\frac{\partial}{\partial x} \right)^2 + 4\mathrm{h}\theta \left(\frac{H'}{H} \right) \circ \frac{\partial}{\partial x} + V.
\end{aligned}
$$

The next task is to calculate the RHS in (5.8) in coordinate $(\mathcal{K}, x)$. The first summand is calculated by using the functional identity $\frac{\partial}{\partial \widetilde{x}} = H^{-1} \frac{\partial}{\partial x}$:

$$
H^2 \circ \frac{\kappa}{2} \left(\frac{\partial}{\partial \widetilde{x}} \right)^2 = \frac{\kappa}{2} \left[\left(\frac{\partial}{\partial x} \right)^2 - \frac{H'}{H} \frac{\partial}{\partial x} \right]. \tag{5.18}
$$

Lemma 5.4. *Vector field*

$$
\widetilde{V} = \frac{2}{\gamma_{\widetilde{\mathcal{K}}} - \widetilde{x}} \frac{\delta}{\delta \gamma_{\widetilde{\mathcal{K}}}}
$$

acts in coordinate $(\mathcal{K}, x)$ *as*

$$
H^{-2} \circ V + 3 \frac{H'}{H^3} \frac{\partial}{\partial x}. \tag{5.19}
$$

Proof of Lemma 5.4. An argument similar to that in the proof of Lemma 5.2, shows that

$$
\widetilde{V} = H^{-2} V + \Phi \frac{\partial}{\partial x},
$$

where Φ is a function on $\widehat{Hull}$. This function Φ is calculated by using the identity $\widetilde{V}(\widetilde{x}) = \widetilde{V}(h(x)) = 0$. This identity yields that

$$
\left[H^{-2} V(h) \right] (z) \Big|_{z=x} + (\Phi H)(z) \Big|_{z=x} = 0.
$$

The value $[V(h)](z)\big|_{z=x}$ is the zeroth coefficient in the RHS of (5.16):

$$
[V(h)](z)\Big|_{z=x} = -2a_2 - 4a_2 = -3H'.
$$

This proves Lemma 5.4. □

Combining Eqns (5.10)–(5.19), we obtain the assertion of Proposition 3. $\square$

Proof of Proposition 4. Taking into account the above facts, the proof is concise. We have to calculate $\widetilde{V}(Time^\alpha)$. The result follows directly from Lemma 5.4 as $\frac{\partial}{\partial x}(Time^\alpha) = 0$ and $V(Time^\alpha) = 1$. This concludes the proof of Proposition 4. $\square$

Theorem 2 has now been proved. $\square$

Remark 5.1. In [W3], Werner consctructed a local martingale for the SLE_κ process given by the formula

$$r(\mathcal{K}_t, w_t) = \left[h_t(w_t)\right]^{\mathrm{h}} \exp\left[\frac{\mathrm{c}}{6}\int_0^t \mathcal{S}_{h_s}(w_s)\mathrm{d}s\right], \tag{5.20}$$

where $(\mathcal{K}_s, w_s)_{s\geq 0}$ is a path of the SLE_κ process starting at $(\emptyset, 0)$ and h_s stands for the mapping h associated with $(\mathcal{K}_s, w_s)$. It follows from Lemma 5.3 that (5.20) coincides with $r(\mathcal{K}_t, w_t)$, modulo the constant factor $\alpha'(0)^{\mathrm{h}}$. An advantage of our formula (5.6) is that it refers to the final point of the path $(\mathcal{K}_s, w_s)$, at $s = t$. $\blacksquare$

5.2 End of proof of Theorem 1

To complete the proof of Theorem 1, it remains to check that measures $\boldsymbol{\mu}^\kappa_\infty$ on $Int_{0,\infty}$ have the RC property; see Lemma 4.1. We will check this property in the special case where embedding α is such that the closure $\overline{\alpha(\mathbb{H})}$ of $\alpha(\mathbb{H})$ in $\overline{\mathbb{H}}$ contains either $[0, +\infty]$ or $[-\infty, 0]$. The general case will follow by composition of two embeddings with the above property.

Set $\mathcal{A} = \mathbb{H} \setminus \alpha(\mathbb{H})$; it is a hull touching $\partial\overline{\mathbb{H}}$ either strictly to the left or strictly to the right of 0.

Proposition 5. *For μ^κ_∞-almost every trajectory $(\widetilde{\mathcal{K}}_s, \widetilde{x}_s)_{s\geq 0}$ avoiding $\mathcal{A}$, and the associate trajectory $(\mathcal{K}_t, x_t)_{t\geq 0}$, where*

$$(\mathcal{K}_t, x_t) = \alpha_*(\widetilde{\mathcal{K}}_s, \widetilde{x}_s) \;\; \text{and} \;\; t = Time^\alpha(\widetilde{\mathcal{K}}_s, \widetilde{x}_s), \tag{5.21}$$

the Radon-Nikodym derivative $r_t(\mathcal{K}_t, x_t)$ (cf. Theorem 2) has a limit as $t \to \infty$ (and $s \to \infty$). Namely,

$$\lim_{t\to\infty} r_t(\mathcal{K}_t, x_t) = \left(q_\alpha^{\tan}\right)^{\mathrm{h}} \left(q_\alpha^{\det}\right)^{\mathrm{c}} (\widetilde{\mathcal{K}}_\infty). \tag{5.22}$$

Here we use the fact that

$$\widetilde{\mathcal{K}}_\infty = \lim_{s\to\infty} \widetilde{\mathcal{K}}_s \tag{5.23}$$

is an element of $Int_{0,\infty}$ (see subsection 3.3.2), and constant $q_\alpha^{\tan}$ and function $q_\alpha^{\det}$ are defined in (4.12) and (4.15), respectively.

Proof of Proposition 5. By definition, $q_\alpha^{\tan}(\widetilde{\mathcal{K}}_\infty)$ coincides with $h_0'(0) = \dfrac{1}{\alpha'(0)}$.

Lemma 5.5. *In the assumptions of the Proposition* 1 *one has*

$$\lim_{t\to\infty} h_t'(x_t) = 1. \tag{5.24}$$

Proof of Lemma 5.5. As was mentioned in Remark 4.2, with probability one the trajectory $(\widetilde{\mathcal{K}}_s, \widetilde{x}_s)$ is a growing family of truncations $\mathcal{I}_s$ of an interval $\mathcal{I} \in Int_{0,\infty}$. It is easy to see that for any interval $\mathcal{I} \in Int_{0,\infty}$ avoiding $\mathcal{A}$ there exists a function $b(s) > 0$, $s > 0$, such that the uniformisation

$$b(s)\gamma_{\mathcal{I}_s} : \ \mathbb{H} \setminus \mathcal{I}_s \to \mathbb{H} \tag{5.25}$$

has the following properties. (i) $b(s)\gamma_{\mathcal{I}_s}$ maps ∞ to ∞ and the tip of $\mathcal{I}_s$ to 0, and (ii) $b(s)\gamma_{\mathcal{I}_s}$ maps hull $\mathcal{A}$ to a domain $\mathcal{A}_s$ lying in an ϵ_s-neighborhood of point $1 \in \overline{\mathbb{H}}$ or point $-1 \in \overline{\mathbb{H}}$, depending on the position of $\mathcal{A}$, where $\lim_{s\to\infty} \epsilon_s = 0$.

Obviously, the uniformising coordinate $w(z)$ on $\mathbb{H} \setminus \mathcal{A}_s$, normalised so as $w(z) = z + O(1)$ and $w(0) = 0$, approaches, together with its first derivative $w'(z)$, to the standard coordinate on $\mathbb{H}$ at $z = 0$ as $s \to \infty$. This proves Lemma 5.5. □

Finally, by Proposition 2 from section 3.4, we have

$$\lim_{t\to\infty} r_\alpha^{\det}(\mathcal{K}_t) = q_\alpha^{\det}(\widetilde{\mathcal{K}}_\infty)\,. \tag{5.26}$$

As was mentioned earlier, Eqn (5.26) establishes the relation between (4.15) and (5.5).

Thus the limit (5.21) is established. This completes the proof of Proposition 5. □

Now we are ready to finish the proof of Theorem 1.

Proposition 6. *Consider the probability measure $\boldsymbol{\mu}_\infty^\kappa$ on $Int_{0,\infty}$ generated by process SLE_κ. Then $\boldsymbol{\mu}_\infty^\kappa$ is* (c,h)*-RC.*

Proof of Proposition 6. Proposition 5 implies that measure $\boldsymbol{\mu}_\infty^\kappa$ is invariant under any embedding α such that $q_{\alpha,\infty}^{\tan} = 1$, in the notation from subsection 3.2.2. The invariance of $\boldsymbol{\mu}_\infty^\kappa$ under dilations follows from the scaling covariance property of SLE_κ; see (4.24). The assertion of Proposition 6 then follows. □

The invariance of measure $\boldsymbol{\mu}_\infty^\kappa$ under the complex conjugation $z \mapsto -\bar{z}$ is obvious. This completes the proof of Theorem 1. □

5.3 Concluding remarks

Remark 5.2. The first remark is that for $\mathrm{c} = 0$, the assignment $(\Sigma, x, y) \mapsto \boldsymbol{\lambda}_{\Sigma,x,y}$ is unique, up to a scalar factor. This can be verified by using an argument similar to that from [W4]. ■

Remark 5.3. One can show that the assignment $(\Sigma, x, y) \mapsto \boldsymbol{\lambda}_{\Sigma,x,y}$ constructed in sections 4.1–4.5 is covariant under the exchange $x \leftrightarrow y$ of the endpoints. It follows from the time reversal symmetry of SLE_κ process established in [W1]. ■

Remark 5.4. By using our construction of the LCC assignment $(\Sigma, x, y) \mapsto \boldsymbol{\lambda}_{\Sigma,x,y}$, we can define a multi-interval assignment

$$(\Sigma, \underline{x}, \underline{y}) \mapsto \boldsymbol{\lambda}_{\Sigma,\underline{x},\underline{y}} \tag{5.27}$$

satisfying the corresponding LCC property: for any embedding $\xi : \Sigma \hookrightarrow \Sigma'$,

$$\xi^*\big(\boldsymbol{\lambda}_{\Sigma',\xi(\underline{x}),\xi(\underline{y})}\big) = \boldsymbol{\lambda}_{\Sigma,\underline{x},\underline{y}}, \tag{5.28}$$

Here, $\underline{x}$ and $\underline{y}$ are two disjoint ordered collections of distinct points from $\partial\Sigma$ and $\xi(\underline{x})$ and $\xi(\underline{x})$ are their images under ξ:

$$\begin{gathered} \underline{x} = (x_1, \dots, x_n), \quad \underline{y} = (y_1, \dots, y_n), \\ \xi(\underline{x}) = \big(\xi(x_1), \dots, \xi(x_n)\big), \quad \xi(\underline{y}) = \big(\xi(y_1), \dots, \xi(y_n)\big). \end{gathered} \tag{5.29}$$

Further, measure $\boldsymbol{\lambda}_{\Sigma,\underline{x},\underline{y}}$ is supported by n-tuples of disjoint intervals $(\mathcal{I}_1, \dots, \mathcal{I}_n) \in \mathop{\times}\limits_{i=1}^{n} Int_{x_i,y_i}(\Sigma)$, with values in the tensor product

$$\bigotimes_{i=1}^{n} \left(\big|\mathrm{Tan}\big|_{\Sigma,x_i,y_i}^{\otimes(-\mathrm{h})} \otimes \big|\mathrm{Det}\big|_{\Sigma,x_i,y_i}^{\otimes \mathrm{c}} \right). \tag{5.30}$$

of corresponding bundles (4.4).

Namely, the set of n-tuples of disjoint intervals $(\mathcal{I}_1,\dots,\mathcal{I}_n)$ is an open subset, $Int^{rmdisj}_{\underline{x},\underline{y}}(\Sigma)\subset\mathop{\times}\limits_{i=1}^{n}Int_{x_i,y_i}(\Sigma)$, and $\boldsymbol{\lambda}_{\Sigma,\underline{x},\underline{y}}$ is the restriction of the product-measure $\mathop{\times}\limits_{i=1}^{n}\boldsymbol{\lambda}_{\Sigma,x_i,y_i}$ on $Int^{\rm disj}_{\underline{x},uy}(\Sigma)$. If set $Int_{\underline{x},\underline{y}}(\Sigma)$ is non-empty then assignment (5.27) is non-zero.

Again, in the case $\mathrm{c}=0$, it is possible to check that such an assignment is unique, up to a scalar factor. However, for a general $\mathrm{c}=0$ the uniqueness of the LCC assignment remains open. ■

Remark 5.5. Now consider assignments $(\Sigma,x,y)\mapsto\boldsymbol{\lambda}_{\Sigma,x,y}$ where one of the two endpoints lies strictly in the interior $\Sigma\setminus\partial\Sigma$. The line bundle where the measure should take its values is modified for the corresponding endpoint. Suppose for definiteness that $x\in\Sigma\setminus\partial\Sigma$ and $y\in\partial\Sigma$. Then we replace, in (4.3), (4.4), the factor $(|T_x\partial\Sigma|)^{\otimes(-\mathrm{h})}$ by

$$\big|\det T_x\Sigma\big|^{\otimes(-\mathrm{h})}=\big(|\wedge^2 T_x\Sigma|\big)^{\otimes(-\mathrm{h})}\,. \tag{5.31}$$

Constructions from sections 4.1–4.3 can be extended to cover this case, but instead of chordal, one will have to use radial SLE_κ processes; see [BF], [LSW], [W2]. Again it will yield an LCC assignment, which for $\mathrm{c}=0$ is unique up to a scalar factor.

In the case where two endpoints lie in the interior $\Sigma\setminus\partial\Sigma$, the question of existence and uniqueness of an LCC assignment remains open. ■

Remark 5.6. It is possible to define LCC assignments $\Sigma\mapsto\boldsymbol{\lambda}_{\Sigma,\mathrm{free}}$ on spaces of intervals $\bigcup\limits_{x,y\in\partial\Sigma:\,x\neq y}Int_{x,y}(\Sigma)$ with non-fixed endpoints. The measure $\boldsymbol{\lambda}_{\Sigma,\mathrm{free}}$ will take values in the line bundle with a fiber at point $\mathcal{I}\in Int_{x,y}(\Sigma)$ equal to

$$\big(|T_x\partial\Sigma|\otimes|T_t\partial\Sigma|\big)^{\otimes(1-\mathrm{h})}\otimes|\det|^{\otimes\mathrm{c}}_{\mathcal{I},\Sigma}. \tag{5.32}$$

The reason is that there is a canonical ('tautological') measure $\boldsymbol{\tau}_{\partial\Sigma}$ on $\partial\Sigma$ with values in $|T\partial\Sigma|$. Measure $\boldsymbol{\lambda}_{\Sigma,\mathrm{free}}$ is the product of measure

$$\boldsymbol{\tau}_{\partial\Sigma}\times\boldsymbol{\tau}_{\partial\Sigma}\ \text{ on }\ \big(\partial\Sigma\times\partial\Sigma\big)\setminus\mathrm{diag}\big(\partial\Sigma\times\partial\Sigma\big)$$

and the family of measures

$$\boldsymbol{\lambda}_{\Sigma,x,y}\ \text{ on }\ Int_{x,y}(\Sigma),\ \text{ where }\ x,y\in\partial\Sigma,\ x\neq y.\quad ■$$

6 Applications to statistical physics

This section follows some parts of a talk given by one of us at the Arbeitstagung (Bonn, 2003), see [K2].

6.1 Phase boundaries

It is believed that the conformal field theory (CFT) helps to describe a large-scale behaviour of lattice models near phase transition points. In particular, the CFT (and its massive perturbations by relevant fields) are credited with predictions of asymptotics of correlators of local observables. However, there is a different part of the picture, not reduced directly to local observables and related to statistics of phase boundaries. See, e.g., [C1].

The basic example here is the two-dimensional Ising model on the square lattice, with the zero magnetic field and at a temperature $T = T_{\text{crit}} - \delta T$ with small $\delta T > 0$. Here, in the thermodynamic limit we will have with probability $1/2$ the 'sea' of spins $+1$ with 'islands' of spins -1, or vice versa. Typical 'large' islands will have size $\simeq (\delta T)^{-\mu}$ for some critical exponent $\mu > 0$. Inside islands of, say spins -1, the system is 'confused' about the global phase, and one expects that there will be yet smaller 'second-order' islands of spins $+1$, etc. Passing to the limit $\delta T \to 0$ and rescaling simultaneously the distance on $\mathbb{R}^2 \supset \mathbb{Z}^2$ by factor $(\delta T)^{\mu}$, one obtains, hypothetically, a random collection of closed pairwise disjoint Jordan curves on $\mathbb{R}^2$, called phase boundaries (or domain walls). This collection is, with probability 1, everywhere dense, but there will be 'very few' curves of a large size $\gg 1$. Furthermore, there will be many curves of size $\simeq 1$ covering a positive part of the total area.

This picture is *not* conformally invariant and should be associated in general with a massive perturbation of a CFT with two vacua.

Next, consider the behavior of phase boundaries at small distances, i.e. rescale again the distance in $\mathbb{R}^2$. By general heuristic arguments, one can show that the limiting distribution of collections of phase boundaries is not degenerate, i.e. there are many curves of size (diameter) $\simeq 1$, and the distribution is now *scale invariant*. One can also expect that this distribution is also *conformally invariant*.

Many people, e.g. the late colleagues Roland Dobrushin and Claude Itzykson, asked about how to derive from a CFT the description of the probabilistic ensemble of loops. A strong motivation for works in this direction was provided by recent spectacular development connected with the SLE processes. In this context, a hypothetical picture of the phase boundaries was outlined in the last chapter in [F] and in an earlier presentation [FK]. In these publications, a description was given, of a probability measure on intervals, which connect two phase changing points on the boundary of a surface. We will discuss this approach in section 6.2. We note that a possibility of a connection between the subjects of the CFT and the SLE was earlier discussed in [BB].

6.2 The Malliavin measures and the CFT

In this section we describe a new approach to the ensemble of phase separating loops based on the Malliavin measures. We remind some basic facts about the CFTs in two dimensions. The basic parameter characterising a CFT is a *central charge* $\mathrm{c} \in \mathbb{R}$. Next, with any surface Σ there is associated a partition function $Z_\Sigma \in |\det|_\Sigma^{\otimes \mathrm{c}} \otimes_{\mathbb{R}} \mathbb{C}$. The usual axiomatics of the CFT assumes that the theory is unitary and oriented towards the quantum field theory on surfaces with Lorentzian metrics. (Recently, there appeared non-unitary versions of the CFT, with discrete spectrum and logarithmic operator product expansion (OPE).) To our knowledge, there is yet no systematic approach proposed to CFTs based on the probability theory, despite the common belief that the simplest unitary CFT with $\mathrm{c} = 1/2$ must describe the large scale behavior of the Ising model at the critical temperature. A recent work [SW], [W5] indicates that for any value of the central charge $\mathrm{c} \in (0, 1]$ there should exists a probabilistic CFT which has a natural Markov property and gives a random field of disjoint Jordan loops describing (hypothetically) a picture of phase boundaries in a stochastic particle system. This differs sharply from the the unitary CFTs, where all theories with $\mathrm{c} < 1$ were classified, and only a discrete set of values of $\mathrm{c} = 1 - 6/(k(k + 1))$, $k = 2, 3, \ldots$, is allowed.

In a probabilistic model of CFT one should expect Z_Σ to be a positive point of $|\det|_\Sigma^{\otimes \mathrm{c}}$. In a sense, Z_Σ is a regularised value of the partition function for a lattice approximation.

Similarly, in probabilistic CFT models with *boundary conditions* (in short, boundary CFTs (BCFTs); see [C2]) one should have positive points $Z_{\Sigma,\omega} \in |\det|_\Sigma^{\otimes \mathrm{c}}$ where Σ is a surface of finite type and ω is a specified boundary condition. To be concrete, let us focus from now on on the continuous limit of the critical Ising model. In this case, a boundary condition ω (in the microscopic description) is a locally constant map assigning values $+$ or $-$ to each connected component of the boundary $\partial\Sigma$. (In the case where Σ is a surface without a boundary, we have a positive point $Z_{\Sigma,\mathrm{free}} \in |\det|_{\Sigma}^{\otimes c}$ and speak about free boundary conditions.) Consider a loop $\mathcal{L} \in Loop(\Sigma)$, and also attach sign $+$ to one side of $\mathcal{L}$ in Σ, and the sign $-$ to the opposite side. We are interested in the probability that in the critical Ising model with boundary condition ω there will be a phase separating loop close to $\mathcal{L}$ (with phases near $\mathcal{L}$ specified by our choices of signs).

Thus we will talk about 'cooriented' loops, i.e., pairs $\overline{\mathcal{L}} = (\mathcal{L}, \vartheta)$ where ϑ indicates the $\pm$ signs on both sides of $\mathcal{L}$. Denote by $\Sigma'(= \Sigma'_{\mathcal{L}})$ the complement $\Sigma \setminus \mathcal{L}$ with canonically attached boundaries so that the canonical conformal structure in the interior of Σ' extends smoothly to the boundary $\partial\Sigma'$. Given an 'initial' boundary condition ω and an attachment ϑ, we obtain a boundary condition ω' for Σ'. (In the case of surface Σ without a boundary, ω' is reduced to ϑ.) There is a canonical isomorphism between the oriented real lines

$$|\det|_{\mathcal{L},\Sigma} \simeq |\det|_{\Sigma}/|\det|_{\Sigma'}. \tag{6.1}$$

Hence the ratio

$$\frac{Z_{\Sigma',\omega'}}{Z_{\Sigma,\omega}} \left(\text{or } \frac{Z_{\Sigma',\omega'}}{Z_{\Sigma,\mathrm{free}}} \right) \tag{6.2}$$

can be interpreted (as a function of $\mathcal{L}$) as a section of the line bundle $|\mathrm{Det}|_{\Sigma}^{-\otimes c}$ on $Loop(\Sigma)$. Therefore, the product

$$\rho_{\Sigma,\omega} \left(= \rho^{(1)}_{\Sigma,\omega}\right) = \boldsymbol{\lambda}_{\Sigma} \frac{Z_{\Sigma',\omega'}}{Z_{\Sigma,\omega}} \left(\text{or } \rho_{\Sigma,\mathrm{free}} \left(= \rho^{(1)}_{\Sigma,\mathrm{free}}\right) = \boldsymbol{\lambda}_{\Sigma} \frac{Z_{\Sigma',\omega'}}{Z_{\Sigma,\mathrm{free}}} \right) \tag{6.3}$$

is a scalar measure on the space $\overline{Loop}\,(\Sigma)$ of cooriented loops in Σ (it is a double covering of $Loop(\Sigma)$). Superscript (1) in notation $\rho^{(1)}_{\Sigma,\omega}$ (or $\rho^{(1)}_{\Sigma,\mathrm{free}}$) has a straightforward (and important) meaning which we explain below. For simplicity, we will not treat the case of a surface without boundary separately; in this case the reader should substitute the subscript free in place of ω.

Our prediction is that measure (6.3) is proportional to the rate measure (more precisely, the first-order rate measure), of the random field of phase separating loops. Formally, for any Borel subset $\overline{\mathbf{U}} \subset \overline{Loop}\,(\Sigma)$, the quantity

$$\zeta \int\limits_{Loop(\Sigma)} \mathbf{1}_{\overline{\mathbf{U}}}\, \rho_{\Sigma,\omega} = \zeta\, \rho_{\Sigma,\omega}(\overline{\mathbf{U}}) \tag{6.4}$$

gives the expected value of the random number of (equipped) loops falling in $\overline{\mathbf{U}}$. (Here and below, $\mathbf{1}_{\mathbf{W}}$ stands for the indicator function of a subset $\mathbf{W}$, in a given (topological) space). The constant $\zeta > 0$ standing in front in (6.4) depends on the specification of the BCFT (recall that assignment $\Sigma \mapsto \boldsymbol{\lambda}_\Sigma$ is determined up to a scalar factor).

Formula (6.3) for the rate measure is very natural. Indeed, it expresses the infinitesimal probability of having $\overline{\mathcal{L}} = (\mathcal{L}, \vartheta)$ as a phase-separating curve in the form of probability of the event specified by the requirement of having spin + on one side of $\mathcal{L}$ and spin − on the other side, specified by ϑ. The probability of such an event (in the lattice approximation) is then written as the ratio of two sums of Boltzmann weights. The numerator is the sum of Boltzmann weights over configurations with boundary conditions ω', and the denominator is that over configurations with boundary condition ω.

This description can be generalised directly to the case of several cooriented loops, leading to a sequence of 'higher-order' rate measures $\rho^{(n)}_{\Sigma,\omega}$, $n = 1, 2, \ldots$. Here $\rho^{(n)}_{\Sigma,\omega}$ is a scalar measure on $\left[\overline{Loop}\ (\Sigma)\right]^{\times n}_{\rm disj}$, the set of ordered n-tuples of disjoint cooriented equipped loops $(\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_n)$. $\Big(\left[\overline{Loop}\ (\Sigma)\right]^{\times n}_{\rm disj}$ is an open subset in the Cartesian product $[Loop(\Sigma)]^{\times n} = \overline{Loop}\ (\Sigma) \times \ldots \times \overline{Loop}\ (\Sigma).\Big)$ The meaning of $\rho^{(n)}_{\Sigma,\omega,\omega'}$ is, as above, that $\forall$ (Borel) $\overline{\mathbf{U}}^{(n)} \subseteq \left[\overline{Loop}\ (\Sigma)\right]^{\times n}_{\rm disj}$, the quantity

$$\zeta^n \int\limits_{[Loop(\Sigma)]^{\times n}_{\rm disj}} \mathbf{1}_{\overline{\mathbf{U}}^{(n)}}\ \rho^{(n)}_{\Sigma,\omega} = \zeta^n\ \rho^{(n)}_{\Sigma,\omega}\big(\overline{\mathbf{U}}^{(n)}\big) \tag{6.5}$$

gives the expected value of the random number of n-tuples $(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ falling in $\overline{\mathbf{U}}^{(n)}$ where $\overline{\mathbf{U}}^{(n)}$ is a Borel subset in $\left[\overline{Loop}\ (\Sigma)\right]^{\times n}_{\rm disj}$. Measure $\rho^{(n)}_{\Sigma,\omega}$ is invariant under the action, on $\left[\overline{Loop}\ (\Sigma)\right]^{\times n}_{\rm disj}$, of the permutation group of the nth order.

The sequence of measures $\rho^{(n)}_{\Sigma,\omega}$ would eventually lead to a *random point field* $\underline{\rho}_{\Sigma,\omega}$ on $Loop(\Sigma)$ whose sample realisation is a countable collection of disjoint Jordan cooriented loops from $\overline{Loop}\ (\Sigma)$, compatible with each other and with boundary condition ω and everywhere dense in Σ. (For brevity, we will refer simply to a sample realisation, having in mind all above-listed properties.) A way to identify $\underline{\rho}_{\Sigma,\omega}$ is discussed below.

A consequence of this proposal is a collection of inequalities on partition functions $Z_{\Sigma,\omega}$ involving alternate values of measures $\rho^{(n+k)}_{\Sigma,\omega}$. More precisely, consider the class $\mathfrak{f}$ of (Borel) subsets $\overline{\mathbf{V}} \subset \overline{Loop}(\Sigma)$ such that the series

$$\sum_{k\geq 1} \zeta^k \rho^{(k)}_{\Sigma,\omega} \left(\overline{\mathbf{V}}^{\times k}_{\neq}\right) < \infty. \tag{6.6}$$

Then, $\forall\ \overline{\mathbf{V}} \in \mathfrak{f}$ and $n \geq 0$, we will have:

$$0 \leq \pi_{\Sigma,\omega}(\overline{\mathbf{V}}, n) \leq 1, \tag{6.7}$$

where

$$\pi_{\Sigma,\omega}(\overline{\mathbf{V}}, n) = \frac{1}{n!}\sum_{k\geq 0}(-1)^k \frac{1}{k!} x\zeta^{n+k}\rho^{(n+k)}\left(\overline{\mathbf{V}}^{\times(n+k)}_{\neq}\right), \tag{6.8}$$

and, for $n = k = 0$, $\rho^{(0)}\left(\overline{\mathbf{V}}^{\times(0)}_{\neq}\right)$ is set to be equal to 1.

The quantity $\pi_{\Sigma,\omega}(\overline{\mathbf{V}}, n)$ has a transparent probabilistic meaning: it gives the probability that in the sample realisation, there will be exactly n disjoint cooriented loops falling in set $\overline{\mathbf{V}}$. Furthermore, these quantities, for different $\overline{\mathbf{V}}$ and n, will satisfy obvious compatibility properties.

We predict that in the BCFT corresponding to the critical Ising model (for c $= 1/2$), $\forall\ \zeta > 0$, surface Σ and boundary condition ω, there exists a unique probability distribution $\underline{\rho}_{\Sigma,\omega}$ on the space of sample realisations compatible with ω, with the following properties.

(i) $\forall\ n \geq 0$ and set $\overline{\mathbf{V}} \in \mathfrak{f}$, the $\underline{\rho}_{\Sigma,\omega}$-probability

$$\underline{\rho}_{\Sigma,\omega}\Big(\text{sample realisation contains exactly } n \text{ loops } \mathcal{L} \in \overline{\mathbf{V}}\Big) = \pi_{\Sigma,\omega}(\overline{\mathbf{V}}, n). \tag{6.9}$$

(ii) $\forall\ n \geq 1$ and set $\overline{\mathbf{U}}^{(n)} \subset [\overline{Loop}]^{\times n}_{\text{disj}}$, the expected value (relative to $\underline{\rho}_{\Sigma,\omega}$)

$$\begin{aligned}\mathbb{E}_{\underline{\rho}_{\Sigma,\omega}}\Big(&\text{the number of ordered } n\text{-tuples of disjoint cooriented loops,}\\ &\qquad\text{from the sample realisation, which fall in } \overline{\mathbf{U}}^{(n)}\Big)\\ &= \zeta^n \rho^{(n)}_{\Sigma,\omega}(\overline{\mathbf{U}}^{(n)}).\end{aligned} \tag{6.10}$$

By construction, $\underline{\rho}_{\Sigma,\omega}$ satisfies the following Markov property. For any cooriented loop $\overline{\mathcal{L}} = (\mathcal{L}, \vartheta) \in \overline{Loop}\ (\Sigma)$, the distribution of the sample realisation, conditional on the fact that it contains $\overline{\mathcal{L}}$ is decomposed into a product of two marginal distributions, one on the sample realisations inside $\mathcal{L}$, the other on the sample realisations outside $\mathcal{L}$, both collections being compatible with the boundary condition ω' on $\Sigma' = \Sigma \setminus \mathcal{L}$ induced by ω and ϑ as explained above.

6.3 On a proposal by Friedrich and the SLE measures

One can also consider a CFT with boundary conditions which change their nature at some points on the boundary $\partial\Sigma$ for a given surface Σ. E.g., one can divide $\partial\Sigma$ into a finite number of intervals and put on each of these intervals boundary condition $+$ or $-$. In this case there is no canonical way to define partition function, and correlators depend on certain insertions at boundary changing points. Such insertions form an infinite-dimensional vector space $\mathcal{H}_{+-}$. Friedrich's proposal [F] is that there exists a canonical vector $\psi \in \mathcal{H}_{+-}$ (unique up to a positive scalar factor) which is the highest vector for the natural Virasoro action and satisfies the property

$$L_n\psi = 0,\ n \geq 1,\ \ L_0\psi = \mathrm{h}\psi,\ \ (\theta(L_{-1})^2 - L_{-2})\psi = 0, \tag{6.11}$$

where θ and h are determined by the central charge c via Eqns (4.6).

Vector ψ plays a role of a 'vacuum vector' in $\mathcal{H}_{+-}$ and has the lowest conformal dimension. The correlators $\langle\psi(x_1)\dots\psi(x_{2n})\rangle$, where $x_1, \dots, x_{2n} \in \partial\Sigma$, are points of change of the boundary condition, should be positive and equal to the renormalised partition functions in the lattice approximation.

Suppose we are given a two-dimensional connected closed smooth manifold S with non-empty boundary S and an ordered collection $\underline{x}$ $= (x_1, \dots x_m)$ of m points in ∂S. Denote by $\mathcal{M}_{S,\underline{x}}$ the space of moduli of pairs $(\Sigma, \underline{y})$ diffeomorphic to $(S, \underline{x})$, where Σ is a surface (endowed with a conformal structure), and $\underline{y}$ are marked points on the boundary $\partial\Sigma$. This is a finite-dimensional orbifold. We will assume that we are in a hyperbolic case, with $m > 2\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. We define the oriented real line bundles $|T_i|$, $i = 1, \dots, m$, and $\big|\mathrm{Det}\big|$ on $\mathcal{M}_{S,\underline{x}}$ as follows. The fiber of $|T_i|$ at point $[(\Sigma, \underline{y})]$ (the equivalence class represented by $(\Sigma, \underline{y})$) is defined as $|T_{y_i}\partial\Sigma|$. The fiber of $\big|\mathrm{Det}\big|$ at point $[(\Sigma, \underline{y})]$ is $|\det|_\Sigma$. The element $(\theta(L_{-1})^2 - L_{-2})$ of the envelopping algebra of the Virasoro algebra gives rise to a collection of second-order hypolelliptic differential operators Δ_i on $\mathcal{M}_{S,\underline{x}}$, $i = 1, \dots, m$:

$$\begin{aligned}\Delta_i :\ & \Gamma(\mathcal{M}_{S,\underline{x}}, \otimes_{j=1}^m |T_j|^{\otimes(-\mathrm{h})} \otimes \big|\mathrm{Det}\big|^{\otimes \mathrm{c}}) \\ & \to \Gamma(\mathcal{M}_{S,\underline{x}}, \otimes_{j=1}^m |T_j|^{\otimes(-\mathrm{h})} \otimes \big|\mathrm{Det}\big|^{\otimes \mathrm{c}} \otimes |T_i|^{\otimes(-2)}).\end{aligned} \tag{6.12}$$

This fact can be deduced from the well-known Virasoro uniformisation of moduli spaces (see [K1]), as explained in [K2], [F] and [FK].

It follows from (6.11) that the correlator $\langle\psi(y_1)\dots\psi(y_{2n})\rangle$ (with an even number of points $m = 2n$) is a harmonic section of line bundle $\otimes_{j=1}^{2n}|T_j|^{\otimes(-\mathrm{h})}\otimes$

$\left|\mathrm{Det}\right|^{\otimes c}$) with respect to each operator Δ_i. Every Δ_i gives rise, after division by $\langle\psi(y_1)\dots\psi(y_{2n})\rangle$, to the generator of a Brownian motion on $\mathcal{M}_{S,\underline{x}}$, defined modulo a time change. Friedrich's remark is that, for $n=1$, the random path of the Brownian motion on $\mathcal{M}_{S,\underline{x}}$, associated with Δ_1, corresponds to a self-avoiding curve growing in Σ, from point $y_1 \in \partial\Sigma$, and eventually reaching $y_2 \in \partial\Sigma$. Such a random interval should correspond to a (random) phase boundary.

Considerations from section 6.2 can be extended in a straightforward way to incorporate both phase-separating loops and intervals. Namely, one should replace the partition function $Z_{\Sigma,\omega}$ by the correlator $\langle\psi(y_1)\dots\psi(y_{2n})\rangle$ and use a joint measure whose marginals are the corresponding Malliavin and SLE measures. For the critical Ising model this would give a description of a 'joint' ensemble of phase separating lines combining loops and intervals.

If we focus on intervals only then the corresponding prediction will give the *same* measure as in Friedrich's proposal. This can be deduced from Girsanov's formula. In a sense, our construction gives a justification of Friedrich's proposal, as our approach is physically more transparent.

6.4 Operadic structure and quadratic identities for partition functions

The conjectured assignments $\Sigma \mapsto \boldsymbol{\lambda}_\Sigma$ can be used to construct a nice algebraic structure called the modular operad [GK]. The prototype is the collection of homology groups $H_*(\overline{\mathcal{M}}_{g,n})$ of moduli stacks of stable curves with marked points, where $g, n \geq 0$ and $2-2g-n<0$. There exist polylinear operations on these spaces given by pushforward maps from the boundary strata of moduli stacks.

In this section we assume that all surfaces are oriented. Modifications needed in the non-oriented case are straightforward. Without stressing it every time again, we assume that we are given an c-LCC assignment $\Sigma \mapsto \boldsymbol{\lambda}_\Sigma$.

Let us define (for given $c \in (-\infty, +1]$) an infinite-dimensional real vector space $V_{g,n}$ as the space of measurable sections of the line bundle $|\det|^{\otimes c}$ on the moduli space $\mathcal{M}^{\text{holes}}_{g,n}$ of puncture-free conformal structures on surfaces of genus g with n enumerated holes. Here we assume that $g \geq 1$, $n \geq 0$ or $g=0$, $n \geq 2$. Space $V_{g,n}$ contains a convex cone $V^+_{g,n}$ consisting of non-negative sections.

Our goal is to define certain polylinear maps between spaces $V_{g,n}$. These maps will be only partially defined and preserve cones $V_{g,n}^+$.

Suppose we are given a two-dimensional connected C^∞-manifold S, of finite topological type, and a collection of disjoint loops $L_1, \ldots, L_k$ in S. Let $S_1, \ldots, S_m$ be the connected components of $S \setminus (\sqcup_i L_i)$. We assume that none of $S_1, \ldots, S_m$ is a sphere or a disk. We associate with these topological data a map

$$\mathfrak{a}_{S,L_1,\ldots,L_k} : \otimes_{i=1}^m V_{g,n_i} \to V_{g,n}.$$

Here g_i and g are the genera of, and n_i and n the numbers of holes in, S_i and S, respectively. Namely, given sections $s_i \in V_{g_i,n_i}$ and a surface Σ representing point $[\Sigma] \in \mathcal{M}_{g,n}^{\text{holes}}$, the value of the section $\mathfrak{a}_{S,L_1,\ldots,L_k}(s_1 \otimes \ldots \otimes s_k)$ at the point $[\Sigma]$ is given by

$$\begin{aligned} &\big(\mathfrak{a}_{S,L_1,\ldots,L_k}(s_1 \otimes \ldots \otimes s_k)\big)([\Sigma]) \\ &= \int\limits_{\mathcal{U}(S,L_1,\ldots,L_k)} \otimes_{i=1}^m s_i([\Sigma \setminus (\sqcup_{j=1}^k \mathcal{L}_j)])\, \mathrm{d}\lambda_\Sigma(\mathcal{L}_1) \cdots \mathrm{d}\lambda_\Sigma(\mathcal{L}_k) \end{aligned} \tag{6.13}$$

Here, $\mathcal{U}(S, L_1, \ldots, L_k)$ consists of disjoint k-tuples of loops $(\mathcal{L}_1, \ldots, \mathcal{L}_k) \in [Loop(\Sigma)]_{\text{disj}}^k$ such that $(\Sigma, \mathcal{L}_1, \ldots, \mathcal{L}_k)$ is homeomorphic to $(S, L_1, \ldots, L_k)$. (We use here an obvious identification of line bundles.)

In general, convergence of the integral in the RHS of (6.13) is not guaranteed; in the case of non-negative sections $s_1, \ldots, s_m$, the integral is finite or equal to $+\infty$.

The set of polylinear maps $\mathfrak{a}_{S,L_1,\ldots,L_k}$, for varying $S, L_1, \ldots, L_k$, is closed under composition. In particular, in the case where S is an open cylinder, $k = 1$ and L_1 is a single-winding loop, we obtain a bilinear operation $\star$ on $V_{0,2}$. Operation $\star$ is associative, as follows from the composition property. This operation is also commutative: this follows from the symmetry of the cylinder under swapping the boundary circles with each other.

The conclusion is that we obtain a partially defined commutative associative product $\star$ on $V_{0,2}$, depending on the value c. By using the conformal parameter of a cylinder, and the canonical vector v_Σ from subsection 2.2.4, space $V_{0,2}$ can be identified with the set of measurable functions on the interval $(0, 1) \simeq \mathcal{M}_{2,0}^{\text{holes}}$.

An easy combinatorial argument shows, heuristically, that our prediction for phase boundaries implies the following identity. Let $Z_{++} = Z_{--}$ and $Z_{+-} = Z_{-+}$ be the elements of $V_{0,2}$ corresponding to the partition functions of the critical Ising model on a cylinder with boundary conditions $+, +$ on both boundary components (+ on one components and − on another component respectively). Then one has

$$Z_{+-} = Z_{++} \star Z_{--} - Z_{+-} \star Z_{+-}. \tag{6.14}$$

The reason is that for a configuration of ±-spins on a cylinder, with the boundary condition + on the left and − on the right, one should always have an *odd* number of single-winding phase-separating loops. Moreover, the number of such loops with attachment $+|-$ (plus to the left, minus to the right) equals one plus the number of loops with attachment $-|+$ (plus to the right, minus to the left). See the figure below.

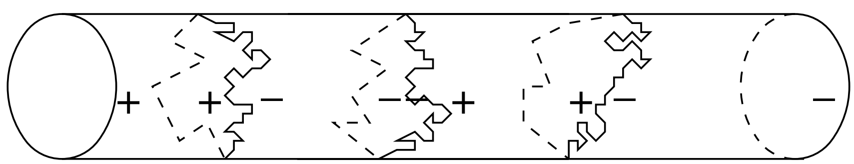

Figure 22: Signle-winding phase-deparating loops

The conjectured random point field on the cylinder Σ with the $+/-$ boundary condition should be supported by sample realisations containing finitely many single-winding loops. The expected number of $+|-$ loops equals

$$\frac{(Z_{++} \star Z_{--})([\Sigma])}{Z_{+-}([\Sigma])}.$$

Similarly, the expected number of $-|+$ loops equals

$$\frac{(Z_{+-} \star Z_{+-})([\Sigma])}{Z_{+-}([\Sigma])}.$$

The identity (6.14) follows from the aforementioned relation that the number of the $+|-$ loops is one more than that of the $-|+$ ones.

In this regard, we state the following problem:

Fix c $\in (-infty, 1]$ *and consider the corresponding product* $\star$ *on functions on* $(0, 1)$. *It is natural to expect that it has the representation*

$$(f \star g)(r) = \int_0^1 \int_0^1 K(r; r_1, r_2) f(r_1) g(r_2) \frac{\mathrm{d}r_1}{r_1} \frac{\mathrm{d}r_2}{r_2} \tag{6.15}$$

Calculate kernel $K(r; r_1, r_2)$ in a closed form.

Similar questions may be posed for general compositions $\mathfrak{a}_{S,L_1,\dots,L_k}$.

Equations of a type similar to (6.14) can be derived in the case of a surface Σ of a higher genus. This results in an infinite system of integral equations on partition functions $Z_{\Sigma,\omega}$ in the critical Ising model.

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