# Symplectic geometry of homological algebra

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### Derived non-commutative algebraic geometry

With any scheme $X$ over ground field $\mathbf{k}$ we can associate a $\mathbf{k}$-linear triangulated category $\mathrm{Perf}(X)$ of perfect complexes, i.e. the full subcategory of the unbounded derived category of quasi-coherent sheaves on $X$, consisting of objects which are locally (in Zariski topology) quasi-isomorphic to finite complexes of free sheaves of finite rank.

The category $\mathrm{Perf}(X)$ is essentially small, admits a natural enhancement to a differential graded (dg in short) category up to a homotopy equivalence, and is Karoubi (e.g. idempotent) closed. The main idea of derived non-commutative algebraic geometry is to treat any Karoubi closed small dg category as the category of perfect complexes on a "space".

By a foundamental result of A. Bondal and M. Van den Bergh, any separated scheme of finite type is *affine* in the derived sense, i.e. $\mathrm{Perf}(X)$ is generated by just one object. Equivalently,

$$\mathrm{Perf}(X) \sim \mathrm{Perf}(A)$$

for some dg algebra $A$, where prefect A-modules are direct summands in the homotopy sense of modules $M$ which are free finitely generated $\mathbb{Z}$-graded $A$-modules, with generators $m_1, \dots, m_N$ of certain degree $\deg(m_i) \in \mathbb{Z}$, such that $dm_i \in \oplus_{j<i} A \cdot m_j$ for all $i$. Algebra $A$ associated with $X$ is not unique, it is defined up to a derived Morita equivalence.

Some basic properties of schemes one can formulate purely in derived terms.

**Definition 1.** *Dg algebra $A$ is called smooth if $A \in \mathrm{Perf}(A \otimes A^{op})$. It is compact if $\dim H^\bullet(A, d) < \infty$. This properties are preserved under the derived Morita equivalence.*

For a separated scheme $X$ of finite type the properties of smoothness and properness are equivalent to the corresponding properties of a dg algebra $A$ with $\mathrm{Perf}(A) \sim \mathrm{Perf}(X)$. Smooth and compact dg algebras are expected to be the "ideal" objects of derived geometry, similar to smooth projective varieties in the usual algebraic geometry. For a smooth algebra $A$ the homotopy category $\mathrm{Fin}(A)$ of dg-modules with finite-dimensional total cohomology is contained in $\mathrm{Perf}(A)$, and for compact $A$ the category $\mathrm{Perf}(A)$ is contained in $\mathrm{Fin}(A)$. One can define two notions of a Calabi-Yau algebra of dimension $D \in \mathbb{Z}$. In the smooth case it says that $A^! := \mathrm{Hom}_{A\otimes A^{op}-mod}(A, A \otimes A^{op})$ is quasi-isomorphic to $A[-D]$ as $A \otimes A^{op}$-module (it corresponds to the triviality of the canonical bundle for smooth schemes). Similarly, in the compact case we demand that $A^* = \mathrm{Hom}_{\mathbf{k}-mod}(A, \mathbf{k})$ is quasi-ismorphic to $A[D]$, as a bimodule (it corresponds for schemes to the condition that $X$ has Gorenstein singularities and the dualizing sheaf is trivial).

The notion of smoothness for dg algebras is itself not perfect, as e.g. it includes somewhat pathological example $\mathbf{k}[x, (1/(x - x_i)_{i\in S}]$ where $S \subset \mathbf{k}$ is an *infinite* subset. It seems that the right analog of smooth shemes (of finite type) is encoded in the following notion of dg algebra of finite type due to B. Toën and M. Vaquie:

**Definition 2.** *A dg algebra $A$ is called of finite type if it is a homotopy retract in the homotopy category of dg algebras of the free finitely generated algbera $\mathbf{k}\langle x_1, \ldots, x_N\rangle$, $\deg(x_i) \in \mathbb{Z}$ with the differential of the form*

$$dx_i \in \mathbf{k}\langle x_1, \ldots, x_{i-1}\rangle, \ \ i = 1, \ldots, N \ .$$

Any dg algebra of finite type is smooth, and any smooth compact dg algebra is of finite type. It is also convenient to replace a free graded algebra in the definition of finite type by the algebra of paths in a finite $\mathbb{Z}$-graded quiver.

A large class of small triangulated categories (including many examples from representation theory) can be interpreted as the categories of perfect complexes on a space of finite type with a given "support". In terms of dg algebras, in order to specify the support one should pick a perfect complex $M \in \mathrm{Perf}(A)$. The corresponding category is the full subcategory of $\mathrm{Perf}(A)$ generated by $M$, and is equivalent to $\mathrm{Perf}(B)$ where $B = \mathrm{End}_{A-mod}(M, M)^{op}$. One can say in non-commutative terms what is the "complement" $X - \mathrm{Supp}(M)$ and the "formal completion" $\widehat{X}_{\mathrm{Supp}\,M}$ of $X$ at $\mathrm{Supp}(M)$. The complement is given by the localization of $\mathrm{Perf}(X) = \mathrm{Perf}(A)$ at $M$, and is again of finite type. By Drinfeld's construction, in terms of dg quivers it means that we add a new free generator $h_M \in$

$\mathrm{Hom}^{-1}(M,M)$ with $dh_M = \mathrm{id}_M$. The formal completion is given by algebra $C = \mathrm{End}_{B-mod}(M,M)^{op}$. E.g. when $A = \mathbf{k}[x]$ and $M = \mathbf{k}$ with $x$ acting trivially, we have $B = H^{\bullet}(S^1, \mathbf{k})$ (the exterior algebra in one variable in degree $+1$), and $C = \mathbf{k}[[x]]$.

**Examples of categories of finite type**

**Algebraic geometry**: For any smooth scheme $X$ the category $\mathrm{Perf}(X) \simeq D^b(Coh(X))$ is of finite type.

**Topology:** Let $X$ be now a space homotopy equivalent to a finite connected CW complex. Define $A_X := \mathrm{Chains}(\Omega(X, x_0))$, the dg algebra of chains (graded in non-positive degrees) of the monoid of based loops in $X$, with the product induced from the composition of loops. This algebra is of finite type as can be seen directly from the following description of a quasi-isomorphic algebra.

Let us assume for simplicity that $X$ is simplicial subcomplex in a standard simplex $\Delta^K$ for some $K \in \mathbb{Z}_{\geq 0}$. We associate with such $X$ a finite dg quiver $Q_X$. Its vertices are $v_i$, $i = 0, \ldots, K$ for $i \in X$. The arrows are $a_{i_0,\ldots,i_k}$ for $k > 0$, where $(i_0, \ldots, i_k)$ is a face of $X$, and $i_0 < i_1 < \cdots < i_k$. The arrow $a_{i_0,\ldots,i_k}$ has degree $(1-k)$ and goes from $v_{i_0}$ to $v_{i_k}$. We define the differential in $Q_X$ by

$$da_{i_0,\ldots,i_k} = \sum_{j=1}^{k-1} (-1)^j a_{i_0,\ldots,i_j} \cdot a_{i_j,\ldots,i_k}$$

Then we have to "invert" all arrows of degree 0, i.e. add inverse arrows $a_{i_0,i_1}$ for all egdes $(i_0, i_1)$ in $X$. It can be done either directly (but then we obtain a non-free quiver), or in a more pedantical way which gives a free quiver. In general, if want to invert a arrow $a_{EF}$ in a dg quiver connecting verices $E$ and $F$, with $\deg a_{EF} = 0$ and $da_{EF} = 0$, one can proceed as follows. To say that $a_{EF}$ is an isomorphism is the same as to say that the cone $C := \mathrm{Cone}(a_{EF} : E \to F)$ is zero. Hence we should add an endmorphism $h_C$ of the cone of degree $-1$ whose differential is the identity morphism. Describing $h_C$ as $2 \times 2$ matrix one obtains the following. One has to add 4 arrows

$$h^0_{FE}, h^{-1}_{EE}, h^{-1}_{FF}, h^{-2}_{EF}$$

with degrees indicated by the upper index, with differentials

$$dh^0_{FE} = 0,\ dh^{-1}_{EE} = \mathrm{id}_E - a_{EF} \cdot h^0_{FE},$$
$$dh^{-1}_{FF} = \mathrm{id}_F - h^0_{FE} \cdot a_{EF},\ dh^{-2}_{EF} = a_{EF} \cdot h^{-1}_{FF} - h^{-1}_{EE} \cdot a_{EF}\,.$$

**Theorem 1.** *The quiver $Q_X$ localized in either way, is dg equivalent to $A_X$.*

In particular, if $X$ is space of type $K(\Gamma, 1)$ then $A_X$ is homotopy equivalent to an ordinary algebra in degree 0, the group ring $\mathbf{k}[\Gamma]$. In particular, such an algebra is of finite type. In the case $\mathrm{char}(\mathbf{k}) = 0$ one can also allow torsion, i.e. consider orbispaces, hence $\Gamma$ can be an arithmetic group, a mapping class group, etc.

The full subcategory of finite-dimensional dg modules $\mathrm{Fin}(A_X) \subset \mathrm{Perf}(A_X)$ is the triangulated category of sheaves whose cohomology are finite rank local systems on $X$. If we invert not all arrows of degree 0 in $Q_X$ for simplicial $X \subset \Delta^K$, we can obtain categories of complexes of sheaves with cohomology constructible with respect to a given CW-stratification, and even more general categories.

**Algebraic geometry II:** The last example of a category of finite type is somewhat paradoxical.

**Theorem 2.** *(V.Lunts) For any separated scheme $X$ of finite type the category $D^b(Coh(X))$ (with its natural dg enhancement) is of finite type.*

Morally one should interpret $\mathrm{Perf}(X)$ as the category of perfect complexes on a smooth derived noncommutative space $Y$ with support on a closed subset $Z$. Then the category $D^b(Coh(X))$ can be thought as the category of perfect complexes on the formal neighborhood $\widehat{X}_Z$. It turns out that for the case of usual schemes this neighbourhood coincides with $Y$ itself. The informal reason is that the "transversal coordinates" to $Z$ in $Y$ are of strictly negative degrees, hence the formal power series coincide with polynomials in $\mathbb{Z}$-graded sense.

**Fukaya categories**

Let $(X, \omega)$ be a compact symplectic $C^\infty$ manifold with $c_1(T_X) = 0$

The idea of K. Fukaya is that one should associate with $(X, \omega)$ a compact $A_\infty$ Calabi-Yau category over a non-archimedean field (Novikov ring)

$$Nov := \sum_i a_i T^{E_i}, \;\; a_i \in \mathbb{Q}, \;\; E_i \in \mathbb{R}, \;\; E_i \to +\infty \;,$$

where numbers $E_i$ have the meaning of areas of pseudo-holomorphic discs. The objects of $\mathcal{F}(X)$ in the classical limit $T \to 0$ should be oriented Lagrangain spin manifolds $L \subset X$ (maybe endowed with a local system). There are several modifications of the original definition:

- one can allow manifolds with $c_1 \neq 0$ (in this case one get only a $\mathbb{Z}/2\mathbb{Z}$-graded category),
- on can allow $X$ to have a pseudo-convex boundary (see the discussion of the Stein case below),
- (Landau-Ginzburg model), $X$ is endowed with a potential $W : X \to \mathbb{C}$ satisfying some conditions at infinity (then the corresponding Fukaya-Seidel category is not a Calabi-Yau one),
- allow $X$ to have holes inside, then one get so called "wrapped" Fukaya category with infinite-dimensional Hom-spaces.

**Fukaya categories of Stein manifolds**

The simplest and the most important case is when $X$ is compact complex manifold with real boundary such that there exists a strictly plurisubharmonic function $f : X \to \mathbb{R}_{\leq 0}$ with $f_{|\partial X} = 0$ and no critical points on $\partial X$.

Seidel in his book gave a complete definition of the Fukaya category of Stein manifold in terms of Lefschetz fibrations. The additional data necessary for $\mathbb{Z}$-grading is a trivialization of the square of the canonical bundle. One can analyze his construction and associate certain algebra $A$ of finite type (over $\mathbb{Z}$) such that the Fukaya category constructed by Seidel is a full subcategory of $\mathrm{Fin}(A)$. We propose to consider $A$ (or category $\mathrm{Perf}(A)$ and *not* $\mathrm{Fin}(A)$) to be a more foundamental object, and to formulate all the theory in such terms. For example, for $X = T^*Y$ where $Y$ is a compact oriented manifold, the algebra $A$ is $\mathrm{Chains}(\Omega(Y, y_0))$ contains information about the foundamental group of $Y$, whereas the category of finite-dimensional representations could be very poor for non-residually finite group $\pi_1(Y)$.

Also we propose a slightly different viewpoint on $A_X$. Namely, one can make $X$ smaller and smaller without changing $A$, and eventually contract $X$ to a singular Lagrangian submanifold $L \subset X$. Hence we can say that $A = A_L$ depends only on $L$ (up to derived Morita equivalence). One can think for example about $L$ being a 3-valent graph embedded in an open complex curve $X$ as a homotopy retract. If $X$ is endowed with a potential, we should contract $X$ to a noncompact $L$ such that $\mathrm{Re}(W)_{|L} : L \to \mathbb{R}$ is a proper map to $[c, +\infty)$, $c \in \mathbb{R}$, e.g. $L = \mathbb{R}^n$ for $X = \mathbb{C}^n$ with the holomorphic potential $\sum_{i=1}^n z_i^2$.

We expect that $\mathrm{Fin}(A_X)$ is the global category associated with a constructible sheaf (in homotopy sense) $\mathcal{E}_L$ of smooth compact dg categories on

$L$ depending only on the local geometry. In terms of dg algebras, $A_X$ is a homotopy colimit of a finite diagram of local algebras. For example, if $L$ is smooth and oriented and spin, the sheaf $\mathcal{E}_L$ is the constant sheaf of $\mathrm{Perf}(\mathbb{Z})$, and the global algebra is the algebra $\mathrm{Chains}(\Omega(L, x_0))$ considered before.

In terms of topological field theory, the stalks of $\mathcal{E}_L$ are possible boundary terms for the theory of pseudo-holomorphic discs in $X$ with boundary on $L$.

In codimension 1 singular Lagrangian $L$ looks generically as the product of a smooth manifold with the union of three rays $\{z \in \mathbb{C} \,|\, z^3 \in \mathbb{R}_{\geq 0}\}$, endowed with a natural cyclic order. The stalk of the sheaf $\mathcal{E}_L$ at such a point is $\mathrm{Perf}(A_2)$, the category of representations of quiver $A_2$ (two vertices and one directed egde). The symmetry group of $\mathrm{Perf}(A_2)$ after factoring by the central subgroup of shifts by $2\mathbb{Z}$ is equal to $\mathbb{Z}/3\mathbb{Z}$. Explicitly it can be done by the following modification of the quiver at triple points. Namely, consider the quiver with three vertices (corresponding to 3 objects $E, F, G$), a closed arrow $F \to G$ of degree 0, two arrows $E \to F$, $E \to G$ of degreess $-1$ and 0 respectively (with differential saying that we have a morphism $E \to \mathrm{Cone}(F \to G)$. We say that $E$ is quasi-isomorohic to $\mathrm{Cone}(F \to G)$, i.e.

$$\mathrm{Cone}(E \to \mathrm{Cone}(F \to G)) = 0 \ .$$

This can be done explicitly by constructing a homotopy to the identity of the above object, which is a $3 \times 3$-matrix. Combining all equations together we get a quiver with 3 vertices and 12 arrows which gives a heavy but explicit finite type model for exact triangles.

A natural example of a Lagrangian submanifold with triple point singularities comes from any union of transversally intersecting Lagarngain submanifolds $L_i \subset X$, $i = 1, \ldots, k$. For any point $x$ of intersection (or self-intersection) we should remove small discs in two branches of Lagrangian manifolds intersecting at $x$, and glue a small ball with two collars. The set of triple points forms a sphere.

Global algebra $A_L$ of finite type is Calabi-Yau if $L$ is compact, and not Calabi-Yau in general for non-compact $L$. There are many examples of (compact and non-compact) sungular Lagrangian manifolds such that

$$\mathrm{Perf}(A_L) \simeq D^b(Coh(X))$$

for some scheme $X$ of finite type over $\mathbb{Z}$ (maybe singular and/or non-compact). In the pictures at the end we collected several examples of this "limiting mirror symmetry". Categories of type $A_L$ one can consider as "non-commutative spaces of finite type" defined combinatorially, without parameters. Among other examples one can list toric varieties, maximally degenerate stable curves, etc.

**Deforming degenerate Fukaya categories**

Let us assume that $X$ is compact, in fact a complex projective manifold, and take a complement $X^o$ to an ample divisor. New manifold $X^o$ is Stein, and can be contracted to a singular Lagrangian $L \subset X^o$. The advantage of $X^o$ is that is has no continuous parameters as a symplectic manifold. As was advocated by P. Seidel several years ago, one can think of $\mathcal{F}(X)$ as a deformation of $\mathcal{F}(X^o)$. For example, if $X$ is a two-dimensional torus (elliptic curve) and $X^o$ is the complement to a finite set, then $\mathcal{F}(X^o)$ is equivalent to $\mathrm{Perf}(Y_0)$ where $Y_0$ is a degenerate elliptic curve, a chain of copies of $\mathbb{P}^1$.

In algebraic terms, holomorphic discs in $X$ give a solution of the Maurer-Cartan equation

$$d\gamma + [\gamma, \gamma]/2 = 0, \ \ \gamma \in C^\bullet(A_L, A_L)\widehat{\otimes}\mathfrak{m}_{Nov}$$

where $C^\bullet(A_L, A_L) = \mathrm{Cone}(A_L \to \mathrm{Der}(A_L))$ is the cohomological Hochschild complex of smooth algebra $A_L$, and $\mathfrak{m}_{Nov}$ is the maximal ideal in the ring of integers in the Novikov field $Nov$.

Analogy with algebraic geometry suggests that different choices of open $X^o \subset X$ should lead to dg algebras of finite type endowed with deformations over $\mathfrak{m}_{Nov}$ such that algebras became (in certain sense) derived Morita equivalent after the localization to $Nov$. We expect that such a formulation will handle the cases when the deformed Fukaya category is too small, e.g. when the mirror family consists of non-algebraic varieties (e.g. non-algebraic K3 surfaces or complex tori).

**Question about automorphisms**

The group of connected components of $X$ (with appropriate modications for the potential/Landau-Ginzburg/wrapped cases) acts by automorphisms of dg category $\mathcal{F}(X)$ over the local field $Nov$. One can ask whetehr this group coincides with the whole group of automorphisms. To our knowledge, there is no counterexamples to it! In principle, one can extend the group by taking the product of $X$ with the Landau-Ginzburg model $(\mathbb{C}^n, \sum_{i=1}^n z_i^2)$ which is undistinguishable categorically from a point. So, a more realistic conjecture is that the automorphism group of Fukaya category coincides with the stabilized symplectomorphism group. Why anything like this should be true?

There is an analogous statement in the (commutative) algebraic geometry. The group of automorphisms of a maximally degenerating Calabi-Yau variety $Y$ over a local non-archimedean field $K$ maps naturally to the group of integral piece-wise linear homeomorphisms of certain polytope (called the skeleton, and usally homeomorphic to a sphere). The skeleton lies intrisically in the Berkovich spectrum $Y^{an}$ where the latter is defined as the colimit of sets of points $X(K')$ over all non-archimedean field extensions $K' \supset K$. The Berkovich spectrum is a very hairy but Hausdorff topological space, and the skeleton is a naturally defined homotopy retract of $Y^{an}$.

We expect that one can define some notion of analytic spectrum for a dg algebra over a non-archimedean field, and its skeleton should be probably a piecewise symplectic manifold (maybe infinite-dimensional). For Fukaya type categories this skeleton should be the original symplectic manifold.