

Affine structures and non-archimedean analytic spaces

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Abstract

In this paper we propose a way to construct an analytic space over a non-archimedean field, starting with a real manifold with an affine structure which has integral monodromy. Our construction is motivated by the junction of Homological Mirror conjecture and geometric Strominger-Yau-Zaslow conjecture. In particular, we glue from “flat pieces” an analytic K3 surface. As a byproduct of our approach we obtain an action of an arithmetic subgroup of the group $SO(1, 18)$ by piecewise-linear transformations on the 2-dimensional sphere S^2 equipped with naturally defined singular affine structure.

1 Introduction

1.1

An integral affine structure on a manifold of dimension n is given by a torsion-free flat connection with the monodromy reduced to $GL(n, \mathbf{Z})$. There are two basic situations in which integral affine structures occur naturally. One is the case of classical integrable systems described briefly in Section 3. Most interesting for us is a class of examples arising from analytic manifolds over non-archimedean fields which is discussed in Section 4. It is motivated by the approach to Mirror Symmetry suggested in [KoSo]. We recall it in Section 5. From our point of view manifolds with integral affine structure appear in Mirror Symmetry in two ways. One considers the Gromov-Hausdorff collapse of degenerating families of Calabi-Yau manifolds. The limiting space can be

interpreted either as a contraction (see Section 4.1) of an analytic manifold over a non-archimedean field of Laurent series $\mathbf{C}((t))$, or as a base of a fibration of a Calabi-Yau manifold by Lagrangian tori (with respect to the symplectic Kähler 2-form). On a dense open subset of the limiting space one gets two integral affine structures associated with two interpretations, the non-archimedean one and the symplectic one. Mirror dual family of degenerating Calabi-Yau manifolds should have metrically the same Gromov-Hausdorff limit, with the roles of two integral affine structures interchanged.

Very interesting question arises: how to reconstruct these families of Calabi-Yau manifolds from the corresponding manifolds with integral affine structures? This question was one of the main motivations for present work.

1.2

Our approach to the reconstruction of analytic Calabi-Yau manifolds from real manifolds with integral affine structure can be illustrated in the following toy-model example. Let $S^1 = \mathbf{R}/\mathbf{Z}$ be a circle equipped with the induced from \mathbf{R} affine structure. We equip S^1 with the canonical sheaf $\mathcal{O}_{S^1}^{can}$ of Noetherian $\mathbf{C}((q))$ -algebras. By definition, for an open interval $U \subset S^1$ algebra $\mathcal{O}_{S^1}^{can}(U)$ consists of formal series $f = \sum_{m,n \in \mathbf{Z}} a_{m,n} q^m z^n$, $a_{m,n} \in \mathbf{C}$ such that $\inf_{a_{m,n} \neq 0} (m + nx) > -\infty$. Here $x \in \mathbf{R}$ is any point in a connected component of the pre-image of U in \mathbf{R} , the choice of a different component $x \rightarrow x + k$, $k \in \mathbf{Z}$ corresponds to the substitution $z \mapsto q^k z$. The corresponding analytic space is the Tate elliptic curve (E, \mathcal{O}_E) , and there is a continuous map $\pi : E \rightarrow S^1$ such that $\pi_*(\mathcal{O}_E) = \mathcal{O}_{S^1}^{can}$.

In the case of K3 surfaces one starts with S^2 . The corresponding integral affine structure is well-defined on the set $S^2 \setminus \{x_1, \dots, x_{24}\} \subset S^2$, where x_1, \dots, x_{24} are distinct points. Similarly to the above toy-model example one can construct the canonical sheaf $\mathcal{O}_{S^2 \setminus \{x_1, \dots, x_{24}\}}^{can}$ of algebras, an open 2-dimensional smooth analytic surface X' with the trivial canonical bundle (Calabi-Yau manifold), and a continuous projection $\pi' : X' \rightarrow S^2 \setminus \{x_1, \dots, x_{24}\}$ such that $\pi'_*(\mathcal{O}_{X'}) = \mathcal{O}_{S^2 \setminus \{x_1, \dots, x_{24}\}}^{can}$. The problem is to find a sheaf \mathcal{O}_{S^2} whose restriction to $S^2 \setminus \{x_1, \dots, x_{24}\}$ is locally isomorphic to $\mathcal{O}_{S^2 \setminus \{x_1, \dots, x_{24}\}}^{can}$, an analytic compact K3 surface X , and a continuous projection $\pi : X \rightarrow S^2$ such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_{S^2}$. We call this problem (in general case) the Lifting Problem and discuss it in Section 7. Unfortunately we do not know the conditions one should impose on singularities of the affine

structure, so that the Lifting Problem would have a solution. We consider a special case of K3 surfaces in Sections 8-11. Here the solution is non-trivial and depends on data which are not visible in the statement of the problem. They are motivated by Mirror Symmetry and consist, roughly speaking, of an infinite collection of trees embedded into $S^2 \setminus \{x_1, \dots, x_{24}\}$ with the tail vertices belonging to the set $\{x_1, \dots, x_{24}\}$. The sheaf $\mathcal{O}_{S^2 \setminus \{x_1, \dots, x_{24}\}}^{can}$ has to be modified by means of automorphisms assigned to every edge of a tree and then glued together with certain model sheaf near each singular point x_i .

Informally speaking, we break $S^2 \setminus \{x_1, \dots, x_{24}\}$ endowed with the sheaf $\mathcal{O}_{S^2 \setminus \{x_1, \dots, x_{24}\}}^{can}$ into infinitely many infinitely small pieces and then glue them back together in a slightly deformed way. The idea of such a construction was proposed several years ago independently by K. Fukaya and the first author. The realization of this idea was hindered by a poor understanding of singularities of the Gromov-Hausdorff collapse and by the lack of knowledge of certain open Gromov-Witten invariants (“instanton corrections”). The last problem is circumvented here (and in fact solved) with the use of some pro-nilpotent Lie group (see Section 10).

1.3

The relationship between K3 surfaces and singular affine structures on S^2 is of very general origin. Starting with a projective analytic Calabi-Yau manifold X over a complete non-archimedean local field K one can canonically construct a PL manifold $Sk(X)$ called the skeleton of X . If X is a generic K3 surface then $Sk(X)$ is S^2 . We discuss skeleta in Section 6.6. The group of birational automorphisms of X acts on $Sk(X)$ by integral PL transformations. For $X = K3$ we obtain an action of an arithmetic subgroup of $SO(1, 18)$ on S^2 . Further examples should come from Calabi-Yau manifolds with large groups of birational automorphisms.

1.4

We have already discussed the content of the paper. Let us summarize it. The paper is naturally divided into three parts. Part 1 is devoted to generalities on integral affine structures and examples, including Mirror Symmetry. Motivated by string theory we use term A-model (resp. B-model) for examples arising in symplectic (resp. analytic) geometry.

In Part 2 we discuss the concept of singular integral affine structure, including an affine version of Gauss-Bonnet theorem. The latter implies that if all singularities of an integral affine structure on S^2 are standard (so-called focus-focus singularities) then there are exactly 24 singular points. Part 2 also contains a statement of the Lifting Problem and discussion of flat coordinates on the moduli space of complex Calabi-Yau manifolds. We expect that under mild conditions on the singular integral affine structure there exists a solution of the Lifting Problem, which is unique as long as we fix periods (see Sections 7.3 and 7.4 for more details).

Most technical Part 3 contains a solution of the Lifting Problem for K3 surfaces. We construct the corresponding analytic K3 surface as a ringed space. The sheaf of analytic functions is defined differently near a singular point and far from the singular set. It turns out that the “naive” candidate for the sheaf on the complement of the singular set has to be modified before we can glue it with the model sheaf near each singular point. This modification procedure involves a new set of data (we call them *lines*). We also discuss the group of automorphisms of the canonical sheaf which preserve the symplectic form. We use this group in order to modify the “naive” sheaf along each line.

The paper has two Appendices. First one contains some background on analytic spaces, while the second one is devoted to Torelli theorem.

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Part I

2 \mathbf{Z} -affine structures

2.1 Definitions

Let us recall that an affine structure on manifold Y (smooth, of dimension n) is given by a torsion-free flat connection ∇ on the tangent bundle TY .

We will give below three equivalent definitions of the notion of an integral affine structure.

Definition 1 *An integral affine structure on Y (\mathbf{Z} -affine structure for short)*

is an affine structure ∇ together with a ∇ -covariant lattice of maximal rank $T^{\mathbf{Z}} = (TY)^{\mathbf{Z}} \subset TY$.

It is easy to see that if Y carries a \mathbf{Z} -affine structure then for any point $y \in Y$ there exist small neighborhood U , local coordinate system (x_1, \dots, x_n) in U such that $\nabla = d$ in coordinates (x_1, \dots, x_n) , and the lattice $(T_x Y)^{\mathbf{Z}}, x \in U$ is a free abelian group generated by the tangent vectors $\partial/\partial x_i \in T_x Y, 1 \leq i \leq n$. Let us call \mathbf{Z} -affine such a coordinate system in U (sometimes we will call such U a \mathbf{Z} -affine chart). For a covering of Y by \mathbf{Z} -affine charts the transition functions belong (locally) to $GL(n, \mathbf{Z}) \times \mathbf{R}^n$. Explicitly, a change of coordinates is given by the formula

$$x'_i = \sum_{1 \leq j \leq n} a_{ij} x_j + b_i ,$$

where $(a_{ij}) \in GL(n, \mathbf{Z}), (b_i) \in \mathbf{R}^n$.

Hence, Definition 1 is equivalent to the following

Definition 2 *A \mathbf{Z} -affine structure on Y is given by a maximal atlas of charts such that the transition functions belong locally to $GL(n, \mathbf{Z}) \times \mathbf{R}^n$.*

In the above definition Y is just a topological manifold, C^∞ -structure on it can be reconstructed canonically from \mathbf{Z} -affine structure.

We can restate the notion of \mathbf{Z} -affine structure in the language of sheaves of affine functions.

We say that a real-valued function f on \mathbf{R}^n is \mathbf{Z} -affine if it has the form

$$f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n + b ,$$

where $a_1, \dots, a_n \in \mathbf{Z}$ and $b \in \mathbf{R}$. We will denote by $Aff_{\mathbf{Z}, \mathbf{R}^n}$ the sheaf of functions on \mathbf{R}^n which are locally \mathbf{Z} -affine.

Definition 3 *A \mathbf{Z} -affine structure (of dimension n) on a Hausdorff topological space Y is a subsheaf $Aff_{\mathbf{Z}, Y}$ of the sheaf of continuous functions on Y , such that the pair $(Y, Aff_{\mathbf{Z}, Y})$ is locally isomorphic to $(\mathbf{R}^n, Aff_{\mathbf{Z}, \mathbf{R}^n})$.*

Equivalence of the last two definitions follows from the observation that a homeomorphism between two open domains in \mathbf{R}^n preserving the sheaf $Aff_{\mathbf{Z}, \mathbf{R}^n}$ is given by the same formula $x' = A(x) + b, A \in GL(n, \mathbf{Z}), b \in \mathbf{R}^n$ as the change of coordinates between two \mathbf{Z} -affine coordinate systems.

2.2 Monodromy representation and its invariant

With a given affine structure on Y we can associate a flat affine connection ∇^{aff} (see [KN]). The corresponding parallel transport acts on tangent spaces by affine transformations. For a \mathbf{Z} -affine structure the monodromy of ∇^{aff} belongs to $GL(n, \mathbf{Z}) \times \mathbf{R}^n$, i.e. $\forall y \in Y$ we have a monodromy representation

$$\rho : \pi_1(Y, y) \rightarrow GL(n, \mathbf{Z}) \times \mathbf{R}^n .$$

Alternatively, we can define the monodromy representation by covering a loop in Y by \mathbf{Z} -affine coordinate charts and composing the corresponding transition functions.

Notice that a \mathbf{Z} -affine structure on Y gives rise to a class

$$[\rho] \in H^1(Y, T^{\mathbf{Z}} \otimes \mathbf{R}) = H^1(Y, T_Y^{\nabla}) ,$$

where $T_Y^{\nabla} \subset T_Y$ is the subsheaf of ∇ -flat sections¹. De Rham representative of class $[\rho]$ is given by a differential 1-form $\theta \in \Omega^1(Y, T_Y)$ such that $\theta(v) = v$ for any tangent vector v . In affine coordinates one has $\theta = \sum_i \partial/\partial x_i \otimes dx_i$. Clearly $\nabla(\theta) = 0$.

We will need later an explicit formula for the \mathbf{R} -valued pairing of $[\rho]$ with a closed singular 1-chain with coefficients in the local system $(T^*)^{\mathbf{Z}} = (T^*Y)^{\mathbf{Z}}$, the dual covariant lattice in T^*Y . With any singular 1-chain c with values in $(T^*)^{\mathbf{Z}}$ we associate a real number $j(c)$ in the following way. Suppose that c is given by a continuous map $\gamma : [0, 1] \rightarrow Y$ and a section $\alpha \in \Gamma([0, 1], \gamma^*(T^*)^{\mathbf{Z}})$. Parallel transport via the connection ∇^{aff} gives rise to a map $\bar{\gamma} : [0, 1] \rightarrow T_{\gamma(0)}Y$, $\bar{\gamma}(0) = 0$. Let $\alpha_0 = \alpha(0) \in (T_{\gamma(0)}^*)^{\mathbf{Z}} \subset T_{\gamma(0)}^*Y$. We define $j(c) = \langle \alpha_0, \bar{\gamma}(1) \rangle$. We extend $j(c)$ to an arbitrary singular 1-chain c by additivity. Then the class $[\rho]$ can be calculated as $\langle [\rho], [c] \rangle = j(c)$ for any closed 1-chain $c \in C_1(Y, (T^*)^{\mathbf{Z}})$.

3 A-model construction

3.1 Integrable systems

Let (X, ω) be a smooth symplectic manifold of dimension $2n$, B_0 a smooth manifold of dimension n , $\pi : X \rightarrow B_0$ a smooth map with compact fibers,

¹Here we slightly abuse notations because Y is not necessarily connected.

such that $\{\pi^*(f), \pi^*(g)\} = 0$ for any $f, g \in C^\infty(B_0)$. Here $\{\cdot, \cdot\}$ denotes the Poisson bracket on X . We assume that π is a submersion on an open dense subset $X' \subset X$. Such a triple (X, π, B_0) is called an *integrable system*. In applications it is typically given by a collection of smooth functions (H_1, \dots, H_n) on X (these functions are called Hamiltonians) such that $\{H_i, H_j\} = 0, 1 \leq i, j \leq n$. Usually first Hamiltonian $H = H_1$ is identified with the energy of mechanical system.

Let us consider the case when π is proper. It is a natural restriction, because in applications the energy H_1 is already a proper map $H_1 : X \rightarrow \mathbf{R}$.

Let $x \in B_0$ be a point such that the restriction of π to $\pi^{-1}(x)$ is a submersion. We call such points π -smooth. According to Sard theorem π -smooth points form an open dense subset of B_0 . The fiber $\pi^{-1}(x)$ is a compact Lagrangian submanifold of X . The Liouville integrability theorem (see [Ar]) says that $\pi^{-1}(x)$ is a disjoint union of finitely many tori T_α^n . Moreover, for each torus T_α^n there exists a local coordinate system $(\varphi_1, \dots, \varphi_n, I_1, \dots, I_n)$ in a neighborhood W_α of T_α^n such that $\varphi_i \in \mathbf{R}/2\pi\mathbf{Z}, (I_1, \dots, I_n) \in \mathbf{R}^n$ and $\omega = \sum_{1 \leq i \leq n} dI_i \wedge d\varphi_i$. These coordinates are called action-angle coordinates. The map π in action-angle coordinates is given by the projection $(\varphi_1, \dots, \varphi_n, I_1, \dots, I_n) \mapsto (I_1, \dots, I_n)$. There is an ambiguity in the choice of action-angle coordinates. In particular action coordinates $I = (I_1, \dots, I_n)$ are defined up to a transformation $I' = A(I) + b, A \in GL(n, \mathbf{Z}), b \in \mathbf{R}^n$. Indeed, the free abelian group generated by 1-forms $dI_i, 1 \leq i \leq n$ in each cotangent space $T_x^*B_0$ admits an invariant description. It is the free abelian group generated by the restrictions of 1-forms $\int_\gamma \omega$ to $T_x^*B_0$, where γ runs through closed singular 1-chains in $\pi^{-1}(x) \cap W_\alpha$. In this way we obtain a \mathbf{Z} -affine structure on $\pi(W_\alpha)$.

Let B be the set of connected components of fibers of π . Endowed with the natural topology it becomes a locally compact Hausdorff space, projection from X to B will be denoted by the same letter π . The natural continuous map $B \rightarrow B_0$ is a kind of “ramified finite covering”. Let us define $B^{sm} \subset B$ as the set of connected components on which π is a submersion (i.e. the set of all Liouville tori). Then B^{sm} is an open dense subset in B . Hence it carries a \mathbf{Z} -affine structure given by the action coordinates.

The singular part $B^{sing} = B \setminus B^{sm}$ consists of projections of singular fibers. Typically the codimension of B^{sing} is greater or equal to 1. The codimension 1 stratum consists of the boundary of the image of π and of the ramification locus of the map $B \rightarrow B_0$. The structure of singularities of the

integral affine structure in higher codimensions is less understood. It seems that the following property is always satisfied:

Fixed Point property . For any $x \in B^{sing}$ there is a small neighborhood U such that the monodromy representation $\pi_1((U \setminus B^{sing})_\alpha) \rightarrow GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$ for any connected component $(U \setminus B^{sing})_\alpha$ of $U \setminus B^{sing}$ has a fixed vector in \mathbf{R}^n in the natural representation by affine transformations.

We will discuss this property in Section 6 devoted to compactifications.

3.1.1 Cohomological interpretation of class $[\rho]$

In Section 2.2 we introduced an invariant $[\rho] \in H^1(B^{sm}, T^{\mathbf{Z}} \otimes \mathbf{R})$ of a \mathbf{Z} -affine structure. Here we will give an interpretation of $[\rho]$ for integrable systems.

Let us consider $X' = \pi^{-1}(B^{sm})$ which is a Lagrangian torus fibration over B^{sm} (i.e. fibers are Lagrangian tori such that the fiber over $x \in B^{sm}$ is isomorphic up to a shift to the torus $T_x^*B^{sm}/(T_x^*B^{sm})^{\mathbf{Z}}$).

Any singular closed 1-chain c on B^{sm} with values in the local system

$$(T_x^*B^{sm})^{\mathbf{Z}} \simeq H_1(T_x^*B^{sm}/(T_x^*B^{sm})^{\mathbf{Z}}, \mathbf{Z})$$

gives a 2-chain \bar{c} on X' with the boundary belonging to a finite collection of fibers $\pi^{-1}(x^{(i)})$, $1 \leq i \leq N$ of the fibration $\pi : X' \rightarrow B^{sm}$. Moreover, for every point $x^{(i)}$ the part of $\partial\bar{c}$ over $x^{(i)}$ is homologous to zero in $\pi^{-1}(x^{(i)})$. Therefore, there exists a collection of 2-chains \bar{c}_i , $1 \leq i \leq N$ supported on $\pi^{-1}(x^{(i)})$ such that the 2-chain $\bar{c} + \sum_{1 \leq i \leq N} \bar{c}_i$ is closed. In this way we obtain a group homomorphism $J_s : H_1(B^{sm}, (T^*)^{\mathbf{Z}}) \rightarrow H_2(X', \mathbf{Z})/H_2^0(X', \mathbf{Z})$, where $H_2^0(X', \mathbf{Z}) \subset H_2(X', \mathbf{Z})$ denotes the sum of images of $H_2(\pi^{-1}(y), \mathbf{Z})$ where $y \in B^{sm}$ (it is enough to pick one base point y for any connected component of B^{sm}). It is easy to see that $\langle [\rho], [c] \rangle = \langle [\omega], J_s([c]) \rangle$, where $[\omega]$ is the class of the symplectic form ω .

3.2 Examples of integrable systems

We describe here few examples related to the rest of the paper.

3.2.1 Flat tori

First example is the triple (X, π, B_0) where $X = \mathbf{R}^{2n}/\Lambda$, $B_0 = \mathbf{R}^n/\Lambda'$ are tori (here $\Lambda \simeq \mathbf{Z}^{2n}$, $\Lambda' \simeq \mathbf{Z}^n$ are lattices), projection $\pi : X \rightarrow B_0$ is an affine map of tori, and X carries a constant symplectic form. Assuming that fibers

of π are connected we have $B_0 = B = B^{sm}$. The monodromy representation is a homomorphism $\rho : \pi_1(B) \rightarrow \mathbf{R}^n \subset GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$. Integral affine structure on B depends on n^2 real parameters, which are coefficients of an invertible $n \times n$ matrix expressing a basis of the lattice $\Lambda' \subset T_x B$ as a linear combination of generators of the lattice $(T_x B)^{\mathbf{Z}} \subset T_x B$, where $x \in B$ is an arbitrary point.

3.2.2 Surfaces

Let (X, ω) be a surface and $\pi : X \rightarrow B_0 = \mathbf{R}$ be an arbitrary smooth proper function with isolated critical points. Then (X, π, B_0) is an integrable system. Space B of connected components of fibers is a graph, and \mathbf{Z} -affine structure on $B^{sm} \subset B$ gives a length element on edges of B .

3.2.3 Moment map

Consider a compact connected symplectic manifold (X, ω) of dimension $2n$ together with a Hamiltonian action of the torus T^n . Then one has an integrable system $\pi : X \rightarrow B_0$, where π is the moment map of the action and $B_0 = (Lie(T^n))^* \simeq \mathbf{R}^n$. Furthermore, it is well-known that $B = \pi(X)$ is a convex polytope and B^{sm} is the interior of B .

3.2.4 K3 surfaces

Before considering this example let us remark that one can define integrable systems in the case of complex manifolds. More precisely, assume that X is a complex manifold of complex dimension $2n$, $\omega_{\mathbf{C}}$ is a holomorphic closed non-degenerate 2-form on X , $B = B_0$ is a complex manifold of dimension n and $\pi : X \rightarrow B$ is a surjective proper holomorphic map such that generic fibers of π are connected complex Lagrangian submanifolds of X . With a complex integrable system one can associate a real one by forgetting complex structures on X and B and taking $\omega := Re(\omega_{\mathbf{C}})$ as a symplectic form on X . It is easy to see that the image of the monodromy representation belongs to $Sp(2n, \mathbf{Z}) \ltimes \mathbf{R}^{2n} \subset GL(2n, \mathbf{Z}) \ltimes \mathbf{R}^{2n}$.

Let (X, Ω) be a complex K3 surface equipped with a non-zero holomorphic 2-form $\omega_{\mathbf{C}} = \Omega$ and $\pi : X \rightarrow \mathbf{C}P^1$ a holomorphic fibration such that the generic fiber of π is an elliptic curve. For example, X can be represented as a surface in $\mathbf{C}P^2 \times \mathbf{C}P^1$ given by a general equation $F(x_0, x_1, x_2, y_0, y_1) = 0$

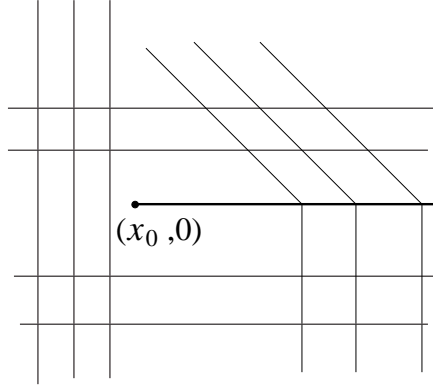


Figure 1: Focus-focus singularity. All lines are straight in the modified \mathbf{Z} -affine structure.

of bidegree $(3, 2)$ in homogeneous coordinates. Map π is the projection to the second factor. Holomorphic form Ω is given by

$$\Omega = i_{Euler_x \wedge Euler_y} \frac{dx_0 \wedge dx_1 \wedge dx_2 \wedge dy_0 \wedge dy_1}{dF},$$

where $Euler_p$ denotes the Euler vector field along coordinates $p = (x_i)$ or (y_i) . Such an elliptic fibration gives an integrable system. Namely, we set $X := X(\mathbf{C})$, $\omega := Re(\Omega)$, $B := \mathbf{C}P^1 \simeq S^2$. Generically B^{sing} is a set of $24 = \chi(X)$ points in S^2 . Singularity of the affine structure near each of 24 points is well-known in the theory of integrable systems where it is called focus-focus singularity (see e.g. [Au], [Zu]). We will discuss it in Section 6.4. Here we give a short description of this singularity. We take \mathbf{R}^2 with the standard integral affine structure, remove the point $(x_0, 0)$ on the horizontal axis. Then we modify the affine structure (and also the C^∞ -structure!) on the ray $\{(x, 0) \mid x > x_0\}$. New local integral affine coordinates near points of this ray will be functions y and $x + \max(y, 0)$ (see Figure 1). The monodromy of the resulting integral affine structure around removed singular point $(x_0, 0)$ is given by the transformation $(x, y) \mapsto (x + y, y)$.

3.3 Families of integrable systems and PL actions

In many examples an integrable system depends on parameters. It often happens that the parameter space \mathcal{P} carries a natural foliation \mathcal{F} such that the fundamental group $\pi_1(\mathcal{F}_p, p)$, $p \in \mathcal{P}$ of any leaf acts on the base space B_p

of the corresponding torus fibration. This action is given by piecewise-linear homeomorphisms with integral linear parts.

Let us illustrate this phenomenon in the case of the family of integrable systems associated with a K3 surface discussed above.

Here the parameter space \mathcal{P} has dimension 38, which is twice of the complex dimension of the space of polynomials F modulo unimodular linear transformations. On the other hand, the miniversal family of representations (up to a conjugation)

$$\{ \rho : \pi_1(S^2 - \{24 \text{ points}\}) \rightarrow SL(2, \mathbf{Z}) \rtimes \mathbf{R}^2 \}$$

such that the monodromy around each puncture is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, has dimension 20.

Thus, we obtain a foliation \mathcal{F} of \mathcal{P} of rank $18 = 38 - 20$. It is defined by the following property: if we continuously vary parameters $p \in \mathcal{P}$ along leaves of \mathcal{F} then the conjugacy class of the monodromy representation ρ remains unchanged.

Notice that in the local model described above we can move the position $(x_0, 0)$ at which we start the cut. Then we have on the sphere S^2 a set of 24 “worms” (singular points, each of them can move in its preferred direction, which is the line invariant under the local monodromy). One can show easily that any continuous deformation of \mathbf{Z} -affine structure satisfying Fixed Point property (see Section 3.1) and preserving the conjugacy class of ρ , corresponds to a movement of worms.²

Moving “worms” we get a canonical identification of manifolds with integral affine structures far enough from singular points. We will see later in Section 6.4 that we also have a canonical PL identification of manifolds near singular points. Therefore we obtain a local system along leaves of \mathcal{F} with the fiber over $p \in \mathcal{P}$ being a manifold $B_p \simeq S^2$ with the above \mathbf{Z} -affine structure. In this way we get a homomorphism from $\pi_1(\mathcal{F}_p, p)$ to $Aut_{\mathbf{Z}PL}(S^2)$, where $\mathbf{Z}PL$ denotes the group of integral PL transformations of S^2 equipped with the above \mathbf{Z} -affine structure. We will return to this action in Section 6.7 where it will be compared with another PL action on the same space.

²Notice that in our example $rk(\mathcal{F}) = 18$ is less than 24. This means that there are 6 constraints on moving worms.

4 B-model construction

4.1 Z-affine structure on smooth points

Here we are going to define an analog of the notion of integrable system in the framework of rigid analytic geometry. Roughly speaking, it is a triple (X, π, B) , where X is a variety defined over a non-archimedean field (see [Be1] and Appendix A), B a CW complex and $\pi : X \rightarrow B$ a continuous map. More precisely, let K be a field with non-trivial valuation, X an irreducible algebraic variety over K of dimension n , $f = (f_1, \dots, f_N)$ a collection of non-zero rational functions on X . Then we have a multivalued map

$$X(\overline{K}) \rightarrow [-\infty, +\infty]^N, \quad x \mapsto \text{val}_{\overline{K}}(f(x)) := (\text{val}_{\overline{K}}(f_1(x)), \dots, \text{val}_{\overline{K}}(f_N(x))) .$$

Here \overline{K} is the algebraic closure of K , and $\text{val}_{\overline{K}}$ denotes valuation on \overline{K} .

Let $\psi : [-\infty, +\infty]^N \rightarrow B$ be a continuous map such that the composition $\pi = \psi \circ \text{val}(f)$ is single-valued. Our map π will always be of this form. More generally, we can take X to be a (not necessarily algebraic) compact smooth K -analytic space, and $\pi : X \rightarrow B$ be a continuous map which factorizes as the composition of the projection $p_{\mathcal{X}} : X \rightarrow S_{\mathcal{X}}$ to the Clemens polytope $S_{\mathcal{X}}$ of some model \mathcal{X} of X and a continuous map $\pi' : S_{\mathcal{X}} \rightarrow B$ (see Section 4.2.3).

Now we would like to be more precise. Let K be a complete non-archimedean field, with valuation val and the corresponding norm $|x| := \exp(-\text{val}(x)) \in \mathbf{R}_{\geq 0}$. Before giving next definition we observe that there is a canonical continuous map $\pi_{\text{can}} : (\mathbf{G}_m^{\text{an}})^n \rightarrow \mathbf{R}^n$ (see Section A.2 in Appendix A). Here \mathbf{G}_m^{an} is a multiplicative group (considered as an analytic space over K) and the restriction of π_{can} to $(\overline{K}^{\times})^n$ is given by the formula

$$\pi_{\text{can}}(z_1, \dots, z_n) = (\log |z_1|, \dots, \log |z_n|) .$$

The sheaf $\mathcal{O}_{\mathbf{R}^n}^{\text{can}} := (\pi_{\text{can}})_*(\mathcal{O}_{(\mathbf{G}_m^{\text{an}})^n})$ of K -algebras is called the *canonical sheaf*.

Let X be a smooth K -analytic space of dimension n , $\pi : X \rightarrow B$ a continuous map of X into a Hausdorff topological space B .

Definition 4 *We call a point $x \in B$ smooth (or π -smooth) if there is a neighborhood U of x such that the fibration $\pi^{-1}(U) \rightarrow U$ is isomorphic to a fibration $\pi_{\text{can}}^{-1}(V) \rightarrow V$ for some open subset $V \subset \mathbf{R}^n$. Here the isomorphism*

$\pi^{-1}(U) \simeq \pi_{can}^{-1}(V)$ is taken in the category of K -analytic spaces while $U \simeq V$ is a homeomorphism.

In this case we will call π (or the triple $(\pi^{-1}(U), \pi, U)$) an analytic torus fibration.

Let B^{sm} denotes the set of smooth points of B . It is a topological subspace of B (in fact a topological manifold of dimension n).

Theorem 1 *The space B^{sm} carries a sheaf of \mathbf{Z} -affine functions, which is locally isomorphic to the canonical sheaf of \mathbf{Z} -affine functions on \mathbf{R}^n .*

Proof. We start with the following Lemma.

Lemma 1 *Let $V \subset \mathbf{R}^n$ be a connected open set, $\varphi \in \mathcal{O}_{(\mathbf{G}_m^{an})^n}^\times(\pi_{can}^{-1}(V))$ be an invertible analytic function. Then the function $val_x(\varphi(x))$ is constant along fibers of π_{can} , and it is a pull-back of a \mathbf{Z} -affine function on \mathbf{R}^n .*

In order to prove Lemma we observe that any analytic function $\psi \in \mathcal{O}_{(\mathbf{G}_m^{an})^n}^\times(\pi_{can}^{-1}(V))$ can be decomposed into Laurent series:

$$\psi = \sum_{I=(i_1, \dots, i_n) \in \mathbf{Z}^n} c_I z^I, \quad c_I \in K$$

satisfying certain convergence conditions (see Section A.2).

Then for a non-zero analytic function ψ on $\pi_{can}^{-1}(V)$ we introduce a real-valued function $Val(\psi)(x) := \inf_{I \in \mathbf{Z}^n} (val(c_I) - \langle I, x \rangle)$, $x \in V$. It is a concave, locally piecewise-linear function on V . It is easy to see that

- a) there is a dense open subset $V_1 \subset V$ such that for any $x \in V_1$ the infimum in the definition of $Val(\psi)$ is achieved for a single multi-index I ;
- b) $Val(\psi_1 \psi_2) = Val(\psi_1) + Val(\psi_2)$.

For an invertible function φ we have $0 = Val(1) = Val(\varphi \varphi^{-1}) = Val(\varphi) + Val(\varphi^{-1})$. Since both $Val(\varphi)$ and $Val(\varphi^{-1})$ are concave, their sum can be equal to zero iff they are both affine. Moreover they are both \mathbf{Z} -affine since the linear part of $Val(\varphi)$ is given by the integer vector I for some single multi-index I . Finally, observe that $val_x(\varphi(x)) \geq \pi_{can}^*(Val(\varphi))(x)$, $x \in \pi_{can}^{-1}(V)$. Therefore $val_x(\varphi(x)) = Val(\varphi)(\pi_{can}(x))$ for invertible φ . ■

Now we can finish the proof of the Theorem. The above formula gives us a coordinate-free description of $\pi_{can}^*(Val(\varphi))$. It is easy to see that any \mathbf{Z} -affine function on V is of the form $Val(\varphi) + c, c \in \mathbf{R}$ for some invertible φ (in the case of \mathbf{R}^n it suffices to take monomials as φ). We can identify $\pi^{-1}(U) \rightarrow U$ with $\pi_{can}^{-1}(V) \rightarrow V$ for some small open $U \subset X$ and $V \subset \mathbf{R}^n$. Then we can define $Val(\varphi)$ for any invertible $\varphi \in \mathcal{O}_X(\pi^{-1}(U))$ by the above formula. Finally we define a sheaf of \mathbf{Z} -affine functions on B^{sm} by taking all functions of the form $Val(\varphi) + c, c \in \mathbf{R}$. It follows from the above discussion that in this way we obtain a \mathbf{Z} -affine structure on B^{sm} , which is locally isomorphic to the standard one on \mathbf{R}^n . ■

We will denote by $Aff_{\mathbf{Z}, B^{sm}}^{can}$ the sheaf of \mathbf{Z} -affine functions constructed in the proof.

4.2 Examples

4.2.1 Logarithmic map

This is a basic example

$$\pi = \pi_{can} = \log |\cdot| : X = (\mathbf{G}_m^{an})^n \rightarrow B_0 = B = \mathbf{R}^n$$

described in details in Appendix A. For any algebraic (or analytic) subvariety $Z \subset (\mathbf{G}_m^{an})^n$ of dimension $m \leq n$ its image $\pi(Z)$ is a non-compact piecewise-linear closed subset of \mathbf{R}^n of real dimension m . Smooth points for $\pi|_Z$ are dense in $\pi(Z)$.

In particular, if Z is a curve then $\pi(Z)$ is a graph in B with straight edges having rational directions. One can try to make a dictionary which translates the properties of the algebraic variety $Z \subset \mathbf{G}_m^n$ to the properties of the PL set $\pi(Z^{an})$ which is the closure of $\pi(Z(\overline{K}))$ in \mathbf{R}^n . This circle of ideas is a subject of the so-called “tropical geometry” (see e.g. [Mi]).

4.2.2 Tate tori

Let $\rho : \mathbf{Z}^n \rightarrow (K^\times)^n$ be a group homomorphism such that the image of the composition $val \circ \rho : \mathbf{Z}^n \rightarrow \mathbf{R}^n$ is a rank n lattice in \mathbf{R}^n . Group $(K^\times)^n$ acts by translations on the analytic space $(\mathbf{G}_m^{an})^n$. Restriction of this action to \mathbf{Z}^n (via ρ) is discrete and cocompact. The quotient is a K -analytic space X called *Tate torus*. There is an obvious map $\pi : X \rightarrow B := \mathbf{R}^n / (val \circ \rho)(\mathbf{Z}^n)$. All points of B are smooth. The space X depends on n^2 parameters taking values in K^\times (cf. with the flat tori example in Section 3.2.1).

4.2.3 Clemens polytopes and their contractions

For any smooth projective variety X of dimension n , and a snc model \mathcal{X} of it (see Appendix A) we have a canonical projection to the corresponding Clemens polytope

$$p_{\mathcal{X}} : X^{an} \rightarrow S_{\mathcal{X}} .$$

All interior points of n -dimensional simplices of $S_{\mathcal{X}}$ are $p_{\mathcal{X}}$ -smooth, although there might be other smooth points too. More generally, one can compose projection $p_{\mathcal{X}}$ with a continuous surjection $\pi' : S_{\mathcal{X}} \twoheadrightarrow B$ where B is a finite CW complex and map π' is a cell map for some cell subdivision of $S_{\mathcal{X}}$. We assume that fibers of the composition $\pi := \pi' \circ p_{\mathcal{X}} : X^{an} \rightarrow B$ are connected. This seems to be the most general case of maps from projective varieties over complete local fields to CW complexes relevant for our purposes.

4.2.4 Curves

Let X/K be a connected smooth projective curve of genus $g > 1$. After passing to a finite extension K' of K we may assume that X has a canonical model \mathcal{X} with stable reduction. The graph Γ' corresponding to the special fiber \mathcal{X}_0 is a retraction of $(X \otimes_K K')^{an}$. The quotient graph $\Gamma = \Gamma'/Gal(K'/K)$ is a retraction of the analytic curve X^{an} (see [Be1]). We define $B := \Gamma$. Then B^{sm} is a complement to a finite set. As in Section 3.2.2, a \mathbf{Z} -affine structure on a graph is the same as a length element (i.e. a metric). Therefore Γ is a metrized graph. Notice also that the maximal number of edges of the graph corresponding to a genus g curve is $3g - 3$, which is the dimension of the moduli space of genus g curves.

Notice that if in Section 4.2.1 subvariety Z is a curve then its projection is a noncompact metrized graph with unbounded edges corresponding to punctures $\overline{Z} \setminus Z$.

4.2.5 K3 surfaces

Here we will describe a particular case of the construction from Section 4.2.3 (a contraction of a Clemens polytope).

Let field K be $\mathbf{C}((t))$ and $X \subset \mathbf{P}_K^3$ be a formal family of complex K3 surfaces given by the equation

$$x_0x_1x_2x_3 + tP_4(x_0, x_1, x_2, x_3) = 0 ,$$

where P_4 is a generic homogeneous polynomial of degree four, and t is a formal parameter.

The special fiber at $t = 0$ of this family is singular, it is given by the equation $x_0x_1x_2x_3 = 0$. Let us denote by $\widetilde{\mathbf{P}^3}$ the blow-up of the total space of the trivial \mathbf{P}^3 -bundle over $\text{Spec}(\mathcal{O}_K)$ at 24 points $p_\alpha, 1 \leq \alpha \leq 24$ of the special fiber, where each p_α is a solution of the equation

$$P_4(x_0, x_1, x_2, x_3) = 0, \quad x_i = x_j = 0, \quad 1 \leq i < j \leq 4 .$$

The closure \mathcal{X} of X in $\widetilde{\mathbf{P}^3}$ is a model with simple normal crossings. The associated Clemens polytope $S_{\mathcal{X}}$ has 28 vertices. Four of them correspond to coordinate hyperplanes $x_i = 0$ in \mathbf{P}^3 , and 24 other correspond to divisors sitting at the pre-images of the points p_α . Therefore $S_{\mathcal{X}}$ is the union of the boundary $\partial\Delta^3$ of the standard 3-simplex Δ^3 with 24 copies of the standard 2-simplex Δ^2 . Those 24 triangles $\Delta_\alpha^2, 1 \leq \alpha \leq 24$ are decomposed into six groups of four triangles in each. All triangles from the same group have a common edge, which is identified with an edge of $\partial\Delta^3$ (tetrahedron with 24 “wings”). As we mentioned in the previous example, there is a continuous map $p : X^{an} \rightarrow S_{\mathcal{X}}$. We are going to construct B as a retraction of $S_{\mathcal{X}}$.

In order to do this we observe that for an edge $e \subset \Delta^2$ and a point $a \in e$ one has the canonical retraction $p_{a,e} : \Delta^2 \rightarrow e$. Namely, let us identify the edge e with the interval $[-1, 1]$ of the real line, so that a is identified with the point $a = (a_0, 0)$, and Δ^2 is bounded by e and the segments $0 \leq y \leq 1 - |x|$. Then we define $p_{a,e}$ by the formulas (see Figure 2)

$$\begin{aligned} (x, y) &\mapsto (x + y, 0), & x + y \leq a_0 ; \\ (x, y) &\mapsto (x - y, 0), & x - y \geq a_0 ; \\ (x, y) &\mapsto (a_0, 0), & \text{otherwise.} \end{aligned}$$

Now we choose a point $q_{ij}, 0 \leq i < j \leq 3$ in the interior of each edge e_{ij} of $\partial\Delta^3$ (here i, j are identified with the vertices of $\partial\Delta^3$). There are four “wings” Δ_α^2 having e_{ij} as a common edge. Then we retract each Δ_α^2 to e_{ij} by the map $p_{q_{ij}, e_{ij}}$. This gives us a retraction $\pi' = \pi'_{(q_{ij})} : S_{\mathcal{X}} \rightarrow \partial\Delta^3$. Let $\pi := p_{\mathcal{X}} \circ \pi'_{(q_{ij})} : X^{an} \rightarrow B$ be the composition of the projection $p_{\mathcal{X}} : X^{an} \rightarrow S_{\mathcal{X}}$ with the above retraction. One can show that all points of $B := \partial\Delta^3$ are π -smooth except of the chosen six points $q_{ij}, 0 \leq i < j \leq 3$. According to Theorem 1 we obtain a \mathbf{Z} -affine structure on $S^2 \setminus \cup_{1 \leq i < j \leq 3} \{q_{ij}\}$. One can show that the local monodromy around each point q_{ij} is conjugate to the

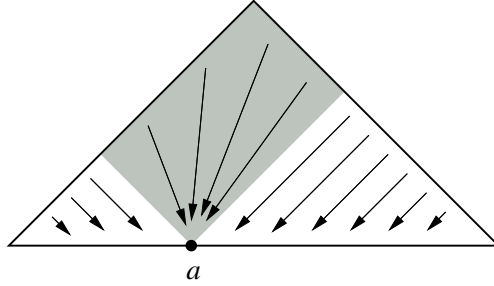


Figure 2: Triangle contracted to one side. The dashed area maps to point a .

matrix

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

We skip the computations here.

4.3 Stein property

A K -analytic space X is called *Stein* if the natural map

$$X \rightarrow \text{Spec}^{an}(\Gamma(X, \mathcal{O}_X))$$

is a homeomorphism. Here $\Gamma(X, \mathcal{O}_X)$ is considered as a topological K -algebra. This definition is equivalent to the standard one. Let us call the projection $\pi : X \rightarrow B$ *Stein* if for any $b \in B$ there exists a fundamental systems of neighborhoods U_i of x such that $\pi^{-1}(U_i) \subset X$ is a Stein domain. If π is Stein then we can reconstruct (X, \mathcal{O}_X) and π from the space B endowed with the sheaf $\pi_*(\mathcal{O}_X)$ of topological K -algebras.

Proposition 1 *Let B be a contraction of Clemens polytope $S_{\mathcal{X}}$ of some model \mathcal{X} of X as in Section 4.2.3, and π a Stein map. Then B^{sm} is dense in B .*

*Proof.*³ It suffices to prove that n -dimensional cells are dense in B , where $n = \dim X$. For any open $U \subset B$, $U \neq \emptyset$ we have $H_c^n(U, \pi_*(\Omega_X^n)) \simeq H_c^n(\pi^{-1}(U), \Omega_X^n)$.

The last group is nontrivial, because for any non-empty open $V \subset X^{an}$ the integration map $\int : H_c^n(V, \Omega_X^n) \rightarrow K$ is onto. Therefore $\dim(U) \geq n$. ■

³We thank to Ofer Gabber for suggesting the proof below

All the examples in Sections 4.2.1–4.2.5 (except Section 4.2.3) have Stein property.

5 Z-affine structures and mirror symmetry

5.1 Gromov-Hausdorff collapse of Calabi-Yau manifolds

We recall that a Calabi-Yau metric on a complex manifold X is a Kähler metric with vanishing Ricci curvature. If such a metric exists then $c_1(TX) = 0 \in H^2(X, \mathbf{R})$ and hence the class of the canonical bundle $\bigwedge^{\dim X}(T^*X)$ is torsion in $Pic(X)$. According to the famous Yau theorem, for any compact Kähler manifold X such that $c_1(TX) = 0 \in H^2(X, \mathbf{R})$, and any Kähler class $[\omega] \in H^2(X, \mathbf{R})$ there exists a unique Calabi-Yau metric g_{CY} with the class $[\omega]^4$. Up to now, there is no explicitly known non-flat Calabi-Yau metric on a compact manifold.

In Mirror Symmetry one studies the limiting behavior of g_{CY} as the complex structure on X approaches a “cusp” in the moduli space of complex structures (“maximal degeneration”). Well-known conjecture of Strominger, Yau and Zaslow (see [SYZ]) claims a torus fibration structure of Calabi-Yau manifolds near the cusp. A metric approach to the maximal degeneration (see [GW], [KoSo]) explains the structure of such Calabi-Yau manifolds in terms of their Gromov-Hausdorff limits. We recall this picture below following [KoSo].

We start with the definition of a maximally degenerating family of algebraic Calabi-Yau manifolds.

Let $\mathbf{C}_t^{mer} = \{f = \sum_{n \geq n_0} a_n t^n\}$ be the field of germs at $t = 0$ of meromorphic functions in one complex variable, and X_{mer} be an algebraic n -dimensional Calabi-Yau manifold over \mathbf{C}_t^{mer} (i.e. X_{mer} is a smooth projective manifold over \mathbf{C}_t^{mer} with the trivial canonical class: $K_{X_{mer}} = 0$). We fix an algebraic non-vanishing volume element $\Omega \in \Gamma(X_{mer}, K_{X_{mer}})$. The pair (X_{mer}, Ω) defines a 1-parameter analytic family of complex Calabi-Yau manifolds (X_t, Ω_t) , $0 < |t| < \epsilon$, for some $\epsilon > 0$.

⁴Notice that there is a discrepancy in terminology. In algebraic situation one usually calls Calabi-Yau a projective variety with the trivial canonical class in $Pic(X)$, and the polarization is not considered as a part of data.

Let $[\omega] \in H_{DR}^2(X_{mer})$ be a cohomology class in the ample cone. Then for every t , such that $0 < |t| < \epsilon$ it defines a Kähler class ω_t on X_t . We denote by g_{X_t} the unique Calabi-Yau metric on X_t with the Kähler class $[\omega_t]$.

It follows from the resolution of singularities, that as $t \rightarrow 0$ one has

$$\int_{X_t} \Omega_t \wedge \bar{\Omega}_t = C(\log |t|)^m |t|^{2k} (1 + o(1))$$

for some $C \in \mathbf{C}^\times, k \in \mathbf{Z}, 0 \leq m \leq n = \dim(X_{mer})$.

Definition 5 *We say that X_{mer} has maximal degeneration at $t = 0$ if in the formula above we have $m = n$.*

Let us rescale the Calabi-Yau metric: $g_{X_t}^{new} = g_{X_t} / \text{diam}(X_t, g_{X_t})^{1/2}$. In this way we obtain a family of Riemannian manifolds $X_t^{new} = (X_t, g_{X_t}^{new})$ of diameter 1.

Conjecture 1 *If X_{mer} has maximal degeneration at $t = 0$ then*

$$\text{diam}(X_t, g_{X_t}) = (\log |t|)^{-1} \exp(O(1))$$

and there is a limit (B, g_B) of X_t^{new} in the Gromov-Hausdorff metric as $t \rightarrow 0$, such that:

- a) *(B, g_B) is a compact metric space, which contains a smooth oriented Riemannian manifold $(B^{sm}, g_{B^{sm}})$ of dimension n as a dense open metric subspace. The Hausdorff dimension of $B^{sing} = B \setminus B^{sm}$ is less than or equal to $n - 2$.*
- b) *B^{sm} carries a \mathbf{Z} -affine structure.*
- c) *The metric $g_{B^{sm}}$ has a potential. This means that it is locally given in affine coordinates by a symmetric matrix $(g_{ij}) = (\partial^2 F / \partial x_i \partial x_j)$, where F is a smooth function (defined modulo adding an affine function).*
- d) *In affine coordinates the metric volume element is constant, i.e.*

$$\det(g_{ij}) = \det(\partial^2 F / \partial x_i \partial x_j) = \text{const}$$

(real Monge-Ampère equation).

There is a more precise conjecture (see [KoSo] for the details) which says that outside of B^{sing} the space X_t^{new} is metrically close to a torus fibration with flat Lagrangian fibers (integrable system). This torus fibration can be canonically reconstructed (up to a locally constant twist) from the limiting data a)-d).

Conjecture 1 holds for abelian varieties (since $B = B^{sm}$ is a flat torus in this case). It is non-trivial for K3 surfaces (see [GW] for the proof). In 3-dimensional case there is now a substantial progress (see [LYZ]).

Definition 6 *A Monge-Ampère manifold is a triple (Y, g, ∇) , where (Y, g) is a smooth Riemannian manifold with the metric g , and ∇ is a flat connection on TY such that:*

- a) ∇ defines an affine structure on Y .
- b) *Locally in affine coordinates (x_1, \dots, x_n) the matrix (g_{ij}) of g is given by $(g_{ij}) = (\partial^2 F / \partial x_i \partial x_j)$ for some smooth real-valued function F .*
- c) *The Monge-Ampère equation $\det(\partial^2 F / \partial x_i \partial x_j) = \text{const}$ is satisfied.*

The following easy Proposition is well-known.

Proposition 2 *For a given Monge-Ampère manifold (Y, g_Y, ∇_Y) there is a canonically defined dual Monge-Ampère manifold $(Y^\vee, g_Y^\vee, \nabla_Y^\vee)$ such that (Y, g_Y) is identified with (Y^\vee, g_Y^\vee) as Riemannian manifolds, and the local system (TY^\vee, ∇_Y^\vee) is naturally isomorphic to the local system dual to (TY, ∇_Y) (dual local system is constructed via the metric g_Y).*

Corollary 1 *If ∇_Y defines an integral affine structure on Y with the covariantly constant lattice $(TY)^\mathbf{Z}$ then ∇_Y^\vee defines an integral affine structure on Y^\vee such that for all $x \in Y^\vee = Y$ the lattice $(T_x Y^\vee)^\mathbf{Z}$ is dual to $(T_x Y)^\mathbf{Z}$ with respect to the Riemannian metric g_Y on Y .*

We will call *integral* a Monge-Ampère manifold with \mathbf{Z} -affine structure.

In Mirror Symmetry one often has a so-called dual family of Calabi-Yau manifolds associated with the given one. There is no general definition of the dual family, but there are many examples. The following Conjecture (see [KoSo]) formalizes Strominger-Yau-Zaslow picture of Mirror Symmetry:

Conjecture 2 *Smooth parts of Gromov-Hausdorff limits of dual families of Calabi-Yau manifolds are dual integral Monge-Ampère manifolds.*

One can say that Monge-Ampère manifolds with integral affine structures are real analogs of Calabi-Yau manifolds. Conversely, having an integral Monge-Ampère manifold $(Y, g_Y, \nabla_Y, (TY)^{\mathbf{Z}})$ one can construct a torus fibration $TY/(TY)^{\mathbf{Z}} \rightarrow Y$. It is easy to see that the total space of this fibration is in fact a Calabi-Yau manifold (typically non-compact as Y is non-compact too). Rescaling the covariant lattice we can make fibers small (of the size $O((\log |t|)^{-1})$). As we already mentioned, the extended version of Conjecture 1 says that this torus fibration is close (after a locally constant twist) to X_t^{new} outside of a “singular” subset.

5.1.1 K3 example

In the case of collapsing K3 surfaces the corresponding integral Monge-Ampère manifold has an explicit description.

Let S be a complex surface endowed with a holomorphic non-vanishing volume form Ω_S , and $\pi : S \rightarrow C$ be a holomorphic fibration over a complex curve C , such that fibers of π are non-singular elliptic curves.

We define a metric g_C on C as the Kähler metric associated with the $(1, 1)$ -form $\pi_*(\Omega_S \wedge \overline{\Omega}_S)$. Let us choose (locally on C) a basis (γ_1, γ_2) in $H_1(\pi^{-1}(x), \mathbf{Z}), x \in C$. We define two closed 1-forms on C by the formulas

$$\alpha_i = \text{Re} \left(\int_{\gamma_i} \Omega_S \right), \quad i = 1, 2 .$$

It follows that $\alpha_i = dx_i$ for some functions $x_i, i = 1, 2$. We define a \mathbf{Z} -affine structure on C , and the corresponding connection ∇ , by saying that (x_1, x_2) are \mathbf{Z} -affine coordinates (compare with 3.2.4). One can check directly that (C, g_C, ∇) is a Monge-Ampère manifold. In a typical example of elliptic fibration of a K3 surface, one gets $C = \mathbf{C}P^1 \setminus \{x_1, \dots, x_{24}\}$, where $\{x_1, \dots, x_{24}\}$ is a set of distinct 24 points in $\mathbf{C}P^1$. M. Gross and P. Wilson (see [GW]) proved that there exists a family of K3 surfaces with Calabi-Yau metrics collapsing to $S^2 \simeq \mathbf{C}P^1$ with the integral Monge-Ampère structure described above.

5.2 Non-archimedean picture for the space B

Here we would like to formulate a conjecture which relates the Gromov-Hausdorff limit with non-archimedean geometry, thus giving a pure algebraic description of \mathbf{Z} -affine structure on B^{sm} . Let $\overline{\mathbf{C}}_t^{mer} = \cup_{m \geq 1} \mathbf{C}_{t^{1/m}}^{mer}$ be the

algebraic closure of \mathbf{C}_t^{mer} . We denote by $\pi_{mer} : X(\overline{\mathbf{C}_t^{mer}}) \rightarrow B$ the map which associates the limiting point (in Gromov-Hausdorff metric) of points $x(t^{1/m}) \in X_{t^{1/m}}(\mathbf{C})$ as $t^{1/m} \rightarrow 0$.

Let $K = \mathbf{C}((t))$ be the field of Laurent formal series. Then, by extending scalars we obtain an algebraic Calabi-Yau manifold X over K . We denote by X^{an} the corresponding smooth K -analytic space.

Conjecture 3 *The map π_{mer} is well-defined and extends by continuity to the map $\pi : X^{an} \rightarrow B$. The set B^{sm} (defined as the maximal open subset of B on which the limiting metric is smooth) coincides with the set of π -smooth points. Two \mathbf{Z} -affine structures on B^{sm} , one coming from the collapse picture, another coming from non-archimedean picture, coincide with each other.*

Also we make the following conjecture (or better a wish, because it is based on a very thin evidence):

Conjecture 4 *Map π is Stein.*

Part II

6 Compactifications of \mathbf{Z} -affine structures

6.1 Properties of compactifications

Assume that we are given a non-compact manifold B^{sm} with a \mathbf{Z} -affine structure. We would like to “compactify” it, i.e. to find a compact Hausdorff topological space B such that $B^{sm} \subset B$ is an open dense subset. We do not require an extension of the \mathbf{Z} -affine structure to B . The question is: what kind of properties one should expect from such a compactification? We cannot give a complete list of such properties at the moment. Instead, we formulate two of them and illustrate the notion of compactification in PL case. Similarity between examples in Sections 3.2 and 4.2 suggests that the class of singularities which appear in integrable systems should be more or less the same as the class of singularities appearing in non-archimedean geometry.

Let $x \in B^{sing} := B \setminus B^{sm}$ be a singular point of some compactification of B^{sm} . Then we require the following

Finiteness property. There is a fundamental system of neighborhoods $U \subset B$ of x such that the number of connected components of $U \cap B^{sm}$ is finite.

Let $U \cap B^{sm} = \sqcup_{1 \leq i \leq N} U_i$ be the disjoint union of the connected components. Let us pick a point $x_i \in U_i$ and consider a continuous path $\gamma : [0, 1] \rightarrow B$ such that $\gamma(0) = x_i, \gamma(1) = x, \gamma([0, 1]) \subset U_i$. Using the affine structure we can canonically lift this path to a path $\gamma' : [0, 1] \rightarrow T_{x_i}B$. We assume that the lifted path γ' extends to time $t = 1$ and is analytic at $t = 1$ (it is a technical assumption, helping to avoid some pathologies). Then we require the following

Independence property. Path γ with the properties as above exists, and point $\gamma'(1) \in T_{x_i}B$ does not depend on the choice of γ .

Independence property implies the existence of a fixed vector for the monodromy representation restricted to $\pi_1(U \cap B^{sm}, x_i)$ (this implies the Fixed Point property from Section 3.1).

6.2 PL compactifications

Let V be a finite set, $S \subset 2^V$ belongs to the set of $(n + 1)$ -element subsets of V . Then we have a n -dimensional simplicial complex $B = \cup_{Y \in S} \Delta^Y \subset \Delta^V$.

Let us choose a \mathbf{Z} -affine structure on n -dimensional faces of B which is compatible with the standard affine structure, and consider all $(n - 1)$ -dimensional faces which enjoy the following property: they belong to exactly two n -dimensional faces. For any two such n -dimensional faces σ and τ we choose a \mathbf{Z} -affine structure on $\sigma \cup \tau$ which is compatible with the already chosen \mathbf{Z} -affine structures on σ and τ (such a choice is equivalent to a choice of \mathbf{Z} -affine structure in a neighborhood of the $(n - 1)$ -dimensional face $\sigma \cap \tau$). In this way we obtain a \mathbf{Z} -affine structure on the union U of the interior points of all n -dimensional simplices and also the interior points of $(n - 1)$ -dimensional faces belonging to exactly two top-dimensional cells.

Proposition 3 *There exists (and unique) maximal extension of this \mathbf{Z} -affine structure to an open subset $U_{\max} \subset B$ containing U .*

Proof. Let us proceed inductively by codimension of faces. The induction step reduces to the obvious remark that the extension of the standard \mathbf{Z} -affine structure on $\mathbf{R}^n \setminus L$ to a neighborhood of point $p \in L$ in \mathbf{R}^n is unique in the case when $L \subset \mathbf{R}^n$ is an affine subspace, $\dim L \leq n - 2$. ■

It is easy to see that $B^{sm} := U_{\max}$ with \mathbf{Z} -affine structure on it, compactified by B satisfies both Finiteness and Independence properties.

We introduce PL compactifications both as a “toy model”, and also (as we hope, see Conjecture 6 in Section 6.3) as a sufficiently representative class for applications. In this case we can try to formulate additional desired properties. One of goals is to find a good substitution for the algebro-geometric notion of a canonical singularity (which is, morally, a singularity of a non-collapsing limit of a family of Calabi-Yau manifolds with fixed Kähler class).

For a large class of maximally degenerating Calabi-Yau manifolds there is a proposal by several authors (see [GS] and [HZh]) for a PL compactification B conjecturally related to the Gromov-Hausdorff limit. Space B is topologically a sphere S^n , it carries two dual cell decompositions. Each of these decompositions is identified with the boundary ∂P_1 or ∂P_2 of a convex $(n + 1)$ -dimensional polytope. Moreover, on each n -dimensional face of each polytope we have a \mathbf{Z} -affine structure compatible with the natural affine structure. The assumption is that for any two open n -cells U, U' from the first and the second cell decompositions, two induced \mathbf{Z} -affine structures on $U \cap U'$ coincide. This gives a \mathbf{Z} -affine structure on $B \setminus (Sk_{n-1} \cap Sk'_{n-1})$ where Sk_{n-1} and Sk'_{n-1} are $(n - 1)$ -skeletons of two CW-structures.

6.3 Some conjectures about singular sets

Our conjectures are in fact rather “wishes”, i.e. they are desired properties of $B^{sing} = B \setminus B^{sm}$. For simplicity we assume that B^{sing} is a stratified set (say, CW complex) of dimension less or equal than $n - 1$.

Conjecture 5 *We have a decomposition $B^{sing} = B_{n-1}^{sing} \cup B_{\leq n-2}^{sing}$, where $B_{\leq n-2}^{sing}$ consists of strata of dimension less or equal than $n - 2$, B_{n-1}^{sing} is the union of strata of dimension $n - 1$, and locally near every point $x \in B_{n-1}^{sing}$ the \mathbf{Z} -affine structure is modeled by the “book” $\cup_{i \in I} \mathbf{R}^{n-1} \times \mathbf{R}_{\geq 0}$. Here I is a finite set, all half-spaces have a common plane $\mathbf{R}^{n-1} \times \{0\}$ and x belongs to this plane. \mathbf{Z} -affine structure on $B^{sm} = \sqcup_{i \in I} \mathbf{R}^{n-1} \times \mathbf{R}_{>0}$ is the natural one.*

This conjecture gives a local model for a singular \mathbf{Z} -affine structure at a singular component of codimension one. Let us discuss the case of higher codimension. We start with the following definition.

Definition 7 *A \mathbf{Z} -affine structure with singularities on B is given by:*

1. a closed subset $B^{pre-sing} \subset B$ of a compact space B ;
2. a \mathbf{Z} -affine structure on the open set $B \setminus B^{pre-sing}$.

One can think about closed set of “potential singularities” $B^{pre-sing}$ as containing the actual set of singularities B^{sing}).

Definition 8 A continuous path $\gamma(t), t \in [0, 1]$ in the space of \mathbf{Z} -affine structures with singularities on a given compact space B is given by:

1. a continuous path $B_t^{pre-sing}$ in the space of all compact subsets of B ,
2. a \mathbf{Z} -affine structure on $B \setminus B_t^{pre-sing}$ for all $t \in [0, 1]$

Notice that for each $t_0 \in (0, 1)$ and $x_0 \in B \setminus B_{t_0}^{pre-sing}$ we can choose neighborhoods U_{t_0} of t_0 and U_{x_0} of x_0 such that $U_{x_0} \subset B \setminus B_t^{pre-sing}$ for all $t \in U_{t_0}$. Then we require that:

3. if U_{t_0} and U_{x_0} are sufficiently small then the induced \mathbf{Z} -affine structure on U_{x_0} does not depend on $t \in U_{t_0}$.

Notice that in the case when the homotopy type of $B \setminus B_t^{pre-sing}$ remains unchanged the representation $\rho_t : \pi_1(B \setminus B_t^{pre-sing}) \rightarrow GL(n, \mathbf{Z}) \times \mathbf{R}^n$ stays the same.

We are going to give an example of a non-trivial path in the next subsection. We expect that singularities which appear in the collapse of Calabi-Yau manifolds satisfy the following

Conjecture 6 If $B^{sing} = B^{pre-sing}$ is of codimension at least two in B , then there is a continuous path $\gamma(t)$ in the space of \mathbf{Z} -affine structures with singularities which connects a given structure with the one coming from a PL compactification, and such that for all $t \in [0, 1]$ we have $\text{codim}(B_t^{pre-sing}) \geq 2$ and $\gamma(t)$ has Finiteness and Independence properties.

6.4 Standard singularities in codimension two

Let us remove the angle $\{(x, y) \in \mathbf{R}^2 \mid 0 < x < y\}$ from \mathbf{R}^2 . After that we identify sides of the angle by the affine transformation $(x, y) \mapsto (x + y, y)$.

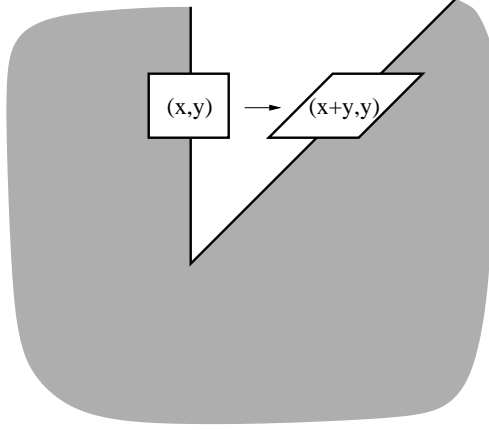


Figure 3: Glue both sides of the dashed area. White parallelograms are identified.

In this way we introduce a new \mathbf{Z} -affine structure on $\mathbf{R}^2 \setminus \{(0,0)\}$ with the monodromy around $(0,0)$ given by the unipotent matrix (see Figure 3)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This \mathbf{Z} -affine structure does not admit a continuation to \mathbf{R}^2 . Therefore we obtain a \mathbf{Z} -affine structure with singularities on \mathbf{R}^2 . We will call *standard* the singularity at $(0,0)$.

Equivalently, we can describe this \mathbf{Z} -affine structure on $\mathbf{R}^2 \setminus \{(0,0)\}$ by taking a cut along the ray $\{(x,0) \mid x > 0\}$ in \mathbf{R}^2 and glue the standard \mathbf{Z} -affine structure above and below the cut by means of the affine transformation $(x,y) \mapsto (x+y,y)$ (see Figure 1 in Section 3.2.4). In this description it is clear that we can start the cut at *arbitrary* point $(x_0,0)$ on the x -axes. The resulting singularity will be also called the standard one.

Remark 1 *We can vary a position of $(x_0,0)$, thus obtaining a continuous path in the space of \mathbf{Z} -affine structures with singularities in \mathbf{R}^2 .*

More generally, suppose that B is equipped with a \mathbf{Z} -affine structure which has standard singularities at points p_1, \dots, p_m . Then we can slightly move

each point p_i in the direction invariant under the local monodromy around p_i . This gives a new \mathbf{Z} -affine structure which is \mathbf{ZPL} -isomorphic to the initial one.

Standard singularity is called focus-focus singularity in the theory of integrable systems (see [Zu]). In non-archimedean geometry it appears as a singular value of some map $f : X^{an} \rightarrow \mathbf{R}^2$, where X is an algebraic surface in 3-dimensional affine space \mathbf{A}_K^3 (see Section 8).

Let us consider the Cartesian product of $\mathbf{R}^2 \setminus \{(0, 0)\}$ equipped with the above \mathbf{Z} -affine structure with the standard (non-singular) \mathbf{Z} -affine structure on \mathbf{R}^{n-2} . Let us choose a continuous function $f(z_1, \dots, z_{n-2})$ and start the cuts at all points $(f(z_1, \dots, z_{n-2}), 0, z_1, \dots, z_{n-2})$. This means that we introduce the standard non-singular \mathbf{Z} -affine structure in the region $y \neq 0$ as well as in the region $(y = 0, x < f(z_1, \dots, z_{n-2}))$. Near points $(y = 0, x > f(z_1, \dots, z_{n-2}))$ we introduce a modified \mathbf{Z} -affine structure by declaring functions

$$(y, x + \max(y, 0), z_1, \dots, z_{n-2})$$

to be \mathbf{Z} -affine coordinates. This gives an example of a “curved” singular set B^{sing} of codimension 2. Since function f can be approximated by PL functions, the above \mathbf{Z} -affine structure can be deformed to a PL one.

6.5 \mathbf{Z} -affine version of Gauss-Bonnet theorem

Let B be a connected compact oriented topological surface, $B^{sing} \subset B$ a finite set. Assume that $B^{sm} = B \setminus B^{sing}$ carries a \mathbf{Z} -affine structure such that for any $x \in B^{sing}$ there exists a small neighborhood U such that $U = \cup_{i \in I} U_i$ where I is a finite set and each U_i is affine equivalent to a germ of an angle in \mathbf{R}^2 , with x being the apex of each angle.

The aim of this section is to define a map $i_{loc} : B^{sing} \rightarrow \frac{1}{12}\mathbf{Z}$ (which depends only on \mathbf{Z} -affine structure near B^{sing}) and prove the following result (a kind of Gauss-Bonnet theorem).

Theorem 2 *The following equality holds*

$$\sum_{x \in B^{sing}} i_{loc}(x) = \chi(B) ,$$

where $\chi(B)$ is the Euler characteristic of B .

We start with the construction of i_{loc} . Let us denote by $\widetilde{SL}(2, \mathbf{Z})$ the pre-image of $SL(2, \mathbf{Z})$ in the universal covering $\widetilde{SL}(2, \mathbf{R})$ of the group $SL(2, \mathbf{R})$. The group $\widetilde{SL}(2, \mathbf{R})$ contains $\pi_1(SL(2, \mathbf{R})) \simeq \mathbf{Z}$. Let u be a generator of the latter (it belongs also to $\widetilde{SL}(2, \mathbf{Z})$).

We have an exact sequence of groups

$$1 \rightarrow \mathbf{Z} \rightarrow \widetilde{SL}(2, \mathbf{Z}) \rightarrow PSL(2, \mathbf{Z}) \rightarrow 1 .$$

Notice that $PSL(2, \mathbf{Z})$ is a free product $\mathbf{Z}/2 * \mathbf{Z}/3$. Moreover, in the above exact sequence \mathbf{Z} is embedded into the center of $\widetilde{SL}(2, \mathbf{Z})$. Notice that u is the image of $2 \in \mathbf{Z}$. One can choose representatives a_2, a_3 of the standard generators of $PSL(2, \mathbf{Z})$ in such a way that $\widetilde{SL}(2, \mathbf{Z})$ is generated by u, a_2, a_3 subject to the relations $a_2^2 = a_3^3, a_2^4 = a_3^6 = u$. This gives a homomorphism of groups $\phi : \widetilde{SL}(2, \mathbf{Z}) \rightarrow \mathbf{Z}$ such that $\phi(a_2) = 3, \phi(a_3) = 2, \phi(u) = 12$. Dividing by 12 we obtain a homomorphism $i : \widetilde{SL}(2, \mathbf{Z}) \rightarrow \frac{1}{12}\mathbf{Z}$ such that $i(u) = 1$.

Let us consider a topological S^1 -bundle E over B , such that the fiber over $x \in B$ is the union of all affine rays outcoming of x . Then the restriction of E to B^{sm} is just the spherical bundle. Let us pick $x_0 \in B^{sm}$ and remove it from B together with small neighborhoods of all points B^{sing} . We denote by B_1 the topological space obtained in this way. We can trivialize the tangent bundle over B_1 (we choose a C^∞ -trivialization, compatible with $SL(2, \mathbf{R})$ -structure) in such a way that it extends to a continuous trivialization of S^1 -bundle E over $B \setminus \{x_0\}$. Let $\alpha \in \Omega^1(B_1) \otimes sl(2, \mathbf{R})$ be a 1-form defined by means of the affine structure ∇ on B_1 . Then $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$ and α defines a flat connection on a trivial $\widetilde{SL}(2, \mathbf{R})$ -bundle on B_1 . It gives a a monodromy representation $\pi_1(B_1) \rightarrow \widetilde{SL}(2, \mathbf{Z})$ defined up to a conjugation. Composing it with the homomorphism i we obtain a homomorphism $\pi_1(B_1) \rightarrow \frac{1}{12}\mathbf{Z}$. Since $\frac{1}{12}\mathbf{Z}$ is an abelian group, the latter homomorphism is a composition $\pi_1(B_1) \rightarrow H_1(B_1, \mathbf{Z}) \rightarrow \frac{1}{12}\mathbf{Z}$. Let us pick small circles $[\gamma_x] \in H_1(B, \mathbf{Z})$ for each $x \in B^{sing}$. Then the above homomorphism gives us a number denoted by $i_{loc}(x) \in \frac{1}{12}\mathbf{Z}$.

Proof of the Theorem. Let us pick up a small circle $[\gamma_{x_0}] \in H_1(B_1, \mathbf{Z})$ around x_0 . Then $\sum_{x \in B^{sing}} [\gamma_x] + [\gamma_{x_0}] = 0$. The monodromy around x_0 can be easily computed via the winding number of the induced vector field (section of $E|_{\gamma_{x_0}}$) and is equal to $-\chi(B)u$. Applying homomorphism i we obtain the result. ■

Corollary 2 *Suppose that the monodromy for each point $x \in B^{sing}$ is the standard one (see Section 6.4). Then one has two possibilities:*

- a) $B^{sing} = \emptyset$ and $B = B^{sm}$ is a 2-dimensional torus;
- b) the set B^{sing} consists of 24 distinct points on the sphere S^2 .

Proof. It is easy to see that for each point $x \in B^{sing}$ one has $i_{loc}(x) = \frac{1}{12}$. Then from Gauss-Bonnet theorem one deduces that $\chi(B) = 2 - 2g$, where g is the genus of the Riemann surface B . Then we have

$$\frac{|B^{sing}|}{12} = 2 - 2g .$$

Since LHS is non-negative we conclude that either $g = 1$ or $g = 0$. In the first case $|B^{sing}| = 0$ and we have a \mathbf{Z} -affine structure on a torus. In the second case we have $|B^{sing}| = 24$ and $g = 0$. ■

Remark 2 *This corollary was proved in [LeS] by different methods.*

Similarly, for the affine structure with the monodromy at each point conjugate to

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

one has $i_{loc}(x) = \frac{1}{3}$, and we have six singular points on S^2 (see Section 4.2.5).

6.6 Skeleton of a non-archimedean Calabi-Yau variety

Let $X = X/K$ be a smooth proper algebraic variety over a non-archimedean field K , $\dim X = n$ and $\Omega \in \Gamma(X, \Omega_X^n)$ be a non-zero top degree form on X . We will associate canonically with the pair (X, Ω) a piecewise-linear compact space $Sk(X, \Omega) \subset X^{an}$ such that $\dim_{\mathbf{R}} Sk(X, \Omega) \leq n$.

Let us assume for simplicity that $K = \mathbf{C}((t))$ and X is defined over $\mathbf{C}_t^{mer} \subset K$. Analytic space X^{an} contains a dense subset X_{Div} of divisorial points corresponding to irreducible components of special fibers of all snc models \mathcal{X} of X (see Appendix A):

$$X_{Div} = \cup_{\mathcal{X}} i_{\mathcal{X}}(V_{S_{\mathcal{X}}}) ,$$

where $V_{S_{\mathcal{X}}}$ is the set of vertices of the Clemens polytope $S_{\mathcal{X}}$.

Top degree form Ω gives rise to a map $\psi_\Omega : X_{Div} \rightarrow \mathbf{Q}$. Namely, if $p : \mathcal{X} \rightarrow \text{Spec}(\mathbf{C}_t^{mer})$ is a snc model and $D \subset \mathcal{X}_0$ is an irreducible divisor of the special fiber then we define

$$\psi_\Omega(D) = \frac{\text{ord}_D(\Omega \wedge dt/t)}{\text{ord}_D(p^*(t))}.$$

Here $\Omega \wedge dt/t$ is a meromorphic top degree form on \mathcal{X}/\mathbf{C} .

It is easy to show that $\psi_\Omega(D)$ depends only on the point $i_{\mathcal{X}}(D) \in X^{an}$. Function ψ_Ω is (globally) bounded from below.

Definition 9 A divisorial point $i_{\mathcal{X}}(D)$ is called *essential* if

$$\psi_\Omega(D) = \inf_{x \in X_{Div}} \psi_\Omega(x).$$

Definition 10 Skeleton $Sk(X, \Omega)$ is the closure in X^{an} of the set of essential points.⁵

Let \mathcal{X} be a snc model. We will explain how to describe $Sk(X, \Omega)$ in terms of \mathcal{X} and Ω . In fact it is a nonempty simplicial subcomplex of $i_{\mathcal{X}}(S_{\mathcal{X}})$.

Let us call \mathcal{X} -essential a divisor $D_i \subset \mathcal{X}_0$ such that

$$\psi_\Omega(D_i) = \min_{D_j \in \mathcal{X}_0} \psi_\Omega(D_j).$$

A nonempty collection D_{i_1}, \dots, D_{i_l} of divisors in \mathcal{X}_0 is called \mathcal{X} -essential if all D_{i_k} are \mathcal{X} -essential, the intersection $D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_l}$ is non-empty and does not belong to the closure of the divisor of zeros of Ω is $\mathcal{X} \setminus \mathcal{X}_0$.

Theorem 3 The skeleton $Sk(X, \Omega)$ is the image under $i_{\mathcal{X}}$ of the subcomplex $Sk(\mathcal{X}, \Omega) \subset S_{\mathcal{X}}$ consisting of simplices corresponding to \mathcal{X} -essential collections of divisors.

Sketch of the proof. Notice that for any snc model \mathcal{X} the subset $S_{\mathcal{X}}(\mathbf{Q}) \subset S_{\mathcal{X}}$ consisting of points with rational barycentric coordinates is mapped by $i_{\mathcal{X}}$ into X_{Div} . Namely, we can modify \mathcal{X} by blowing up at nonempty intersections of irreducible components of the special fiber and then continue this

⁵Our notion of a skeleton should not be mixed with the one introduced in [Be3]. The latter is related to the Clemens polytope $S_{\mathcal{X}}$ of a snc model \mathcal{X} .

process indefinitely. Divisorial points obtained in this way exhaust all points of $i_{\mathcal{X}}(S_{\mathcal{X}}(\mathbf{Q}))$.

We will prove that the set of essential points in X^{an} coincides with $i_{\mathcal{X}}(S_{\mathcal{X}}(\mathbf{Q}) \cap Sk(\mathcal{X}, \Omega))$. First of all, a direct computation shows that ψ_{Ω} being restricted to $i_{\mathcal{X}}(S_{\mathcal{X}}(\mathbf{Q}))$ achieves its absolute minimum on $i_{\mathcal{X}}(S_{\mathcal{X}}(\mathbf{Q}) \cap Sk(\mathcal{X}, \Omega))$. Secondly, another straightforward computation shows that the latter set does not change under blow-ups of first and second type (see Section A.5 in Appendix A). This concludes the proof. ■

For Calabi-Yau manifold X we will denote $Sk(X, \Omega)$ simply by $Sk(X)$, as there exists only one (up to a scalar) non-zero top-degree form Ω on X and $Sk(X, \lambda\Omega) = Sk(X, \Omega) \quad \forall \lambda \in K^{\times}$.

One can prove that the PL space $Sk(X, \Omega)$ is in fact a birational invariant. Moreover the group $Aut^{brt}(X)$ of birational automorphisms of X acts on the skeleton by **ZPL** transformations. In order to obtain non-trivial examples of such actions we need Calabi-Yau manifolds with large groups of birational automorphisms. An example of a **ZPL**-action is considered in the next subsection.

6.7 K3 surfaces and **ZPL**-actions on S^2

6.7.1 Integrable systems

Recall that in Section 3.3 we constructed a 38-dimensional space \mathcal{P} parameterizing integrable systems $(X, \omega) \rightarrow B$ with $B \simeq S^2$. The space \mathcal{P} carries a codimension 20 foliation \mathcal{F} corresponding to small deformations of integrable systems which do not change the invariant $[\rho]$ of the local system $\rho : \pi_1(B^{sm}) \rightarrow SL(2, \mathbf{Z}) \ltimes \mathbf{R}^2$. We explained that the fundamental group of a leaf of \mathcal{F} acts by PL homeomorphisms of S^2 . Here we are going to give a (partial) description of \mathcal{P} and \mathcal{F} in cohomological terms using Torelli theorem (see Appendix B).

An algebraic polarized K3 surface X/\mathbf{C} elliptically fibered over $\mathbf{C}P^1$, equipped with a holomorphic volume form Ω can be encoded by the data $(\Lambda, (\cdot, \cdot), [\omega], [\Omega], [\gamma], \mathcal{K}_X)$, where

1. $(\Lambda, (\cdot, \cdot), \mathbf{C}[\Omega], \mathcal{K}_X)$ is a K3 period data;
2. $[\omega], [\gamma] \in \Lambda, \Omega \in \Lambda \otimes \mathbf{C}$;
3. $[\omega] \in \mathcal{K}_X, \gamma \in \partial\mathcal{K}_X, ([\omega], [\Omega]) = ([\gamma], [\Omega]) = ([\gamma], [\gamma]) = 0$;

4. γ is a non-zero primitive lattice vector.

Here $[\omega]$ is the class of polarization (projective embedding) of X , $[\gamma]$ is dual to the class of generic fiber of the elliptic fibration $\pi : X \rightarrow \mathbf{C}P^1$.

Perhaps one can express in cohomological terms the fact that π has exactly 24 critical values. The latter is an open condition.

Let $L \subset H_2(X, \mathbf{Z})$ be a subgroup consisting of homology classes which can be represented by cycles which are projected into graphs in B^{sm} (such cycles are circle fibrations over graphs). When we move along a leaf of \mathcal{F} then the pairing of $Re([\Omega])$ with L remains unchanged (see Section 3.1.1). Clearly $L \subset [\gamma]^\perp$, and moreover, one can check that $L = [\gamma]^\perp \simeq \mathbf{Z}^{21}$. The pairing with $Re([\Omega])$ gives a map $\Lambda_{2,18} := [\gamma]^\perp / \mathbf{Z}[\gamma] \rightarrow \mathbf{R}$, where $\Lambda_{2,18}$ is the following even unimodular lattice of signature $(2, 18)$:

$$\Lambda_{2,18} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \oplus \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \oplus (-E_8) \oplus (-E_8) ,$$

where $-E_8$ is the Cartan matrix for Dynkin diagram E_8 taken with the minus sign.

The functional $(Re[\Omega], \cdot)$ on $\Lambda_{2,18}$ can be represented as $(v_{Re[\Omega]}, \cdot)$ where $v_{Re[\Omega]} \in \Lambda_{2,18} \otimes \mathbf{R}$ is a vector with the strictly positive square norm. One can show that the (non-Hausdorff) space of leaves of \mathcal{F} is canonically identified with the set $\{v \in \Lambda_{2,18} \otimes \mathbf{R} | (v, v) > 0\} / Aut(\Lambda_{2,18})$.

The fundamental group of the leaf \mathcal{F}_v corresponding to a vector $v \in \Lambda_{2,18} \otimes \mathbf{R}$ maps onto the group $\Gamma_v \subset Aut(\Lambda_{2,18})$. This group is (up to a conjugation) the stabilizer in $(Aut(\Lambda_{2,18}), (\cdot, \cdot)_{2,18}, v)$ of the cone K_v , which is a connected component of the set

$$\{w \in \Lambda_{2,18} \otimes \mathbf{R} | (w, v) = 0, (w, w) > 0\} \setminus \bigcup_{\gamma \in \Lambda_{2,18}, (\gamma, \gamma) = -2, (\gamma, v) = 0} H_\gamma$$

and $H_\gamma \in \Lambda_{2,18} \otimes \mathbf{R}$ is the hyperplane orthogonal to γ (cf. Appendix B). Let us denote by $Aut_{\mathbf{Z}PL, v}(S^2)$ the group of piecewise-linear transformations of S^2 with integer linear part. Index v signifies the dependence of $\mathbf{Z}PL$ -structure on S^2 on v .

Conjecture 7 *The homomorphism $\pi_1(\mathcal{F}_v) \rightarrow Aut_{\mathbf{Z}PL, v}(S^2)$ arising from the monodromy of the local system along the leaf \mathcal{F}_v (see Sections 3.3, 6.4) is equal to the composition*

$$\pi_1(\mathcal{F}_v) \twoheadrightarrow \Gamma_v \rightarrow \text{Aut}_{\mathbf{Z}PL,v}(S^2) ,$$

where the homomorphism $\phi_v : \Gamma_v \rightarrow \text{Aut}_{\mathbf{Z}PL,v}(S^2)$ is uniquely determined by this property.

One can consider the whole moduli space \mathcal{M}_{44} of \mathbf{Z} -affine structures on S^2 with 24 standard singularities. This space is a Hausdorff orbifold (with a natural \mathbf{Z} -affine structure!) of dimension 44, and it carries a foliation of codimension 20 as before. It seems that using our main result (Theorem 5 in Part III) together with certain natural assumption (see Conjecture 11 in Section 11.6) one can show that the action by $\mathbf{Z}PL$ transformations of S^2 of the fundamental group of leaves of the foliation on the larger space \mathcal{M}_{44} is again reduced to the action of Γ_v .

6.7.2 Analytic surfaces

Let $X = (X_t)_{t \rightarrow 0}$ be a maximally degenerate K3 surface over the field \mathbf{C}_t^{mer} (see Section 5.1). We denote by Λ_X the quotient group $[\gamma_0]^\perp / \mathbf{Z}[\gamma_0]$ where $[\gamma_0] \in H_2(X_t, \mathbf{Z})$ is the vanishing cycle. Then $\Lambda_X \simeq \Lambda_{2,18}$. Let us assume that the monodromy acts trivially on Λ_X .

We define a natural homomorphism $\rho_X : \Lambda_X \rightarrow (\mathbf{C}_t^{mer})^\times$ by the formula

$$\rho_X([\gamma]) = \exp \left(2\pi i \frac{\int_\gamma \Omega_t}{\int_{\gamma_0} \Omega_t} \right), \quad [\gamma] \in [\gamma_0]^\perp .$$

One can give a more abstract definition of ρ_X in terms of the variation of Hodge structure. It is easy to see that $(\text{val}_{\mathbf{C}_t^{mer}} \circ \rho_X)([\gamma]) = (v_X, [\gamma])$ where $v_X \in \Lambda_X$ is a vector such that $(v_X, v_X) > 0$, and $\text{val}_{\mathbf{C}_t^{mer}}$ is the standard valuation on the field $\mathbf{C}_t^{mer} \subset \mathbf{C}((t))$.

Let X^{an} be the corresponding analytic K3 surface over the field $K = \mathbf{C}((t))$. We have an analytic torus fibration over $S^2 \setminus \{x_1, \dots, x_{24}\}$ which can be extended to a continuous map $X^{an} \rightarrow S^2$. Let us call such an extension a *singular analytic torus fibration* with standard singularities.

Conjecture 8 *For any analytic K3 surface X^{an}/K admitting an analytic torus fibration $X^{an} \rightarrow S^2$ with standard singularities, one can define intrinsically the lattice $\Lambda_{X^{an}}$ and the homomorphism $\rho_{X^{an}} : \Lambda_{X^{an}} \rightarrow K^\times$.*

Notice that for K3 surfaces any birational automorphism is biregular. Hence the group of birational automorphisms $Aut^{brt}(X)$ acts by a \mathbf{Z} PL-transformations of the sphere S^2 which is equipped with a singular \mathbf{Z} -affine structure (see Section 6.6), i.e. we have a homomorphism

$$Aut^{brt}(X) = Aut(X) \rightarrow Aut_{\mathbf{Z}PL, v_X}(Sk(X^{an}, \Omega)) \simeq Aut_{\mathbf{Z}PL, v_X}(S^2) .$$

Conjecture 9 1) *The image Γ_{ρ_X} of $Aut(X)$ in $Aut(\Lambda_X, \rho_X)$ is a subgroup of Γ_{v_X} where $v_X := val_K \circ \rho_X : \Lambda_X \rightarrow \mathbf{R}$.*

2) *The homomorphism $Aut(X) \rightarrow Aut_{\mathbf{Z}PL, v_X}(S^2)$ is conjugate to the restriction to Γ_{ρ_X} of the homomorphism ϕ_{v_X} defined in the previous subsection.*

6.7.3 Lattice points

Let us consider the special case when vector v is a lattice vector, i.e. $v \in \Lambda_{2,18}$. In A-model picture it corresponds to the integrality of the class $[\omega]$ of symplectic 2-form. In B-model this means that the non-archimedean field K has valuation in $\mathbf{Z} \subset \mathbf{R}$. In terms of \mathbf{Z} -affine structures it means that the monodromy of the affine connection is reduced to $SL(2, \mathbf{Z}) \ltimes \mathbf{Z}^2$. Group Γ_v is a subgroup (and also a quotient group) of an arithmetic subgroup in the Lie group $SO(1, 18)$. Also in this case there is a Γ_v -invariant notion of a point with integer coordinates on $B \simeq S^2$, as well of points with coordinates in $\frac{1}{N}\mathbf{Z}$ for any integer $N \geq 1$. The number $M_{v,N}$ of such points is finite. It is not hard to see that $M_{v,N} = Area_v + 2 = \frac{(v,v)}{2}N^2 + 2$ where $Area_v$ is the area of B with a \mathbf{Z} -PL structure corresponding to v . This is analogous to the Riemann-Roch formula $rk(\Gamma(X_{\mathbf{C}}, L^{\otimes N})) = \int_X \frac{c_1(L)^2}{2} + 2$ for an ample line bundle L on a complex K3 surface $X_{\mathbf{C}}$.

The action of Γ_v on S^2 gives rise to a homomorphism $\Gamma_v \rightarrow S_{M_{v,n}}$ where $S_{M_{v,n}}$ is the symmetric group. Also the action gives a homomorphism from Γ_v to the mapping class group $\pi_1(\mathcal{M}_{0, M_{v,N}}^{unord})$, the fundamental group of the moduli space of genus zero complex curves with $M_{v,N}$ unordered distinct marked points. The last group is closely related to the braid group. The conclusion is that we have constructed homomorphisms from arithmetic groups to a tower of braid groups.

One can deduce from Torelli theorem an interpretation of Γ_v as a quotient group of the fundamental group of a neighborhood U of a cusp in 19-dimensional moduli space of polarized complex algebraic K3-surfaces, where vector v corresponds to the polarization. Therefore the homomorphism $\Gamma_v \rightarrow S_{M_{v,n}}$ gives a finite covering U' of U . One may wonder whether

there exists a line bundle over U' whose direct image to U coincides with the direct image of the sheaf $L^{\otimes N}$ from the universal family of K3 surfaces (this question is in spirit of some ideas of Andrey Tyurin, see e.g. [Tyu]).

6.8 Further examples

There are many families of Calabi-Yau varieties with huge groups of birational automorphisms. The following example we learned from D.Panov and D.Zvonkine. For any real numbers $l_1, \dots, l_n > 0$ we can consider the space of planar n -gons with the length of edges equal to l_1, \dots, l_n , modulo the group of orientation-preserving motions. This space can be identified with the space of solutions of the following system of equations

$$\sum l_i z_i = 0, \quad \sum l_i z_i^{-1} = 0$$

where $(z_1 : \dots : z_n) \in \mathbf{C}P^{n-1}$ is a point satisfying the reality condition $|z_i| = 1$, $i = 1, \dots, n$. Hence we obtain a singular subvariety of $\mathbf{C}P^{n-1}$ of codimension 2, depending on parameters l_1, \dots, l_n . One can check that this variety is birationally isomorphic to a non-singular Calabi-Yau variety. For any proper set $I \subset \{1, \dots, n\}$, $2 \leq |I| \leq n-2$ we have a birational involution σ_I defined by the formula

$$\sigma_I^*(z_i) = \begin{cases} c/z_i & \text{if } i \in I \\ z_i & \text{if } i \notin I \end{cases}$$

where $c := \frac{\sum_{i \in I} l_i z_i}{\sum_{i \in I} l_i / z_i}$.

We do not know at the moment the structure of the group G_n generated by involutions σ_I . One can obtain easily explicit formulas for the action of G_n by piecewise-linear homomorphisms of S^{n-3} . Length parameters l_i should be replaced by elements of a non-archimedean field K with “generic” norms $\lambda_i = \text{val}_K(l_i) \in \mathbf{R}$. Denote by ζ_i , $i = 1, \dots, n$ real variables which have the meaning of valuations of variables $z_i \in K$. Sphere S^{n-3} is obtained in the following way. In \mathbf{R}^n we consider the intersection of two subsets:

$$\{(\zeta_1, \dots, \zeta_n) \mid \min_i (\lambda_i + \zeta_i) \text{ is achieved at least twice}\}$$

and

$$\{(\zeta_1, \dots, \zeta_n) \mid \min_i (\lambda_i - \zeta_i) \text{ is achieved at least twice}\}$$

and then take the quotient by the action of \mathbf{R} :

$$(\zeta_1, \dots, \zeta_n) \rightarrow (\zeta_1 + c, \dots, \zeta_n + c)$$

corresponding to the projectivization. For appropriately chosen $(\lambda_1, \dots, \lambda_n)$ we obtain a set which is the union of S^{n-3} with several “wings” going to infinity. The action of the involution σ_I is obtained from algebraic formulas from above, in which one replace non-archimedean variables by real ones, addition by minimum and multiplication (division) by addition (subtraction).

7 K -affine structures

7.1 Definitions

Let B^{sm} be a manifold with \mathbf{Z} -affine structure. The sheaf of \mathbf{Z} -affine functions $Aff_{\mathbf{Z}} := Aff_{\mathbf{Z}, B^{sm}}$ gives rise to an exact sequence of sheaves of abelian groups

$$0 \rightarrow \mathbf{R} \rightarrow Aff_{\mathbf{Z}} \rightarrow (T^*)^{\mathbf{Z}} \rightarrow 0 .$$

Let K be a complete non-archimedean field with a valuation map val . We give two equivalent definitions of a K -affine structure on B^{sm} compatible with a given \mathbf{Z} -affine structure.

Definition 11 *A K -affine structure on B^{sm} compatible with the given \mathbf{Z} -affine structure is a sheaf Aff_K of abelian groups on B^{sm} , an exact sequence of sheaves*

$$0 \rightarrow K^\times \rightarrow Aff_K \rightarrow (T^*)^{\mathbf{Z}} \rightarrow 0 ,$$

together with a homomorphism Φ of this exact sequence to the exact sequence of sheaves of abelian groups

$$0 \rightarrow \mathbf{R} \rightarrow Aff_{\mathbf{Z}} \rightarrow (T^*)^{\mathbf{Z}} \rightarrow 0 ,$$

such that $\Phi = id$ on $(T^)^{\mathbf{Z}}$ and $\Phi = val$ on K^\times .*

Since B^{sm} carries a \mathbf{Z} -affine structure, we have an associated $GL(n, \mathbf{Z}) \times \mathbf{R}^n$ -torsor on B^{sm} , whose fiber over a point x consists of all \mathbf{Z} -affine coordinate systems at x .

Definition 12 A K -affine structure on B^{sm} compatible with the given \mathbf{Z} -affine structure is a $GL(n, \mathbf{Z}) \ltimes (K^\times)^n$ -torsor on B^{sm} such that the application of $val^{\times n}$ to $(K^\times)^n$ gives the initial $GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$ -torsor.

Equivalence of two definitions from above is obvious in local \mathbf{Z} -affine coordinates. The reason is that the set of automorphisms of the exact sequence of groups

$$0 \rightarrow K^\times \rightarrow K^\times \times \mathbf{Z}^n \rightarrow \mathbf{Z}^n \rightarrow 0$$

identical on K^\times coincides with the group $GL(n, \mathbf{Z}) \ltimes (K^\times)^n$.

Finally, we can formulate the Fixed Point Property for K -affine structures (see Section 3.1 for \mathbf{Z} -affine case):

Fixed Point Property for K -affine structures. In the notation of the end of Section 3.1, for any $b \in B^{sing}$ and sufficiently small neighborhood U of b the lifted monodromy representation $\pi_1(U) \rightarrow GL(n, \mathbf{Z}) \ltimes (K^\times)^n$ has fixed vectors in $K^{\times n}$, and the \mathbf{R} -affine span of the corresponding (under the valuation map) vectors in \mathbf{R}^n coincides with the set of fixed points of the monodromy representation $\pi_1(U) \rightarrow GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$.

7.2 K -affine structure on smooth points

Starting from this section till the end of the paper (except of the Section 11.7) we will assume the following

Zero Characteristic Assumption. K is a complete non-archimedean local field such that its residue field has characteristic zero.

Let X be a K -analytic manifold of dimension n and we are given a continuous map $\pi : X \rightarrow B$, where B is a topological space. Then B^{sm} carries a \mathbf{Z} -affine structure (Theorem 1). Suppose that there is an open K -analytic submanifold $U \subset X$ such that $\pi^{-1}(B^{sm}) \subset U$ and there is a nowhere vanishing analytic form $\Omega \in \Gamma(U, \Omega_X^n)$. We are going to define a \mathbf{Z} -affine function $Val(\Omega)$ similarly to the definition of the function $Val(\varphi)$ in Section 4.1. Namely, in local coordinates (z_1, \dots, z_n) we consider the expression $\varphi := \Omega / \bigwedge_{1 \leq i \leq n} (dz_i/z_i)$. This is an invertible function, and we define $Val(\Omega)$ as $Val(\varphi)$. The independence on the choice of coordinates follows from the following lemma

Lemma 2 Let $(z_i)_{i=1, \dots, n}$, $(z'_i)_{i=1, \dots, n}$ be two systems of invertible coordinates

on $\pi^{-1}(U)$ for some connected open $U \subset B^{sm}$. Then

$$\left| \left(\bigwedge_{1 \leq i \leq n} (dz_i/z_i) \right) / \left(\bigwedge_{1 \leq i \leq n} (dz'_i/z'_i) \right) \right|_x = 1 \quad \forall x \in \pi^{-1}(U) .$$

Proof: By Lemma 1 from Section 4.1 we know that z'_i as any invertible function can be written in form $c_i z^{I^{(i)}} (1 + o(1))$ for some nonzero $c_i \in K$ and a multi-index $I^{(i)} \in \mathbf{Z}^n$. Vectors $I^{(1)}, \dots, I^{(n)}$ form a basis of \mathbf{Z}^n , as follows from the condition that z'_1, \dots, z'_n form a coordinate system. Therefore, after applying the change of coordinates $z_i \mapsto c_i z^{I^{(i)}}$ preserving form $\bigwedge_i dz_i/z_i$ up to sign, we may assume that $z'_i = (1 + o(1))z_i$. The Jacobian matrix of the transformation $(z_i) \rightarrow (z'_i)$ is the identity matrix plus terms of size $o(1)$. Therefore its determinant has norm equal to 1. ■

Now we make the following

Constant Norm Assumption. The function $Val(\varphi)$ is locally constant.

Theorem 4 *If the Constant Norm Assumption is satisfied then there is a K -affine structure on B^{sm} compatible with the \mathbf{Z} -affine structure $Aff_{\mathbf{Z}, B^{sm}}^{can}$ (see Section 4.1).*

Proof. Let us write in local coordinates $\Omega = \varphi(z_1, \dots, z_n) \bigwedge_{1 \leq i \leq n} \frac{dz_i}{z_i}$. Define residue $Res(\Omega) \in K$ as the constant term φ_0 in the Laurent expansion $\varphi(z_1, \dots, z_n) = \sum_{I \in \mathbf{Z}^n} \varphi_I z^I$. It is easy to see that $Res(\Omega)$ does not depend (up to a sign) on the choice of local coordinates. For non-vanishing everywhere Ω satisfying Constant Norm Assumption we have $\exp(-Val(\varphi)) = |\varphi| = |\varphi_0|$. Therefore we have $Res(\Omega) \neq 0$.

Let us return to the proof of the Theorem. Let F be the sheaf of abelian groups $F \subset \pi_*(\mathcal{O}_X^\times)$ consisting of f such that $Val(f) = 0$. Then we have an exact sequence of sheaves

$$0 \rightarrow K^\times / \mathcal{O}_K^\times \rightarrow \pi_*(\mathcal{O}_X^\times) / F \rightarrow (T_X^*)^{\mathbf{Z}} \rightarrow 0 ,$$

where \mathcal{O}_K denotes the constant sheaf with the fiber being the ring of integers of K . Indeed we embed $K^\times / \mathcal{O}_K^\times$ into $\pi_*(\mathcal{O}_X^\times) / F$ as constant functions. The projection $\pi_*(\mathcal{O}_X^\times) / F \rightarrow (T_X^*)^{\mathbf{Z}}$ assigns to the function f the linear part of the corresponding \mathbf{Z} -affine function $Val(f)$.

Notice that if $U \subset B^{sm}$ is a connected domain then any $f \in \Gamma(U, F)$ can be written (non-canonically) as $f = a(1 + r)$, where $a \in \mathcal{O}_K^\times$ and $r = o(1)$ in $\pi^{-1}(U)$.

We define an epimorphism of sheaves $p_\Omega : F \rightarrow \mathcal{O}_K^\times$ by formula

$$p_\Omega(f) = p_\Omega(a(1+r)) = a \exp\left(\frac{\text{Res}(\Omega \log(1+r))}{\text{Res}(\Omega)}\right).$$

Here \exp and \log are understood as infinite convergent series (in order to make sense of them we use Zero Characteristic Assumption).

It is easy to see that p_Ω is well-defined. Then the exact sequence of sheaves

$$1 \rightarrow K^\times \rightarrow \pi_*(\mathcal{O}_X^\times)/\ker(p_\Omega) \rightarrow (T_X^*)^{\mathbf{Z}} \rightarrow 1$$

defines a K -affine structure on B^{sm} compatible with $Aff_{\mathbf{Z}, B^{sm}}^{can}$. This concludes proof of the Theorem. ■

Notice that the above proof gives an explicit construction of the K -affine structure. We will denote it by $Aff_{K, B^{sm}}^\Omega$. It is easy to see that this K -affine structure does not change if we make a rescaling $\Omega \mapsto c\Omega, c \in K^\times$.

7.3 Lifting Problem

Let K be as in Section 7.2, $B \supset B^{pre-sing}$ be a space with singular \mathbf{Z} -affine structure (see Section 6.3), and an extension of \mathbf{Z} -affine structure on $B \setminus B^{pre-sing}$ to a K -affine structure satisfying fixed point property (see 7.1). We assume that \mathbf{Z} -affine structure cannot be extended to a larger open set $U \supset B \setminus B^{pre-sing}, U \neq B \setminus B^{pre-sing}$. Slightly abusing notation we will denote $B \setminus B^{pre-sing}$ simply by B^{sm} . We want to have a K -analytic space X , meromorphic non-zero top degree form Ω and a continuous proper (and maybe also Stein) map $\pi : X \rightarrow B$ such that:

1. $B^{pre-sing}$ coincides with B^{sing} , and \mathbf{Z} -affine structure on B^{sm} arising from the projection π coincides with the given one;
2. the restriction $\Omega|_{\pi^{-1}(B^{sm})}$ is a nowhere vanishing analytic form which satisfies the Constant Norm Assumption;
3. the K -affine structure on B^{sm} arising from the pair (X, Ω) coincides with the initial one.

We call the problem of finding such data *Lifting Problem*.

Remark 3 *If a solution of the Lifting Problem exists then B^{sm} is orientable. Indeed, $Res(\Omega)$ is locally a constant defined up to a sign which depends on the orientation of B^{sm} . Global choice of the constant gives an orientation. For oriented B^{sm} we can rescale Ω canonically in such a way that $Res(\Omega) = 1$*

Question. What restrictions on the behavior of the K -affine structure near $B^{pre-sing} = B^{sing}$ should we impose in order to guarantee the existence of a solution of the Lifting Problem?

Let $B = B^{sm}$ be a flat torus (see Section 3.2.1). Then the Lifting Problem has a solution (canonical up to rescaling of Ω) for any compatible K -affine structure. More precisely, the groupoid of Tate tori and isomorphisms between them is equivalent to the groupoid of K -affine structures on real flat tori.

In Sections 8-11 we are going to discuss a solution of the Lifting Problem for K3 surfaces. In that case $B^{sing} \neq \emptyset$.

If we restrict ourselves only to the smooth part B^{sm} (i.e. we allow non-compact X) then there is a canonical solution of this “reduced” Lifting Problem. In other words one can construct a smooth K -analytic space X' with an analytic top degree form Ω' and a map $\pi' : X' \rightarrow B^{sm}$ satisfying the above conditions 1–3. Let us explain this construction assuming that B^{sm} is oriented.

First of all we notice that the orientation of B^{sm} gives a reduction to $SL(n, \mathbf{Z}) \ltimes (K^\times)^n$ of the structure group of the torsor defining the K -affine structure. The reduced group naturally acts by automorphisms of the fibration $\pi_{can} : (\mathbf{G}_m^{an})^n \rightarrow \mathbf{R}^n$ preserving the form $\bigwedge_{1 \leq i \leq n} \frac{dz_i}{z_i}$. The action on $(\mathbf{G}_m^{an})^n$ is induced from the action on monomials. Namely, the inverse to an element $(A, \lambda_1, \dots, \lambda_n) \in SL(n, \mathbf{Z}) \ltimes (K^\times)^n$ acts on monomials as

$$z^I = z_1^{I_1} \dots z_n^{I_n} \mapsto \left(\prod_{i=1}^n \lambda_i^{I_i} \right) z^{A(I)} .$$

The action of the same element on \mathbf{R}^n is given by the similar formula

$$x = (x_1, \dots, x_n) \mapsto A(x) - (val(\lambda_1), \dots, val(\lambda_n)) .$$

Let $B^{sm} = \cup_\alpha U_\alpha$ be an open covering by coordinate charts $U_\alpha \simeq V_\alpha \subset \mathbf{R}^n$ such that for any α, β we are given elements $g_{\alpha, \beta} \in SL(n, \mathbf{Z}) \ltimes (K^\times)^n$ satisfying the 1-cocycle condition for any triple α, β, γ . Then the space X' is obtained from $\pi_{can}^{-1}(V_\alpha)$ by gluing by means of the transformations $g_{\alpha, \beta}$. The form $\bigwedge_{1 \leq i \leq n} \frac{dz_i}{z_i}$ gives rise to a nowhere vanishing analytic top degree form

Ω' on X' . Thus we have obtained a solution of the reduced Lifting Problem. The sheaf $\pi_*(\mathcal{O}_{X'}) := \mathcal{O}_{B^{sm}}^{can}$ is called the *canonical sheaf*.

In the case $B^{pre-sing} \neq \emptyset$ this solution seems to be a “wrong” one, i.e. it cannot be extended to a solution $\pi : X \rightarrow B$, where X and B are compact. In the case of K3 surfaces we will show later how to modify it in order to obtain a “true” solution of the Lifting Problem.

7.4 Flat coordinates and periods

Here we are going to discuss a relation between K -affine structures and so-called flat coordinates on the moduli space of complex structures on Calabi-Yau manifolds. We assume the picture of collapse from Section 5.1.

7.4.1 Flat coordinates for degenerating complex Calabi-Yau manifolds

Let $X_{mer} = (X_t)_{t \rightarrow 0}$ be a maximally degenerating algebraic Calabi-Yau manifold of dimension n over \mathbf{C}_t^{mer} . We denote by B the Gromov-Hausdorff limit of our family (see Conjecture 1, Section 5.1). Its connected oriented open dense part B^{sm} carries a \mathbf{Z} -affine structure with the covariant lattice $T^{\mathbf{Z}}$.

Recall that according to the picture of collapse presented in Section 5.1 there is a canonical isotopy class of embeddings from a torus bundle $p : X'_t \rightarrow B^{sm}$ to the complex manifold X_t for all sufficiently small $t \neq 0$. Let us denote by $[\gamma_0] \in H_n(X'_t, \mathbf{Z})$ the fundamental class of the fiber of p . This is the homology class of a singular chain in X'_t which projects to a point by p .

Let $H_n^{\leq 1}(X'_t, \mathbf{Z}) \subset H_n(X'_t, \mathbf{Z})$ be the subgroup generated by homology classes of chains which are projected into graphs in B^{sm} . It follows from the definition that we have an epimorphism

$$J_a : H_1(B^{sm}, \wedge^{n-1} T^{\mathbf{Z}}) \twoheadrightarrow H_n^{\leq 1}(X'_t, \mathbf{Z}) / \mathbf{Z}[\gamma_0]$$

similar to the homomorphisms J_s defined in the symplectic case (see Section 3.1.1). The following formula defines a homomorphism of groups

$$P : H_n^{\leq 1}(X'_t, \mathbf{Z}) / \mathbf{Z}[\gamma_0] \rightarrow (\mathbf{C}_t^{mer})^\times, \quad [\gamma] \mapsto \exp \left(2\pi i \frac{\int_{[\gamma]} \Omega_t}{\int_{[\gamma_0]} \Omega_t} \right).$$

We will call P the *period map*. Notice that $\mathbf{Z}[\gamma_0] := H_n^{\leq 0}(X'_t, \mathbf{Z}) \subset H_n^{\leq 1}(X'_t, \mathbf{Z})$ is a low degree part of the limiting Hodge filtration on the homology of Calabi-Yau manifold X_t . Non-zero complex numbers

$$\exp \left(2\pi i \frac{\int_{[\gamma_i]} \Omega_t}{\int_{[\gamma_0]} \Omega_t} \right) ,$$

where γ_i is a set of generators of $H_n^{\leq 1}(X'_t, \mathbf{Z})/H_n^{\leq 0}(X'_t, \mathbf{Z})$ are called *flat coordinates* in Mirror Symmetry (see e.g. [Mor]). Those are local coordinates near a point close to the “cusp” of the moduli space of complex structures (local Torelli theorem).

The orientation of B^{sm} gives rise to an isomorphism $\bigwedge^{n-1} T^{\mathbf{Z}} \simeq (T^*)^{\mathbf{Z}}$. Therefore, combining maps J_a, P and the above isomorphism we obtain a homomorphism

$$\tilde{P} : H^1(B^{sm}, (T^*)^{\mathbf{Z}}) \rightarrow (\mathbf{C}_t^{mer})^\times .$$

7.4.2 Non-archimedean periods

Let X^{an} be a smooth analytic Calabi-Yau manifold associated with X_{mer} . Assuming the equivalence of Gromov-Hausdorff and non-archimedean pictures of collapse presented in Section 5 we have a continuous map $\pi : X^{an} \rightarrow B$. It gives a K -affine structure on B^{sm} . The corresponding exact sequence

$$0 \rightarrow K^\times \rightarrow Aff_K \rightarrow (T^*)^{\mathbf{Z}} \rightarrow 0$$

represents a class in $H^1(B^{sm}, T^{\mathbf{Z}} \otimes K^\times) \simeq Ext^1((T^*)^{\mathbf{Z}}, K^\times)$. Pairing with this class gives another homomorphism

$$P' : H_1(B^{sm}, (T^*)^{\mathbf{Z}}) \rightarrow K^\times = H_0(B^{sm}, K^\times) .$$

Conjecture 10 *Homomorphism P' is equal to the composition of \tilde{P} with the embedding $(\mathbf{C}_t^{mer})^\times \hookrightarrow K^\times$.*

Part III

We fix field K satisfying Zero Characteristic Assumption.

Let B be a compact oriented surface, $B^{sing} \subset B$ a finite set, and $Aff_K = Aff_{K,Y}$ a sheaf defining a K -affine structure on $Y := B^{sm} = B \setminus B^{sing}$. We assume that all singularities of the underlying \mathbf{Z} -affine structure are standard (see Section 6.4), and local monodromy around each $b \in B^{sing}$ acts on $(K^\times)^2$ with a fixed point (see the Fixed Point property at the end of Section 7.1). Main result of Part 3 of the paper can be formulated such as follows.

Theorem 5 *There exist a compact K -analytic surface X^{an} , a top degree analytic form $\Omega = \Omega_{X^{an}}$ and a continuous proper Stein map $\pi : X^{an} \rightarrow B$ such that the set of π -smooth points coincides with Y and the induced K -affine structure coincides with the one given by Aff_K .*

In other words, the triple (X^{an}, π, Ω) is a solution of the Lifting Problem.

By Stein property it suffices to construct the sheaf $\mathcal{O}_B = \pi_*(\mathcal{O}_{X^{an}})$ of K -algebras on B . We will see that outside of the finite singular set $S = \{x_1, \dots, x_{24}\}$ the sheaf \mathcal{O}_B is locally isomorphic to \mathcal{O}_Y^{can} . In the next section we will describe the local model for the sheaf \mathcal{O}_B near each singular point. It will be glued together with a modification of the canonical sheaf \mathcal{O}_Y^{can} . This modification depends on the data called *lines*. Appearance of lines is motivated by Homological Mirror Symmetry (see [Ko], [KoSo])⁶. Roughly speaking, lines correspond (for mirror dual K3 surface) to collapsing holomorphic discs with boundaries belonging to fibers of the dual torus fibration (see Section 5.1 and [KoSo]). Such “bad” fibers are Lagrangian tori, but they do not correspond to objects of the Fukaya category (A-branes in terminology of physicists). There are infinitely many such fibers and hence infinitely many lines. We will axiomatize this piece of data in Section 9. Subsequently, with each line l we will associate an automorphism of the restriction of \mathcal{O}_Y^{can} to l . This will give us the above-mentioned modified canonical sheaf.

8 Model near a singular point

Here we will construct an analytic torus fibration corresponding to standard singularity (see Sections 3.2.4 and 6.4).

Let $X \subset \mathbf{A}^3$ be the algebraic surface given by equation $(\alpha\beta - 1)\gamma = 1$ in coordinates (α, β, γ) , and X^{an} be the corresponding analytic space. We define a continuous map $f : X^{an} \rightarrow \mathbf{R}^3$ by the formula $f(\alpha, \beta, \gamma) = (a, b, c)$ where $a = \max(0, \log |\alpha|_p)$, $b = \max(0, \log |\beta|_p)$, $c = \log |\gamma|_p = -\log |\alpha\beta - 1|_p$. Here $|\cdot|_p = \exp(-val_p(\cdot))$ denotes the multiplicative seminorm corresponding to the point $p \in X^{an}$ (see Appendix A).

Proposition 4 *The map f is proper. Moreover*

⁶The main idea is that X is a component of the moduli space of certain objects (skyscraper sheaves) in the derived category $D^b(Coh(X))$. These objects correspond to $U(1)$ -local systems on Lagrangian tori in the Fukaya category of the mirror dual symplectic manifold.

- a) Image of f is homeomorphic to \mathbf{R}^2 .
b) All points of the image except of $(0, 0, 0)$ are f -smooth.

Proof. Here is the plan of the proof.

1. We define three open domains T_i , $i = 1, 2, 3$ in three copies of the standard two-dimensional analytic torus $(\mathbf{G}_m^{an})^2$, and continuous maps $\pi_i : T_i \rightarrow \mathbf{R}^2$ such that all points of the image $U_i = \pi_i(T_i)$ are π_i -smooth (i.e. each π_i is an analytic torus fibration). Domains U_i cover $\mathbf{R}^2 \setminus \{(0, 0)\}$.
2. For each i , $1 \leq i \leq 3$ we construct an open embedding $g_i : T_i \hookrightarrow X^{an}$.
3. We construct an embedding $j : \mathbf{R}^2 \hookrightarrow \mathbf{R}^3$ such that each open set U_i is homeomorphically identified with $f(g_i(T_i))$ and $j((0, 0)) = (0, 0, 0)$. Moreover, π_i -smooth points are mapped into f -smooth points.

The Proposition will follow from 1)-3).

Let us describe the constructions and formulas. We start with open sets U_i , $1 \leq i \leq 3$. Let us fix a number $0 < \varepsilon < 1$ and define

$$\begin{aligned} U_1 &= \{(x, y) \in \mathbf{R}^2 \mid x < \varepsilon|y|\} \\ U_2 &= \{(x, y) \in \mathbf{R}^2 \mid x > 0, y < \varepsilon x\} \\ U_3 &= \{(x, y) \in \mathbf{R}^2 \mid x > 0, y > 0\} \end{aligned}$$

Clearly $\mathbf{R}^2 \setminus \{(0, 0)\} = U_1 \cup U_2 \cup U_3$. We define also a slightly modified domain U'_2 as $\{(x, y) \in \mathbf{R}^2 \mid x > 0, y < \frac{\varepsilon}{1+\varepsilon}x\}$.

We define $T_i := \pi_{can}^{-1}(U_i) \subset (\mathbf{G}_m^{an})^2$, $i = 1, 3$ and $T_2 := \pi_{can}^{-1}(U'_2) \subset (\mathbf{G}_m^{an})^2$. Then the projections $\pi_i : T_i \rightarrow U_i$ are given by the formulas

$$\begin{aligned} \pi_i(\xi_i, \eta_i) &= \pi_{can}(\xi_i, \eta_i) = (\log |\xi_i|, \log |\eta_i|), \quad i = 1, 3, \\ \pi_2(\xi_2, \eta_2) &= \begin{cases} (\log |\xi_2|, \log |\eta_2|) & \text{if } |\eta_2| < 1 \\ (\log |\xi_2| - \log |\eta_2|, \log |\eta_2|) & \text{if } |\eta_2| \geq 1 \end{cases}. \end{aligned}$$

In these formulas (ξ_i, η_i) are coordinates on T_i , $1 \leq i \leq 3$.

We define inclusion $g_i : T_i \hookrightarrow X$, $1 \leq i \leq 3$ by the following formulas:

$$\begin{aligned} g_1(\xi_1, \eta_1) &= \left(\frac{1}{\xi_1}, \xi_1(1 + \eta_1), \frac{1}{\eta_1}\right) \\ g_2(\xi_2, \eta_2) &= \left(\frac{1+\eta_2}{\xi_2}, \xi_2, \frac{1}{\eta_2}\right) \\ g_3(\xi_3, \eta_3) &= \left(\frac{1+\eta_3}{\xi_3\eta_3}, \xi_3\eta_3, \frac{1}{\eta_3}\right) \end{aligned}$$

Let us decompose $X^{an} = X_- \cup X_0 \cup X_+$ according to the sign of $\log|\gamma|_p$ where $p \in X^{an}$ is a point. It is easy to see that

$$\begin{aligned} f(X_-) &= \{ (a, b, c) \in \mathbf{R}^3 \mid c < 0, a \geq 0, b \geq 0, ab(a + b + c) = 0 \} \\ f(X_0) &= \{ (a, b, c) \in \mathbf{R}^3 \mid c = 0, a \geq 0, b \geq 0, ab = 0 \} \\ f(X_+) &= \{ (a, b, c) \in \mathbf{R}^3 \mid c > 0, a \geq 0, b \geq 0, ab = 0 \} \end{aligned}$$

From this explicit description we see that f is proper and the image of f is homeomorphic to \mathbf{R}^2 .

Let us consider the embedding $j : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ given by formula

$$j(x, y) = \begin{cases} (-x, \max(x + y, 0), -y) & \text{if } x \leq 0 \\ (0, x + \max(y, 0), -y) & \text{if } x \geq 0 \end{cases}$$

One can easily check that the image of j coincides with the image of f , $j \circ \pi_i = f \circ g_i$ and $f^{-1}(j(U_i)) = g_i(T_i)$ for all $1 \leq i \leq 3$. This concludes the proof of Proposition. ■

We can derive more from explicit formulas given in the proof.

Let us denote by $\pi : X^{an} \rightarrow \mathbf{R}^2$ the map $j^{(-1)} \circ f$. It is an analytic torus fibration outside of point $(0, 0)$. The induced \mathbf{Z} -affine structure on $\mathbf{R}^2 \setminus \{(0, 0)\}$ is in fact the standard singular \mathbf{Z} -affine structure described in Sections 3.2.4 and 6.4, as follows immediately from formulas for projections π_i , $i = 1, 2, 3$.

Let us introduce another sheaf \mathcal{O}^{can} on $\mathbf{R}^2 \setminus \{(0, 0)\}$. It is defined as $(\pi_i)_*(\mathcal{O}_{T_i})$ in each domain U_i , with identifications

$$\begin{aligned} (\xi_1, \eta_1) &= (\xi_2, \eta_2) && \text{on } U_1 \cap U_2 \\ (\xi_1, \eta_1) &= (\xi_3, \eta_3) && \text{on } U_1 \cap U_3 \\ (\xi_2, \eta_2) &= (\xi_3, \eta_3) && \text{on } U_2 \cap U_3 \end{aligned}$$

Let us consider the direct image sheaf $\pi_*(\mathcal{O}_{X^{an}})$. It is easy to see that on the sets U_1 and $U_2 \cup U_3$ this sheaf is canonically isomorphic to \mathcal{O}^{can} . The isomorphism is given by the identification of coordinates (ξ_1, η_1) on U_1 , and of coordinates (ξ_2, η_2) and (ξ_3, η_3) on $U_2 \cup U_3$. Therefore on the intersection $U_1 \cap (U_2 \cup U_3)$ we identify two copies of the canonical sheaf by certain automorphism φ of \mathcal{O}^{can} which preserves one coordinate (namely, the coordinate η). We will develop the theory of such transformations and their analytic continuations in Section 11. The explicit formulas for φ is

$$\varphi(\xi, \eta) = \begin{cases} (\xi(1 + \eta), \eta) & \text{on } U_1 \cap U_2 \\ (\xi(1 + 1/\eta), \eta) & \text{on } U_1 \cap U_3 \end{cases}$$

We would like to say now few words about analytic volume forms. Notice that each $T_i \subset (\mathbf{G}_m^{an})^2$ carries a nowhere vanishing top degree analytic form given by the formula $\Omega_i = \frac{d\xi_i \wedge d\eta_i}{\xi_i \eta_i}$. Then a straightforward calculation shows that Ω^{T_i} is the pullback under g_i of nowhere vanishing on X^{an} analytic top degree form

$$\Omega = -\gamma d\alpha \wedge d\beta .$$

Form Ω satisfies Constant Norm Assumption, hence it gives a K -affine structure on $\mathbf{R}^2 \setminus \{(0,0)\}$. On the other hand, the sheaf \mathcal{O}^{can} of algebras is also endowed with top-degree form Ω^{can} , equal to $\frac{d\xi_i \wedge d\eta_i}{\xi_i \eta_i}$ in local coordinates.

Lemma 3 *The K -affine structure on $\mathbf{R}^2 \setminus \{(0,0)\}$ associated with Ω coincides with the one associated with Ω^{can} .*

Proof: Using definitions from Section 7.2 one sees immediately that the statement of the Lemma follows from the equality $p_{\Omega^{can}}(1 + \eta) = 1$, which is straightforward: $p_{\Omega^{can}}(1 + \eta) = \exp(\text{Res}(\Omega^{can} \log(1 + \eta))) = 1 \in \mathcal{O}_K^\times$. ■

In all the definitions and formulas in this section on can shift domains $U_i, i = 1, 2, 3$ by vector $(x_0, 0) \in \mathbf{R}^2$ for arbitrary $x_0 \in \mathbf{R}$, thus giving a map $X^{an} \rightarrow \mathbf{R}^2$ with singularity at the point $(x_0, 0)$.

Finally, we denote $\pi_*(\mathcal{O}_{X^{an}})$ by $\mathcal{O}_{\mathbf{R}^2}^{model}$. This will be our model for the sheaf \mathcal{O}_B near each point of the singular set B^{sing} .

9 Lines on surfaces

In this section we are going to describe axiomatically the notion of collection of lines on a surface.

9.1 Data

- a) A compact oriented surface B , a finite subset $B^{sing} \subset B$.
- b) A \mathbf{Z} -affine structure on $Y = B^{sm} = B \setminus B^{sing}$ with the standard singularities near each $b \in B^{sing}$.
- c) A set \mathcal{L} of *lines*. With each line $l \in \mathcal{L}$ there is an associated continuous map $f_l : (0, +\infty) \rightarrow Y$. We assume that \mathcal{L} is decomposed into a disjoint union of two subsets $\mathcal{L} = \mathcal{L}_{in} \sqcup \mathcal{L}_{com}$. Lines belonging to \mathcal{L}_{in} are called

initial, while those in \mathcal{L}_{com} are called *composite*. We assume that for any $l \in \mathcal{L}$ there exists a continuous extension $f_l : [0, +\infty) \rightarrow B$ such that $f_l(0) \in B^{sing}$ if $l \in \mathcal{L}_{in}$ and $f_l(0) \in Y = B^{sm}$ if $l \in \mathcal{L}_{com}$.

- d)** A collection of covariantly constant nowhere vanishing integer-valued 1-forms $\alpha_l \in \Gamma((0, +\infty), f_l^*((T^*)^{\mathbf{Z}}))$, $l \in \mathcal{L}$. We assume that for $l \in \mathcal{L}_{in}$ in the standard coordinates (x, y) near singular point $f_l(0)$ we have: $f_l(t) = (0, t)$ or $f_l(t) = (0, -t)$ for all sufficiently small $t > 0$, and $\alpha_l(t) = \pm f_l^*(dy)$.
- e)** A map $\mathcal{L} \rightarrow \mathcal{L} \times \mathcal{L}$, $l \mapsto (p_{left}(l), p_{right}(l))$ (the letter p stands for “parent”: one can think about these lines as “generating l in a collision”).

Notice that since the form dy is invariant with respect to the monodromy, the condition in **d)** is coordinate-independent. The covector $\alpha_l(t)$ will be called a *direction covector* of l at time t . It gives rise to a half-plane

$$P_{l,t}^{(0)} = \{v \in T_{f_l(t)}Y \mid \langle \alpha_l(t), v \rangle > 0\} .$$

9.2 Axioms

To every $l \in \mathcal{L}_{in}$ we assign a pair $(f_l(0), sgn(\alpha_l(0))) \in B^{sing} \times \{\pm 1\}$, where $sgn(\alpha_l(0))$ is a choice of sign in $\pm f_l^*(dy)$ (see data **d)** in the previous subsection). In this way we obtain a map $r : \mathcal{L}_{in} \rightarrow B^{sing} \times \{\pm 1\}$.

Axiom 1. Map r is one-to-one.

Let $U \subset Y$ be a simply-connected domain, and line l intersects U . Let $I \subset \mathbf{R}_+$ be an interval such that $f_l(I) \subset U$. Then there exists a covariantly constant closed non-zero 1-form β_U in U (with constant integer coefficients), such that $f_l^*(\beta_U) = \alpha_l$, when both sides are restricted to I .

Axiom 2. For any $t_1, t_2 \in I$ one has

$$\int_{f_l(t_1)}^{f_l(t_2)} \beta_U = t_2 - t_1 .$$

Let $l_1, l_2 \in \mathcal{L}$, $t_1, t_2 > 0$ satisfy the condition $f_{l_1}(t_1) = f_{l_2}(t_2) = x \in Y$. In this case we say that lines l_1 and l_2 have a collision at x at the times t_1 and t_2 respectively.

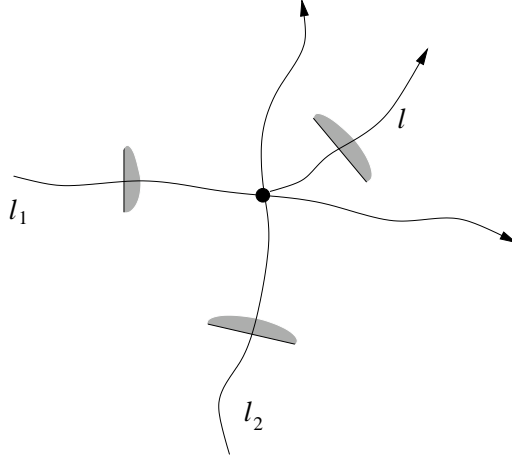


Figure 4: Line l and its two parents l_1, l_2 . Dashed half-planes are domains in tangent planes where 1-forms α take positive values.

Axiom 3. Under the above assumptions there are only two possibilities:

3a) either $l_1 = l_2$ and $t_1 = t_2$, or

3b) covector $\alpha_{l_1}(t_1)$ is not proportional to $\alpha_{l_2}(t_2)$. Then we may assume that $\alpha_{l_1}(t_1) \wedge \alpha_{l_2}(t_2) > 0$. Under these conditions we require that for any coprime positive integers n_1, n_2 there exists a unique line $l \in \mathcal{L}$ such that $l_1 = p_{left}(l), l_2 = p_{right}(l), f_l(0) = x$ and $\alpha_l(0) = n_1\alpha_{l_1}(t_1) + n_2\alpha_{l_2}(t_2)$.

In other words, l_1 and l_2 are “parents of l ”, and the direction covector of l at the intersection point is a primitive integral linear combination of those for l_1 and l_2 (see Figure 4).

Axiom 4. For every line $l \in \mathcal{L}_{com}$ there exist l_1 and l_2 such that they satisfy the condition **3b**).

Axiom 5. For any $x \in Y$ there are no more than two pairs $(l, t) \in \mathcal{L} \times (0, +\infty)$ such that $x = f_l(t)$. In other words, there are no more than two lines intersecting at a point in Y .

Let l_1, l_2, t_1, t_2, x mean the same as in the Axiom 3, and assume that $\alpha_{l_1}(t_1) \wedge \alpha_{l_2}(t_2) > 0$. Let us consider the set $\mathcal{L}_{(x)}$ of germs of all $l \in \mathcal{L}_{com}$ starting at x (i.e. such that $f_l(0) = x$).

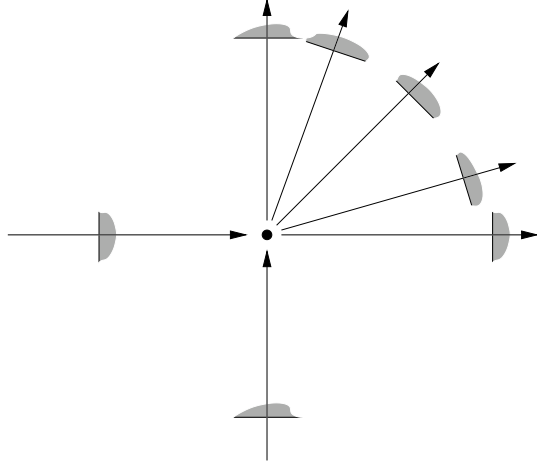


Figure 5: Two intersecting lines and some of new lines obtained as a result of collision. All lines are straightened by a homeomorphism of \mathbf{R}^2 .

Axiom 6. For any finite subset $\mathcal{L}' \subset \mathcal{L}_{(x)}$ there is an orientation preserving homeomorphism of a neighborhood of x onto a neighborhood of $(0, 0) \in \mathbf{R}^2$ such that:

6a) Germs of oriented curves which are images of l_1 and l_2 get transformed into the germs at $(0, 0)$ of coordinate axes $(x, 0)$ and $(0, y)$ respectively.

6b) Germ of the image of $l \in \mathcal{L}'$ gets transformed into the germ of the ray $\{(n_1 t, n_2 t) \mid t > 0\}$ where $\alpha_l(0) = n_1 \alpha_{l_1}(t_1) + n_2 \alpha_{l_2}(t_2)$.

Figure 5 illustrates this axiom.

Axiom 7. Let p_i denotes either p_{left} or p_{right} . Then for any $l \in \mathcal{L}$ there exists $N \geq 1$ such that if the line $p_1(p_2(\dots p_N(l) \dots))$ is well-defined then it belongs to \mathcal{L}_{in} .

This axiom says that any composed line $l \in \mathcal{L}_{com}$ appears as a result of finitely many collisions. The tree of ancestors of a given line form a tree embedded in B , see Figure 6.

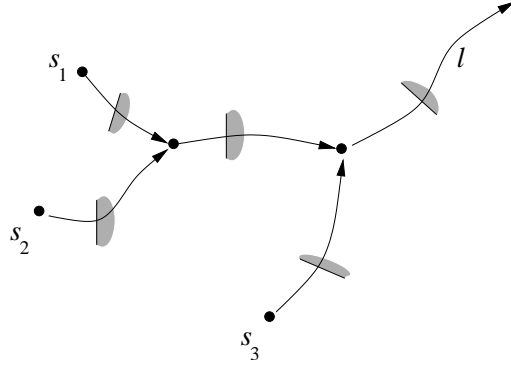


Figure 6: Tree of ancestors of line l starting from 3 singular points $s_1, s_2, s_3 \in B^{sing}$.

9.3 Example: gradient lines

Here we offer a construction of the set of lines satisfying the above axioms.

Let us use the standard \mathbf{R}^2 as a model around each $b \in B^{sing}$ in order to fix a structure of smooth manifold on the whole surface B . Let \tilde{Y} denotes the covering of Y such that the fiber over $y \in Y$ is $(T_y^*Y)^{\mathbf{Z}} \setminus \{0\}$.

Let us fix a *generic* smooth metric on B . By the pull-back it gives a metric on \tilde{Y} . Notice that there is a canonical closed 1-form β on \tilde{Y} such that $\beta_{|(y,\mu)} = \mu$, where $y \in Y$, $\mu \in (T_y^*Y)^{\mathbf{Z}}$. Using the metric we obtain dual to β gradient vector field v on \tilde{Y} .

For any $s \in B^{sing}$ and a choice of 1-form $\alpha(0) = \pm dy$ in local coordinates, we take the unique integral line of v starting at $(s, \alpha(0))$. Set \mathcal{L}_{in} will be the set of all lines obtained in this way. Each line $l \in \mathcal{L}_{in}$ carries a covariantly constant closed 1-form α_l . Using Axiom 2 as a definition, we obtain a canonical parametrization of each line by the time parameter t . Since the metric is generic, a line cannot return to a point in B^{sing} .

Then we proceed inductively. If two already constructed lines $l_1, l_2 \in \mathcal{L}$ meet at $x \in Y$ we produce a new integral line l of v with the direction covector satisfying the condition **3b)** for any pair of coprime positive integers n_1, n_2 . In this way we construct a set of lines \mathcal{L} satisfying all the axioms. The only non-trivial thing to check is that for each line values of the parameter t are in one-to-one correspondence with the interval $(0, +\infty)$. In order to see this we observe that the length of each line is infinite. Indeed, an integral curve of v cannot have a limiting point in Y (since the flow generated by v is smooth,

and the lengths of tangent vectors are bounded from below because of the integrality of 1-forms).

We conclude that there exists a set \mathcal{L} of lines satisfying Axioms 1-7.

10 Groups and symplectomorphisms

In this section we are going to discuss the sheaf of groups of symplectomorphisms $Symp := Symp(\mathcal{O}_Y^{can})$ of the sheaf \mathcal{O}_Y^{can} . Let $U \subset Y$ be an open convex subset. By definition, a *symplectomorphism* of $\mathcal{O}_Y^{can}(U)$ is an automorphism of K -algebra $\mathcal{O}_Y^{can}(U)$ preserving projection to Y and the canonical symplectic form $\Omega = \frac{d\xi \wedge d\eta}{\xi\eta}$ (the latter is understood as an element of the algebra of Kähler differential forms). To each line l we will assign a symplectomorphism of the restriction of \mathcal{O}_Y^{can} to l , so that the assignment will be compatible with the collision of lines. Then we are going to modify the sheaf \mathcal{O}_Y^{can} using symplectomorphisms, associated with lines and obtain the sheaf \mathcal{O}_Y^{modif} . This sheaf will be glued with the sheaf $\mathcal{O}_{\mathbf{R}^2}^{model}$ near each point of B^{sing} .

10.1 Pro-nilpotent Lie algebra

Here it will be convenient to work in local coordinates $(x, y) = (\log |\xi|, \log |\eta|)$ on Y .

Let $(x_0, y_0) \in \mathbf{R}^2$ be a point, $\alpha_1, \alpha_2 \in (\mathbf{Z}^2)^*$ be 1-covectors such that $\alpha_1 \wedge \alpha_2 > 0$. Denote by $V = V_{(x_0, y_0), \alpha_1, \alpha_2}$ the closed angle

$$\{(x, y) \in \mathbf{R}^2 \mid \langle \alpha_i, (x, y) - (x_0, y_0) \rangle \geq 0, i = 1, 2\}.$$

Let $\mathcal{O}(V)$ be a K -algebra consisting of series $f = \sum_{n, m \in \mathbf{Z}} c_{n, m} \xi^n \eta^m$, such that $c_{n, m} \in K$ and for all $(x, y) \in V$ we have:

1. if $c_{n, m} \neq 0$ then $\langle (n, m), (x, y) - (x_0, y_0) \rangle \leq 0$, where we identified $(n, m) \in \mathbf{Z}^2$ with a covector in $(T_p^*Y)^{\mathbf{Z}}$;
2. $\log |c_{n, m}| + nx + my \rightarrow -\infty$ as long as $|n| + |m| \rightarrow +\infty$.

Algebra $\mathcal{O}(V)$ is a Poisson algebra with respect to the bracket $\{\xi, \eta\} = \xi\eta$.

For an integer covector $\mu = adx + bdy \in (\mathbf{Z}^2)^*$ we denote by R_μ the monomial $\xi^a \eta^b$.

Let us consider a pro-nilpotent Lie algebra $\mathfrak{g} := \mathfrak{g}_{\alpha_1, \alpha_2, V} \subset \mathcal{O}(V)$ consisting of series

$$f = \sum_{n_1, n_2 \geq 0, n_1 + n_2 > 0} c_{n_1, n_2} R_{\alpha_1}^{-n_1} R_{\alpha_2}^{-n_2}$$

satisfying the condition

$$\log |c_{n, m}| - n_1 \langle \alpha_1, (x, y) \rangle - n_2 \langle \alpha_2, (x, y) \rangle \leq 0 \quad \forall (x, y) \in V .$$

The latter condition is equivalent to $\log |c_{n, m}| - \langle n_1 \alpha_1 + n_2 \alpha_2, (x, y) \rangle \leq 0$.

Lie algebra \mathfrak{g} admits a filtration by Lie subalgebras $\mathfrak{g}^{\geq k}$, $k \in \mathbf{Z}$, $k \geq 1$, $\mathfrak{g} = \mathfrak{g}^{\geq 1}$, such that $\mathfrak{g}^{\geq k}$ consists of the above series which satisfy the condition $n_1 + n_2 \geq k$. Clearly $[\mathfrak{g}^{\geq k_1}, \mathfrak{g}^{\geq k_2}] \subset \mathfrak{g}^{\geq k_1 + k_2}$, and $\mathfrak{g} = \varprojlim_{k \rightarrow +\infty} \mathfrak{g}/\mathfrak{g}^{\geq k}$.

Thus, \mathfrak{g} is a topological complete pro-nilpotent Lie algebra over K . We denote by G the corresponding pro-nilpotent Lie group $\exp(\mathfrak{g})$. It inherits the filtration by normal subgroups $G^{\geq k}$ obtained from the corresponding Lie algebras.

10.2 Lie groups G_λ

For each $\lambda \in [0, +\infty]_{\mathbf{Q}} := \mathbf{Q}_{\geq 0} \cup \infty$ we define a Lie subalgebra

$$\mathfrak{g}_\lambda = \left\{ \sum_{n_1, n_2} c_{m, n} R_{\alpha_1}^{-n_1} R_{\alpha_2}^{-n_2} \in \mathfrak{g} \mid c_{n_1, n_2} \in K, \frac{n_2}{n_1} = \lambda \right\} .$$

Each \mathfrak{g}_λ is an abelian Lie algebra. It carries the induced filtration by Lie algebras $\mathfrak{g}_\lambda^{\geq k} = \mathfrak{g}_\lambda \cap \mathfrak{g}^{\geq k}$. Denote by $G_\lambda = \exp(\mathfrak{g}_\lambda)$ the corresponding pro-nilpotent group.

Lemma 4 *For any given $k \geq 1$ there exist finitely many $\lambda_1 < \lambda_2 < \dots < \lambda_{N_k}$ such that $\mathfrak{g}_\lambda/\mathfrak{g}_\lambda^{\geq k} = 0$ for $\lambda \neq \lambda_i$, $1 \leq i \leq N_k$.*

Proof. Indeed, for the monomial $R_{\alpha_1}^{-n_1} R_{\alpha_2}^{-n_2} \in \mathfrak{g}_\lambda$ which maps non-trivially to the quotient $\mathfrak{g}_\lambda/\mathfrak{g}_\lambda^{\geq k}$ we have: $n_1 + n_2 \leq k$, $n_1/n_2 = \lambda$, where n_1, n_2 are non-negative integers. There are finitely many such non-negative integers n_1 and n_2 . ■

It follows from the Lemma that we have a natural isomorphism of vector spaces $\prod_{\lambda \in [0, +\infty]_{\mathbf{Q}}} \mathfrak{g}_\lambda/\mathfrak{g}_\lambda^{\geq k} \rightarrow \mathfrak{g}/\mathfrak{g}^{\geq k}$, hence the map

$$(f_\lambda)_{\lambda \in [0, +\infty]_{\mathbf{Q}}} \mapsto \sum_{\lambda} f_\lambda = \sum_{i=1}^{N_i} f_{\lambda_i}, \quad \text{where } f_\lambda \in \mathfrak{g}_\lambda/\mathfrak{g}_\lambda^{\geq k} \quad \forall \lambda \in [0, +\infty]_{\mathbf{Q}}$$

is well-defined and gives rise (after taking the projective limit as $k \rightarrow +\infty$) to the isomorphism $\mathfrak{g} \simeq \prod_{\lambda \in [0, +\infty]_{\mathbf{Q}}} \mathfrak{g}_{\lambda}$.

In a similar way we define the map $\prod_{\rightarrow} : \prod_{\lambda \in [0, +\infty]_{\mathbf{Q}}} G_{\lambda} \rightarrow G$, the product is taken with respect to the natural order on \mathbf{Q} . Namely, for any $k \geq 1$ we define

$$\prod_{\rightarrow}^{(k)} : \prod_{i=1}^{N_k} G_{\lambda_i} / G_{\lambda_i}^{\geq k} \rightarrow G / G^{\geq k}, \quad (g_1, \dots, g_{N_k}) \mapsto g_1 \dots g_{N_k}, \quad \text{for } g_i \in G_{\lambda_i} / G_{\lambda_i}^{\geq k}$$

and then set $\prod_{\rightarrow} := \varprojlim_k \prod_{\rightarrow}^{(k)}$.

Theorem 6 *Map \prod_{\rightarrow} is a bijection of sets.*

Proof. Let $k \geq 1$ be an integer. We claim that $\prod_{\rightarrow}^{(k)}$ is a bijection of sets (this implies the proposition by taking the projective limit as $k \rightarrow +\infty$). We will prove the bijection by induction in k . Case $k = 1$ is obvious because all the groups under considerations are trivial.

We would like to prove that $\prod_{\rightarrow}^{(k+1)}$ is a bijection assuming that $\prod_{\rightarrow}^{(k)}$ is a bijection. Let h be an element of $G / G^{\geq k+1}$ and \bar{h} its image in $G / G^{\geq k}$. By the induction assumption there exist unique $\bar{h}_i \in G_{\lambda_i} / G_{\lambda_i}^{\geq k+1}$, $1 \leq i \leq N_{k+1}$ such that $\bar{h}_1 \dots \bar{h}_{N_{k+1}} = \bar{h}$. Let h_i , $1 \leq i \leq N_{k+1}$ be any liftings of \bar{h}_i to $G_i / G_i^{\geq k}$. Then $h_1 \dots h_{N_{k+1}} = h \pmod{G^{\geq k}}$, hence $c := h_1 \dots h_{N_{k+1}} h^{-1}$ belongs to $G^{\geq k} / G^{\geq k+1} \subset \text{Center}(G / G^{\geq k+1})$. The last inclusion holds because $[\mathfrak{g}, \mathfrak{g}^{\geq k}] = [\mathfrak{g}^{\geq 1}, \mathfrak{g}^{\geq k}] \subset \mathfrak{g}^{\geq k+1}$.

Next we observe that the isomorphism of abelian Lie algebras

$$\bigoplus_{1 \leq i \leq N_{k+1}} \mathfrak{g}_{\lambda_i}^{\geq k} / \mathfrak{g}_{\lambda_i}^{\geq k+1} \simeq \mathfrak{g}^{\geq k} / \mathfrak{g}^{\geq k+1}$$

implies an isomorphism of the corresponding abelian groups

$$\prod_{1 \leq i \leq N_{k+1}} G_i^{\geq k} / G_i^{\geq k+1} \simeq G^{\geq k} / G^{\geq k+1}.$$

Hence we can write uniquely $c = c_1 \dots c_{N_{k+1}}$, where $c_i \in G^{\geq k} / G^{\geq k+1} \subset \text{Center}(G / G^{\geq k+1})$. It follows that $\prod_{\rightarrow}^{(k+1)} ((h_i c_i^{-1})) = h$. Also it is now clear that this decomposition of h is unique. This concludes the proof. ■

10.3 Function ord_l

For $l \in \mathcal{L}$ we will define an *order* function

$$ord_l \in \Gamma((0, +\infty), f_l^*(Aff_{\mathbf{Z}, Y}))$$

(its meaning will become clear later) by the following inductive procedure:

1. Let $l \in \mathcal{L}_{in}$ and $t > 0$ be sufficiently small. Then in the standard affine coordinates near $s = f_l(0)$ one has $\alpha_l = \pm f_l^*(dy)$. We define $ord_l = \pm f_l^*(y)$. Then $d(ord_l) = \alpha_l$, and we can extend uniquely ord_l for all $t \in (0, +\infty)$.
2. Let $l \in \mathcal{L}_{com}$ and l_1, l_2 be parents of l . In the notation of Axiom 3 we have $f_{l_1}(t_1) = f_{l_2}(t_2) = f_l(0)$ and $\alpha_l(0) = n_1\alpha_{l_1}(t_1) + n_2\alpha_{l_2}(t_2)$. Then we define $ord_l(0) := n_1ord_{l_1}(t_1) + n_2ord_{l_2}(t_2)$. Again, using the condition $d(ord_l) = \alpha_l$ and the knowledge of $ord_{l_i}(0)$ we can extend ord_l for $t > 0$.

Notice that $ord_l(t)$ can be thought of as affine function on the tangent space $T_{f_l(t)}Y$ (in the induced integral affine structure). In particular, we have a half-plane $P_{l,t} \subset T_{f_l(t)}Y$ defined by the inequality $ord_l(t) > 0$. The family of half-planes $P_{l,t}$ is covariantly constant with respect to ∇^{aff} .

Each half-plane $P_{l,t}$ contains $0 \in T_{f_l(t)}Y$ strictly in its interior. Recall that at the end of Section 9.1 we defined another half-plane $P_{l,t}^{(0)} \subset T_{f_l(t)}Y$. It is easy to see that $P_{l,t}^{(0)}$ is the half-plane parallel to $P_{l,t}$ such that $0 \in T_{f_l(t)}Y$ is on the boundary of $P_{l,t}^{(0)}$.

10.4 Symplectomorphisms assigned to lines

In this section we are going to assign to each line $l \in \mathcal{L}$ a symplectomorphism

$$\varphi_l \in \Gamma((0, +\infty), f_l^*(Symp))$$

giving for each $t > 0$ a transformation $\varphi_l(t) : \mathcal{O}_{Y, f_l(t)}^{can} \rightarrow \mathcal{O}_{Y, f_l(t)}^{can}$. This symplectomorphism in local coordinates will belong to the subgroup G_λ where λ is the slope of $\alpha_l(t)$. More precisely, we demand that $\varphi_l(t)$ is of the form

$$\varphi_l(t) = \exp\{F_{l,t}(\xi^{-a}\eta^{-b}), \cdot\},$$

where $\alpha_l(t) = adx + bdy$, operation $\{\cdot, \cdot\}$ is the Poisson bracket on $\mathcal{O}_{Y, f_l(t)}^{can}$ and $F_{l,t}(z) \in zK[[z]]$ is an analytic function of *one* variable satisfying the

following condition. Let us consider the pullback (by the exponential map) of the function $F_{l,t}(\xi^{-a}\eta^{-b})$ to a section of the sheaf \mathcal{O}^{can} on vector space $T_{f_l(t)}Y \simeq \mathbf{R}^2$ considered as a manifold with \mathbf{Z} -affine structure. Then this pullback should admit an analytic continuation from $0 \in T_{f_l(t)}$ to the half-plane $P_{l,t}$, and obey there the bound

$$|F_{l,t}(\xi^{-a}\eta^{-b})| \leq \exp(-ord_l(t)) .$$

Let us explain the construction of $\varphi_l(t)$, leaving the justification for the next sections.

Symplectomorphisms φ_l are constructed by an inductive procedure. Let $l = l_+ \in \mathcal{L}_{in}$ be (in standard affine coordinates) a line in the half-plane $y > 0$ emerging from $(0, 0)$ (there is another such line l_- in the half-plane $y < 0$). Assume that t is sufficiently small. Then we define $\varphi_l(t) \in \text{Sym}p_{f_l(t)}$ on topological generators ξ, η by the formula (as in Section 8)

$$\varphi_l(t)(\xi, \eta) = (\xi(1 + 1/\eta), \eta) .$$

Notice that $\varphi_l(t) = \exp\{F(\eta^{-1}), \cdot\}$, where $F(z) = \sum_{n>0} (-1)^n z^n / n^2$ is convergent for $|z| < 1$.

In order to extend $\varphi_l(t)$ to the interval $(0, t_0)$, where t_0 is not small, we cover the corresponding segment of l by open charts. Notice that change of affine coordinates transforms η into a monomial multiplied by a constant from K^\times . Therefore η extends analytically in a unique way to a global section over $(0, +\infty)$ of the sheaf $f_l^*((\mathcal{O}^{can})^\times)$. Moreover, the norm $|\eta|$ strictly decreases as t increases, and remains strictly smaller than 1. Hence $F(\eta)$ can be canonically extended for all $t > 0$.

Each symplectomorphism $\varphi_l(t)$ is defined by a series which converges in the half-plane $P_{l,t}$. Using the exponential map associated with the affine structure as well as estimates of $ord_l(t)$, we can extend analytically $\varphi_l(t)$ into a neighborhood of $f_l(t)$.

Let us now assume that l_1 and l_2 collide at $p = f_{l_1}(t_1) = f_{l_2}(t_2)$, generating the line $l \in \mathcal{L}_{com}$. Then $\varphi_l(0)$ is defined with the help of factorization theorem in the group G . More precisely, we set $\alpha_i := \alpha_{l_i}(t_i)$, $i = 1, 2$ and the angle V to be the intersection of half-planes $P_{l_1, t_1} \cap P_{l_2, t_2}$. By construction elements $g_0 := \varphi_{l_1}(t_1)$ and $g_{+\infty} := \varphi_{l_2}(t_2)$ belong respectively to G_0 and $G_{+\infty}$. Then we can use the factorization Theorem 6 and write down the formula

$$g_{+\infty}g_0 = \prod_{\rightarrow} ((g_\lambda)_{\lambda \in [0, +\infty]_{\mathbf{Q}}}) = g_0 \dots g_{1/2} \dots g_1 \dots g_{+\infty} ,$$

where $g_\lambda \in G_\lambda$ and the product on the right is in the *increasing* order. There is no clash of notations because it is easy to see that the boundary factors in the decomposition from above are indeed equal to g_0 and $g_{+\infty}$. Each term g_λ with $0 < \lambda = n_1/n_2 < +\infty$ corresponds to the newborn line l with the direction covector $n_1\alpha_{l_1}(t_1) + n_2\alpha_{l_2}(t_2)$. Then we set $\varphi_l(0) := g_\lambda$. This transformation is defined by a series which is convergent in a neighborhood of p , and using the analytic continuation as above, we obtain $\varphi_l(t)$ for $t > 0$. The decomposition identity can be rewritten as

$$g_0 \cdots g_{1/2} \cdots g_1 \cdots g_{+\infty} g_0^{-1} g_{+\infty}^{-1} = id$$

where each factor corresponds to half-lines at the collision point (see Figure 5), and the meaning of the identity is that the infinite composition of symplectomorphisms in the natural cyclic order on half-lines, is trivial.

11 Modification of the sheaf \mathcal{O}^{can}

11.1 Pieces of lines and convergence regions

Definition 13 *A neighborhood U of a point $x \in Y$ is convex if there exists an open convex $U_1 \in T_x Y, 0 \in U_1$ which is isomorphic to U by means of the exponential map $\exp_x : T_x Y \rightarrow Y$ associated with the affine structure on Y .*

For $x \in Y$ let $U \subset U'$ be convex neighborhoods of x such that U is relatively compact in U' . Let $l \in \mathcal{L}$. Then there is a natural embedding $f_l^{-1}(U) \rightarrow f_l^{-1}(U')$.

Definition 14 *A piece of l defined by the pair (U, U') is an element of the image of the set of connected components $\pi_0(f_l^{-1}(U))$ into $\pi_0(f_l^{-1}(U'))$ under the above embedding.*

In plain words a piece L of l is an equivalence class of a connected interval of $l \cap U$. Two connected intervals are equivalent if they are contained in a larger connected interval of $l \cap U'$. The sole purpose of the introduction of the notion of a piece is to avoid some pathology. Namely, for any pair (U, U') as above, any $l \in \mathcal{L}$ and any $T \in \mathbf{R}_{>0}$, there is only a finite number of pieces of l in (U, U') which have points with time parameter $t \in (0, T)$.

Let L be a piece of l defined by a pair (U, U') . Then one can define an affine function $ord_L \in Aff_{\mathbf{Z}, Y}(U')$ in the following way. Let $t > 0$ be such

that $f_l(t)$ belongs to L . Since U' is convex, there is a unique continuation of $ord_l(t)$ to U' . This is an affine function which does not depend on the choice of t . We will denote it by ord_L .

For any germ of a symplectomorphism $\varphi \in \text{Symp}_p$ at a point $p \in Y$ we define its convergence region as the maximal convex subset $\Omega(\varphi) \subset T_p Y$ such that the pullback $\exp_p^*(\varphi)$ extends to $\Omega(\varphi)$. Since the definition of φ_l (and hence its convergence region) is covariant with respect to the affine connection we have the following result:

Proposition 5 *Let $p = f_l(t)$ belongs to a line l . Then the convergence region of $\varphi_l(t)$ at p contains an open half-plane $P_{l,t}$.*

It is clear that one can define convergence regions for symplectomorphisms associated with pieces of lines, and a similar property holds for them.

11.2 Main assumptions, and an apology

Let us suppose that our collection of lines satisfies the following assumptions:

Assumption A1 There is a smooth metric $g = g_B$ and a collection of balls $D(s, r_s)$ with centers at $s \in B^{sing}$ such that each ball $D(s, r_s)$ contains exactly two lines $l_{\pm} \in \mathcal{L}_{in}$ outcoming of s .

Assumption A2 There exists $\varepsilon > 0$ such that for any $p = f_l(t) \in Y' := B \setminus \cup_{s \in B^{sing}} D(s, r_s)$ the distance in $T_p Y$ between $0 \in T_p Y$ and the boundary of $P_{l,t}$ is greater or equal to ε .

We are going to show that such a collection does exist in Section 11.

Assumptions **A1** and **A2** are very artificial, they do not hold in physical picture which is the main motivation for the construction. It is quite possible that they can be weakened or even omitted. The main purpose of introducing them here is the possibility to define the sheaf of analytic functions by simple gluing. In complex geometry it is similar to the gluing of closed Riemann surfaces with boundaries by the mean of real-analytic identifications of the boundaries. It is well-known that one can replace real-analytic maps by smooth ones (or even by quasi-symmetric continuous maps). Maybe the rest of this section is unnecessary, and unpleasant technical arguments in Section 11.5 can be avoided.

11.3 Infinite product and its convergence

Denote by $W_{\mathcal{L}} := \cup_{l \in \mathcal{L}} f_l([0, +\infty))$ the set of all points of all lines. It has measure zero. Let p be a point of Y . We consider two convex neighborhoods $U \subset U'$ of p such that U is relatively compact in U' .

For any two points x, y belonging to $U \setminus W_{\mathcal{L}}$, and a path γ joining x and y in U , we would like to define an infinite ordered product $i_{x,y}^{\gamma}$ of transformations $\varphi_L^{\pm 1}$, where factors correspond to the intersection points of γ with all possible pieces L relative to (U, U') . Factors in the infinite product are ordered according to the time parameter of γ , the sign corresponds to the mutual position of orientations of γ and a piece L at the intersection point.

In order to give a precise meaning to the infinite product the neighborhood U of p should be sufficiently small. Then we will have an analytic continuation of symplectomorphisms φ_L to U , and the convergence of the infinite product. We are also going to prove that the product is independent of the choice of path γ . In order to achieve these goals it suffices to assume:

C1 for any l, t such $f_l(t) \in U$ the set $(\exp_{f_l(t)})^{-1}(U)$ is contained in $P_{l,t}$;

C2 for any $C \in \mathbf{R}$ there is only a finite number of pieces L of lines in U such that $\inf_{x \in U} \text{ord}_L(x) < C$.

Theorem 7 *Assume two above conditions. Then the product defining $i_{x,y}^{\gamma}$ converges at every point of U and in fact gives an element of $\text{Symp}(U)$. Moreover, the product does not depend on the choice of path γ , and for any $x, y, z \in U \setminus W_{\mathcal{L}}$ satisfies the relation $i_{x,y} i_{y,z} = i_{x,z}$.*

Proof. Condition **C1** implies that all transformations φ_L admit an analytic continuation to U . Let us introduce a decreasing filtration by positive real numbers $\text{Symp}^{\geq r}(U)$, $r \in \mathbf{R}, r \geq 0$ on group $\text{Symp}(U)$ by the formula

$$\{g \in \text{Symp}(U) \mid \log |\xi'/\xi - 1|, \log |\eta'/\eta - 1| < -r \text{ where } (\xi', \eta') = g((\xi, \eta))\}$$

This is a complete filtration, and condition **C2** implies that in any quotient $\text{Symp}(U)/\text{Symp}^{\geq r}(U)$ only a finite number of elements φ_L are non-trivial. Therefore we can define the product in the quotient group.

In order to prove independence of γ , we consider the quotient group $\text{Symp}(U)/\text{Symp}^{\geq r}(U)$, and the finite 1-dimensional CW-complex (graph) consisting of finitely many pieces L , such that $\varphi_L \neq 1$ in the quotient. For each vertex v of the graph there is a natural cyclic order on the edges incident

to v . The product $\varphi_v = \prod_L \varphi_L^{\pm 1}$ taken in the cyclic order over the set of edges incident to v is equal to id (this follows from the construction of φ_l via factorizations). Since U is simply-connected, we conclude that the image of $i_{x,y}^\gamma$ in $Symp(U)/Symp^{>r}(U)$ does not depend on γ . Using completeness of the filtration we see that $i_{x,y} := i_{x,y}^\gamma$ does not depend on γ . Proof of the identity $i_{x,y}i_{y,z} = i_{x,z}$ is similar. ■

Theorem 8 *Assumptions **A1** and **A2** imply that for any $p \in Y$ there exist neighborhood U (and also U') satisfying conditions **C1** and **C2**.*

Proof. Assumption **A1** implies that the result near any singular point $s \in B^{sing}$, as there are only two lines near s . If we are far from B^{sing} then obviously **A2** implies **C1**.

In order to check **C2** we prove the following lemma

Lemma 5 *Under Assumptions **A1** and **A2**, for any $C > 0$ the set*

$$\{(l, t) \mid ord_l(t)(f_l(t)) < C\} \subset \mathcal{L} \times (0, +\infty)$$

consists of a finite number of intervals.

Proof: We proceed by induction in “complexity of the line”. Let $\delta \in \mathbf{R}_{>0}$ be the infimum of $ord_l(t)(f_l(t))$ where $l \in \mathcal{L}_{in}$ has a collision at time t . This number is strictly positive because the number of initial lines is finite, and by **A1** there is no collisions at small times. Observe that the value of $ord_l(0)$ at the beginning of any composite line l is greater or equal to the sum $ord_{l_1}(t_1) + ord_{l_2}(t_2)$. Therefore the inequality in the lemma implies that the number of collisions is bounded from above by C/δ . Also we have an upper bound on integer coefficients (n_1, n_2) in each collision (see Axiom **3b**) in Section 9.2). Let us observe that the length of each edge of the ansector tree of l is also bounded from above by $A ord_l$, for some absolute constant $A > 0$. Hence we have only finitely many possibilities for intersections. ■

For point $p \in Y$ which is far from B^{sing} we chose as U a neighborhood of radius $\epsilon' \ll \epsilon$ where $\epsilon > 0$ is constant from Assumption **A2**. Then for any point of a line $f_l(t) \in U$ we will have the inclusion

$$U \subset \exp_{f_l(t)} \left(\frac{1}{2} P_{l,t} \right) .$$

This implies that ord_L in U for the corresponding piece L is bounded below by

$$\frac{1}{2}ord_l(t)(f_l(t)) .$$

Since (by the last lemma) there exists only a finite number of pieces L intersecting such U , we obtain convergence condition **C2**. ■

11.4 Construction of the modified sheaf \mathcal{O}_B^{modif}

For any point $p \in Y$ and a neighborhood U satisfying conditions **C1** and **C2** we define the sheaf \mathcal{O}_U^{modif} as the result of the identification of copies of the sheaf $(\mathcal{O}_Y^{can})|_U$ labeled by points $x \in U \setminus W_{\mathcal{L}}$, by isomorphisms $i_{x,y}$. It follows from formulas in Section 8 that near singular points one can identify canonically this sheaf with the restriction of the sheaf $\mathcal{O}_{\mathbf{R}^2}^{model}$ to a punctured neighborhood of $(0,0) \in \mathbf{R}^2$.

Proposition 6 *For the modified sheaf \mathcal{O}^{modif} one has a canonical nowhere vanishing section Ω of the associated sheaf of K -analytic 2-forms.*

The K -affine structure $Aff_{K,Y}^{\Omega}$ on Y associated with Ω coincides with the initial one $Aff_{K,Y}$.

Proof. Existence of Ω follows from the fact that all modifications associated with lines are symplectomorphisms. In order to finish the proof it suffices to check that the modification associated with a line does not change the K -affine structure on Y . In local coordinates we may assume that $\Omega = \frac{d\xi}{\xi} \wedge \frac{d\eta}{\eta}$ and the modification is of the form $\varphi(\xi, \eta) = (\xi f(\eta^{-1}), \eta)$, where $f(z) = 1 + \sum_{n \geq 1} c_n z^n \in K[[z]]$ is convergent in an appropriate domain. We need to check that the automorphism φ acts trivially on the quotient sheaf $\pi_*(\mathcal{O}_X^\times) / \ker p_\Omega$ (see Section 7.2 for the notation). This check reduces to the calculation of

$$p_\Omega \left(\frac{\xi f(\eta)}{\xi} \right) = \exp \left(\frac{Res(\Omega \log(\xi f(\eta)/\xi))}{Res(\Omega)} \right) .$$

The latter is equal to $\exp(Res(\Omega \log(f(\eta)))) = 1$ because $\log(f(\eta^{-1}))$ belongs to $\eta^{-1}K[[\eta^{-1}]]$ and therefore has no constant term. ■

Thus, we have a solution of the Lifting Problem under Assumptions **A1** and **A2**.

11.5 Construction of the collection of lines

We would like to show that there exists a smooth metric g and a collection of lines satisfying the Assumptions **A1** and **A2**.

Let g_0 be an arbitrary smooth metric, flat near singular points. We define germs of lines $l \in \mathcal{L}_{in}$ in such a way that for each $s \in B^{sing}$ in local coordinates these lines are given by $\{(0, y) | y > 0\}$ and $\{(0, y) | y < 0\}$. The metric g will coincide with g_0 in a sufficiently small neighborhood $U = \cup_{s \in B^{sing}} D(s, r_s)$ of the singular set. Hence Assumption **A1** will be satisfied.

In order to construct the whole family of lines we introduce a 3-dimensional manifold \mathcal{M} consisting of pairs (x, P) where $x \in B \setminus \overline{U}_1$ and P is a half-plane in $T_x B$ whose boundary contains zero. Here $U_1 := \cup_{s \in B^{sing}} D(s, 2r_s)$ is a larger neighborhood of B^{sing} .

We would like to construct a smooth section $v : (x, P) \mapsto v_{(x,P)} \in T_x B$ of the pull-back to \mathcal{M} of the tangent bundle TB satisfying the following conditions:

1. for any $(x, P) \in \mathcal{M}$ one has $v_{(x,P)} \in \text{int}(P)$;
2. for any $x \in B \setminus \overline{U}_1$ the map $(x, P) \mapsto \mathbf{R}_{>0}^\times \cdot v_{(x,P)}$ is an orientation-preserving diffeomorphism

$$S^1 \simeq (T_x B^* \setminus \{0\}) / \mathbf{R}_{>0}^\times \rightarrow S^1 \simeq (T_x B \setminus \{0\}) / \mathbf{R}_{>0}^\times ;$$

3. for every $l \in \mathcal{L}_{in}$ there exists a smooth extension of the piece of l in U_1 to a larger piece intersecting ∂U_1 such that

$$\dot{f}_l(t) \in \mathbf{R}_{>0}^\times \cdot v_{(f_l(t), P_{l,t})} ,$$

for such $t > 0$ that $f_l(t) \in B \setminus \overline{U}_1$;

Let us associate with the section v a nowhere vanishing vector field \hat{v} on $T^*(B \setminus \overline{U}_1) \setminus (\text{Zero Section})$ in the following way:

- For each $(x, \alpha) \in T_x^* B$ the vector $\hat{v}(x, \alpha)$ is tangent to the horizontal distribution associated with the flat connection ∇ (the one which defines the affine structure on $B \setminus B^{sing}$).
- Projection of $\hat{v}(x, \alpha)$ to B coincides with $v_{(x, P_\alpha)}$, where $P_\alpha = \{\gamma | (\alpha, \gamma) > 0\}$.

Clearly these conditions determines \hat{v} uniquely. Now we formulate last condition:

4. there exists $r'_s > 2r_s$ such that for almost all (in the sense of Baire category) initial values $(x_0, P_0) \in \mathcal{M}$ the integral curve of \hat{v} starting at (x_0, P_0) reaches the pullback of $B \setminus \cup_{s \in B^{sing}} D(s, r'_s)$ in finite time.

Using the vector field \hat{v} we will construct (under certain genericity assumptions) a set \mathcal{L} of lines satisfying Assumption **A1**. Namely, the data consisting of a line l and an integer-valued 1-form α_l (see Section 9) will be an integral line of \hat{v} .

We are going to construct lines by induction by the number of collisions. Lines $l \in \mathcal{L}_{in}$ will be constructed using condition 3. The genericity assumption mentioned after the condition 4 is the assumption that no more than two lines collide and that initial values for newborn lines will be sufficiently generic. Conditions 1 and 4 plus genericity imply that one can parametrize any line $l \in \mathcal{L}$ by the new “time” $t > 0$ such that the Axiom 2 is satisfied. Axiom 6 follows from the condition 2. Other axioms and the Assumption **A1** will be satisfied automatically.

Now we would like to discuss Assumption **A2**.

Proposition 7 *Suppose that the metric g and field v described above are such that for any $(x, P) \in \mathcal{M}$ there exists $C > 0$ such that*

$$(\nabla_{v_{(x,P)}} g)(n_P, n_P) \leq C g(n_P, v_{(x,P)}) ,$$

where n_P is the normal unit vector to P directed inside and $\nabla_{v_{(x,P)}} g$ is the covariant derivative of the metric g considered as a symmetric tensor on the cotangent bundle.

*Then the Assumption **A2** is satisfied.*

Proof. In order to satisfy Assumption **A2** it suffices to find such $\varepsilon > 0$ that for any $x \in B \setminus \bar{U}_1$ and any half-plane $P_x \subset T_x B$, $0 \in \text{int}(P_x)$ with the distance $\text{dist}_{g_x}(0, \partial P_x) = \varepsilon$, and another half-plane $P'_x \subset T_x B$ parallel to P_x such that $0 \in \partial P'_x$, one has the following property: if $P_{x+\delta t v_{(x,P'_x)}}$ is the half-plane obtained from P_x by a small covariant (with respect to the *affine* connection ∇^{aff}) shift δt in the direction of $v_{(x,P'_x)}$, then

$$\text{dist}_{g_{x+\delta t v_{(x,P'_x)}}}(0, P_{x+\delta t v_{(x,P'_x)}}) \geq \text{dist}_{g_x}(0, \partial P_x) .$$

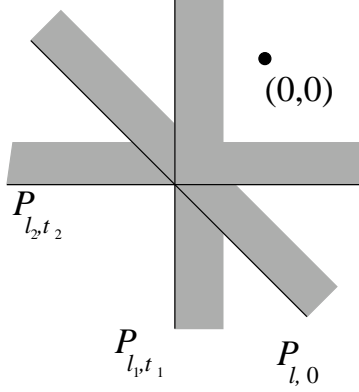


Figure 7: Three half-planes containing zero.

Here g_x etc. denotes the induced flat metric on the tangent space $T_x B$. This property guarantees that the condition $dist_{g_x}(0, \partial P_{l,t}) \geq \varepsilon$ will propagate along the line. For a new line obtained as a result of collision of l_1 and l_2 at the times t_1 and t_2 respectively one has

$$dist_{g_x}(0, \partial P_{l,0}) \geq \min\{dist_{g_x}(0, \partial P_{l_1,t_1}), dist_{g_x}(0, \partial P_{l_2,t_2})\}$$

since $\partial P_{l,0}$ contains the intersection point $\partial P_{l_1,t_1} \cap \partial P_{l_2,t_2}$, see Figure 7.

One can easily see that the infinitesimal inequality from above is equivalent to

$$\delta t g_x(v_{(x,P'_x)}, n_{P'_x}) + \varepsilon/2(g_{x+\delta t v_{(x,P'_x)}} - g_x)(n_{P'_x}, n_{P'_x}) \geq 0$$

(the change of the distance consists of two summands: one corresponds to the shift along $\delta t v_{(x,P'_x)}$ with the fixed metric, and the other one corresponds to the change of the metric). Taking the limit $\delta t \rightarrow 0$ we arrive to the inequality for the covariant derivative of the metric with $C = 2/\varepsilon$. ■

Now our goal is to construct the field of directions v and the metric g satisfying the conditions 1–4 and the inequality from the last Propostion. This will conclude the construction of the set \mathcal{L} of lines satisfying the Assumptions **A1** and **A2**.

Since ∂U_1 is a boundary of the convex set, we can locally model it by the graph of function $y = f(x)$ such that $f''(x) > 0$, $f'(x_0) = 0$. We may assume that $P = P_0$ is the upper half-plane. Then we take

$$v_{((x,y),P)} = \partial/\partial y + \frac{(f(x) - f(x_0))/f'(x)}{f(x) - f(x_0) + f(x) - y} \partial/\partial x .$$

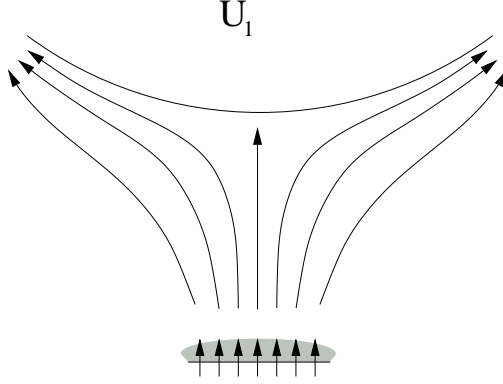


Figure 8: Vector field near ∂U_1 for $P =$ the upper half-plane.

We extend this local model of v near ∂U_1 to $B \setminus \overline{U}_1$ in such a way that conditions 1 and 2 are satisfied. It is clear that we can satisfy conditions 3,4 as well by taking a small perturbation of v . On Figure 8 there is a picture of the field $(x, y) \mapsto v_{((x,y), P_0)}$.

For an arbitrary choice of the metric g we have $g(n_P, v_{z,P}) > 0$ for all $(z, P) \in \mathcal{M}$. The problem with inequality

$$(\nabla_{v_{(z,P)}} g)(n_P, n_P) \leq Cg(n_P, v_{(z,P)})$$

arises only as the point z approaches ∂U_1 . Indeed, in this case the vector $v_{(z,P)}$ can be very close to the tangent vector to $\partial P_z \subset T_z B$.

Lemma 6 *With the above choice of v assume that the metric satisfies for any $z \in \partial U_1$ the condition*

$$(\nabla_{e_z} g)(n_z, n_z) = 0 ,$$

where $e_z \in T_z B$ is the unit tangent vector to ∂U_1 and n_z is the normal vector to ∂U_1 (all scalar products and lengths are taken with respect to the metric g).

Then there exists $C > 0$ such that

$$(\nabla_{v_{(z,P)}} g)(n_P, n_P) \leq Cg(n_P, v_{(z,P)})$$

for all $(z, P) \in \mathcal{M}$.

Proof. We need to check that the ratio

$$\frac{(\nabla_{v_{(z,P)}} g)(n_P, n_P)}{g(n_P, v_{(z,P)})}$$

is bounded for $(z, P) \in \mathcal{M}$.

It suffices to prove the Lemma assuming that U_1 is the parabolic domain $\{(x, y) \in \mathbf{R}^2 | y > x^2\}$ and P is the upper half-plane. The vector field $v_{(z,P)}$ is given for $z = (x, y)$ by the formulas

$$v_{(z,P)} = \partial/\partial y + \frac{x}{4x^2 - 2y} \partial/\partial x .$$

The denominator is equal to $g(n_P, v_{(z,P)}) = \langle dy, v_{(z,P)} \rangle \cdot \sqrt{g(\partial/\partial y, \partial/\partial y)} = \sqrt{g(\partial/\partial y, \partial/\partial y)} = \exp(O(1))$ near $(0, 0)$.

The numerator is equal to

$$\frac{x}{4x^2 - 2y} f_1(x, y) + f_2(x, y) ,$$

where $f_1(x, y) = (\nabla_{\partial/\partial x} g)(n_P, n_P)$ and $f_2(x, y) = (\nabla_{\partial/\partial y} g)(n_P, n_P)$ are two C^∞ -functions.

By assumption of the Lemma we have $f_1(0, 0) = 0$. Therefore $|f_1(x, y)| \leq \text{const} \max\{|x|, |w|\}$ where $w = x^2 - y$ is a convenient local coordinate near the point $(0, 0)$. Notice also that $f_2(x, y) = O(1)$.

Now we can estimate first summand of the numerator assuming that $|x|$ and $|w|$ are sufficiently small. As we have seen, it is bounded by

$$I := \frac{x}{x^2 + w} O(\max\{|x|, |w|\}) .$$

There are three cases which we need to consider.

- a) If $0 < w < x^2$ then $I = \frac{x}{x^2} O(|x|) = O(1)$.
- b) if $x^2 \leq w < x$ then $I = \frac{x}{w} O(|x|) = O(1)$.
- c) If $x \leq w \leq 1$ the $I = \frac{x}{w} O(|w|) = O(1)$.

We see that the numerator is bounded. This concludes the proof of Lemma. ■

Finally, we have the following result.

Lemma 7 *There exists metric g satisfying the conditions of Lemma 6.*

Proof: First of all, the condition on g from Lemma 6 is the condition on a loop $g|_{T_z B}$ of scalar products on 2-dimensional spaces, here $z \in \partial U_1 \simeq S^1$. We can write $g = \exp(\psi)g_0$ where $\det(g_0) = 1$ and ψ is a smooth function. Then we have

$$\nabla_{e_z}(\exp \psi g_0) = \exp(\psi)\nabla_{e_z}g_0 + \exp(\psi)\partial_{e_z}(\psi)g_0 .$$

The equation of Lemma 6 gives $\partial_{e_z}\psi = -(\nabla_{e_z}g_0)(n_z, n_z)/g_0(n_z, n_z)$. The RHS of this expression is known as long as we know g_0 . Hence we can say that $d\psi = \beta_{g_0}$, where β_{g_0} is a 1-form depending on the restriction $(g_0)|_{\partial U_1}$. We see that it suffices to find such g_0 that $\int_{\partial U_1} \beta_{g_0} = 0$ (then ψ and hence g does exist).

Let us consider the functional $I(g_0) = \int_{S^1} \beta_{g_0}$. We can interpret a metric g_0 as a point in the Lobachevsky plane $\mathcal{H} = SL(2, \mathbf{R})/SO(2)$. More precisely, let us consider the space S of pairs (g_0, P) where g_0 is a positive quadratic form on \mathbf{R}^2 such that $\det(g_0) = 1$ and P is a half-plane in \mathbf{R}^2 (the meaning of P is the inward oriented tangent half-plane to ∂U_1 at point $z \in \partial U_1$). This space is naturally diffeomorphic to $S^*(\mathbf{R}^2) \times \mathcal{H}$. The latter manifold can be identified in $SL(2, \mathbf{R})$ -equivariant way with the manifold consisting of pairs (x, y) , where $x \in \mathcal{H}$ and y belongs to the absolute. Hence $(g_0)|_{\partial U_1}$ is (locally) a non-parametrized path in S (it would be a global path, if the bundle over S^1 given by the all metrics on S^1 with the determinant 1 was trivial).

Next we observe that the variation $\delta I(g_0) = \int_N \omega$, where N is a 2-dimensional surface bounded by the paths defined by g_0 and $g_0 + \delta g_0$, and ω is a canonical $SL(2, \mathbf{R})$ -invariant 2-form on S . One can show that even by a small variation of the path defined by g_0 we can make $I(g_0)$ an arbitrary real number. In particular, we can find g_0 such that $I(g_0) = 0$. This concludes the proof of Lemma 6. ■

Summarizing, we have constructed a set of lines satisfying the Assumptions **A1** and **A2**. This concludes the proof of Theorem 5. Thus we have obtained a solution of the Lifting Problem, which is a K -analytic K3 surface.

11.6 Independence and uniqueness

It is natural to ask how the above construction of the K -analytic K3 surface (X^{an}, Ω) depends on the choice of the set \mathcal{L} of lines. We know that the ‘‘periods’’ of Ω (they are encoded in the initial K -affine structure) do not depend on \mathcal{L} (see Sections 7.3, 10.4). In the light of Torelli theorem (see Appendix B) it is natural to formulate the following conjecture.

Conjecture 11 *The isomorphism class of the pair (X^{an}, Ω) does not depend on the choice of the set \mathcal{L} of lines.*

More precisely, the change of \mathcal{L} corresponds to the change of the projection $\pi := \pi_{\mathcal{L}} : X^{an} \rightarrow B$ (see Section 7.3).

Remark 4 *For $B = S^2$ and $B^{sing} = \{x_1, \dots, x_{24}\}$ with the standard singular \mathbf{Z} -affine structure we have constructed a K -analytic K3 surface depending on 20 parameters in K^\times . More precisely, we have a 20-dimensional K -analytic space of conjugacy classes of representations*

$$\pi_1(S^2 \setminus B^{sing}) \rightarrow SL(2, \mathbf{Z}) \ltimes (K^\times)^2$$

such that the monodromy around each singular point is conjugate to the pair $(A, (1, 1))$ where $A \in SL(2, \mathbf{Z})$ is equal to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(compare with Section 3.3).

11.7 Remark on the case of positive and mixed characteristic

Our construction of (X^{an}, Ω) works even without the assumption $char k = 0$ where k is the residue field of K . This can be explained from the point of view of factorization theorem (see Section 10.4). It turns out that symplectomorphisms which appear in the infinite product in the RHS of the factorization theorem are infinite series whose coefficients are integer polynomials in the coefficients of the “parent” symplectomorphisms.

For example, let $f_0(z) = 1 + \sum_{n \geq 1} c_n z^n$ and $f_\infty(z) = 1 + \sum_{n \geq 1} d_n z^n$ be two power series convergent when $|z| < 1$. Let us consider two symplectomorphisms: $F_0(\xi, \eta) = (\xi, \eta f_0(\xi^{-1}))$ and $F_\infty(\xi, \eta) = (\xi f_\infty(\eta^{-1}), \eta)$ and decompose $F_\infty \circ F_0$ into the infinite ordered product $\prod_{\rightarrow} (F_\lambda)$. Here

$$F_{p/q}(\xi, \eta) = (\xi f_{p/q}(\xi^{-p} \eta^{-q})^q, \eta f_{p/q}(\xi^{-p} \eta^{-q})^{-p})$$

where $f_{p/q}(z) = 1 + \sum_{n \geq 1} c_n^{p/q} z^n$. Then one can check that for any coprime $p, q \in \mathbf{Z}_{>0}$ and any $n \geq 1$ one has

$$c_n^{p/q} \in \mathbf{Z}[c_1, c_2, \dots, d_1, d_2, \dots].$$

This implies that our construction works when one replaces K by arbitrary commutative ring R endowed with a complete non-trivial valuation $val : R \rightarrow (-\infty, +\infty]$.

11.8 Further generalizations

First of all, one can introduce a small parameter $\hbar \in K, |\hbar| < 1$ of non-commutativity in the picture, coordinates ξ, η will not commute but instead satisfy the relation

$$\eta\xi = \xi\eta \exp(\hbar) .$$

For such a noncommutative analytic torus one can still define sheaf $\mathcal{O}_{\hbar}^{can}$ on \mathbf{R}^2 by the “same” formula as in the commutative case:

$$\mathcal{O}_{\hbar}^{can}(U) = \left\{ \sum_{n,m \in \mathbf{Z}} c_{n,m} \xi^n \eta^m \mid \forall (x,y) \in U \sup_{n,m} (\log |c_{n,m}| + nx + my) < \infty \right\}$$

where $U \subset \mathbf{R}^2$ is connected. Also one can construct a non-commutative deformation of the model sheaf near the singular point. All arguments with the groups work as well. In this way we will obtain a kind of quantized K3 surface over a non-archimeden field.

Secondly, we believe that one can generalize our construction to higher dimensions. Instead of lines there will be codimension one walls which should be flat hypersurfaces with respect to \mathbf{Z} -affine structure and carry foliations by parallel lines. Generically on the intersection of two such foliated hypersurfaces one can “separate” variables into the product of a purely 2-dimensional situation studied in the present paper, and $n - 2$ dummy variables. Presumably everywhere except a countable union of codimension 2 subsets one can use 2-dimensional factorization and define gluing volume preserving maps. One can hope that by a kind of Hartogs principle the sheaf will have a canonical extension to the whole space B .

A Analytic geometry

In this section we collect several facts and definitions about rigid analytic spaces and Clemens polytopes. Some of them are well-known, the rest is borrowed from [KoT].

We always work over a complete non-archimedean local field K . The field K carries a valuation map $val_K := val : K \rightarrow \mathbf{R} \cup \{+\infty\}$ such that $val(0) = +\infty$, $val(1) = 0$, $val(xy) = val(x) + val(y)$, $val(x + y) \geq \min(val(x), val(y))$.

We will assume that the valuation is non-trivial. The ring

$$\mathcal{O}_K = val_K^{-1}(\mathbf{R}_{\geq 0} \cup \{+\infty\})$$

is called the ring of integers of K . The residue field is defined as $k = \mathcal{O}_K/m_K$, where $m_K = val_K^{-1}(\mathbf{R}_{>0} \cup \{+\infty\})$ is the maximal ideal in \mathcal{O}_K .

Our main example is the field $K = \mathbf{C}((t))$ of Laurent series in one variable. In this case $val_K(\sum_{n \geq n_0} c_n t^n) = n_0$, as long as $c_{n_0} \neq 0$.

A.1 Berkovich spectrum

We refer the reader to [Be1] for the general definition of an analytic space and more details. In this Appendix we restrict ourselves to analytic spaces associated with algebraic varieties (although we use the general definition in the paper as well).

Let $R = R/K$ be a commutative unital finitely generated K -algebra. The underlying set of the Berkovich spectrum $Spec^{an}(R) := Spec^{an}(R/K)$ can be defined in two ways. First one uses valuations (or, equivalently, multiplicative seminorms).

Definition 15 (*Valuations*) *A point x of $Spec^{an}(R/K)$ is an additive valuation*

$$val_x : R \rightarrow \mathbf{R} \cup \{+\infty\}$$

extending $val := val_K$, i.e. it is a map satisfying the conditions

- $val_x(r + r') \leq \max(val_x(r), val_x(r'))$;
- $val_x(rr') = val_x(r) + val_x(r')$;
- $val_x(\lambda) = val_K(\lambda)$

for all $r, r' \in R$ and all $\lambda \in K$.

Having a valuation and a real number $q_0 \in (0, 1)$ one can define the *multiplicative seminorm* $|a| = q_0^{val_K(a)}$, $a \in R$. In particular, in the previous definition one can take seminorms $|\cdot|_x$ instead of valuations $val_x(\cdot)$. The

reader has noticed that in the main body of the paper, for $R = K$ we often took $|a| = e^{-val(a)}$. It is easy to translate the definition of Berkovich spectrum to the language of multiplicative seminorms. We use it freely in the paper.

The second way to define X^{an} uses evaluations (characters).

Definition 16 (*Evaluation maps*) *A point x of $Spec^{an}(R/K)$ is an equivalence class of homomorphisms of K -algebras*

$$eval_x : R \rightarrow K_x ,$$

where $K_x \supset K$ is a complete field equipped with a non-archimedean valuation, which extends the valuation val_K , and such that K_x is generated by the closure of the image of $eval_x$.

The field K_x is determined by $x \in X^{an}$ in a canonical way. We define for $r \in R$ and $x \in X^{an}$ the “value” $r(x) \in K_x$ as the image $eval_x(r)$.

In order to pass from the first description of $Spec^{an}(R/K)$ to the second, starting with a valuation val_x one defines the field K_x as the completion of the field of fractions of R/I_x , where $I_x = (val_x)^{-1}(\{+\infty\})$.

Definition 17 *The topology on $Spec^{an}(R/K)$ is the weakest topology such that for all $r \in R$ the map*

$$\begin{array}{ccc} Spec^{an}(R/K) & \rightarrow & \mathbf{R} \cup \{+\infty\}, \\ x & \mapsto & val_x(r) \end{array}$$

is continuous.

An element $f \in R$ defines a function $f : Spec^{an}(R) \rightarrow K_x$, where K_x is the non-archimedean valuation field, which is the completion of the field of fractions of the domain $R/\ker(val_x)$. Since each K_x carries a seminorm, we obtain a function $|f| : Spec^{an} \rightarrow \mathbf{R}_{\geq 0}$, $x \mapsto |f(x)|$.

A fundamental system of neighborhoods $U = U_x \subset Spec^{an}(R)$ of a point x is parametrized by the following data: a finite collections of functions

$$(f_i)_{i \in I}, (g_j)_{j \in J} \in R$$

and numbers

$$\beta_i^+, \beta_i^-, \gamma_j \in \mathbf{R}_{>0}$$

such that $\beta_i^- < |f_i(x)| < \beta_i^+$, $|g_j(x)| = 0$, The corresponding neighborhood consists of points x' such that $\beta_i^- < |f_i(x')| < \beta_i^+$, $|g_j(x')| < \gamma_j$ for all $i \in I, j \in J$ and $x' \in U$.

Let us assume that elements $(f_i)_{i \in I}, (g_j)_{j \in J}$ generate R , i.e.

$$R = K[(f_i)_{i \in I}, (g_j)_{j \in J}]/I$$

where I is an ideal. Let us consider the algebra of series

$$s = \sum_{n_I \in \mathbf{Z}^I, m_J \in \mathbf{N}^J} c_{I,J} f_I^{n_I} g_J^{m_J}$$

with constants $c_{I,J} \in K$, absolutely convergent when variables $(f_i)_{i \in I}, (g_j)_{j \in J}$ satisfy the above inequalities. The quotient of this algebra by the topological closure of ideal I is the algebra $\mathcal{O}_{\text{Spec}^{an}(R/K)}(U)$.

As in the case of schemes we can glue $\text{Spec}^{an}(R/K)$ into ringed spaces called *analytic spaces* (or rigid analytic spaces). Moreover we get a functor

$$\begin{array}{ccc} (\text{Schemes}/K) & \rightarrow & (K - \text{analytic spaces}) \\ X & \mapsto & (X^{an}, \mathcal{O}_{X^{an}}). \end{array}$$

Proposition 8 *The space X^{an}*

- a) *is a locally compact Hausdorff space as long as X is separated;*
- b) *has the homotopy type of a finite CW-complex;*
- c) *is contractible if X has good reduction with irreducible special fiber.*

Example 1 *Let $X = \mathbf{A}^1 = \text{Spec}(K[x])$ be the affine line. The analytic space X^{an} contains, among others, points of the following types:*

- $X(K) \hookrightarrow X(\overline{K})/\text{Gal}(\overline{K}/K) \hookrightarrow X^{an}$;
- for $r \in \mathbf{R}_{\geq 0}$ define

$$\left| \sum_{j=0}^d c_j z^j \right|_r := \max_j (|c_j| r^j) .$$

This gives an embedding $\mathbf{R}_{\geq 0} \hookrightarrow X^{an}$.

We see that X^{an} contains, in a sense, both p -adic and real points.

Define the *cone* over X^{an} as

$$C_{X^{an}}(\mathbf{R}) := X^{an} \times \mathbf{R}_{>0} .$$

We interpret a point $\mathbf{x} = (x, \lambda)$ of $C_{X^{an}}(\mathbf{R})$ as a K_x -point of X , where $K_x \supset K$ is a complete field with the \mathbf{R} -valued valuation

$$val_{\mathbf{x}} := \lambda val_x ,$$

whose restriction to K is proportional to val_K . The set of points $\mathbf{x} \in C_{X^{an}}(\mathbf{R})$ such that the valuation $val_{\mathbf{x}}$ is \mathbf{Z} -valued is denoted by $C_{X^{an}}(\mathbf{Z})$.

A.2 Algebraic torus and the logarithmic map

Here we will describe explicitly the main example for our paper. Let $X = \mathbf{G}_m^n = Spec(K[z_i^{\pm 1}])$, $1 \leq i \leq n$ be an algebraic torus. and $X^{an} = (\mathbf{G}_m^{an})^n$ the corresponding analytic space.

Firstly, we define an embedding $i_{can} : \mathbf{R}^n \hookrightarrow X^{an}$. For real vector $(x_i)_{1 \leq i \leq n} \in \mathbf{R}^n$ the corresponding point $p := i_{can}(x_1, \dots, x_n) \in X^{an}$ will be described in terms of valuations.

For every Laurent polynomial $f = \sum_{I \in \mathbf{Z}^n} c_I z^I$, $c_I \in K$ we set

$$val_p(f) := \min_{I \in \mathbf{Z}^n} \left(val(c_I) - \sum_{i=1}^n x_i I_i \right) .$$

Secondly, we define a projection $\pi_{can} : X^{an} \rightarrow \mathbf{R}^n$ by formula

$$\pi_{can}(y) = (-val_y(z_1), \dots, -val_y(z_n)) = (\log |z_1|_y, \dots, \log |z_n|_y) .$$

The fiber over a point $(x_1, \dots, x_n) \in \mathbf{R}^n$ can be identified with the set of such seminorms $|\cdot|_y$ that $|z_i|_y = \exp(x_i)$, $1 \leq i \leq n$. We see π_{can} is a kind of torus fibration⁷. Moreover, $\pi_{can} \circ i_{can} = id_{\mathbf{R}^n}$.

For any open connected $U \in (\mathbf{R}^n)$ the K -algebra of analytic functions on $\pi_{can}^{-1}(U)$ consists of series $f = \sum_{I \in \mathbf{Z}^n} c_I z^I$ with coefficients $c_I \in K$ such that for any $p = (x_1, \dots, x_n) \in U$ we have $\log |c_I| + \sum_{i=1}^n x_i I_i \rightarrow +\infty$ when $|I| \rightarrow +\infty$. It is easy to see that $\pi_{can}^{-1}(U) = \pi_{can}^{-1}(Conv(U))$ where $Conv(U)$ is the convex hull of U .

The sheaf $(\pi_{can})_*(\mathcal{O}_{X^{an}}) := \mathcal{O}_{\mathbf{R}^n}^{can}$ (canonical sheaf) plays an important role in the paper (see Sections 4.1, 7.3, 8).

⁷This is the origin of the term “analytic torus fibration” introduced in Section 4.1.

A.3 Clemens polytopes

Let X be a smooth proper scheme over the non-archimedean field K . We assume that K carries a discrete valuation val such that $val(K^\times) = \mathbf{Z}$.

Definition 18 *A model of X is a scheme of finite type $\mathcal{X}/\mathcal{O}_K$ flat and proper over \mathcal{O}_K , together with an isomorphism $\mathcal{X} \times_{Spec(K)} Spec(\mathcal{O}_K) \simeq X$. Denote the special fiber of \mathcal{X} by*

$$\mathcal{X}^0 := \mathcal{X} \times_{Spec(K)} Spec(k) .$$

A model has no nontrivial automorphisms. Thus, the stack of equivalence classes of models is in fact a set, which we denote by Mod_X . It carries a natural partial order. Namely, we say that $\mathcal{X}_1 \geq \mathcal{X}_2$ if there exists a map $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ over $Spec(\mathcal{O}_K)$. Such a map is automatically unique.

Definition 19 *A model \mathcal{X} has normal crossings if the scheme \mathcal{X} is regular and the reduced subscheme \mathcal{X}_{red}^0 is a divisor with normal crossings.*

By the resolution of singularities, in the case $char\ k = 0$ we know that every model is dominated by a model with normal crossings.

Definition 20 *A model \mathcal{X} has simple normal crossings (snc model for short) if*

- *it has normal crossings;*
- *all irreducible components of \mathcal{X}_{red}^0 are smooth and*
- *all intersections of irreducible components of \mathcal{X}_{red}^0 are either empty or irreducible.*

The set of equivalence classes of snc models will be denoted by Mod_X^{snc} . It is a filtered partially ordered set. The order is given by dominating maps of models which give the identity automorphism on the generic fiber.

It is easy to show that starting with any model with normal crossings and applying blow-ups centered at certain self-intersection loci of the special fiber we can get a snc model. In what follows we use snc models only. This choice is dictated by convenience and not by necessity. Working with snc models has the advantage that all definitions and calculations can be made very

transparent. The reader can consult [Be2] for the approach in the general case, without the use of the resolution of singularities.

Let \mathcal{X} be an snc model and $I = I_{\mathcal{X}}$ the set of irreducible components of \mathcal{X}_{red}^0 . Denote by $D_i \subset \mathcal{X}$ the divisor corresponding to $i \in I$. For any finite non-empty subset $J \subset I$ put

$$D_J := \bigcap_{j \in J} D_j .$$

By the snc property the set D_J is either empty or is a smooth connected proper variety over k of dimension $\dim(D_J) = (n - |J| + 1)$. For a divisor $D_i \subset \mathcal{X}^0$ we denote by $d_i \in \mathbf{Z}_{>0}$ the order of vanishing of u at D_i , where $u \in K$ is a uniformizing element, $val_K(u) = 1$. Equivalently, d_i is the multiplicity of D_i in \mathcal{X}^0 .

Definition 21 *The Clemens polytope $S_{\mathcal{X}}$ is the finite simplicial subcomplex of the simplex Δ^I such that Δ^J is a face of $S_{\mathcal{X}}$ iff $D_J \neq \emptyset$.*

Clearly, $S_{\mathcal{X}}$ is a nonempty connected CW-complex. We will also consider the cone over $S_{\mathcal{X}}$:

$$C_{\mathcal{X}}(\mathbf{R}) := \left\{ \sum_{i \in I} a_i \langle D_i \rangle \mid a_i \in \mathbf{R}_{\geq 0}, \bigcap_{i: a_i > 0} D_i \neq \emptyset \right\} \setminus \{0\} \subset \mathbf{R}^I .$$

Analogously, we can define $C_{\mathcal{X}}(\mathbf{Z})$.

We identify $S_{\mathcal{X}}$ with the following subset of $C_{\mathcal{X}}(\mathbf{R})$:

$$\left\{ \sum_{i \in I} a_i \langle D_i \rangle \in C_{\mathcal{X}}(\mathbf{R}) \mid \sum_i a_i d_i = 1 \right\} .$$

Obviously, we can also describe $S_{\mathcal{X}}$ as a quotient of $C_{\mathcal{X}}(\mathbf{R})$:

$$S_{\mathcal{X}} = C_{\mathcal{X}}(\mathbf{R}) / \mathbf{R}_+^{\times} .$$

A.4 Simple blow-ups

Let \mathcal{X} be an snc model, $J \subset I_{\mathcal{X}}$ a non-empty subset and $Y \subset D_J$ a smooth irreducible variety of dimension less or equal than n . Let us assume that Y intersects transversally (in D_J) all subvarieties $D_{J'}$ of D_J (for $J' \supset J$), and that all intersections $Y \cap D_J$ are either empty or irreducible. It is obvious that the blow-up $\mathcal{X}' := Bl_Y(\mathcal{X})$ of \mathcal{X} with the center at Y is again a snc model.

Definition 22 For a pair of snc models $\mathcal{X}' \geq \mathcal{X}$ as above we say that \mathcal{X}' is obtained from \mathcal{X} by a simple blow-up. If $Y = D_J$ we say that we have a simple blow-up of the first type. Otherwise (when $\dim(Y) < \dim(D_J)$), we have a simple blow-up of the second type.

Let us describe the behavior of $S_{\mathcal{X}}$ under simple blow-ups. To the set of vertices we add a new vertex corresponding to the divisor \tilde{Y} obtained from Y :

$$I_{\mathcal{X}'} = I_{\mathcal{X}} \sqcup \{\text{new}\}, \quad D_{\text{new}} := \tilde{Y} .$$

The degree of the new divisor is (for both the first and the second type)

$$d_{\text{new}} := \sum_{i \in J} d_j .$$

For blow-ups of the first type we have automatically $\#J > 1$. Here is the list of faces of $S_{\mathcal{X}'}$:

- 1) I' for $I' \in \text{Faces}(S_{\mathcal{X}})$, $I' \not\subset J$;
- 2) $I' \sqcup \{\text{new}\}$ for $I' \in \text{Faces}(S_{\mathcal{X}})$, $I' \neq J$, $I' \cup J \in \text{Faces}(S_{\mathcal{X}})$;
- 3) the vertex $\{\text{new}\}$.

For blow-ups of the second type the list of faces of $S_{\mathcal{X}'}$ is

- 1) I' for $I' \in \text{Faces}(S_{\mathcal{X}})$;
- 2) $I' \sqcup \{\text{new}\}$ for $I' \in \text{Faces}(S_{\mathcal{X}})$, $I' \supset J$, $Y \cap D_{I'} \neq \emptyset$;
- 3) the vertex $\{\text{new}\}$.

One can deduce from results [AKMW] the following

Theorem 9 (*Weak factorization*) Assume that $\text{char } k = 0$. Then for any two snc models $\mathcal{X}, \mathcal{X}'$ there exists a finite alternating sequence of simple blow-ups

$$\mathcal{X} < \mathcal{X}_1 > \mathcal{X}_2 < \cdots < \mathcal{X}_{2m+1} > \mathcal{X}' .$$

Corollary 3 Simple homotopy type of $S_{\mathcal{X}}$ does not depend on the choice of a snc model \mathcal{X} .

A.5 Clemens cones and valuations

Let \mathcal{X} be a snc model of X . We define a map

$$i_{\mathcal{X}} : C_{\mathcal{X}}(\mathbf{R}) \rightarrow C_{X^{\text{an}}}(\mathbf{R})$$

such as follows. For $J = \{j_1, \dots, j_k\} \subset I_{\mathcal{X}}$ such that $D_J \neq \emptyset$ let us consider a point $x \in C_{\mathcal{X}}(\mathbf{R})$

$$x = \sum_{i=1}^k a_i \langle D_{j_i} \rangle, \quad a_i \in \mathbf{R}_{>0} \quad \forall i \in \{1, \dots, k\}$$

and an affine Zariski open subset $U \subset \mathcal{X}$ containing the generic point of D_J . One can embed $\mathcal{O}(U)$ into the algebra of formal series $K_J[[z_1, \dots, z_k]]$ where K_J is the field of rational functions on D_J and $z_i = 0$ are equations of divisors D_{j_i} , $i = 1, \dots, k$. We define a valuation v_x of $\mathcal{O}(U)$ by the formula

$$v_x \left(\sum_{n_1, \dots, n_k \geq 0} c_{n_1, \dots, n_k} \prod_{i=1}^k z_i^{n_i} \right) = \inf \left\{ \sum a_i n_i \mid c_{n_1, \dots, n_k} \neq 0 \right\} .$$

We define $i_{\mathcal{X}}(x)$ to be the image of the point $v_x \in \text{Spec}^{an}(\mathcal{O}(U)/K)$ in X^{an} . It is easy to check that the element $i_{\mathcal{X}}(x)$ does not depend on the choice of the open subset U .

The following proposition is obvious:

Proposition 9 *The map $i_{\mathcal{X}}^{\mathbf{R}}$ is an embedding.*

We will denote also by $i_{\mathcal{X}}$ the induced embedding $S_{\mathcal{X}} \hookrightarrow X^{an}$.

A.6 Clemens cones and paths

For a model \mathcal{X} we can interpret elements of $C_{X^{an}}(\mathbf{Z})$ as *paths* in \mathcal{X} , i.e. equivalence classes of maps

$$\phi : \text{Spec}(\mathcal{O}_L) \rightarrow \mathcal{X},$$

where \mathcal{O}_L is the ring of integers in a field L with discrete valuation in \mathbf{Z} , such that the image of ϕ does not lie in \mathcal{X} . We define the map

$$p_{\mathcal{X}}^{\mathbf{Z}} : C_{X^{an}}(\mathbf{Z}) \rightarrow C_{\mathcal{X}}(\mathbf{Z})$$

as

$$p_{\mathcal{X}}^{\mathbf{Z}}([\phi]) := \sum_i a_i \langle D_i \rangle,$$

where $a_i \in \mathbf{Z}_{\geq 0}$ is the multiplicity of the intersection of the path ϕ with the divisor D_i , $i \in I_{\mathcal{X}}$.

The following proposition can be derived from [Be1].

Proposition 10 *The map $p_{\mathcal{X}}^{\mathbf{Z}}$ extends uniquely to a continuous \mathbf{R}_+^{\times} -equivariant map $p_{\mathcal{X}}^{\mathbf{R}} : C_{X^{an}}(\mathbf{R}) \rightarrow C_{\mathcal{X}}(\mathbf{R})$. The map $p_{\mathcal{X}}^{\mathbf{R}}$ is a surjection.*

We denote by $p_{\mathcal{X}} : X^{an} \rightarrow S_{\mathcal{X}}$ the map induced by $p_{\mathcal{X}}^{\mathbf{R}}$.

Let $f : \mathcal{X}' \rightarrow \mathcal{X}$ be a dominating map of models. Let us denote by $m_{i,i'} \in \mathbf{Z}_{\geq 0}$ the multiplicity of a divisor $D_{i'}$, $i' \in I_{\mathcal{X}'}$ in the proper pull-back of D_i , $i \in I_{\mathcal{X}}$. We define $p_{\mathcal{X}',\mathcal{X}}^{\mathbf{Z}} : C_{X^{an}}(\mathbf{Z}) \rightarrow C_{\mathcal{X}}(\mathbf{Z})$ by the formulas $\sum_{i'} a_{i'} \langle D_{i'} \rangle \mapsto \sum_i m_{i,i'} a_{i'} \langle D_i \rangle$. Let $p_{\mathcal{X}',\mathcal{X}} : S_{X^{an}}(\mathbf{R}) \rightarrow S_{\mathcal{X}}(\mathbf{R})$ be the corresponding by map of Clemens polytopes.

Then we have the following result, which is easy to prove.

Lemma 8 *For any dominating map of models $\mathcal{X}' \rightarrow \mathcal{X}$ we have*

$$p_{\mathcal{X}}^{\mathbf{Z}} = p_{\mathcal{X}',\mathcal{X}}^{\mathbf{Z}} \circ p_{\mathcal{X}'}^{\mathbf{Z}} .$$

Corollary 4 *For dominating maps $\mathcal{X}'' \geq \mathcal{X}' \geq \mathcal{X}$ we have*

$$p_{\mathcal{X}'',\mathcal{X}} = p_{\mathcal{X}',\mathcal{X}} \circ p_{\mathcal{X}'',\mathcal{X}'} .$$

Theorem 10 *For any algebraic X the analytic space X^{an} is a projective limit over the partially ordered set of snc models \mathcal{X} of Clemens polytopes $S_{\mathcal{X}}$. The connecting maps are $p_{\mathcal{X}',\mathcal{X}}$.*

With any meromorphic at $t = 0$ family of smooth complex projective varieties X_t , $0 < |t| < \epsilon$ one can associate a variety X over the field $\mathbf{C}((t))$. It is easy to see that for any snc model \mathcal{X} one can canonically complete the family X_t by adding $S_{\mathcal{X}}$ as the fiber over $t = 0$. The total space is not a complex manifold by just a Hausdorff locally compact space which maps properly to the dick $\{t \in \mathbf{C} \mid |t| < \epsilon\}$. Passing to the projective limit we see that one can compactify the family X_t at $t = 0$ by X^{an} .

B Torelli theorem for K3 surfaces

Here we recall the classification theory of complex K3 surfaces (see [PSS] and its extension to non-algebraic case in [LP]). Let X be a complex K3 surface, i.e. smooth connected complex manifold with $\dim_{\mathbf{C}} X = 2$ which admits a nowhere vanishing holomorphic 2-form Ω , and such that $H^1(X, \mathbf{Z}) = 0$.

It is known that the group $H^2(X, \mathbf{Z})$ endowed with the Poincare pairing (\cdot, \cdot) is isomorphic to the lattice

$$\Lambda_{K3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (-E_8) \oplus (-E_8)$$

of signature $(3, 19)$.

Complex 1-dimensional vector space $H^{2,0}(X) = \mathbf{C} \cdot [\Omega] \subset H^2(X, \mathbf{Z}) \otimes \mathbf{C}$ satisfies the condition $(v, v) = 0, (v, \bar{v}) > 0$ for any non-zero vector v . Finally, it is known that X admits a Kähler metric, and Kähler cone $\mathcal{K}_X \subset H^2(X, \mathbf{R})$ of all Kähler metrics on X is an open subset of $C_X := \{[\omega] \in H^2(X, \mathbf{Z}) \otimes \mathbf{R} \mid ([\omega], [\Omega]) = 0, ([\omega], [\omega]) > 0\}$. In fact \mathcal{K}_X is a connected component of the set $C_X \setminus \cup_{v \in H^2(X, \mathbf{Z}), (v, v) = -2, (v, [\Omega]) = 0} H_v$, where H_v is the hyperplane orthogonal to v .

Axiomatizing these data we arrive to the following definition.

Definition 23 *K3 period data is a quadruple $(\Lambda, (\cdot, \cdot), H^{2,0}, \mathcal{K})$ consisting of a free abelian group Λ , a symmetric pairing $(\cdot, \cdot) : \Lambda \times \Lambda \rightarrow \mathbf{Z}$, a 1-dimensional complex vector subspace $H^{2,0} \subset \Lambda \otimes \mathbf{C}$ and a set $\mathcal{K} \subset \Lambda \otimes \mathbf{R}$ satisfying the following conditions:*

1. $rk \Lambda = 22$;
2. $(\Lambda, (\cdot, \cdot))$ is isomorphic to Λ_{K3} ;
3. for any $v \in H^{2,0} \setminus \{0\}$ one has $(v, v) = 0$ and $(v, \bar{v}) > 0$;
4. the set \mathcal{K} is a connected component of $C \setminus \cup_{v \in \Lambda, (v, v) = -2, (v, H^{2,0}) = 0} H_v$, where $C = \{w \in \Lambda \otimes \mathbf{R} \mid (w, H^{2,0}) = 0, (w, w) > 0\}$ and H_v is the hyperplane orthogonal to v .

The K3 period data form a groupoid. On the other hand, K3 surfaces also form a groupoid (morphisms are isomorphisms of K3 surfaces). Then classical global Torelli theorem can be formulated in the following way.

Theorem 11 *Groupoid of K3 surfaces is equivalent to the groupoid of K3 period data.*

In particular the automorphism group of a K3 surface is isomorphic to the automorphism group of its period data.

More generally one can speak about holomorphic families of K3 surfaces over complex analytic spaces. For a K3 surface over an analytic space M the period data consist of a local system of integral lattices $(\Lambda, (\cdot, \cdot))$ pointwise isomorphic to Λ_{K3} , a holomorphic line subbundle $H^{2,0}$ of $\Lambda \otimes_{\mathbf{Z}} \mathcal{O}_M$ which is isotropic with respect to the symmetric pairing (\cdot, \cdot) , and satisfies pointwise the condition $(v, \bar{v}) > 0, v \in H_x^{2,0} \setminus \{0\}, x \in M^{red}$, and an open subset of the total space of the bundle over M^{red} with the fibers $\Lambda_x \otimes \mathbf{R} \cap (H^{2,0})^\perp$ ($(H^{2,0})^\perp$ is the orthogonal complement) satisfying pointwise the condition 4) from the definition of K3 period data. Then Torelli theorem holds for families as well.

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