NON-COMMUTATIVE MOTIVES

Plan:

- Non-commutative algebraic geometry
- Examples of saturated spaces
- Hodge & de Rham cohomology
- NC pure Hodge structures, pure & mixed motives $\mathbb{C}$
- Frobenius isomorphism, Euler factors, L-functions $\mathbb{Z}_p$

1° Basic NC algebraic geometry

$\underbrace{}$ "derived"   started by A. Bondal, ~90'

Definition A **NC-space** X

:= small triangulated category $\mathcal{C}_X$ Karoubi closed ($\Leftrightarrow \forall$ projector splits) & appropriately enriched

by spectra: $\mathrm{Hom}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F}[i]) = \pi_{-i}\, \underline{\mathrm{Hom}}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F})$

or

by complexes of k-linear vector spaces: $\mathrm{Hom}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F}[i]) = H^i(\underline{\mathrm{Hom}}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F}))$

X is **k-linear**, write X/k. k is $\forall$ field

Remark One can define X/R where R is $\forall$ commutative ring; complexes of R-modules should be cofibrant.

Defn $X/_k$ is **algebraic** if $\exists$ dg algebra $A/_k$ s.t.

$C_X \sim \mathrm{Perf}(A\text{-mod})$ (with enrichment

:= closure of one-object full subcategory $\{A\}$ by shifts, cones, direct summands in appropriate triangulated category A-mod.

Thm (Bondal - van den Bergh)

If $X/_k$ is a scheme of finite type then X is algebraic in NC sense

$C_X := \mathrm{Perf}(X)$ perfect complexes of quasi-coherent sheaves

has a split-generator $\mathcal{E}$ $A = R\mathrm{Hom}(\mathcal{E}, \mathcal{E})^{op}$

Example (A. Beilinson)

$X = \mathbb{P}^n_k$ $A = \mathrm{End}(\mathcal{O}(0) \oplus \ldots \oplus \mathcal{O}(n))^{op}$

$D^b(\mathrm{Coh}\, X) = \mathrm{Perf}(X) = D^b(A\text{-mod}^{\text{fin.gen.}}) = \mathrm{Perf}(A\text{-mod})$

Defn Algebraic NC space $X/_k$ is

A) **proper** if $\boxed{\sum_{i\in\mathbb{Z}} \operatorname{rk} H^i(A,d) < +\infty}$

B) **smooth** if $\boxed{A \in \operatorname{Perf}(A\otimes A^{op}\text{-mod})}$

Thm These notions do not depend on the choice of A (= of generator), coincide with the usual $\begin{cases}\text{properness}\\ \text{smoothness}\end{cases}$ for schemes of finite type.

Examples of algebras A (in degree $=0$) s.t. "Spec A" : $\mathcal{C}_{\text{"Spec}A\text{"}} = \operatorname{Perf}(A\text{-mod})$ is smooth

$\mathcal{O}_X$ $\quad$ X smooth affine scheme $/_k$

$T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$, $\operatorname{rk} V < \infty$ $\quad$ free fin. gen. algebra

$U_q\mathfrak{g}$ $\quad$ Drinfeld-Jimbo envelopping algebra of a quantum group

Finiteness for sheaves:

$X/_k$ is <u>proper</u> $\Rightarrow$

$\forall\ \mathcal{E}, \mathcal{F} \in \mathcal{C}_X \quad \sum_{i\in\mathbb{Z}} \mathrm{rk}\ \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i]) < +\infty$

gives bilinear form $\chi_{\mathrm{RHom}} : K_0(\mathcal{C}_X) \otimes K_0(\mathcal{C}_X) \to \mathbb{Z}$

neither symmetric nor skew-symmetric

a NC version of Riemann-Roch

$\{\text{Objects in } \mathcal{C}_X / \text{iso}\}$ = k-points in a $\bigsqcup_{\text{countable}}$ formal schemes$/_k$ of finite type / equivalence relation of similar nature

Finiteness ~~for~~ spaces:

$\left\{ \text{smooth proper } X/_k \;/\; \text{equivalences of categories } \mathcal{C}_X \simeq \mathcal{C}_{X'} \right\}$ = k-points in

. . .

Defn **Saturated** := smooth & proper

(the name comes from saturated categories of Bondal and Kapranov)

Manipulations with saturated spaces

$X \mapsto X^{op}$ | $C_{X^{op}} = C_X^{op}$, $A_{X^{op}} = A_X^{op}$

$X, Y \mapsto X \otimes Y$ | $A_{X\otimes Y} = A_X \otimes_k A_Y$

$X, Y \mapsto \underline{Maps}(X,Y) := X^{op} \otimes Y$ | $C_{\underline{Maps}(X,Y)} =$ Functors $C_X \to C_Y$

glueing
$f: X \to Y \mapsto (X \xrightarrow{f} Y)$ | f is given by $M \in A_Y\text{-mod-}A_X$

$$A_{(X \xrightarrow{f} Y)} = \begin{pmatrix} A_X & 0 \\ M & A_Y \end{pmatrix}$$

Glueing is analogous to cones in Δ-d categories
$\mathbb{P}^n_k$ is glued iteratively from $(n+1)$ points.
Braid group acts on {iterated glueings}.

Duality theory

$\forall$ saturated $X \to$ canonical **Serre functor** $S_X \in \underline{Maps}(X, X)$

$$\boxed{Hom(\mathcal{E},\mathcal{F})^* = Hom(\mathcal{F}, S_X(\mathcal{E}))}$$

in schematic case $S_X = K_X[\dim X](\otimes)$.

2° Examples of saturated spaces

smooth proper

- schemes
- algebraic spaces
- Deligne-Mumford stacks
- (X,α) where $X/_k$ scheme

$\left\{\begin{array}{l}\text{locally} \\ \text{cross-products} \\ k[\Gamma]\ltimes \mathcal{O}_X \\ \Gamma: \text{finite group} \\ \text{acting on } X\end{array}\right.$

$\alpha \in Br(X)$ class of an Azumaya algebra $\mathcal{A}/X$

$C_{(X,\alpha)} := Perf(\mathcal{A}\text{-mod})$

- deformation quantization of smooth projective variety $X/_k$

$\mathcal{L} = \mathcal{O}(1) \to X$ ample line bundle (char $k=0$)

$\gamma \in \Gamma(\mathcal{L}-X, \Lambda^2 T_{\mathcal{L}})^{\mathbb{G}_m}$ homogeneous Poisson structure

assume $H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$

$\Rightarrow$ quantized space $X_q / k((\hbar))$

$$f * g = fg + \hbar\langle \gamma, df\otimes dg\rangle + \dots$$

$\supset$ Feigin-Odessky "elliptic" projective spaces

quantized del Pezzo surfaces, ...

- Artin-Zhang non-commutative projective spaces.

Landau-Ginzburg models

(name comes from topological B-strings)

<u>Defn</u> A **$\mathbb{Z}/2$-graded** NC space X is $\mathcal{C}_X$ together with iso $[0] \sim [2]$.

All notions $\left\{\begin{array}{l}\text{algebraic}\\ \text{smooth}\\ \text{proper}\end{array}\right.$ extend to $\mathbb{Z}/2\mathbb{Z}$-gr. case.

$$f : X \to \mathbb{A}^1 \qquad f \in \mathcal{O}(X)$$

X : smooth scheme $/k$

$\rightsquigarrow$ $\mathbb{Z}/2$-graded space (X, f)

<u>Locally</u> $\mathcal{C}_{(X,f)} =$ super vector bundles $\mathcal{E} = \mathcal{E}^0 \oplus \mathcal{E}^1$ with "differential" $d_\mathcal{E} \in \mathcal{E}nd(\mathcal{E})^{odd}$ $\boxed{d_\mathcal{E}^2 = f \cdot Id_\mathcal{E}}$

$\underline{Hom}((\mathcal{E}, d_\mathcal{E}), (\mathcal{F}, d_\mathcal{F})) :=$

$Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ with differential $d\varphi = \varphi \circ d_\mathcal{E} - d_\mathcal{F} \circ \varphi$, $d^2 = 0$

<u>Globally</u> (D. Orlov) Assume $f \not\equiv 0$ on $\forall$ component of X.

$$\mathcal{C}_{(X,f)} := D^b(Coh\, Z)/Perf\, Z \; . \; Z := f^{-1}(0)$$

Expect: $C_{(X,f)}$ is saturated if

$X_0 := \mathrm{Crit}(f) \cap f^{-1}(0)$ is <u>proper</u>.

Moreover, X can be a formal smooth thickening of X_0.

Example: f: germ of analytic function in $\mathbb{C}^n$

$f(0)=0$ with isolated singularity.

3° (Co)homology theories

A unital associative algebra $/k$

Defn Homological Hochschild complex

$$C_\cdot(A,A) = \dots \to \overset{-2}{A\otimes A\otimes A} \xrightarrow{\partial} \overset{-1}{A\otimes A} \xrightarrow{\partial} \overset{0}{A}$$

$$\boxed{\partial(a_0\otimes\dots\otimes a_n) = \sum_{i=0}^{n-1}(-1)^i a_0\otimes\dots\otimes a_i a_{i+1}\otimes\dots\otimes a_n + (-1)^n a_n a_0\otimes a_1\otimes\dots\otimes a_{n-1}}$$

$$C_\cdot^{red}(A,A) = \dots \to \overset{-2}{A\otimes A/_1\otimes A/_1} \to \overset{-1}{A\otimes A/_1} \to \overset{0}{A}$$

quotient complex, qis $C_\cdot(A,A)$

Thm (Hochschild-Kostant-Rosenberg)
For $A = \mathcal{O}(X)$, $X/_k$ smooth affine variety

$$\boxed{H^{-i}(C_\cdot(A,A)) = \Omega^i(X/_k)}$$

Thm (Ch. Weibel (in another formulation))
For smooth scheme X/k
if char $k=0$ or char $k > \dim X$

$$\boxed{H^n(C_\cdot(A,A)) = \bigoplus_{i-j=n} H^i(X,\Omega^j)}$$

Defn For $\forall$ algebraic NC space X

Hodge cohomology $H^{\bullet}_{Hodge}(X)$

:= Hochschild homology $H_{\bullet}(A,A)$

$\exists$ intrinsic definition in terms of $\mathcal{C}_X$.

For saturated X

$$H^{\bullet}_{Hodge}(X) = R\mathrm{Hom}_{\underline{Maps}(X,X)}(Id_X, S_X)$$

Defn **Connes' operator B** $\begin{matrix} B^2=0 \\ B\partial+\partial B=0 \end{matrix}$

acting on $C^{red}_{\bullet}(A,A) \circlearrowleft$, of $\deg = -1$, is

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni}\, 1_A \otimes \underbrace{a_i \otimes a_{i+1} \otimes \dots \otimes a_{i-1}}_{\text{cyclic permutation}}$$

In the case $A = \mathcal{O}(X)$, X smooth affine scheme $/k$

B induces $\underline{\text{de Rham differential}}$ on

$$\Omega^{-\bullet}(X) = H^{\bullet}_{Hodge}(X)$$

Everything generalises to dg-algebras & $\mathbb{Z}/2$-graded case

$$C_{\bullet}(A,A) = \bigoplus_{n\geq 0} A \otimes A[1]^{\otimes n} \quad \text{etc.}$$

Defn Periodic cyclic (co)homology for algebraic NC space $X/_k$ is

$$\boxed{HP^\cdot(X) := HP_\cdot(A) := H^\cdot\left(C^{red}_\cdot(A,A)((u)),\ \underset{\partial + uB}{\text{differential}}\right)}$$

$k((u))$-module, u is even variable
$\deg u = +2$ in $\mathbb{Z}$-graded case.

In $\mathbb{Z}$-graded case $HP^i(X) \simeq HP^{i+2}(X)$ $\forall i$
$\leadsto \mathbb{Z}/_2$-graded space$/_k$

$$HP^{even}(X) \oplus HP^{odd}(X)$$

$$\boxed{\begin{aligned} HP^{even}(X) &= \bigoplus H^{2i}_{dR}(X) \\ HP^{odd}(X) &= \bigoplus H^{2i+1}_{dR}(X) \end{aligned}}$$

for smooth scheme $X/_k$
char $k = 0$
or
$> \dim X$

$HP^\cdot(X)$ is a NC analog of de Rham cohomology

Finiteness: for saturated $\frac{\mathbb{Z}}{\mathbb{Z}/2}$-graded X

$$+\infty > \mathrm{rk}_{total}\, H^\cdot_{Hodge}(X)_{/k} \geq \mathrm{rk}\, HP^\cdot(X)_{/k((u))} \geq 0$$

Defn For algebraic NC space $X/_k$ spectral sequence Hodge $\Rightarrow$ de Rham collapses at E_1 if for $\forall\, n \geq 1$ $n < +\infty$

$$H^\cdot(C^{red}_\cdot(A,A)[u]/(u^n),\ \partial + uB)$$

is free (= flat) $k[u]/(u^n)$-module.

$\Leftrightarrow$ (for saturated X) $\mathrm{rk}\, H^\cdot_{\text{Hodge, total}}(X) = \mathrm{rk}\, HP^\cdot(X)/_{k((u))}$

Conjecture For saturated X over field k, char k = 0 Hodge $\Rightarrow$ de Rham collapses.

True for schemes, quantum deformations stacks, Azumaya algebras, (X,f) LG models

2 types of proofs in comm. case — Kähler geometry ???
— finite characteristic $\vee$ p-adic
→ (Deligne-Illusie) (Faltings)

There is a good chance in NC case (later)

Assume Conjecture

$\Rightarrow$ H_u super-vector bundle $/ k[[u]]$

Sections $= H^\cdot(C^{red}_\cdot(A,A)[[u]], \partial + uB)$

∇: Canonical connection on H_u, $u \neq 0$

in $\mathbb{Z}$-graded case : comes from

$\mathbb{G}_m$-equivariance $\lambda \in k^*$ $u \to \lambda^2 u$

monodromy $= \begin{cases} id & \text{on } HP^{even} \\ -id & \text{on } HP^{odd} \end{cases}$

has 1st order pole at $u = 0$

$\Leftrightarrow$ filtration by $\frac{1}{2}\mathbb{Z}$ on $H_{dR}(X)$

$$F_a H^n_{dR}(X) = \bigoplus_{n/2 - p \leq a} F^p H^n_{dR}(X)$$

in the case of schemes

in $\mathbb{Z}/2$-graded case : comes from

Gauß-Manin connection on $HP^\cdot(X_\lambda)$

$(X_\lambda)_{\lambda \in \mathbb{G}_m}$: orbit of RG flow acting on $\{\mathbb{Z}/2\text{-gr. spaces}\}$

$A' = A_\lambda$ $= A$ as space $/k$

$a \cdot' b = ab$ $d'(a) = \lambda\, da$

$(\ ,\ ) : H_u \otimes H_{-u} \to k$ non-degenerate ∇-covariant pairing (not symm. or antisymmetric)

In $\mathbb{Z}/2$-graded case connection ∇ has 2nd order pole at $u=0$ $\left(\begin{smallmatrix}\text{from}\\ \text{explicit}\\ \text{formula}\end{smallmatrix}\right)$ still with regular (?) singularity and with quasi-unipotent (?) monodromy

Example:

LG model (X, f)

$\mathrm{Crit}(f) \cap f^{-1}(0)$ proper

$$\boxed{\begin{array}{l}\Gamma(k[[u]], H_u) = \\ \mathbb{H}^{\cdot}(X_{Zar}, \Omega^{\cdot}_{X/k}[[u]], \overset{\text{differential}}{u \cdot d_{dR} + df\wedge \cdot})\end{array}}$$

$$= \mathbb{H}(X_{Zar}, \text{"} e^{f/u} \Omega^{\cdot}_{X/k}[[u]], u\, d_{dR})$$

Degenaration of Hodge $\Rightarrow$ de Rham

3 proofs:
1) S. Barannikov - M.K. using harmonic theory for $e^{f/u}$
2) C. Sabbah, using M. Saito's Hodge modules
3) V. Vologodsky - A. Ogus a la Deligne-Illusie.

An application of collapse Hodge ⇒ de Rham

Construction

- algebraic B-model
- generalization of Deligne's conjecture on cohom. operations

Input: saturated $\mathbb{Z}/2$-graded NC space X
s.t. 1) Hodge ⇒ de Rham collapses
2) X is even or odd Calabi-Yau
$\exists$ iso $S_X \sim Id_X$ or ΠId_X

\+ some choices
1') trivialization of bundle H_u compatible with pairing $(,)$
2') choice of iso $S_X \sim Id_X, \Pi Id_X$ with "higher homotopies"
⟺ purely cohomological data
$\Gamma(\mathbb{A}^1 \setminus 0, H_u) \to k$ satisfying some non-degeneracy.

Output: Cohomological 2d TQFT in the sense M.K. - Y. Manin

$H := H^{\cdot}_{Hodge}(X)$

$\forall\ g, n \geq 0 \quad 2-2g-n < 0$

$$H^{\otimes n} \to H^{\cdot}_{Betti}(\overline{\mathcal{M}}_{g,n}(\mathbb{C}); k)$$

4° NC pure Hodge structures

$k = \mathbb{C}$

<u>Defn</u> (putative)

A NC pure Hodge structure is

- (H_u) Holomorphic super vector bundle over $\{u \in \mathbb{C} \mid |u| \ll 1\}$
- ∇ flat connection on $u \neq 0$ with 2nd order pole at $u=0$. & with regular singularity
- K_u^{top} local system $/ u \neq 0$ of finitely generated $\mathbb{Z}/2$-graded abelian groups
  + ∇-flat iso of super spaces$/\mathbb{C}$
  $$K_u^{top} \otimes \mathbb{C} \simeq H_u$$

MAIN PROBLEM:

How should we define lattice K_u^{top} ?

Answer is clear in (almost)-commutative examples, e.g. in LG model.

VAGUE IDEA: (in $\mathbb{Z}$-graded case)

$\exists$ (?) another "algebraic" NC space X', with nuclear (?) algebra A'

+ map $\varphi : X' \to X$ such that

1) φ induces iso $HP^{\cdot}(X) \xrightarrow{\sim} HP^{\cdot}(X')$

2) K-theory of X' has Bott periodicity
Bott: $K_i(X') \simeq K_{i+2}(X')$

3) $\forall i \geq 0$ Chern character
$ch : K_i(X') \to HP^i(X')$
induces iso $K_i(X') \otimes \mathbb{C} \simeq HP^i(X')$

E.g. for scheme $X/\mathbb{C}$ (smooth proper) $A' := C^{\infty}_{\mathbb{C}}(X(\mathbb{C}))$

Fact: for any C^∞-manifold X

$$HP^{\cdot}_{cont}(X) \simeq H^{\cdot}_{dR}(X)$$

image of $K_0(X)$ in $HP^{even}_{cont}(X)$

= (up to finite torsion)

$$\bigoplus_{n \in \mathbb{Z}} (2\pi\sqrt{-1})^n \cdot H^{2n}(X, \mathbb{Z})$$

image of $K_1(X)$ in $HP^{odd}_{cont}(X)$

<u>divided</u> by $\sqrt{2\pi\sqrt{-1}}$

$$= /_{finite} \bigoplus_{n \in \frac{1}{2}+\mathbb{Z}} (2\pi\sqrt{-1})^n H^{2n}(X, \mathbb{Z})$$

Hodge Conjecture:

For saturated $\mathbb{Z}/2$ NC space $X/\mathbb{C}$

$\mathbb{Q}\otimes$ Image of $K_0(\mathcal{C}_X)$ in $\Gamma(\mathbb{C}[[u]], K_u)$ by Chern character

$$= \mathbb{Q}\otimes \mathrm{Hom}_{\substack{NC \\ \text{pure H.S.}}}\left(\mathbb{1}, H^{\cdot}(X)\right)$$

here I mean formal bundle K_u <u>canonically</u> extended to $|u|\ll 1$ because of regular singularity + putative lattice K_u^{top}

<u>Thm</u> (L. Katzarkov + M.K.)
For LG model (X, f) <u>follows</u> from the usual Hodge conjecture.

Polarizations

Defn A polarization of a NC pure Hodge structure

$$H = (H_u, \nabla, K_u^{top})$$

at radius $r > 0$, $r \in \mathbb{R}$ is an iso

$$\psi : H \xrightarrow{\simeq} (H^{op})^\vee$$

of NC Hodge structures satisfying certain symmetry & positivity condition.

$(H_u^{op})^\vee = H_{-u}^\vee \ldots$
corresponds to $X \mapsto X^{op}$

Suppose: $\begin{array}{c}\mathcal{H}\\ \downarrow\\ \mathbb{CP}^1\end{array}$ holomorphic vector bundle

+ holom. pairing

$$\psi_{\mathcal{H}} : \mathcal{H}_u \otimes \overline{\mathcal{H}}_{\sigma(u)} \to \mathbb{C}$$

$$\sigma(u) = -\frac{r^2}{\overline{u}}$$

such that 1) $\mathcal{H}$ is holomorphically trivial
$\mathcal{H} = \oplus$ copies of $\mathcal{O}$
2) $\psi_{\mathcal{H}}$ induces positive Hermitean form on $\Gamma(\mathbb{CP}^1, \mathcal{H})$

Such $\mathcal{H}$ can be constructed from H & ψ

$$\mathcal{H}|_{|u|\leq r} \simeq H|_{|u|\leq r}$$

pairing $\psi_{\mathcal{H}}|_{S^1 : |u|=r}$ is given by ψ composed with complex conjugation $u\neq 0 : H_u \to \overline{H}_u$, associated with $\mathbb{R}$-structure $K^{top}_u \otimes \mathbb{R} \subset H_u$

Recent result of C. Sabbah
$\Rightarrow$ In LG model $\exists$ polarization for $\forall$ sufficiently small r.

Conjecture For saturated $X/\mathbb{C}$ polarizations on $H^{\cdot}(X)$ exist.

They should come from certain endofunctors
$$F : X \to X \quad (\text{as } ch(F))$$

presumably F is something like
$$\otimes \mathcal{O}(n) \quad n \gg 1.$$

$\exists$ polarization on $H^{\cdot}(X)$

$\Rightarrow$ Image of $K_0(X)$ in $H^{\cdot}(X)$

$= K_0(X) /$ numerical equivalence

$:=$ Ker left $=$ right by S_X

of canonical pairing

$\chi_{RHom}: K_0(X)^{\otimes 2} \to \mathbb{Z}$

Defn. For $\forall$ field k Category of pure motives $/_k$ is Karoubi envelope of "Effective motives":

Objects: saturated $X/_k$

$$\underline{\mathrm{Hom}}_{EM}(X,Y) = \mathbb{Q} \otimes K_0(\underline{\mathrm{Maps}}(X,Y)) \Big/ \text{numerical equivalence}$$

Previous conjectures $\Rightarrow$

NC pure motives / char $k=0$

: semisimple rigid tensor category

Corollary We get pro-reductive group $/\mathbb{Q}$

NC motivic Galois group such that

$$G^{NC}_{mot} \underset{\neq}{\twoheadrightarrow} \mathrm{Ker}\left(G_{mot} \to GL(1)\right)$$

Tate motive representation

usual (pure) motivic Galois group

$\exists$ Interesting things in kernel:
e.g. G. Anderson "t-motives" $H = \bigoplus H^{p,q}$ $p+q \in \mathbb{Z}$
for $\Gamma(t)$, $t \in \mathbb{Q}$, ... $p,q \in \mathbb{Q}$

Defn Triangulated category of NC mixed motives $/_k$
:= triangulated + Karoubi envelope of Category enriched over spectra:

Objects = saturated $X/_k$

Morphisms spaces

$\underline{\mathrm{Hom}}(X,Y) =$ K-theory spectrum of category $C_{\underline{\mathrm{Maps}}(X,Y)}$

5° Frobenius isomorphism

Conjecture: $\forall$ saturated NC $X/\mathbb{Z}_p$ $\exists$ canonical Frobenius isomorphism

$$\boxed{H^\cdot(C_\cdot^{red}((u)), \partial + uB) \sim H^\cdot(C_\cdot^{red}((u)), \partial + puB)}$$

of $\mathbb{Z}_p((u))$-modules, preserving connection ∇. — maybe sign

Using holonomy of ∇ from u to pu we get operator Fr_p with coefficients in $\mathbb{Q}_p$

not 100% canonical if monodromy $\neq id$

Weil Conjecture: (λ_α) eigenvalues of Fr_p are algebraic $\subset \bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$.
& $\forall\, \ell \neq p$ $|\lambda_\alpha|_\ell = 1$ $|\lambda_\alpha|_{\mathbb{C}} = 1$.

Example: $(X, f) = (\mathbb{A}^1, x^2)$

Frobenius $\leftarrow$ intertwining operator $\cdot \exp\left(f + \frac{f^p}{p}\right)$

$\dim H(X, f) = 1$

$Fr_p = \lambda \in \mathbb{Q}_p^\times$

In fact, $\lambda \in \mathbb{Z}_p^\times$

$\lambda = \cancel{(-1)^{\frac{p-1}{2}}}$ $\left(\frac{p-1}{2}\right)!$ mod $p\mathbb{Z}_p$ — unique

$\lambda^4 = 1$

Why I am optimistic?

Observation (D. Kaledin, spring 2005)

$\forall$ associative algebra $A / \mathbb{Z}/p\mathbb{Z}$

$\exists$ canonical linear map

$$H_0(A,A) \circlearrowleft$$

given by $[a] \mapsto [a^p]$ $\quad H_0(A,A) = A/[A,A]$

Moreover, it lifts to

$$H_0(A,A) \to HP_0(A)$$

$[a] \mapsto$ finite sum

$$a^p + \sum_{\substack{i_0+\dots+i_n=p \\ n\equiv 1(2)}} c_{i_0,\dots,i_n}\, a^{i_0}\otimes\dots\otimes a^{i_n}\cdot u^{\frac{n-1}{2}} \qquad p>2$$

$$a^2 + 1\otimes a\otimes a\cdot u \qquad p=2$$

Last term $(p \geq 3)$

$$\dots + \boxed{\left(\tfrac{p-1}{2}\right)!}\ \underbrace{a\otimes\dots\otimes a}_{p\text{ times}}\cdot u^{\frac{p-1}{2}}$$

$\boxed{\neq 0}$

Conjecture For saturated $X/\mathbb{Z}/p\mathbb{Z}$

$$H^\cdot(C_\cdot^{red}[u], \partial + uB)$$

is a <u>coherent</u> $\mathbb{Z}/p\mathbb{Z}[u]$-module

This is completely opposite to char $=0$:

$/k$ char $=0$ $\quad H^\cdot(C_\cdot^{red}[u,u^{-1}], \partial + uB) = 0$.

$H^\cdot(C_\cdot^{red}[u], \partial + uB)$ is ∞ torsion module over $u=0$.

Conjecture For $\forall$ dg algebra $A/\mathbb{Z}$
<u>flat</u> over p

$$A_0 := A \otimes \mathbb{Z}/p\mathbb{Z}$$

$\rightsquigarrow$ canonical iso

$$H^\cdot(C_\cdot^{red}(A_0, A_0)[u,u^{-1}], \partial + uB)$$
$$\wr\wr$$
$$H^\cdot(C_\cdot^{red}(A_0, A_0)[u,u^{-1}], \partial)$$

of $\mathbb{Z}/p\mathbb{Z}[u,u^{-1}]$-modules

no finitness condition!

Two above conjectures together $\Rightarrow$ degeneration Hodge $\Rightarrow$ de Rham ← Kaledin announced the proof !?

Reason in favor of 2nd Conjecture:

Use increasing filtration on $C^{red}_{\cdot}(A,A)$:

$$\boxed{\mathrm{Fil}_{\leq n} := A \otimes \left(A/_1\right)^{\otimes \leq (n-1)} \oplus 1 \otimes \left(A/_1\right)^{\otimes n}}$$

On gr for this fitration we get

$$\boxed{\partial + B : \begin{tikzcd} V^{\otimes n} \arrow[r, bend left, "1-\sigma"] & V^{\otimes n} \arrow[l, bend left, "1+\sigma+\dots+\sigma^{n-1}"] \end{tikzcd}}$$

σ generator of $\mathbb{Z}/n\mathbb{Z}$

$V = H^{\cdot}(A/_1)$

$$\boxed{\partial : V^{\otimes n} \xrightarrow{1-\sigma} V^{\otimes n}}$$

gr, $\partial + B$ is acyclic if $(n,p)=1$

if $n = kp$, canonically qis

gr, ∂ in degree $k = \frac{n}{p}$: $V^{\otimes k} \longrightarrow V^{\otimes k}$.

Works for $\forall$ free $\mathbb{Z}/p^e\mathbb{Z}$ module also.

L-functions

If monodromy $=(-1)^{\text{parity}}$ (e.g. $\mathbb{Z}$-graded case)

$\rightsquigarrow$ L-factors for HP^{odd}, HP^{even}

$=$ usual $L_p(s)$ normalized to have eigenvalues of $Fr_p \in U(1)$

shift $s \mapsto s - \frac{\text{weight}}{2}$

Beilinson conjectures: multiplicity of zero & leading term

	$\mathrm{Re}\, s = \frac{1}{2}$	1	2	3
$L^{ev}(s)$		$\uparrow$	$\uparrow$	$\uparrow$
		$K_1 + K_0/K_0^{(0)}$	K_3	K_5

	$\frac{1}{2}$	$3/2$	$5/2$
$L^{odd}(s)$	$\uparrow$	$\uparrow$	$\uparrow$
	$K_0^{(0)}$	K_2	K_4

$K_0^{(0)} = \mathrm{Ker}\left(\begin{smallmatrix}\text{numerical} \\ \sim\end{smallmatrix}\right)$

It is quite possible that $\forall$ NC motives come from X scheme ($\mathbb{Z}$-graded case)
(X, f) $\mathbb{Z}/2$-graded case ...

Still, potential use,
e.g. in Langlands correspondence
$H^i(GL(n,\mathbb{Z}))$ $\overset{??}{\cdots}$ natural NC spaces