LOCALIZING GAUGE THEORIES I.

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We study $\mathcal{N} = 2$ supersymmetric gauge theories on toric four dimensional manifolds. We recall the notion of Ω-background and calculate the partition function of gauge theory in this background. As a result, we get generalizations of the formulae of the author for the theories on $\mathbb{R}^4$ and Nakajima-Yoshioka for the theories on $\mathbb{R}^4$ blown up at one point. In the case of compact toric space our results give an alternative derivation of Donaldson invariants, and generalize the results of Göttche-Zagier.

1. Introduction

In the past years some progress has been achieved in the quantum field theory calculations of the low-energy effective actions including non-perturbative effects. $\mathcal{N} = 2$ supersymmetric Yang-Mills theory has simple perturbation theory and non-trivial instanton dynamics. By utilizing supersymmetry one is able to calculate the instanton corrections to the low-energy effective action [23]. This is done by placing $\mathcal{N} = 2$ gauge theory in a non-trivial geometric background, involving R-symmetry currents. In this paper we shall generalize the results of [23]. Namely, we study $\mathcal{N} = 2$ gauge theory on a toric four-manifold X. The motivations for this work are manifold. First, we hope to learn about the gauge theory applications of the holomorphic anomaly, which is well-known in the context of topological strings [1,3]. Secondly, the similar formulae are expected to hold in the higher dimensional generalizations of the supersymmetric gauge theories. There, the applications to the black hole counting [30] are expected. Also, the topological vertex and Donaldson-Thomas theory also behave in the similar way. The crucial ingredient missing in all these problems is the detailed understanding of the stability conditions. Since in the case of Donaldson theory lots of information is already available, e.g. in the work [8], we may hope to advance

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2. $\mathcal{N} = 2$ gauge theory

We study pure $\mathcal{N} = 2$ gauge theory. Generalizations involving matter are straightforward. It is well-known that the theory admits the so-called twisting [29], which permits supersymmetric compactifications on arbitrary four-manifold X. The field content of the twisted theory (which turns out to be the integral representation of Donaldson theory) is the following: the gauge field A, the complex adjoint scalar ϕ, and the adjoint fermions: one-form ψ, self-dual two-form χ and a scalar η.

On generic X the twisted theory admits one nilpotent supercharge Q, also called topological charge. If the holonomy of X is reduced then there are extra charges, coming from the original supercharges of the physical theory of the same chirality as Q. If the manifold X admits isometries, then there are extra conserved supercharges G of opposite chirality. Mathematically, the theory on X studies equivariant cohomology of the space of gauge fields. In addition to the gauge group, the equivariance with respect to isometries of X is quite useful in analyzing the theory.

The bosonic part of the $\mathcal{N} = 2$ theory action is given by:

$$S^{bos} = \int \mathrm{Tr}\left(\|F\|^2 + \|D_A \phi\|^2 + \|[\phi, \bar{\phi}]\|^2 \right) \tag{1}$$

3. $\mathbf{\Omega}$-background, local theory: $\mathbb{R}^4$

We start with the theory on $\mathbb{R}^4$. Suppose $\Omega, \bar{\Omega}$ are two commuting infinitesimal rotations of Euclidean space $\mathbb{R}^4$, $[\Omega, \bar{\Omega}] = 0$, $\Omega = \|\Omega_{\mu\nu}\|$. These are generated by the vector fields $V, \bar{V}$, $V^{\mu} = \Omega^{\mu}_{\nu} x^{\nu}$, and the indices are lifted by the Euclidean metric $\delta_{\mu\nu}$. We skip the details and the motivation of the construction [23]. The bosonic part of the action of the theory in the Ω-background is given by:

$$S^{bos} = \int \mathrm{Tr}\left(\|F\|^2 + \|D_A \phi - \iota_V F\|^2 + \|[\phi, \bar{\phi}] - \iota_{\bar{V}} D_A \phi + \iota_V D_A \bar{\phi}\|^2 \right) \tag{2}$$

4. Instanton partition function on $\mathbb{R}^4$

In [23] we calculated the partition function of the gauge theory in the Ω-background. It depends on the skew eigen-values ϵ_1, ϵ_2 of the matrix Ω and the vev $\vec{a}$ of the Higgs field ϕ.

4.1. *Higher times*

The action Eq. (2) is the bosonic part of the integral of $ch_2 = \frac{1}{2}\mathrm{Tr}\phi^2$ over the superspace. Formally one can consider the integral of any function $\mathbf{f}(ch_1, ch_2, ch_3, \dots)$ where

$$ch_p = \frac{1}{p}\mathrm{Tr}\left(\frac{\phi}{2\pi i}\right)^p \tag{3}$$

(only N of ch_p are independent). We expand:

$$\mathbf{f} = \sum_p \tau_p ch_p + \sum_{p_1 p_2} \tau_{p_1 p_2} ch_{p_1} ch_{p_2} + \dots$$

The parameters $\vec{\tau} = (\tau_p | \tau_{p_1 p_2} | \dots)$ are called *times*. All but τ_2 should be viewed as formal parameters.

4.2. *Partition function*

The calculation in [10, 22, 23] gives:

$$Z(\vec{a}, \vec{\tau}, \epsilon_1, \epsilon_2) = \sum_{\vec{\mathfrak{k}}} \mu_{\vec{\mathfrak{k}}}(\vec{a}, \epsilon_1, \epsilon_2) \exp \mathbf{f}_{\vec{\tau}}(\hat{m}_p(\vec{a}, \vec{\mathfrak{k}}, \epsilon_1, \epsilon_2)) \tag{4}$$

$$\hat{m}_p(\vec{a}, \vec{\mathfrak{k}}, \epsilon_1, \epsilon_2) = \sum_l a_l^p +$$

$$+ \sum_{l,i} (a_l + \epsilon_1(i-1))^p - (a_l + \epsilon_1 i)^p - (a_l + \epsilon_1(i-1) + \epsilon_2 k_{li})^p + (a_l + \epsilon_1 i + \epsilon_2 k_{li})^p$$

where $\vec{\mathfrak{k}} = (\mathfrak{k}_1, \dots, \mathfrak{k}_N)$ is the N-tuple of partitions,

$$\mathfrak{k}_l = (k_{l1} \geq k_{l2} \geq k_{l3} \geq \dots \geq k_{l k'_{l1}} > 0)$$

Finally $\mu_{\vec{\mathfrak{k}}}(\vec{a}, \epsilon_1, \epsilon_2)$ is the measure on partitions, derived in [23]:

$$\mu_{\vec{\mathfrak{k}}}(\vec{a}, \epsilon_1, \epsilon_2) = \exp \int dx dy f''_{\vec{\mathfrak{k}}}(x) f''_{\vec{\mathfrak{k}}}(y) \gamma_{\epsilon_1, \epsilon_2}(x-y) \tag{5}$$

and the function $f_{\vec{\mathfrak{k}}}$ (the profile of the colored partition $\vec{\mathfrak{k}}$) , and the kernel γ are defined in [22].

4.3. *Expansion of the partition function on* $\mathbb{R}^4$

In order to proceed with more general four-manifolds we need the partition function on $\mathbb{R}^4$ to be evaluated for the most general parameters, both toric ϵ_1, ϵ_2 and the couplings $\tau_{\vec{n}}$. Expansion in ϵ_1, ϵ_2 has the following form:

$$Z(\vec{a}, \epsilon_1, \epsilon_2; \tau_{\vec{n}}) = \exp \frac{\mathcal{F}_0}{\epsilon_1 \epsilon_2} + \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \mathcal{H}_{\frac{1}{2}} + \mathcal{F}_1 + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} \mathcal{G}_1 + \mathcal{O}(\epsilon_1, \epsilon_2) \quad (6)$$

where $\mathcal{F}_0, \mathcal{F}_1, \mathcal{H}_{\frac{1}{2}}, \mathcal{G}_1$ are functions of $\vec{a}, \tau_{\vec{n}}$.

5. Ω-background, global theory

The standard Donaldson theory in four dimensions can be viewed as the partially twisted six dimensional super-Yang-Mills theory, compacitifed on $\mathbb{T}^2 \times X$, where X is our four-fold, and the torus $\mathbb{T}^2$ has vanishing size.

For X with isometries we modify this construction. Consider flat X-bundle over $\mathbb{T}^2$, which can be viewed as an orbifold of $\mathbb{R}^2 \times X$ by the action of $\mathbb{Z}^2$, which acts by shifts in $\mathbb{R}^2$ direction, and by two commuting isometries g_1, g_2 in the X direction:

$$(z, \bar{z}, x) \mapsto (z + n + m\tau, \bar{z} + n + m\bar{\tau}, g_1^n g_2^m \cdot x) \quad (7)$$

By compactifying the gauge theory on this background, with the partial twist along X, in the limit of vanishing volume of $\mathbb{T}^2$ we end up with the theory in the Ω-background. The bosonic part of its action coincides with Eq. (2).

5.1. *Toric preliminaries*

In this section we collect all the necessary information about toric varieties. We use slightly unconventional language, which is more familiar to physicists.

Let M, d be non-negative integers, and $\mathbb{T}^M, \mathbb{T}^d, \mathbb{T}^{M+d}$ the standard tori $U(1)^{M,d,M+d}$ of the corresponding dimensions. In what follows a runs from a to $M + d$, α runs from 1 to M, μ runs from 1 to d, and l runs from 1 to N. The torus $\mathbb{T}^{M+d}$ acts in the standard fashion on the space $\mathbb{C}^{M+d}$:

$$Z = (Z^a) \mapsto (Z^a \exp\, i\theta^a)$$

Any integral matrix $Q : \mathbb{Z}^M \to \mathbb{Z}^{M+d}$ defines a homomorphism of $\mathbb{T}^M$ into $\mathbb{T}^{M+d}$, and the corresponding action of $\mathbb{T}^M$ on $\mathbb{C}^{M+d}$:

$$Z = (Z^a) \mapsto \left(Z^a \exp\, i \sum_\alpha Q^a_\alpha \varphi^\alpha \right)$$

This action preserves the norms: $p_a = \frac{1}{2}|Z^a|^2$ and the symplectic form

$$\omega = \sum_i dp_a \wedge d\vartheta^a \qquad Z^a = \sqrt{2p_a}e^{i\vartheta^a}$$

Now, fix a collection $\vec{r}$ of M real numbers $r_1, \ldots, r_M$ – Fayet-Illiopoulos terms in the physical language – and consider the Hamiltonian reduction of $\mathbb{C}^{M+d}$ with respect to $\mathbb{T}^M$ at the level of the moment map

$$\vec{\mathbf{m}} = (\mathbf{m}_1, \ldots, \mathbf{m}_M), \qquad \mathbf{m}_\alpha = \sum_a Q^a_\alpha p_a$$

given by $\vec{r}$. Explicitly, we consider the space

$$X = \vec{\mathbf{m}}^{-1}(\vec{r})/\mathbb{T}^M$$

This is our toric variety. By construction, it comes endowed with the map:

$$\vec{p} : X \to \mathbb{R}^{M+d}_+$$

which factors through the projection

$$X \to X/\mathbb{T}^d = \Delta_X$$

where $\mathbb{T}^d = \mathbb{T}^{M+d}/\mathbb{T}^M$. We neglect possible torsion part (it is absent for generic Q). The image of Δ_X under $\vec{p}$ is the convex d-dimensional polyhedron (perhaps non-compact), which fits some d-dimensional affine subspace bL^d in $\mathbb{R}^{M+d}_+$. Another canonical structure on X is a set of complex line bundles L_α and $\mathcal{L}_a$. More precisely, any character χ of the torus $\mathbb{T}^M$ defines a line bundle, which is associated with the help of χ to the principal $\mathbb{T}^M$ bundle:

$$\vec{\mathbf{m}}^{-1}(\vec{r}) \to X \tag{8}$$

All of the above is said under the assumption of genericity of $\vec{r}$, so that the Eq. (8) indeed defines a principal bundle. The line bundles $\mathcal{L}_a$ are associated to the characters $\chi^M_a = \exp iQ^a_\alpha\varphi^\alpha$, while L_α are associated to $\chi^M_\alpha = \exp i\varphi^\alpha$. Hence, topologically:

$$\mathcal{L}_a \approx \otimes_\alpha L_\alpha^{\otimes Q^a_\alpha} \tag{9}$$

The torus $\mathbb{T}^{M+d}$ acts on X. Of course, only $\mathbb{T}^d$ acts faithfully. The full torus $\mathbb{T}^{M+d}$ becomes visible when we start looking at the line bundles over X. Then $\mathcal{L}_a$ become distinct line bundles as $\mathbb{T}^{M+d}$ equivariant bundles. They correspond to the characters $\chi^{M+d}_a = e^{i\theta^a}$. In what follows we shall denote χ^{M+d} simply by χ.

The fixed points v of $\mathbb{T}^d$ action on X are the vertices of Δ_X. Generically, these vertices have d edges emanating from them. These edges correspond to fixed complex projective lines in X (if X is non-compact the edges may go all the way to infinity, in this case they correspond to the fixed complex lines in X).

The tangent space T_vX to X at any fixed point is a representation of $\mathbb{T}^d$, and, consequently, a representation of $\mathbb{T}^{M+d}$. The representation splits as a sum of d one-dimensional irreducibles:

$$T_vX = \bigoplus_{\mu=1}^{d} \chi_{v,\mu}$$

where we took the liberty of identifying the characters with the one dimensional representations. The characters $\chi_{v,\mu}$ define the weights of the $\mathbb{T}^{M+d}$ action on T_vX:

$$\chi_{v,\mu} = \exp a \sum_a w_{v,\mu;a}\theta^a \tag{10}$$

which clearly obey:

$$\sum_a w_{v,\mu;a} Q^a_\alpha = 0, \forall \alpha$$

Each fixed point v defines a subset I_v of the set of indices $\{1, \ldots, M+d\}$ of cardinality d, such that for any

$$\forall a \in I_v \qquad Z^a(v) = 0$$

It is natural to label the elements of I_v by the index $\mu = 1, \ldots, d$. In other words, I_v coincides with the set of weight subspaces of T_vX.

The remaining M coordinates Z^i are completely fixed by the moment map equations, up to the action of $\mathbb{T}^M$. Let $\mathbb{Z}^M_v \subset (\mathbb{Z}^{M+d})^t$ denote the sub-lattice $\{(n_1, \ldots, n_{M+d}) | n_a = 0, a \in I_v\}$, and $\mathbb{R}^M_v = \mathbb{Z}^M_v \otimes \mathbb{R}$. Then $\vec{\partial}(v) \in \mathbb{R}^M_v$. Let Q_v denote the restriction of Q^t onto $\mathbb{Z}^M_v$. It must be invertible over $\mathbb{Z}$ in order for v to be isolated fixed point. Then:

$$w_{v,\mu;a} = \left(\delta_{\mu;a} - Q^\mu_\alpha (Q_v^{-1})^\alpha_a\right) \tag{11}$$

5.1.1. *Second cohomology*

The line bundles $\mathcal{L}_a$ generate the K-group of X. Their first Chern classes generate $H^2(X, \mathbb{Z})$. In the $\mathbb{T}^{M+d}$-equivariant cohomology they are represented by the equivariantly closed forms: $\phi_a + c_1(L_a)$. In the localized cohomology, which is spanned by the fixed points $v \in X^{\mathbb{T}}$, each point contributing a copy of $\mathbb{C}((\varepsilon_1, \ldots, \varepsilon_{M+d}))$, the Chern classes localized to $\phi^{(v)}_a$, which are linear functions of ε_a.

In the applications of gauge theory to four-manifold invariants one studies correlators of the 2-observables, say for single trace operators: $\mathcal{O}_J^{(2)}$. We evaluate integrated 2-observables using localization formula:

$$\int_{\Sigma} \mathcal{O}_{\mathbf{f}}^{(2)} = \sum_v H_{\Sigma}^{(v)} \mathbf{f}(ch_p(\mathcal{E}_v)) \tag{12}$$

where: Σ is the 2-homology class, and $H_{\Sigma} + P.D.[\Sigma]$ is the equivariant version of its Poincare dual, H_{Σ} is a function on X, linear in p_i, and $H_{\Sigma}^{(v)}$ its value at the fixed point v. Finally, $\mathcal{E}_v$ is the localized universal sheaf, i.e. $\iota_v^* \mathcal{E}$, where $\iota^v : \mathfrak{M} \times \{v\} \hookrightarrow \mathfrak{M} \times X$ is the inclusion, where $\mathfrak{M}$ is the moduli space of sheaves (compactified instanton moduli space).

5.2. *Equivariant bundles*

In gauge theory with gauge group G we usually fix the principal G-bundle P, integrate over the space of all connections on P and then sum over topological classes of P. When we formulate the theory on some curved manifold X we have to fix the metric on X. If the manifold X is a toric variety the metric is uniquely specified by the matrix of charges Q and the set of Fayet-Iliopoulos terms $\vec{r}$. If we want to turn on the Ω-background an extra data is needed, namely the lift of the action of the torus $\mathbb{T}^{M+d}$ on P. For the gauge group $G = SU(N)$ this is the same as the choice of N-dimensional representation $\mathbb{E}$ of $\mathbb{T}^{M+d}$. The latter is specified by the matrix

$$k_{l,a} \quad , \quad l = 1, \ldots, N \qquad , a = 1, \ldots M + d \tag{13}$$

of weights. The rank N vector bundle splits as a sum of N line bundles $\mathfrak{L}_l$, of Chern classes:

$$c_1(\mathfrak{L}_l) = \sum_a k_{l,a} Q_{\alpha}^a c_1(L_{\alpha})$$

5.3. *Examples*

(1) Let us consider $X = \mathbb{CP}^d$, which corresponds to $M = 1$. In this case Δ_X is just a simplex,

$$\sum_a p_a = r$$

(if $r > 0$, of course). The fixed points v are in one-to-one correspondence with the vertices of the simplex, i.e. we can set $v = 1, \ldots M+1$, $\vec{p}_v = (0, \ldots, \overset{v}{r}, \ldots, 0)$

(2) The next example is non-compact. X is the total space of the vector bundle $\mathcal{O}(q_1) \oplus \mathcal{O}(q_{d-1})$ over $\mathbb{P}^1$. It corresponds to $M = 1$, the vector of charges:

$$\vec{Q} = (q_1, q_2, \dots, q_{d-1}, 1, 1)^t$$

There are two fixed points v - they belong to the zero section of the vector bundle, and are the North and South poles of the sphere.

5.4. *Stability and holomorphic anomaly*

The partition function of the gauge theory in Ω-background is analytic in the vev of the Higgs field. However the expansion in the parameters ϵ_1, ϵ_2 in the limit $\epsilon \to 0$ leads to holomorphic functions in a, $\mathcal{F}_0(a;\Lambda), \mathcal{F}_1(a;\Lambda)$ etc which are actually not single-valued. They can be made single valued at the expense of adding some non-holomorphic terms (the example of the second Eisenstein series E_2 is helpful here). The precise expression of these non-holomorphic terms is not known. We are proposing an approach below which should fix them, at least partly.

Another, more profound, reason to expect them is to recall the Gromov-Witten interpretation of the $\mathcal{N} = 2$ partition functions. At least in the case $\epsilon_1 = -\epsilon_2$ the expansion coefficients $\mathcal{F}_g(a;\Lambda)$ are (the limits of) the generating functions of the genus g Gromov-Witten invariants of certain (local) Calabi-Yau manifolds, see Ref. [28]. These are well-known to be the large $\bar{a}$ limits of non-holomorphic automorphic forms on the moduli spaces of Kahler structures of these Calabi-Yaus, see Refs. [1,3].

6. MASTER FORMULA

We are now almost ready to state our main formula.

We first fix the vacuum expectation value of the Higgs field Φ,

$$\langle \Phi \rangle = \vec{a} \equiv \mathrm{diag}(a_1, \dots, a_N) \tag{14}$$

In the case of compact X we should take the integral over a, to take into account the tunneling between different vacua.

The second cohomology group of X is isomorphic to $\mathbb{Z}^d$, generically. The $U(N)$ gauge bundle is reduced to the $U(1)^N$ bundle, in the presence of the Higgs vev. We should sum over all equivalence classes of the $U(1)^N$ bundles. They are classified by the vectors $\vec{k}_\alpha = (k_{\alpha,l})$, $\alpha = 1, \dots, d$. In the $SU(N)/\mathbb{Z}_N$ gauge theory one fixes the traces: $\{\vec{k}_\alpha\} = w_\alpha = \sum_{l=1}^N k_{\alpha,l}$. Actually, in the absence of the charged matter, everything depends only on

$w_a$modN. Equivariantly, as we discussed above, topologically equivalent line bundles may differ. So in fact we might sum over equivalence classes of equivariant line bundles. These are labelled by vectors $\vec{k}_e \in \mathbb{Z}^N$, for each edge e of the toric diagram Δ_X. Equivalent information is contained in the vectors $\vec{k}_a = (k_{a,l})$ from Eq. (13). However, the choice of equivariant bundle does not correspond to the fluctuating parameter. So the real sum goes over $\vec{k}_\alpha$ only. Moreover, the actual set of vectors $\vec{k}_\alpha$ is even smaller, because of the so-called stability conditions, to be mentioned below.

The partition function of the $\mathcal{N}=2$ theory in Ω-background on the toric manifold X, in the presence of 2-observables,

$$\langle \exp \sum_{\vec{n}} \int \mathbf{t}_{\vec{p}} \mathcal{O}^{(2)}_{ch_{p_1} ch_{p_2} \ldots} \rangle, \qquad \mathbf{t}_{\vec{n}} \in H^2_c(X,\mathbb{Z})$$

is given by:

$$Z_{\vec{w}}(\vec{a},\vec{\varepsilon};\tau_{\vec{n}},\mathbf{t}_{\vec{n}}) = \sum_{\vec{k}_a \in \mathbb{Z}^N, \{\vec{k}_\alpha\} = w_\alpha} \prod_v Z(\vec{a} + \varepsilon \cdot \vec{k}_v, w_{v1}, w_{v2}; \tau_{\vec{n}} + \mathbf{t}_{\vec{n}} H_v) \quad (15)$$

where $\varepsilon \cdot \vec{k}_v$ is given by:

$$(\sum_a w_{ia} \varepsilon_a p_a)$$

6.1. *Non-equivariant limit*

In the case of compact X the partition function (15) has a finite non-equivariant limit, i.e. the limit where $\vec{\varepsilon} \to 0$. Indeed, the singular terms in the logarithm of partition function vanish thanks to the identity:

$$\sum_v \frac{1}{w_{v1} w_{v2}} = \int_X 1 = 0, \qquad \sum_v \frac{\phi_a^{(v)}(\varepsilon)}{w_{v1} w_{v2}} = \int_X c_1(L_a) = 0 \quad (16)$$

$$\sum_v \frac{w_{v1} + w_{v2}}{w_{v1} w_{v2}} = \int_X c_1(X) = 0$$

and the finite part reduces to:

$$\mathcal{Z}_{\vec{w}}(\vec{a};\tau_{\vec{n}},\mathbf{t}_{\vec{n}}) = \sum_{\vec{k}_a \in \mathbb{Z}^N, \{\vec{k}_a\} = w_a} \exp \int_X \mathcal{F}_0(\vec{a} + \sum_a k_a c_1(L_a)) + \quad (17)$$

$$+ \int_X c_1(X) \mathcal{H}_{\frac{1}{2}}(\vec{a} + \sum_a k_a c_1(L_a)) + \chi(X)\mathcal{F}_1(\vec{a}) + \sigma(X)\mathcal{G}_1(\vec{a})$$

If we now integrate over $\vec{a}$ then the resulting function of $\vec{\tau}, \vec{t}$ (for $\vec{\tau} = (\log\Lambda, 0, 0, \dots))$ is, in fact, the generating function of Donaldson invariants of X, for some specific choice of the metric on X. This choice is correlated with the choice of contour for the $\vec{a}$ integral. Put another way, the sum over k_a's goes over some subset of all possible first Chern classes. This subset is singled out by some *stability* condition, which involves explicitly the Kahler class of X (i.e. the choice of $\vec{r}$).

6.2. *Non-holomorphic speculations*

Looking at the non-equivariant limit of the partition function we recognize the *holomorphic approach* formulae of Ref. [13], see also Ref. [8] . There, the low-energy effective theory of $\mathcal{N} = 2$ super Yang-Mills theory was exploited in order to write down a contribution of the Coulomb branch to the Donaldson invariants of manifolds with $b_2^+ = 1$ (all toric manifolds have this property). More systematic approach, taking into account full multiplet of $\mathcal{N} = 2$ susy leads to the non-holomorphic integrand, see Ref. [13,19], whose integral can be converted to the contour integral of a holomorphic form, similar to what we have above. We hope to infer from this the non-holomorphic modular completion of our master formula.

6.3. *Relation to the results of Nakajima-Yoshioka*

This section is devoted to the simplest non-trivial example of the application of the formula (15): $X = \hat{\mathbb{C}}^2$, the blowup of a point on $\mathbb{C}^2$. It is a toric manifold, $M = 1, d = 2$. In this case the formula (15) reduces to:

$$Z_{X,w}(\vec{a}, \epsilon_1, \epsilon_2, \epsilon_3; \tau_{\vec{n}}, \mathbf{t}_{\vec{n}}) = \tag{18}$$

$$\sum_{\vec{k} \in \mathbb{Z}^N, \{\vec{k}\} = w} Z(\vec{a} + \vec{k}\epsilon_1, \epsilon_1 + \epsilon_3, \epsilon_2 - \epsilon_1; \tau_{\vec{n}} + \mathbf{t}_{\vec{n}}(\epsilon_1 + \epsilon_3)) \times$$

$$Z(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2 + \epsilon_3; \tau_{\vec{n}} + \mathbf{t}_{\vec{n}}(\epsilon_2 + \epsilon_3))$$

which, for $\epsilon_3 = 0$ coincides with the result of Ref. [21].

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