From superstrings to quantum foam using supersymmetry

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Abstract. Hermann Weyl Prize lecture at XXV ICGTMP, Cocoyoc, Mexico, Aug. 4, 2004

Introduction

Since the remarkable realization by J. Scherk and J. Schwarz [22] that string spectrum contains gravitons one is looking for the stringy approach to the quantum gravity. One feature of quantum gravity – strong fluctuations of space-time topology at Planck scales, advocated by S. Hawking [7], was very elusive. The reason is that the conventional approach to string quantization produces S-matrix in the form of the expansion in string coupling constant. The S-matrix describes, at best, the scattering of the gravitational waves in some stringy background, most commonly the Minkowski background (however recent advances in AdS/CFT duality allow to hope for other homogeneous spaces as well). At finite order of string perturbation theory one has a finite number of gravitons, so it is unlikely that the strong fluctuation of the space-time topology would be seen. The only hope is to resum the string perturbation theory, perhaps analytically continuing in the string coupling:

$$\exp \sum_{g=0}^{\infty} \mathfrak{g}_s{}^{2g-2}\mathcal{F}_g(a) + \mathcal{O}\left(e^{-\frac{1}{\mathfrak{g}_s}}\right) = \int_{\text{bndry cndts}\sim a} \mathcal{D}\mathfrak{g}_{\mu\nu}\mathcal{D}(\ldots)\ \exp\ -\ S_{\text{eff}}(\mathfrak{g}_{\mu\nu},\ldots) \tag{1}$$

where $\mathfrak{g}_s$ denotes string coupling, and $\ldots$ stand for superpartners and massive string modes. But to do that one needs an example of all-loop string calculation. For conventional graviton scattering on $\mathbb{R}^{10}$ noone was able to go beyond two loops so far [8]. However, recently, using the advances in the topological string theory, where all-loop exact string amplitudes are not uncommon, the quantum foam picture has indeed emerged. It is the purpose of this lecture to explain how it came about.

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1. Topological strings

In this section we recall the appearence of the topological strings in the physical string calculations. We consider type II superstring compactified on Calabi-Yau threefold. The effective four dimensional supergravity action contains terms which are calculated by the topological string.

1.1. Calabi-Yau compactifications

Critical superstring lives in ten dimensions. We live in four macroscopic dimensions, so the remaining six dimensions should be unobservably small. One option is to have them compactified on a manifold M, which is severely constrained by various symmetry requirements. For example, if we want to have four dimensional supersymmetry, the manifold M has to be Calabi-Yau (if we want $\mathcal{N}=1$ susy we must also turn on some fluxes). In what follows we assume it is Calabi-Yau. It means that M is a complex manifold, of complex dimension three, which can be endowed with Ricci-flat Kahler metric. Yau's theorem states that for given complex structure, and the cohomology class of the Kahler form k, the Ricci-flat metric which has the corresponding Kahler form, exists and is unique. Existence of the Ricci-flat metric implies that $c_1(M)=0$, which, in turn, implies the existence of nowhere vanishing holomorphic three-form Ω, which is unique up to a multiplication by a complex number.

1.2. Effective supergravity in four dimensions

In these circumstances the low energy physics in four dimensions is described by $\mathcal{N}=2$ supergravity (sugra) theory. Such a theory contains gravity supermultiplet, whose bosonic fields are the metric $\mathfrak{g}_{mn}$, $m,n=1,2,3,4$, and the vector field b_m, called graviphoton. In addition there are matter multiplets, vectors and hypers. Vector multiplets contain complex scalars ϕ^a, $a=1,\ldots,r$, and $U(1)$ vector fields A^a_m. The hypermultiplets contain two complex scalars $Q^i, \tilde{Q}_i$, $i=1,\ldots,s$. In IIA string $r=h_{1,1}(M)$, and $s=h_{1,2}(M)$. Mirror symmetry would exchange IIA with IIB and M with $\tilde{M}$ – the mirror Calabi-Yau.

1.2.1. Vector multiplet moduli space Let us work, for definiteness, with IIA string. Then the vector fields in four dimensions come by reduction of the $U(1)$ RR gauge field C_1 in ten dimensions and by reduction of the RR three form C_3 along two-cycles in CY M. The vector field C_1 becomes b and falls into the gravity multiplet. The graviphoton field strength $T=db$ plays a special rôle in what follows. The scalars of the vector multiplets come from the complexified Kahler moduli of Calabi-Yau M – the real part parameterizes the Kahler class k, which fixes some of the metric moduli, while the imaginary part comes from the periods of the NSNS B-field: $\phi^a=\int_{\beta_a}\omega$, $\omega=k+iB$.

The geometry of the vector multiplet moduli space $\mathcal{M}_v$ is governed by the holomorphic function, the prepotential $\mathcal{F}_0(\phi^a)$. The effective Lagrangian for the vector multiplets looks as follows:

$$\begin{gathered}\mathrm{Im}\tau_{ab}d\phi^a\wedge\star d\bar{\phi}^b+\tau_{ab}F^-_a\wedge F^-_b-\bar{\tau}_{ab}F^+_a\wedge F^+_b\\ \tau_{ab}=\frac{\partial^2\mathcal{F}_0}{\partial\phi^a\partial\phi^b}\end{gathered}\tag{2}$$

The geometry encoded in (2) is called (projective) special Kahler.

In addition, there are couplings between the vector multiplets and the gravity multiplet:

$$\sum_{g=1}^{\infty} \mathcal{F}_g(\phi^a) T^{2g-2} R \wedge R \tag{3}$$

where $T^2 = T^-_{mn} T^-_{mn}$. Note that the string dilaton does not show up among the vector multiplets.

1.2.2. Hypermultiplet moduli space The moduli space $\mathcal{M}_h$ of hypermultiplets has quaternionic-Kahler geometry. The hypermultiplet scalars come from: dilaton, axion (dual to the B_{mn} in four dimensions), and the Calabi-Yau C_3 period, corresponding to the $(3,0)$-form Ω: $\int_M C_3 \wedge \Omega$, and $\int_M C_3 \wedge \bar{\Omega}$ (this gives rise to the so-called universal hypermultiplet); and then $h_{1,2}(M)$ multiplets where Q^i corresponds to the complex structure deformations of M, and $\tilde{Q}_i$ to the remaining periods of C_3.

1.3. Topological strings

The vector multiplet couplings $\mathcal{F}_g(\phi)$, $g \geq 0$, can be calculated using a simplified version of string theory. In IIA superstring context it would be the A type topological string, in the IIB context it would be the B type. The topological strings were introduced by E. Witten [23], and their importance for the superstring effective theory calculations was understood in the papers [1, 3]. The type A string calculates $\mathcal{F}_g$'s by summing over the holomorphic maps of the genus g Riemann surfaces into M:

$$\mathcal{F}_g = \sum_{\beta \in H_2(M,\mathbb{Z})} e^{-\int_\beta \omega} \int_{\overline{\mathbb{M}}_g(M,\beta)} 1 \tag{4}$$

where $\overline{\mathbb{M}}_g(M,\beta)$ is the moduli space of stable maps, introduced by M. Kontsevich [11]. The integral of 1 counts these maps, if the moduli space of stable maps consists of isolated points. Its expected dimension is zero, so this is indeed the answer in the generic situation. However, it could happen that the actual dimension is positive. Then one has to integrate the Euler class of the obstruction sheaf over the real moduli space. In this way, say, every isolated rational curve in the primitive homology class β would contribute to $\mathcal{F}_0$ not just a single exponential $e^{-\int_\beta \omega}$ but the whole series of the multicover contributions:

$$\mathcal{F}_0 = \sum_{\beta \in H_2^{\mathrm{primitive}}(M,\mathbb{Z})} n_\beta \mathrm{Li}_3\left(e^{-\int_\beta \omega}\right) \tag{5}$$

The B model definition of the $\mathcal{F}_g$ amplitudes is rather tricky, except for the genus zero part. Suffice it to say that it can be computed by the purely classical means, by calculating the periods of the $(3,0)$ form Ω.

1.3.1. Kodaira-Spencer theory of gravity The great simplification of the topological string compared to the physical string is the reduction, for the B type topological string, of the integrals over the moduli space of Riemann surfaces to the locus of maximally degenerate surfaces [3]. The combinatorics of such surfaces is that of trivalent graphs. Thus the generating function of the B type topological string amplitudes can be written as the sum over Feynman diagrams of some field theory. This theory seems to be the so-called Kodaira-Spencer theory of gravity, introduced in [3]:

$$L_{KS} = \frac{1}{2\mathfrak{g}_s^2}\left[A^\vee \frac{1}{\partial}\bar{\partial} A^\vee + \frac{1}{3}(A \wedge A)^\vee \wedge A^\vee\right] \tag{6}$$

where $A \in \Omega^{-1,1}(M)$, $A^{\vee} = \iota_A \Omega \in \Omega^{2,1}(M)$, $(A \wedge A)^{\vee} = \iota_{A\wedge A}\Omega \in \Omega^{1,2}(M)$.

1.3.2. Kahler gravity The mirror version of the theory (6) is much less understood. The problem is that the worldsheet instantons contribute to the effective action. However, in the large radius limit the theory looks very much like (6) modulo the standard mirror replacements: $A^{\vee} \leftrightarrow \omega$, $\bar{\partial} \leftrightarrow d$, $\partial \leftrightarrow d^c$ [4]:

$$L_{KG} = \frac{1}{2\mathfrak{g}_s^2}\left[\omega \frac{1}{d^c} d\omega + \frac{1}{3}\omega \wedge \omega \wedge \omega\right] \tag{7}$$

2. Localization

The most powerful method of exact calculations in the supersymmetric theories is the localization on the Q-fixed points, where Q is the conserved global supercharge.

Imagine calculating Kahler gravity partition function in some background. Assuming the background has isometries, one can setup an equivariant supercharge, whose fixed points would correspond to the geometries with the same asymptotics as the background, and in addition invariant under the isometry group action. This is in complete parallel with the $\mathcal{N} = 2$ supersymmetric gauge theory partition function calculation [17, 16].

Going back to our gravity problem, we consider toric Calabi-Yau X_0 as our background. The toric CY has $\mathbb{T}^3$ as a group of isometries. The Kahler gravity partition function $Z_{KG}(X_0)$, evaluated by localization with respect to $\mathbb{T}^3$ would be expressed as a sum over all toric X, with the same asymptotics as X_0.

3. Crystal formula

3.1. Type A string on $\mathbb{R}^6$

The simplest, but already inspirational example comes from the Gromov-Witten theory on $\mathbb{C}^3$ – the simplest Calabi-Yau manifold. This theory needs regularization because of the noncompactness of $\mathbb{C}^3$, but this can be cured by using the torus $\mathbb{T}^3$ action. Then the explicit calculation [6] gives:

$$\mathcal{F}_g = \int_{\mathbb{M}_g} c_{g-1}^3(\mathbb{H}) = \frac{B_{2g}B_{2g-2}}{2g(2g-2)(2g-2)!} \tag{8}$$

The all-genus partition function is then:

$$\begin{aligned} Z_{\mathrm{top}}(\mathfrak{g}_s) &= \exp\sum_{g=0}^{\infty} \mathcal{F}_g \mathfrak{g}_s^{2g-2} \sim M^{\frac{1}{2}}(q) \\ M(q) &= \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^n} \\ q &= -e^{-i\mathfrak{g}_s} \end{aligned} \tag{9}$$

To pass from $Z_{\mathrm{top}}(\mathfrak{g}_s)$ to $M(q)$ requires adding some unstable, e.g. $\mathcal{F}_0 = \zeta(3)$, as well as non-perturbative terms, which are fixed in [5, 21].

3.2. Crystal interpretation

3.2.1. MacMahon function The function $M(q)$ which enters (9) is well-known in combinatorics [13, 20]. It counts three dimensional partitions:

$$M(q) = \sum_{\pi - \mathrm{3d\ partitions}} q^{|\pi|} \tag{10}$$

3.2.2. Donaldson-Thomas theory The reason three dimensional partitions pop up in our calculation is explained in [14, 18] – they are $\mathbb{T}^3$-invariant ideal sheaves: the partition π corresponds to the ideal I_π in $\mathbb{C}[x,y,z]$: $(i,j,k) \in \mathbb{Z}^3 \backslash \pi \Leftrightarrow x^{i-1}y^{j-1}z^{k-1} \in I_\pi$. The summation over the partitions is actually the fixed point formula for the integral of 1 over the moduli space of ideal sheaves, weighted by $q^{\mathrm{ch}_3(I_\pi)}$.

For more general toric X_0 the DT theory produces the partition function which is:

$$Z_{DT}(X_0) = \int_{\mathbb{M}_{\mathrm{ideal}}(X_0)} q^{\mathrm{ch}_3(I)} e^{-\int_{X_0} \omega_0 \wedge \mathrm{ch}_2(I)} \tag{11}$$

where $\mathbb{M}_{\mathrm{ideal}}(X_0)$ is the moduli space of ideal sheaves I, $\mathrm{ch}_1(I) = 0$, on X_0.

3.3. Kahler gravity interpretation

Ideal sheaves showed up in physics in some other problems already. In some sense they are singular limits of $U(1)$ instantons. In this same sense the ch_p of a sheaf corresponds to $\frac{1}{p!}\left(\frac{F}{2\pi i}\right)^{\wedge p}$ where F is the curvature of the gauge field.

This point of view allows relating the ideal sheaf counting to the Kahler gravity calculation. The idea is to interpret $\omega_0 - i\mathfrak{g}_s F$ as the Kahler form ω_X on the manifold X, which is obtained from $\mathbb{C}^3$ by the blowup along the ideal I_π. The weight $e^{-\int_{X_0} \omega_0 \wedge \mathrm{ch}_2} q^{\mathrm{ch}_3}$ corresponds to

$$\exp\left(-\frac{1}{6\mathfrak{g}_s^2}\int_X \omega \wedge \omega \wedge \omega\right) \tag{12}$$

which is the value of the Kahler gravity action (7).

Summary and future directions

We have shown that the all-genus topological string partition function can be interpreted as a Kahler gravity partition function [9], which contains the sum over various space-time topologies, in accordance with the quantum foam expectations [7]. It has been shown in [18] that the non-perturbative completion of the type A string on Calabi-Yau manifold M must include the D-brane contributions which in turn can be expressed using the topological B string. The full string partition function would therefore depend on both the Kahler and complex moduli of Calabi-Yau, in other terms, on all the metric moduli. The natural way to combine these moduli is in the G_2 moduli of the seven dimensional manifold $B = S^1 \times M$. Recently, N. Hitchin has proposed a functional on the space of closed three-forms on B whose critical points are the metrics of G_2 holonomy. Its quantization is currently being investigated.

Acknowledgements

I am grateful to A. Losev, G. Moore, A. Okounkov, A. Schwarz, S. Shatashvili, C. Vafa for collaboration and many discussions over the years which led to the results reviewed in this lecture. I would also like to thank the organizers of ICGTMP and the standing committee for awarding me the prize.

My interest in physics was early recognized and amplified by my first physics teacher, Nikolai Kandiorin, who was a replacement teacher in an ordinary school. He left the schooling system soon after I graduated. My education continued further at the 57th school of Moscow, under the guidance of Boris Davidovich. My tutors were Volodya Fock and Yuri Chekanov. I am grateful to them all for their time which they spent with me, helping me to learn mathematics and physics.

The last, but not least, word of gratitude goes to IHES which gave me the best place for quiet and productive work.

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