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Mathematical structures:
On string theory applications in condensed matter physics.
Topological strings and two dimensional electrons

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Quantum field theorists have benefited from ideas originating in the condensed matter physics. In this note we present an interesting model of electrons living on a two dimensional lattice, interacting with random electric field, which can be solved using the knowledge accumulated in the studies of superstring compactifications.

1 Electrons on a lattice, with noisy electric field

Here is the model. Consider the hexagonal lattice with black and white vertices so that only the vertices of the different colors share a common edge. Let B,W denote the sets of black and white vertices, respectively. We can view the edges as the maps $e_i : B \to W$, $e_i^* : W \to B$, $i = 1, 2, 3$. The edge e_1 points northwise, e_2: southeast, and e_3 southwest. The set of edges, connecting black vertices with white ones will be denoted by E. We have two maps: $s : E \to B$ and $t : E \to W$, which send an edge to its source and target.

The free electrons on the lattice are described by the Lagrangian

$$L_0 = \sum_{b\in B} \sum_{i=1,2,3} \psi_b \psi^*_{e_i(b)} = \sum_{w\in W} \sum_{i=1,2,3} \psi_{e_i^*(w)} \psi^*_w \qquad (1.1)$$

The variables ψ_b, ψ^*_w are fermionic variables. Our "electrons" will interact with the $U(1)$ gauge field A_e, where $e \in E$. Introduce three (complex) numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and their sum:$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. We make the free Lagrangian

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(1.1) gauge invariant, by:

$$L_{\psi A} = \sum_{b \in B} \sum_{i=1}^{3} \psi_b e^{i\varepsilon A_{e_i(b)}} \psi^*_{e_i(b)} \tag{1.2}$$

The gauge transformations act as follows:

$$\psi_b \mapsto e^{i\varepsilon\theta_b}, \quad \psi^*_w \mapsto e^{-i\varepsilon\theta_w}\psi^*_w, \quad A_e \mapsto A_e + \theta_{t(e)} - \theta_{s(e)} \tag{1.3}$$

The Lagrangian (1.2) is invariant under (1.3) but the measure $\mathcal{D}\psi\mathcal{D}\psi^*$ is not, there is an "anomaly". It can be cancelled by adding the following Chern-Simons - like term to the Lagrangian (1.2)

$$L_{CS} = -i \sum_{b \in B} \sum_{i=1}^{3} \varepsilon_i A_{e_i(b)} \tag{1.4}$$

In continuous theory in two dimensions one can write the gauge invariant Lagrangian for the gauge field using the first order formalism:

$$\mathcal{L}_{\text{2dYM}} = \int_{\Sigma} \text{tr} E F_A + \sum_k t_k \text{tr} E^k \tag{1.5}$$

where E is the adjoint-valued scalar, the electric field. In the conventional Yang-Mills theory only the quadratic Casimir is kept in (1.5), t_2 playing the role of the (square) of the gauge coupling constant. In our case, the analogue of the Lagrangian (1.5) would be $\mathcal{L}_{\text{latticeYM}} = \sum_f \left(h_f \sum_{e \in \partial f} \pm A_e \right) + \sum_f \mathcal{U}(h_f)$. Note that in the continuous theory one could have added more general gauge invariant expression in E, i.e. involving the derivatives. The simplest non-trivial term would be: $\mathcal{L} = \mathcal{L}_{\text{YM}} + \int \text{tr} g(E) \Delta_A E$ where g is, say, polynomial. Such terms can be generated by integrating out some charged fields. Our lattice model has the kinetic term for the electric field, as well as the lnear potential (it is possible in the abelian theory):

$$L_{Ah} = i \sum_f \left(h_f \sum_{e \in \partial f} \pm A_e \right) - \sum_f \mathcal{U}(h_f)(\Delta h)_f - t \sum_f h_f \tag{1.6}$$

where Δ is the lattice Laplacian, and the "metric" $\mathcal{U}(x)$ is a random field, a gaussian noise with the dispersion law[2]:

$$\langle \mathcal{U}(x)\mathcal{U}(y) \rangle = D(x-y) \equiv \int_0^\infty \mathrm{d}t \frac{e^{-t(x-y)}}{t(1-e^{t\varepsilon_1})(1-e^{t\varepsilon_2})(1-e^{t\varepsilon_3})} \tag{1.7}$$

[2]the integral is regularized via $\int \frac{dt}{t} \to \left. \frac{d}{ds} \right|_{s=0} \frac{1}{\Gamma(s)} \int \frac{dt}{t} t^s$.

The partition function of our model is (we should fix some boundary conditions, see below)

$$Z(t,\varepsilon_1,\varepsilon_2,\varepsilon_3)=\int\mathcal{D}\mathcal{U}e^{-\int\mathcal{U}(x)(D^{-1}\circ\mathcal{U})(x)}\int\mathcal{D}\psi\mathcal{D}\psi^*\mathcal{D}A\mathcal{D}h\ e^{L_{\psi A}+L_{CS}+L_{Ah}} \tag{1.8}$$

2 Dimers and three dimensional partitions

We now proceed with the solution of the complicated model above. The idea is to expand in the kinetic term for the $\psi\psi^*$. The non-vanishing integral comes from the terms where every vertex, both black and white, is represented by the corresponding fermions, and exactly once. Thus the integral over ψ,ψ^* is the sum over dimer configurations [5],[6], weighted with the weight

$$\sum_{\text{dimers}}\prod_{e\in\text{dimer}}e^{i\varepsilon A_e} \tag{2.9}$$

The gauge fields A_e enter now linearly in the exponential, integrating them out we get an equation $dh=\star\omega_{\text{dimer}}$ where ω_{dimer} is the one-form on the hexagonal lattice, whose value on the edge is equal to $\pm\varepsilon_{1,2,3}$ depending on its orientation $\pm\varepsilon$ depending on whether it belongs to the dimer configuration or not. Everything is arranged so the that at each vertex v the sum of the values of ω on the three incoming edges is equal to zero. The solution of the equation on h gives what is called *height function* in the theory of dimers. In our case it is the electric field. If we plot the graph of h_f and make it to a piecewise-linear function of two variables in an obvious way, we get a two dimensional surface – the boundary of a generalized three dimensional partition. In order to make it a boundary of actual three dimensional (or plane) partition, we have to impose certain boundary conditions: asymptotically the graph of h_f looks like the boundary of the positive octant $\mathbf{R}^3_+$[3]. Under these conditions, the final sum over dimers is equivalent to the sum over three dimensional partitions of the so-called *equivariant measure* [3]. The three dimensional partition is a (finite) set $\pi\subset\mathbf{Z}^3_+$ whose complement in $\bar{\pi}=\mathbf{Z}^3_+\backslash\pi$ is invariant under the action of $\mathbf{Z}^3_+$. In other words, the space I_π of polynomials in three variables, generated by monomials $z_1^iz_2^jz_3^k$ where $(i,j,k)\in\bar{\pi}$ is an

[3] i.e. as the function: $h(x,y)=\varepsilon_1 i+\varepsilon_2 j+\varepsilon_3 k$, $x=i-(j+k)/2$, $y=(j-k)/2$, $i,j,k\geq 0$, $ijk=0$

ideal, invariant under the action of the three dimensional torus $\mathbf{T}^3$. Let $ch_\pi = \sum_{(i,j,k)\in\pi} q_1^{i-1}q_2^{j-1}q_3^{k-1}$, $ch_{\bar{\pi}}(q) = \frac{1}{P(q)} - ch_\pi$, $|\pi| = ch_\pi(1)$, $P(q) = (1-q_1)(1-q_2)(1-q_3)$, $q_i = e^{\varepsilon_i}$. Define the "weights" x_α, y_α from $1/P(q) - P(q^{-1})ch_{\bar{\pi}}(q)ch_{\bar{\pi}}(q^{-1}) = \sum_\alpha e^{x_\alpha} - \sum_\alpha e^{y_\alpha}$. Then,

$$\mu_\pi(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \prod_\alpha \frac{y_\alpha}{x_\alpha} \tag{2.10}$$

The partition function of our model reduces to:

$$Z(t,\varepsilon_1,\varepsilon_2,\varepsilon_3) = \sum_\pi \mu_\pi(\varepsilon_1,\varepsilon_2,\varepsilon_3)e^{-t|\pi|} \tag{2.11}$$

3 Topological strings and S-duality

The last partition function arises in the string theory context. The ideals I_π are the fixed points of the action of the torus $\mathbf{T}^3$ on the moduli space of zero dimensional D-branes in the topological string of B type on $\mathbf{C}^3$, bound to a single D5-brane, wrapping the whole space. The equivariant measure μ_π is the ratio of determinants of bosonic and fermionic fluctuations around the solution I_π in the corresponding gauge theory. The parameter t is the (complexified) theta angle, which couples to $\mathrm{tr}F^3$ instanton charge. This model is an infinite volume limit of a topological string on compact Calabi-Yau threefold. The topological string on Calabi-Yau threefold is the subsector of the physical type II superstring on Calabi-Yau $\times\mathbf{R}^4$. It inherits dualities of the physical string, like mirror symmetry and S-duality [4]. It maps the type B partition function (2.11) to the type A partition function. The latter counts holomorphic curves on the Calabi-Yau manifold. In the infinite volume limit it reduces to the two dimensional topological gravity contribution of the constant maps, which can be evaluated to be [3]:

$$\begin{aligned} Z(t,\varepsilon_1,\varepsilon_2,\varepsilon_3) &= \exp\left(\tfrac{(\varepsilon_1+\varepsilon_2)(\varepsilon_3+\varepsilon_2)(\varepsilon_1+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}\right)\sum_{g=0}^\infty t^{2g-2}\tfrac{B_{2g-2}B_{2g}}{2g(2g-2)(2g-2)!} && (3.12)\\ &= M(-e^{-it})^{-\frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_3+\varepsilon_2)(\varepsilon_1+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}} && (3.13)\\ & && (3.14)\end{aligned}$$

where $M(q) = \prod_{n=1}^\infty (1-q^n)^{-n}$ is the so-called MacMahon function.

4 Discussion

We have illustrated in the simple example that the string dualities can be used to solve for partition functions of interesting statistical physics problems. The obvious hope would be that the dualities are powerful enough to provide information on the correlation functions as well. One can consider more general lattices or boundary conditions (they correspond to different toric Calabi-Yau's), more sophisticated noise functions $D(x)$ (e.g. the one coming from Z-theory [7]) . Also, it is tempting to speculate that compact CYs correspond to more interesting condensed matter problems.

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