RESEARCH PROJECT: EQUIVARIANT COHOMOLOGY OF SPHERICAL VARIETIES AND APPLICATIONS TO GROMOV-WITTEN THEORY

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1. Introduction

Let $G$ be a complex connected reductive group. A normal $G$-variety is called spherical if it contains an open $B$-orbit, where $B$ is a Borel subgroup of $G$. Examples include flag varieties, symmetric spaces, toric varieties and group embeddings. For a complete treatment of the theory of spherical varieties, the reader is invited to consult [BLV], [Br6] and [T].

The aim of this project is two-fold: first, we study the equivariant (intersection) cohomology of spherical varieties. In particular, we focus on projective spherical varieties with the additional property that the number of $T$-fixed points and $T$-invariant curves is finite ($T$-skeletal spherical varieties). From this viewpoint, projective group embeddings stand out as a very natural class. Our main topological tool here is GKM theory, a method developed by Goresky, Kottwitz and MacPherson in their seminal paper [GKM]. Secondly, following a suggestion of Nicolas Perrin, we explore the various connections between the cohomology of spherical varieties and Gromov-Witten theory. Many of these relations arise from the quantum to classical principle (Section 3).

The author ([G2], [G1]; see also Section 4) has made some progress in the description of the equivariant cohomology and $K$-theory of rationally smooth group embeddings, extending previous work by Danilov [D] De Concini-Procesi ([DP-1], [DP-2]), Brion ([Br2], [Br1]), Strickland ([St]) and Uma ([U]).

2. Summary of past work

Let $G$ be a complex connected reductive group. A normal irreducible complex algebraic variety $X$ is called an embedding of $G$, or a group embedding, if $X$ is a $G \times G$-variety containing an open orbit isomorphic to $G$. As a consequence of the Bruhat decomposition, group embeddings are spherical $G \times G$-varieties. It is worth emphasizing that this a generalization of the notion of toric varieties, objects that have been studied extensively in algebraic geometry for nearly forty years ([D, DP-1, BCP]). Substantial information about the topology of a group embedding can be obtained by restricting one’s attention to the induced action of a maximal torus $T$ of $G$. 
Let $M$ be a reductive monoid with zero and unit group $G$. Then there exists a central one-parameter subgroup $\epsilon : \mathbb{C}^* \to G$, with image $Z$, such that $\lim_{t \to 0} \epsilon(t) = 0$. Moreover, the quotient space
\[
P_\epsilon(M) := (M \setminus \{0\})/Z
\]
is a normal projective variety on which $G \times G$ acts via $(g, h) \cdot [x] = [gxh^{-1}]$. Hence, $P_\epsilon(M)$ is a normal projective embedding of the quotient group $G/Z$. These varieties were introduced by Renner in his study of algebraic monoids ([R2], [R3], [R4], [R6]). Remarkably, normal projective embeddings of connected reductive groups are exactly the projectivizations of algebraic monoids ([AB], [T]).

On the other hand, Goresky, Kottwitz and MacPherson [GKM] developed a theory, nowadays called GKM theory, that makes it possible to describe the equivariant cohomology of certain $T$-skeletal varieties: projective algebraic varieties upon which an algebraic torus $T$ acts with a finite number of fixed points and weighted invariant curves. Let $X$ be a $T$-skeletal variety and denote by $X^T$ the fixed point set. Cohomology here is considered with rational coefficients. The main purpose of GKM theory is to identify the image of the functorial map $i^* : H^*_T(X) \to H^*_T(X^T)$, assuming $X$ is equivariantly formal. GKM theory has been mostly applied to smooth projective $T$-skeletal varieties, because of the Bialynicki-Birula decomposition [B]. Furthermore, the GKM data issued from the fixed points and invariant curves has been explicitly obtained for some interesting subclasses: flag varieties (Carrell [C]), toric varieties (Brion [Br1]) and regular embeddings of reductive groups (Brion [Br2] and Uma [U]). In the case of singular varieties, GKM theory has been applied to Schubert varieties [C] and, quite recently, to rationally smooth (projective) group embeddings, due to the author’s work [G, G1, G2].

In my thesis [G], we apply GKM theory to the study of rationally smooth group embeddings. These objects satisfy Poincaré duality with rational coefficients and form a class larger than the class of smooth embeddings. The contents of [G] are summarized as follows. Let $X = P_\epsilon(M)$ be a rationally smooth group embedding. Using the theory of reductive monoids, we identify explicitly and combinatorially the salient GKM data that is needed to quantify the sought-after image of $i^* : H^*_T(X) \to H^*_T(X^{T \times T})$. Moreover, building on previous work of Renner ([R2], [R4]), we show that $X$ comes equipped with both a canonical decomposition into rational cells and a filtration by equivariantly formal closed subvarieties. Our major result describes $H^*_T(X)$ and $H^*_G \times G(X)$ explicitly in terms of roots, idempotents, and underlying monoid data. We also characterize those rationally smooth projective embeddings whose equivariant cohomology is obtained via restriction to the associated toric variety $Y = P_\epsilon(T)$. Such characterization is given in terms of the corresponding cross section lattice. Our results increase the effectiveness of GKM theory as a tool in embedding theory.

3. Research Proposal

3.1. Equivariant intersection cohomology of projective group embeddings. Intersection cohomology is a powerful tool for the study of singular algebraic varieties. It was developed by Goresky and MacPherson in [GM1, GM2]. Later on, Brylinski [Bry] and Joshua [Jo] developed Equivariant intersection cohomology, a variant of intersection cohomology that incorporates the group action into its definition. Let $X$ be a $G$-variety. The equivariant intersection cohomology $IH^*_G(X)$, for the middle perversity, is defined to be the hypercohomology $\mathbb{H}^*(EG \times_G X; IC^G(X))$, where $IC^G(X)$ is the equivariant cohomology complex of $X$, and $EG$ is the classifying space of $G$. See [Bry] and [Jo] for more details. $IH^*_G(X)$ is a graded module over $H^*(BG)$, the equivariant cohomology ring of a point. If a variety $X$ is rationally smooth, then both its singular cohomology $H^*(X)$ and its intersection cohomology $IH^*(X)$ agree. There is also a sheaf version of GKM theory for equivariant intersection cohomology, due to the work of Braden and MacPherson [BM].
Our goal is to write down, explicitly and combinatorially, the equivariant intersection cohomology of projective group embeddings, using the GKM theory of [BM], without any assumptions on rational smoothness, so as to allow more general singularities. This is the next step in the program started in my thesis.

The setup: Let $M$ be a reductive monoid with zero, unit group $G$, and maximal torus $T \subset G$. Let $X = \mathbb{P}_\epsilon(M)$ be the associated projective group embedding. It turns out that one can find explicitly the GKM graph of $X$, i.e. the finite graph build from the $T \times T$-fixed points and $T \times T$-invariant curves of $X$, regardless of whether or not $X$ is rationally smooth [G, G2]. Next, consider $IH^*_{T \times T}(X)$, the $T \times T$-equivariant intersection cohomology of $X$. The results of [Bry] and [GKM] imply that $X = \mathbb{P}_\epsilon(M)$ is equivariantly formal for intersection cohomology, that is, $IH^*_{T \times T}(X)$ is a free $H^*_{T \times T}(pt)$-module. Using the results of [BM] one expects the isomorphism

$$IH^*_{T \times T}(X) \simeq \Gamma(X),$$

where $\Gamma(X)$ is the ring of global sections of a GKM sheaf $\Gamma$ defined on the GKM graph of $X$. The problem here is to fully understand $\Gamma$ in terms of the local structure of the underlying monoid $M$. The structure of the various stalks $\Gamma_x$, for $x \in X$, needs to be determined as well. I anticipate that a thorough analysis of the results of [G], [Br5], [R3] and [BM] should shed some light into the innermost structure of the sheaf $\Gamma$. The work of P. Fiebig ([F], [J]) will also play a crucial role in this program, specially on its applications to representation theory.

For toric varieties $X$, the ring $IH^*(X)$ is known, by work of Barthel-Brasselet-Fieseler-Kaup [BBFK]. Our project offers a generalization of their results.

Task 3.1 can be put into a broader context. This is the main motivation for our next goal.

3.2. Spherical varieties, rational smoothness and GKM-theory. An important feature of spherical varieties is that they have only a finite number of $G$-orbits and, consequently, also a finite number of $T$-fixed points, for the induced action of a maximal torus $T$ of $G$. Nevertheless, in general, an spherical variety need not be $T$-skeletal (see [Br1] p. 263). Thus, our main goals are the following.

(1) Characterization of $T$-skeletal spherical varieties and description of the associated GKM-data. Let $X$ be a spherical $G$-variety. First, we need to find suitable conditions on $X$ guaranteeing that the number of $T$-invariant curves is also finite. This should yield a characterization of $T$-skeletal spherical varieties, a large class that encompasses, but is not limited to, flag varieties ([C]), group embeddings ([Br2], [G2]) and spherical varieties of minimal rank ([BJ-2]). The topological study of $T$-skeletal varieties falls within the framework of GKM-theory. Secondly, we compute explicitly the characters associated to the $T$-invariant curves.

Although, as stated, our task seems elusive, we plan to tackle it by first looking at the following concrete situation. Let $X$ be a projective embedding of $G$. By [AB], $X = \mathbb{P}(M)$, for certain reductive monoid $M$. Now let $\sigma : G \to G$ be an involution, and let $G$ act on $X$ via $g \cdot x \mapsto (g, \sigma(g)) \cdot x$. By construction, $O = G/\{1\} = G/G^\sigma$ is a symmetric space contained in $X$, where $G^\sigma = \{g \in G | \sigma(g) = g\}$. Taking $Y_\sigma$ to be the Zariski closure of $O$ in $X$, we get a spherical variety. We then address the question: how are the $T$-curves in $Y_\sigma$ related to the combinatorial data of $M$?, when are they finite?, do they correspond to “twistings” of the rank-two elements of the Renner monoid of $M$? We anticipate that a generalization of the techniques of Brion-Joshua [BJ-2], Renner [R3] and the author [G2] yield the desired answers. It is worth noting that when $G$ is semisimple of adjoint type and $X$ its wonderful compactification, this construction dates back to the work of Littelmann-Procesi [LP].

(2) Classification of rationally smooth spherical varieties. Spherical varieties have been classified combinatorially, in terms of colors and fans, by Brion, Knop, Luna and Vust ([BLV], [T]). On the other hand, a subclass of spherical varieties, namely, rationally smooth group
embeddings, has been characterized combinatorially in recent work of Renner ([R4], [R6]). Recall that a rationally smooth variety is, by definition, a rational cohomology manifold. By extending Renner’s results, we would like to provide a combinatorial characterization of rational smoothness for spherical varieties; in particular, for $T$-skeletal spherical varieties. It is also relevant to determine whether they are $\mathbb{Q}$-filtrable, i.e., whether the cells of the associated BB-decomposition are rationally smooth. If so, by virtue of the GKM-data in step (1), their equivariant (intersection) cohomology is a ring of piecewise polynomial functions or the ring of global sections of a sheaf defined on the associated GKM-graph (Section 3.1).

(3) Computation of structure constants of the equivariant cohomology and $K$-theory of rationally smooth, $T$-skeletal, spherical varieties. Here we aim at a presentation of these algebras in terms of generators and relations (Schubert calculus). Even in the smooth case this has not been completely achieved. Some partial results follow from work of Strickland ([St]) and Uma ([U]). See also Section 3.1 for a report on current work by the author in this direction.

The outcome of Tasks 3.1 and 3.2 has many concrete applications in Gromov-Witten theory, specifically, in the description of quantum cohomology and quantum $K$-theory of projective homogeneous spaces and group embeddings, as we now proceed to explain.

3.3. Quantum cohomology of spherical varieties. (In collaboration with Nicolas Perrin)

Let $X$ be a smooth complex projective variety and let $d \in H_2(X, \mathbb{Z})$. Then there exists an algebraic variety $M_{g,d,n}(X)$ parametrizing the set of all stable maps from a genus $g$ curve $C$, with $n$ marked points $(p_i)_{i \in [1,n]}$, to $X$. This variety is endowed with many combinatorial properties, in particular, there is an evaluation map $ev_i : M_{g,d,n}(X) \to X$, for each marked point $p_i$. For cohomology classes $(\alpha_i)_{i \in [1,n]}$, one can define the Gromov-Witten invariant

$$
\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle_{g,d} = \int_{[M_{g,d,n}(X)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i)
$$

These invariants, for genus 0, built a quantum potential

$$
\Phi_q(\gamma) = \sum_{d,n \geq 3} \frac{q^d}{n!} \langle \gamma^n \rangle_{0,d},
$$

where $\gamma \in H^*(X, \mathbb{C})$. By work of Ruan and Tian (see e.g. [M]), this potential satisfies the DWVV equation, which means that the quantum product $\star$ defined by

$$
(\alpha \star \alpha', \alpha'') = \partial_{\alpha, \alpha', \alpha''} \Phi(\gamma),
$$

where $(, ,)$ is the usual intersection pairing, is associative. The big quantum cohomology ring is the ring obtained from the above quantum product. It is a deformation of the usual cohomology ring. Notice that, for defining quantum cohomology, we only consider rational curves (i.e. we take into account Gromov-Witten invariants of genus 0). Remarkably, Kontsevich and Manin ([KM]) showed that if $H^*(X, \mathbb{C})$ is generated in degree 2, then one can reconstruct from the Gromov-Witten invariants involving only $n = 3$ points, all the Gromov-Witten invariants involving more points. This reduction technique plays a crucial role in applications. The quantum cohomology ring obtained by considering only 3-points Gromov-Witten invariants is called the small quantum cohomology ring of $X$, and we denote it by $QH^*(X)$. In a similar fashion, quantum $K$-theory and the equivariant versions of these rings are defined. Quantum cohomology is intensively studied because of its applications to enumerative problems in algebraic geometry (see [Kon], [M] and the references therein). Recently, quantum $K$-theory has attracted more attention due to its implications in the representation theory of quantum groups.

As Kontsevich pointed out [Kon], if $X$ is also a sufficiently nice $T$-variety, then $M_{0,d,n}(X)$ inherits a natural $T$-action and, by the localization principle ([Bo], [At]), the computation of Gromov-Witten invariants can be reduced to computations on the subspace of fixed points
\(\mathcal{M}_{0,d,n}(X)^T\). Such local computations can be written in terms of equivariant Euler classes. This technique has a wide range of applications. For instance, it was used by the author to provide nice bases for the equivariant cohomology of \(\mathbb{Q}\)-filtrable varieties \([G1]\).

In future collaboration with Nicolas Perrin (Mathematisches Institut, Düsseldorf), we plan to describe the (equivariant) quantum cohomology and quantum \(K\)-theory of certain spherical varieties (e.g., group embeddings, symmetric spaces) and provide explicit formulas for the underlying Schubert calculus. Some important results in this direction have been obtained by Chaput, Manivel and Perrin for the case of minuscule homogeneous spaces and rational homogeneous spaces (see \([\text{CMP08}, \text{CMP07}, \text{CMP10}, \text{CMP09}]\)). A crucial ingredient here is the quantum to classical principle (\([\text{CMP08}, \text{BKT}]\)). Loosely speaking, this principle allows to compute \(QH^*(X)\), where \(X\) is a sufficiently nice space, in terms of the singular cohomology of a suitable replacement \(Y_d\), which, in many cases, turns out to be an spherical variety. Using my previous results (\([G2], [G1]\) and those of Sections 2.2, 3.1 and 3.2, we anticipate a GKM presentation of \(H^*_Y(Y_d)\) and \(H^*(Y_d)\). Such description would be a fundamental step towards implementing a Schubert calculus in \(QH^*(X)\). Indeed, we aim at providing (geometric) bases for these rings and formulas to multiply them.

### 4. Current Related Work

4.1. **Explicit description of the equivariant cohomology ring of projective group embeddings.** (Joint work with K. Aker and Ö. Öztürk). Let \(M\) be a reductive monoid with zero, unit group \(G\), and maximal torus \(T \subset G\). Let \(X = \mathbb{P}_e(M)\) be the associated projective group embedding. Assuming \(X\) is rationally smooth, \([G2]\) gives a precise combinatorial description of \(H^*_{G\times G}(X)\) in terms of finite combinatorial invariants of \(M\). An explicit algebra presentation of \(H^*_{G\times G}(X)\), in terms of generators and relations, is still an open problem. Some partial results have been obtained by Strickland \([\text{St}]\) and Uma \([\text{U}]\), for the case of regular embeddings. However, such polynomial generators are not known for more general (e.g. rationally smooth) embeddings.

Our current project is to compute, explicitly, generators and relations for \(H^*_{G\times G}(X)\) using the abstract tools developed in \([G2]\) and \([R6]\). To successfully achieve our goal, we distinguish two cases, depending on the number of closed \(G \times G\)-orbits in \(X\). It is worth noting that the closed \(G \times G\)-orbits of a projective group embedding are usually of the form \(G/P_e \times G/P_e^-\), where \(P_e\) and \(P_e^-\) are opposite parabolic subgroups of \(G\).

4.1.1. **Simple embeddings.** A projective embedding \(X = \mathbb{P}(M)\) is called *simple* if it contains only one closed \(G \times G\)-orbit \(O\). For instance, the wonderful compactification \(X'\) of a semisimple group \(G'\) is a particular case of a simple embedding (\([\text{DP-1}, \text{BBFK}]\)). In fact, the unique closed orbit of \(X'\) is isomorphic to \(G'/B \times G'/B^-\), for \(B\) a Borel subgroup of \(G'\). Another familiar example is \(\mathbb{P}^{(n+1)^2-1}\), a simple embedding of \(\text{PSL}(n+1, \mathbb{C})\), with unique closed orbit \(\mathbb{P}^n \times \mathbb{P}^n\), a product of Grassmannians.

Let \(X = \mathbb{P}(M)\) be a rationally smooth simple embedding. By the results of \([G2]\), \(H^*_{G\times G}(X)\) is a subring of \(H^*_{G\times G}(O)\). It is known that \(H^*_{G\times G}(O)\) is a polynomial ring. Let \(W\) be the Weyl group of \((G,T)\). The work of Lascoux and Schützenberger \([\text{L-S}]\) provides a finite list of polynomials \(\{\Phi_{\omega,\nu}\}\), indexed over \(W \times W\), that generate \(H^*_{G\times G}(O)\). These polynomials are the Schubert polynomials and play a crucial role in the representation theory of reductive groups. However, \(H^*_{G\times G}(X)\) is a proper subring of \(H^*_{G\times G}(O)\) subject to certain non-trivial relations. That is, a suitable “deformation” of these polynomials is needed. For wonderful embeddings, the results of Strickland \([\text{St}]\) and Uma \([\text{U}]\) yield the right correction factor. Moreover, one can read off the sought-after polynomials from a pair of Bruhat graphs. The natural question...
is: what about other simple embeddings? Except for a few cases, this question has remained unsolved. We plan to solve it as follows:

(1) Construction of generalized Schubert polynomials \( \{\Psi_{v,w}\} \): these polynomials will be indexed by a finite set \( W^J \times W^J \), a quotient of \( W \times W \), which depends on a subset \( J \) of simple reflections. When \( J = \emptyset \), then \( W^J \) is the usual Weyl group. If \( J \) is not empty, then \( W^J \) is no longer a group. Using the results of [R3], one can define a descent set \( D^J(v) \), for each \( v \in W^J \). Our polynomials will include a correction factor coming from the various \( D^J(v) \).

(2) Construction of combined divided differences operators \( \{\partial_s\} \), for each \( s \in S^J \). This is based on ideas of Lascoux and Newton [L]. Using such operators we construct our polynomials \( \{\Psi_{v,w}\} \) by looking at the product of two finite graphs, a procedure that mimics Schubert calculus on flag varieties.

(3) Finally, we provide a neat geometric interpretation of our generalized Schubert polynomials as equivariant pushforwards of characteristic classes associated to the boundary divisors of \( X \). This yields the desired picture for \( H^*_G \times G(X) \) as a \( H^*_G \times G \)-algebra.

4.1.2. More than one closed orbit: Here we address the question: how do we construct explicit polynomial generators for \( H^*_G \times G(X) \) when \( X \) has more than one closed orbit? The consideration of many closed orbits requires a careful analysis of the idempotent set of the underlying monoid \( M \). Our line of attack is to first consider what happens if we assume all the closed orbits to be of the form \( G/B \times G/B^- \). Embeddings with this property are called toroidal embeddings (see [T], [G2] for other characterizations). It includes all regular embeddings studied by De Concini, Procesi, Brion, Strickland and Uma, as well as many singular varieties.

The results of this subsection will appear in the preprint “Group embeddings, divided difference operators and Schubert calculus”, to be submitted by December 2012.

4.2. Equivariant K-theory and rational smoothness. Let \( T = (\mathbb{C}^*)^n \) be an algebraic torus and let \( T = (S^1)^n \) be its compact subtorus. Consider a compact space \( X \) with a \( T \)-action. It is customary to define \( K_T(X) \), the equivariant topological \( K \)-theory of \( X \), as the Grothendieck group of isomorphism classes of \( T \)-equivariant complex vector bundles over \( X \). Similarly, if \( X \) is a locally compact \( T \)-space, one defines \( \tilde{K}_T(X) := \tilde{K}_T(X^+) \), the reduced equivariant \( K \)-theory of the one-point compactification of \( X \) ([AtSe]). Equivariant \( K \)-theory is particularly important in the study of group embeddings. Indeed, it yields a geometric interpretation of various results in representation theory ([AtSe]).

It is an open question whether the techniques developed in [G1] help in understanding the topological equivariant \( K \)-theory of \( \mathbb{Q} \)-filtrable varieties. A normal projective \( T \)-variety \( X \) is called \( \mathbb{Q} \)-filtrable if it has a finite number of \( T \)-fixed points \( x_1, \ldots, x_m \) and the cells

\[
W_i = \{ x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x = x_i \}
\]

of the associated Bialynicki-Birula decomposition are all rationally smooth. The BB-cells of a \( \mathbb{Q} \)-filtrable variety are called rational cells. Rationally smooth projective embeddings are the main examples of \( \mathbb{Q} \)-filtrable varieties. Notice that \( \mathbb{Q} \)-filtrable varieties are equivariantly formal ([G1]). My goal is to provide a GKM-description of \( K_T(X) \). If \( X \) is smooth, this description can be found in [At], [VV] and [U]. However, very little is known when \( X \) is singular. Following a suggestion of W. Graham ([Gr]), we obtain

**Theorem 4.1.** Let \( X \) be a \( T \)-space with a finite number of fixed points. Then \( X \) is equivariantly formal if and only if \( K^*_T(X) \sim K^*_T(X) \otimes_{K^*_T(pt)} \mathbb{Q} \).

If one assumes that \( T \) is one dimensional, then it is possible to reconstruct \( K^*_T(X) \) from the \( K \)-theory of the cells in the context below.
**Theorem 4.2.** Notation being as above, $K^*_T(W_i) \simeq K^*_T(pt)$ if and only if $(W_i)^\alpha$ is rationally smooth for every $\alpha \in T$ of finite order, where $(W_i)^\alpha$ is the subvariety fixed by $\alpha$. Moreover, if $K^*_T(W_i) \simeq K^*_T(pt)$ for every $i = 0, \ldots, m$, then $K^*_T(X)$ is a free $K^*_T$-module of rank $|X^T|$.

This can be thought of as a generalization of the equivariant Bott periodicity [At]. Similar statements hold for higher dimensional tori. The main difficulty here is the presence of singularities on a rational cell and the fact that, even when $X$ is a $T$-variety with no cohomology in odd degrees, $X^\alpha$ might have non-trivial odd cohomology. This is the crucial reason why, in our setting, we cannot apply the topological results of Rosu and Knutson ([RK]) or the algebro-geometric techniques of Edidin-Graham and Vezzosi-Vistoli, as they deal with smooth objects, spaces for which the loci of points fixed by a finite group is always smooth.

It is clear that the conditions of Theorem 4.2 are too rigid, and do not hold in generic situations. For example, letting $T = \mathbb{C}^*$ act on $W = \{z^2 = xy\}$ via $t \cdot (x, y, z) = (t^2 x, t^4 y, t^3 z)$ gives $W^{-1}$ as the union of two coordinate axes, a reducible space which is by no means rationally smooth. In fact, one obtains $K^*_T(W) \simeq R[T] \oplus R[Z_2]$, where $R[Z_2]$ is the representation ring of the cyclic group in two elements. Remarkably, a detailed study of this fundamental example yields

**Theorem 4.3.** Let $(X, x_0)$ be a rational cell. Let $S \subset T$ be a generic one parameter subgroup of $T$ (that is, $X^S = X^T = \{x_0\}$). Assume that $S$ acts freely on $X - \{x_0\}$. Then

$$K^*_S(X) \simeq \mathbb{C}[q, q^{-1}].$$

As the previous theorems illustrate, a good control of the loci of points fixed by a finite group is required. We thus turn to the class of group embeddings for a better understanding of this phenomenon.

Let $M$ and $N$ be two reductive monoids. Following Renner [R2], we write $M \sim_0 N$ if there is a reductive monoid $L$ and finite dominant morphisms $L \to M$ and $L \to N$ of algebraic monoids. It is easy to check that this is indeed an equivalent relation. Moreover, by [R2], $M$ is rationally smooth if and only if $M \sim_0 \prod_i M_{n_i}(\mathbb{C})$.

Now let $X$ be a rationally smooth group embedding. Then the associated BB-cells, say $C_r$, are reductive monoids satisfying $C_r \sim_0 \prod_i M_{n_i}(\mathbb{C})$ (the monoid cells). In ongoing work, we are studying the stability of equivariant $K$-theory under finite surjective morphisms of monoids. This is the key step that allow us to concretely describe $K^*_T \times_T (C_r)$ in terms of the representation rings of the non-trivial isotropy subgroups. Ultimately, we obtain a nice combinatorial description of $K^*_T \times_T (X)$, when $X$ is a rationally smooth projective embedding, after inverting certain primes associated to the non-trivial isotropy subgroups, and the fibre-cardinality of the finite dominant morphisms involved.

In recent correspondence, M. Brion pointed out to the author that determining the structure of the fixed varieties $(G/P \times G/P^-)^{(h,s)}$, where $(h, s)$ are finite-order elements in $T \times T$, and $P, P^-$ form a pair of opposite parabolic subgroups, is of particular interest, especially on the relations to Theorems 4.2 and 4.3. This would not only shed some light into the structure of $X^{(h,s)}$, when $X = \mathbb{P}(M)$, but also provide an interesting setup for applications.

My results are being collected in the preprint “Equivariant $K$-theory, rational smoothness and projective embeddings of algebraic groups”, to be submitted by the end of December 2012. In future work, and using the notion of regular differential forms [Br4], I would like to explore to what extent $K^*_T \times_T (X)$ is isomorphic to the ring of global sections of an equivariant De Rham sheaf defined on the GKM-graph of $X$ (compare [F]).

### 4.3. Equivariant Chow rings of rationally smooth group embeddings

Let $X$ be a rationally smooth group embedding. As explained in the previous section, the associated rational cells $C_r$, appearing in the BB-decomposition $X = \bigcup C_r$, are monoid cells satisfying...
$C_r \sim_0 \prod_i M_{r_i}(\mathbb{C})$. An interesting problem is to determine whether the (rational) equivariant Chow rings $A^*_{T \times T}(X)$ and $A^*_{G \times G}(X)$ (as defined by Edidin-Graham [EG]) are free over $A^*_{T \times T}(pt)$ and $A^*_{G \times G}(pt)$ respectively. Brion has shown that this is indeed the case if $X$ is smooth ([Br1]), and provides a GKM-description of these rings. If $X$ has rationally smooth singularities, the structure of the monoid cells gives a good control of the equivariant cycles. We thank M. Brion for leading us to the following.

**Theorem 4.4.** Let $X$ be a rationally smooth group embedding. Then $A^*_{T \times T}(X)$ and $A^*_{G \times G}(X)$ are free modules over $A^*_{T \times T}(pt)$ and $A^*_{G \times G}(pt)$, respectively. Moreover, $A^*_{T \times T}(X)$ is a ring of piecewise polynomial functions and the cycle map

$$cl_X : A^*_{T \times T}(X) \to H^*_{T \times T}(X)$$

is an isomorphism.

In a forthcoming paper, entitled “Rational smoothness in equivariant Chow rings”, we elaborate on these findings and extend the notions of rational smoothness and Euler classes to the setting of Chow rings. I plan to submit this preprint by mid-December 2012.

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