AN EXTENSION OF THE THEORY OF FREDHOLM DETERMINANTS

by David RUELLE

Abstract. — Analytic functions are introduced, which are analogous to the Fredholm determinant, but may have only finite radius of convergence. These functions are associated with operators of the form $\int \mu(d\omega)\, \mathscr{L}_\omega$, where $\mathscr{L}_\omega \Phi(x) = \varphi_\omega(x) . \Phi(\psi_\omega x)$, Φ belongs to a space of Hölder or C^r functions, φ_ω is Hölder or C^r, and ψ_ω is a contraction or a C^r contraction. The results obtained extend earlier results by Haydn, Pollicott, Tangerman and the author on zeta functions of expanding maps.

1. Assumptions and statement of results

The theory of Fredholm determinants (see for instance [10]) has been extended by Grothendieck [5] and applies to linear operators $\mathscr{K}$ in certain suitable classes. One associates with $\mathscr{K}$ an entire analytic function $d_{\mathscr{K}}$, called the Fredholm determinant, such that

$$(1 - z\mathscr{K})^{-1} = \mathscr{N}(z)/d_{\mathscr{K}}(z)$$

where $\mathscr{N}$ is an entire analytic operator-valued function. In what follows we shall obtain results of the same type. The radius of convergence of the " determinant " will possibly be finite rather than infinite, but larger than the inverse of the spectral radius of $\mathscr{K}$.

The type of extension that we shall obtain concerns operators $\mathscr{K}$ with a kernel $K(x, y)$ which is allowed to have δ-singularities of the type $\varphi(x)\, \delta(y - \psi(x))$, where φ and ψ have certain smoothness properties and ψ is a contraction. Operators of this sort arise in the theory of an expanding map f (or more generally of hyperbolic dynamical systems), and the Fredholm determinants are then related (as we shall see) to dynamical zeta functions which count the periodic points of f, with certain weights. It is desirable to understand the analytic properties of the zeta function and Fredholm determinants because they are closely related to the ergodic properties of the dynamical system defined by f (see [13]). The hyperbolic case of contracting or expanding maps considered here is that for which the most detailed results are known, but extensions to nonhyperbolic situations are possible, as the work of Baladi and Keller [1] on one-dimensional systems indicates.

Let $\alpha > 0$, $0 < \theta < 1$, and let X be a compact metric space. We denote by $C^\alpha = C^\alpha(X)$ the Banach space of (uniformly) α-Hölder functions $X \to \mathbf{C}$ with the usual norm. We assume that $V \subset X$, $\psi : V \to X$ and $\varphi \in C^\alpha$ are given such that ψ is a contraction:

$$d(\psi x, \psi y) \leqslant \theta\, d(x, y)$$

and φ has its support in V. A bounded linear operator $\mathscr{L}$ on C^α is then defined by

$$(\mathscr{L}\Phi)\,(x) = \begin{cases} \varphi(x)\,.\,\Phi(\psi x) & \text{if } x \in V, \\ 0 & \text{if } x \notin V. \end{cases}$$

The operators $\mathscr{K}$ which will interest us are integrals of operators of the form $\mathscr{L}$:

$$\mathscr{K} = \int \mu(d\omega)\, \mathscr{L}_\omega \tag{1.1}$$

where $\mathscr{L}_\omega$ is defined with V_ω, ψ_ω, φ_ω as above, and where μ is a finite positive measure (which we may take to be a probability measure). The following will be standing assumptions:

(i) $\int \mu(d\omega)\, \| \varphi_\omega \| < \infty$

where $\| \quad \|$ is the norm in C^α;

(ii) There is $\delta > 0$ such that, for all ω, V_ω contains the δ-neighborhood of the support of φ_ω;

(iii) $\omega \mapsto V_\omega, \psi_\omega, \varphi_\omega$ are measurable. (Using (ii), and possibly changing δ, we may assume that there are only finitely many different V_ω's, and that they are compact subsets of X. We may take as measurability condition the assumption that $\omega \mapsto V_\omega$, $(\omega, x) \mapsto \psi_\omega(x)$, $\varphi_\omega(x)$ are Borel functions.)

We write

$$\zeta_m = \int \mu(d\omega_1) \ldots \mu(d\omega_m)\, \varphi_{\omega_m}(x(\bar\omega))\, \varphi_{\omega_{m-1}}(\psi_{\omega_m} x(\bar\omega)) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x(\bar\omega)), \tag{1.2}$$

where the integral extends to values of $\omega_1, \ldots, \omega_m$ such that $\psi_{\omega_1}\psi_{\omega_2} \ldots \psi_{\omega_m}$ has a fixed point, which is then necessarily unique, and which we denote by $x(\bar\omega)$. A zeta function is then defined through the following formal power series

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \zeta_m. \tag{1.3}$$

1.1. *Theorem. — Let* $|\mathscr{K}|$ *denote the operator obtained when* φ_ω *is replaced by* $|\varphi_\omega|$ *in the definition of* $\mathscr{K}$, *and let* e^P *be the spectral radius* of* $|\mathscr{K}|$. *The spectral radius of* $\mathscr{K}$ *is then* $\leqslant e^P$, *and the part of the spectrum of* $\mathscr{K}$ *contained in* $\{ \lambda : |\lambda| > \theta^\alpha e^P \}$ *consists of isolated eigenvalues of finite multiplicities. Furthermore,* $1/\zeta(z)$ *converges in*

$$\{ z : |z|\, \theta^\alpha e^P < 1 \} \tag{1.4}$$

* The proof (section 2.5) shows that e^P is also the spectral radius of $|\mathscr{K}|$ taken with respect to the "uniform" norm $\| \quad \|_0$.

and its zeros in this domain are precisely the inverses of the eigenvalues of $\mathscr{K}$, *with the same multiplicities. We may thus write*

$$(1 - z\mathscr{K})^{-1} = \zeta(z)\mathscr{N}(z)$$

where $\mathscr{N}$ *is a holomorphic operator-valued function in* (1.4).

The proof of this theorem is given in Section 2.

1.2. *Remarks.* — *a)* We see that $1/\zeta(z)$ plays the role of a Fredholm determinant. However, $\zeta(z)$ depends on the decomposition (1.1) and not just on the operator $\mathscr{K}$. We shall obtain a "true" determinant in the differentiable case below.

b) Let E be a finite-dimensional α-Hölder vector bundle over X (i.e., E is trivialized by a finite atlas, and the transition between charts uses matrix-valued α-Hölder functions). We assume that $\varphi_\omega : E \to E$ is an adjoint vector bundle map over ψ_ω for every ω (i.e., $\varphi_\omega(x)$: $E(\psi_\omega x) \to E(x)$). We can then define the operator $\mathscr{K}$ as before; it now acts on the Banach space C^α_E of α-Hölder sections of E. We also define

$$\zeta_m = \int \mu(d\omega_1) \ldots \mu(d\omega_m) \operatorname{Tr} \varphi_{\omega_m}(x(\bar\omega))\, \varphi_{\omega_{m-1}}(\psi_{\omega_m} x(\bar\omega)) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x(\bar\omega))$$

where Tr is the trace on $E(x(\bar\omega))$.

Let $|\varphi_\omega(x)|$ be the norm of $\varphi_\omega(x)$ for some metric on E, and $|\mathscr{K}|$ the operator on C^α obtained by the replacement of φ_ω by $|\varphi_\omega|$ in the definition of $\mathscr{K}$. Finally, let e^P be the spectral radius of $|\mathscr{K}|$. It is easily seen from the proofs that, with these new definitions, Theorem 1.1 remains true. [For a sharper result, let $|\mathscr{K}^m|$ be obtained by the replacement of $\varphi_{\omega_m} \ldots \varphi_{\omega_1}$ by $|\varphi_{\omega_m} \ldots \varphi_{\omega_1}|$ in $\mathscr{K}^m$, and take

$$P = \lim \frac{1}{m} \log \| \, |\mathscr{K}^m| \, \|.]$$

Theorem 1.3 below can similarly be extended to differentiable vector bundles. In particular, this permits the treatment of the operators $\mathscr{K}^{(\ell)}$ corresponding to $\mathscr{K}$ but acting on ℓ-forms; see Corollary 1.5.

c) Let $\boldsymbol{r} = (r, \alpha)$ with integer $r \geqslant 0$ and $0 \leqslant \alpha \leqslant 1$. We denote by $C^{\boldsymbol{r}} = C^{\boldsymbol{r}}(X)$ the Banach space (with the usual norm) of functions $X \to \mathbf{C}$ which have continuous derivatives up to order r, the r-th derivative being uniformly α-Hölder. We shall write $\boldsymbol{r} \geqslant 1$ if $r \geqslant 1$, and $|\boldsymbol{r}| = r + \alpha$.

1.3. *Theorem.* — *Let* X *be a smooth compact Riemann manifold. We make the same assumptions as in Theorem* 1.1, *but with* φ_ω, ψ_ω *of class* $C^{\boldsymbol{r}}$, $\boldsymbol{r} \geqslant 1$. *We require that* $\int \mu(d\omega)\, \|\varphi_\omega\| < \infty$, *where* $\|.\|$ *is now the* $C^{\boldsymbol{r}}$ *norm, and let* $\mathscr{K}$ *act on* $C^{\boldsymbol{r}}$. *With these assumptions, the part of the spectrum of* $\mathscr{K}$ *contained in* $\{\lambda : |\lambda| > \theta^{|\boldsymbol{r}|} e^P\}$ *consists of isolated eigenvalues of finite multiplicities.*

Define $\operatorname{tr}\mathcal{K}^m$ *by*

$$\operatorname{tr}\mathcal{K}^m = \int \mu(d\omega_1) \ldots \mu(d\omega_m) \left(\det(1 - D_{x(\bar\omega)}\psi_{\omega_1} \ldots \psi_{\omega_m})\right)^{-1} \varphi_{\omega_m}(x(\bar\omega))\, \varphi_{\omega_{m-1}}(\psi_{\omega_m} x(\bar\omega)) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x(\bar\omega))$$

(where $D_x \psi$ *denotes the derivative of* ψ *at the fixed point* x*), and write*

$$d(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \operatorname{tr}\mathcal{K}^m.$$

Then, $d(z)$ *converges in*

$$\{ z : | z | \theta^{|r|} e^{P} < 1 \} \tag{1.5}$$

and its zeros there are precisely the inverses of the eigenvalues of $\mathcal{K}$*, with the same multiplicities. We may therefore write*

$$(1 - z\mathcal{K})^{-1} = n(z)/d(z)$$

where n *is a holomorphic operator-valued function in* (1.5).

The proof of this theorem is given in Section 3.

1.4. *Remarks.* — *a)* Theorem 1.3 also holds if we take $\boldsymbol{r} = (0, \alpha)$, $\alpha > 0$, but assume that the ψ_ω are differentiable. In that case $z \mapsto \zeta(z)\, d(z)$ is analytic and without zero in (1.4).

b) The assumption that X is compact is for simplicity. It would suffice to assume that $\bigcup_\omega V_\omega$ and $\bigcup_\omega \psi_\omega V_\omega$ are contained in a compact subset of a finite-dimensional (non-compact) manifold.

1.5. *Corollary.* — *Under the conditions of Theorem* 1.3, *define an operator* $\mathcal{K}^{(\ell)}$ *acting on the space of* ℓ*-forms of class* C^{r-1} *on* X *by*

$$\mathcal{K}^{(\ell)} = \int \mu(d\omega)\, \mathcal{L}^{(\ell)}_\omega,$$

where

$$(\mathcal{L}^{(\ell)}_\omega \Phi)(x) = \begin{cases} \varphi(x) . \Lambda^\ell(T^*_x \psi) . \Phi(\psi_\omega x) & \text{if } x \in V_\omega, \\ 0 & \text{if } x \notin V_\omega. \end{cases}$$

Let also

$$\operatorname{tr}\mathcal{K}^{(\ell)m} = \int \mu(d\omega_1) \ldots \mu(d\omega_m) \left[\det(1 - D_{x(\bar\omega)}\psi_{\omega_1} \ldots \psi_{\omega_m})\right]^{-1} \operatorname{Tr}_\ell \Lambda^\ell(D_{x(\bar\omega)}\psi_{\omega_1} \ldots \psi_{\omega_m})\, \varphi_{\omega_m}(x(\bar\omega))\, \varphi_{\omega_{m-1}}(\psi_{\omega_m} x(\bar\omega)) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x(\bar\omega))$$

where Tr_ℓ *is the trace of operators in* $\Lambda^\ell(T_{x(\bar\omega)} X)$ *and*

$$d^{(\ell)}(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \operatorname{tr}\mathcal{K}^{(\ell)m}.$$

With these definitions $\mathcal{K}^{(0)} = \mathcal{K}$, $d^{(0)}(z) = d(z)$, *and the spectral radius of* $\mathcal{K}^{(\ell)}$ *is* $\leqslant \theta^\ell e^P$. *Furthermore, if* $\ell \geqslant 1$, *the essential spectral radius of* $\mathcal{K}^{(\ell)}$ *is* $\leqslant \theta^{|r|+\ell-1} e^P$, $d^{(\ell)}(z)$ *converges in*

$$\{ z : | z | \theta^{|r|+\ell-1} e^P < 1 \}$$

and its zeros there are precisely the inverses of the eigenvalues of $\mathcal{K}^{(\ell)}$*, with the same multiplicities.*

To obtain the corollary, we have to use the extension of Theorem 1.3 to vector bundles (here the cotangent bundle) as explained in Remark 1.2 *b)*. It is clear that the spectral radius of $\mathscr{K}^{(\ell)}$ is $\leqslant \theta^\ell e^{\mathrm{P}}$. Note also that when $\ell \geqslant 1$, the degree of differentiability $\boldsymbol{r}$ has to be replaced by $\boldsymbol{r} - 1$. From this, the corollary follows. (For the case where $\boldsymbol{r} - 1 < 1$, use Remark 1.4 *a)*.)

1.6. *Corollary.* — *Under the conditions of Theorem* 1.3, *we may write*

$$\zeta(z) = \Pi[d^{(\ell)}(z)]^{(-1)^{\ell+1}},$$

where ℓ ranges from 0 *to* $\dim \mathrm{X}$, *so that the zeta function* (1.3) *is meromorphic in* (1.5).

This follows from the identity

$$\zeta_m = \sum_{\ell=0}^{\dim \mathrm{X}} (-1)^\ell \operatorname{tr} \mathscr{K}^{(\ell)m}$$

where ζ_m was defined in (1.2).

1.7. *Corollary.* — *a) Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be operators on* C^{r_1} *and* C^{r_2} *defined by the same* $\mu(d\omega)$, V_ω *and* ψ_ω, φ_ω *of class* C^{r_1}, *with* $\boldsymbol{r}_1 > \boldsymbol{r}_2$. *Then, in the domain,*

$$\{\lambda : |\lambda| > \theta^{|\boldsymbol{r}_2|} e^{\mathrm{P}}\}$$

the operators $\mathscr{K}_1$ and $\mathscr{K}_2$ have the same eigenvalues with the same multiplicities and the same generalized eigenspaces (which consist of C^{r_1} *functions). If* ψ_ω, φ_ω *are* C^∞, *it therefore makes sense to speak of the eigenvalues and eigenfunctions of $\mathscr{K}$ acting on* C^∞, *and $d(z)$ clearly is an entire function*.*

b) If $|\lambda| > \theta^{|\boldsymbol{r}|} e^{\mathrm{P}}$, the elements of the generalized eigenspace of the adjoint $\mathscr{K}^*$ of $\mathscr{K}$ corresponding to the eigenvalue λ are distributions in the sense of Schwartz, of order s for all

$$s > \frac{\mathrm{P} - \log|\lambda|}{|\log\theta|}.$$

To prove *a)* note that the generalized eigenspace of $\mathscr{K}_1$ maps injectively by inclusion in the generalized eigenspace of $\mathscr{K}_2$, but both have the same dimension given by the multiplicity of a zero of $d(z)$. From *a)*, one derives *b)* easily.

1.8. *Expanding maps.* — The case where the ψ_ω are local inverses of a map $f : \mathrm{X} \to \mathrm{X}$ has relations to statistical mechanics and applications to Axiom A dynamical systems and hyperbolic Julia sets. Various aspects of this case have been discussed by Ruelle [12], Pollicott [9], Tangerman [15], and Haydn [6], and a general review has been given in [13]. Note that the conjectures A and B of [13] are proved in the present paper. The real analytic situation, not considered here, has been discussed in Ruelle [11], Mayer [7], and Fried [3], and leads to Fredholm determinants in the sense of Grothendieck [5].

* It would be interesting to estimate the growth of $d(z)$ at infinity.

Note that an erroneous statement about the growth of determinants in [4] and [11] has been corrected by Fried [3]. For piecewise monotone one-dimensional maps see Baladi and Keller [1].

The case of an expanding map f is analysed by using a Markov partition (for which, see Sinai [14] and Bowen [2]). In the more general situation discussed here, there are no Markov partitions. Our proofs will make use, instead, of suitable coverings of X by balls. The present treatment is completely self-contained, but reference to [13] is interesting in providing for instance an interpretation of the spectral radius e^{P} as exponential of a topological pressure.

1.9. *Other examples.* — A class of examples where the results of the present paper apply is described as follows. Let X be a compact manifold, $\hat{\mathrm{X}}$ its universal cover, and $\pi : \hat{\mathrm{X}} \to \mathrm{X}$ the canonical map. We assume that $\hat{\psi} : \hat{\mathrm{X}} \to \hat{\mathrm{X}}$ is a contraction, such that $d(\hat{\psi}x, \hat{\psi}y) \leqslant \theta\, d(x,y)$ and that $\hat{\varphi} : \hat{\mathrm{X}} \to \mathbf{C}$ is of class C^r and suitably tending to zero at infinity. Define

$$(\mathscr{K}\Phi)\ (x) = \Sigma_{y \in \pi^{-1}x}\, \hat{\varphi}(y)\ \Phi(\pi\hat{\psi}y).$$

It is not hard to see that $\mathscr{K}$ is of the form discussed above, and we have

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \Sigma_{(y_1, \ldots, y_m)}\, \hat{\varphi}(y_m) \ldots \hat{\varphi}(y_2)\, \hat{\varphi}(y_1)$$

where the second sum is over m-tuples such that

$$\pi y_1 = \pi\hat{\psi}y_2, \ \ldots, \ \pi y_{m-1} = \pi\hat{\psi}y_m, \qquad \pi y_m = \pi\hat{\psi}y_1.$$

If $\mathrm{X} = \mathbf{R}/\mathbf{Z}$ and $\hat{\psi}y = \theta y$, then $d^{(1)}(z) = d(\theta z)$, so that $\zeta(z) = d(\theta z)/d(z)$.

2. Proof of Theorem 1.1

2.1. *Coverings of* X *by balls.* — The following construction involves the constants θ, δ of Section 1 and a constant κ which will be selected later; for the moment we only assume that $0 < \kappa \leqslant 1$. Let $(x_i)_{i \in \mathrm{I}}$ be a finite $(\kappa/2)\ \delta(1 - \theta)$-dense family of points of X. In particular, the balls

$$\mathrm{X}_i = \{ x : d(x, x_i) < \delta/2 \}$$

cover X. For each j, ω with $\mathrm{X}_j \subset \mathrm{V}_\omega$ we choose measurably $u(j, \omega)$ such that

$$d(\psi_\omega x_j, x_{u(j,\omega)}) < (\kappa/2)\ \delta(1 - \theta)$$

and therefore

$$\psi_\omega \mathrm{X}_j \subset \mathrm{X}_{u(j,\omega)}.$$

For each integer $m \geqslant 0$ we shall now define a finite set $\mathrm{J}^{(m)}$ and a family $(\mathrm{X}(a))_{a \in \mathrm{J}^{(m)}}$ of open balls in X. We choose θ' such that $\theta < \theta' < 1$, and we shall define $\mathrm{J}^{(m)}$ and $(\mathrm{X}(a))$ by induction on m.

First, $J^{(o)} = \{(i) : i \in I\}$, and we let $X_i^o = X_i$ be as before the balls of radius $\delta/2$ and centers $x_i^o = x_i$ forming a $(\kappa/2)\,\delta(1-\theta)$-dense set in X. For $m \geqslant 1$, let similarly (X_k^m) be a finite family of open balls of radius $\delta\theta'^m/2$ and centers x_k^m forming a $(\kappa/2)\,\delta(\theta'-\theta)\,\theta'^{m-1}$-dense set in X. We put

$$J^{(m)} = \{(i, \ldots, k, \ell) : (i, \ldots, k) \in J^{(m-1)} \quad \text{and} \quad d(x_\ell^m, x_k^{m-1}) \leqslant \kappa\,\delta\theta'^{m-1}\}.$$

Choose now $\kappa = (1-\theta')/2$. If $a = (i, \ldots, k, \ell) \in J^{(m)}$ we have $X_\ell^m \subset X_k^{m-1}$, and by induction

$$X_\ell^m \subset X_k^{m-1} \subset \ldots \subset X_i.$$

We shall write $x(a) = x_\ell^m$, $X(a) = X_\ell^m$. We define $p : J^{(m)} \to J^{(m-1)}$ by

$$p(i, \ldots, k, \ell) = (i, \ldots, k).$$

Given $b = (i', \ldots, k') \in J^{(m-1)}$ and ω such that $X_{i'} \subset V_\omega$, we define $v(b, \omega) = (i, \ldots, k, \ell)$ by

$$i = u(i', \omega),$$
$$(i, \ldots, k) = v(pb, \omega) \quad \text{if } m > 1,$$

and ℓ is chosen measurably such that

$$d(\psi_\omega x_{k'}^{m-1}, x_\ell^m) < (\kappa/2)\,\delta(\theta'-\theta)\,\theta'^{m-1}.$$

We have thus

$$pv((i'), \omega) = u(i', \omega),$$
$$pv(b, \omega) = v(pb, \omega) \quad \text{for } m > 1,$$
$$\psi_\omega X(b) \subset X(v(b, \omega)).$$

2.2. *Lemma.* — *We have* $v(b, \omega) \in J^{(m)}$.

We write $b = (i', \ldots, j', k')$. We only have to check that

$$\begin{aligned} d(x_\ell^m, x_k^{m-1}) &\leqslant d(x_\ell^m, \psi_\omega x_{k'}^{m-1}) + d(\psi_\omega x_{k'}^{m-1}, \psi_\omega x_{j'}^{m-2}) + d(\psi_\omega x_{j'}^{m-2}, x_k^{m-1}) \\ &\leqslant (\kappa/2)\,\delta(\theta'-\theta)\,\theta'^{m-1} + \theta\kappa\,\delta\theta'^{m-2} + (\kappa/2)\,\delta(\theta'-\theta)\,\theta'^{m-2} \\ &\leqslant \kappa\,\delta(\theta'-\theta)\,\theta'^{m-2} + \kappa\,\delta\theta\theta'^{m-2} = \kappa\,\delta\theta'^{m-1} \end{aligned}$$

for $m > 1$, and a similar inequality for $m = 1$.

2.3. *The operator* $\mathscr{M}$. — We define

$$\tau_{ji}(\omega) = \begin{cases} 1 & \text{if } X_j \subset V_\omega \text{ and } i = u(j, \omega), \\ 0 & \text{otherwise.} \end{cases}$$

Let Φ_i, $(\mathscr{H}\Phi)_j$ denote the restrictions of Φ, $\mathscr{H}\Phi$ to X_i and X_j respectively. We may then write

$$(\mathscr{H}\Phi)_j(x) = \Sigma_i \int \mu(d\omega)\,\tau_{ji}(\omega)\,\varphi_\omega(x)\,\Phi_i(\psi_\omega x). \tag{2.1}$$

If $\Sigma_{i\in I} X_i$ is the disjoint sum of the X_i, we may write

$$\oplus_i C^\alpha(X_i) = C^\alpha(\Sigma X_i)$$

and define an operator $\mathscr{M}$ on that space by

$$(\mathscr{M}\Phi)_j(x) = \Sigma_i \int \mu(d\omega)\, \tau_{ji}(\omega)\, \varphi_\omega(x)\, \Phi_i(\psi_\omega x).$$

This is the same formula as (2.1), but the Φ_i may now be chosen independently on the various X_i. If we identify $C^\alpha(X)$ with a subspace of $\oplus_i C^\alpha(X_i)$, we see that the restriction of $\mathscr{M}$ to $C^\alpha(X)$ is $\mathscr{K}$. Note that

$$(2.2) \qquad (\mathscr{M}^m \Phi)_{i_m}(x) = \Sigma_{i_0,\ldots,i_{m-1}} \int \mu(d\omega_1) \ldots \mu(d\omega_m)\, \tau_{i_m i_{m-1}}(\omega_m) \ldots \tau_{i_1 i_0}(\omega_1)$$
$$\varphi_{\omega_m}(x) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x)\, \Phi_{i_0}(\psi_{\omega_1} \ldots \psi_{\omega_m} x).$$

2.4. *The operators* $\mathscr{M}^{(m)}$. — For $m \geqslant 1$ we define an operator

$$\mathscr{M}^{(m)} : \oplus_{a\in \mathscr{J}^{(m)}} C^\alpha(X(a)) \to \oplus_{i\in I} C^\alpha(X_i)$$

by the formula

$$(2.3) \qquad (\mathscr{M}^{(m)}\Phi)_j(x) = \int \mu(d\omega_1) \ldots \mu(d\omega_m)$$
$$\varphi_{\omega_m}(x)\, \varphi_{\omega_{m-1}}(\psi_{\omega_m} x) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x)\, \Phi_{v(j,\overline{\omega})}(\psi_{\omega_1} \ldots \psi_{\omega_m} x)$$

where $v(j, \overline{\omega}) = v(v(\ldots v((j), \omega_m), \ldots)\, \omega_1)$. Define

$$Q^{(m)} : \oplus_{i\in I} C^\alpha(X_i) \to \oplus_{a\in \mathscr{J}^{(m)}} C^\alpha(X(a))$$

as the restriction operator such that

$$(Q^{(m)}\Phi)_a = \Phi_i \mid X(a),$$

when $p^m a = (i)$. In view of (2.2), (2.3), we have

$$\mathscr{M}^{(m)} Q^{(m)} = \mathscr{M}^m.$$

We shall also need the operator

$$T^{(m)} : \oplus_{a\in \mathscr{J}^{(m)}} C^\alpha(X(a)) \to \oplus_{a\in \mathscr{J}^{(m)}} C^\alpha(X(a))$$

such that

$$(T^{(m)}\Phi)_a = \Phi(x(a)).$$

We define the norm on $\oplus_i C^\alpha(X_i)$ by

$$\|\Phi\| = \max_{i\in I}\left(\sup_x |\Phi_i(x)| + \sup_{x\neq y} \frac{|\Phi_i(x) - \Phi_i(y)|}{d(x,y)^\alpha}\right)$$

and similarly for $\oplus_{a\in \mathscr{J}^{(m)}} C^\alpha(X(a))$.

Note that, with these norms,

$$\|Q^{(m)}\| \leqslant 1, \qquad \|T^{(m)}\| \leqslant 1.$$

2.5. *Proposition.* — *a) The spectral radius of* $\mathscr{M}$ *(and thus* $\mathscr{K}$*) is* $\leqslant$ *the spectral radius* e^{P} *of* $|\mathscr{K}|$.

b) Given $\varepsilon > 0$, *we have*

$$\| \mathscr{M}^{(m)} - \mathscr{M}^{(m)} \mathrm{T}^{(m)} \| \leqslant \mathrm{const}(\theta'^{\alpha} e^{\mathrm{P}+\varepsilon})^m \tag{2.4}$$

and therefore the essential spectral radius of $\mathscr{M}$ *(and thus* $\mathscr{K}$*) is* $\leqslant \theta^{\alpha} e^{\mathrm{P}}$.

Using (2.2) we have

$$\frac{|(\mathscr{M}^m \Phi)(x) - (\mathscr{M}^m \Phi)(y)|}{d(x,y)^{\alpha}}$$
$$\leqslant \| \mathscr{M}^m \|_0 \| \Phi \| + \mathrm{const} \sum_{k=1}^{m} \| \mathscr{M}^{k-1} \|_0 \| \mathscr{M}^{m-k} \|_0 \| \Phi \|_0,$$

so that

$$\begin{aligned}
\lim_{m\to\infty} (\| \mathscr{M}^m \|)^{1/m} &= \lim_{m\to\infty} (\| \mathscr{M}^m \|_0)^{1/m} \\
&\leqslant \lim_{m\to\infty} (\| \, |\mathscr{M}|^m \|_0)^{1/m} = \lim_{m\to\infty} (\| \, |\mathscr{M}|^m 1 \|_0)^{1/m} \\
&= \lim_{m\to\infty} (\| \, |\mathscr{K}|^m 1 \|_0)^{1/m} = \lim_{m\to\infty} (\| \, |\mathscr{K}|^m \|_0)^{1/m},
\end{aligned}$$

and *a)* follows from the spectral radius formula.

Using the definition (2.3) and the estimate $\| \Phi - \mathrm{T}^{(m)} \Phi \|_0 \leqslant \| \Phi \| (\delta\theta'^m/2)^{\alpha}$, we have also

$$\frac{|(\mathscr{M}^{(m)}(1 - \mathrm{T}^{(m)}) \Phi)(x) - (\mathscr{M}^{(m)}(1 - \mathrm{T}^{(m)}) \Phi)(y)|}{d(x,y)^{\alpha}}$$
$$\leqslant \| \mathscr{M}^{(m)} \|_0 \| \Phi \| (\delta\theta^m)^{\alpha} + \mathrm{const} \sum_{k=1}^{m} \mathrm{C}(k) . \| \Phi \| (\delta\theta'^m/2)^{\alpha}$$

where the const comes from the Hölder norm of φ and $\mathrm{C}(k)$ is estimated, taking absolute values, by

$$\begin{aligned}
\mathrm{C}(k) &\leqslant \| \, |\mathscr{M}|^{k-1} 1 \|_0 . \| \, |\mathscr{M}|^{m-k} 1 \|_0 \\
&\leqslant \| \, |\mathscr{K}|^{k-1} \| . \| \, |\mathscr{K}|^{m-k} \|.
\end{aligned}$$

From this the estimate (2.4) follows, and *b)* results from Nussbaum's essential spectral radius formula [8].

2.6. *The operators* $\mathscr{M}_k$ *and* $\mathscr{M}_k^{(m)}$. — If $k \geqslant 0$, we shall define an operator $\mathscr{M}_k$ on

$$\bigoplus_{(i_0, \ldots, i_k)} \mathrm{C}^{\alpha}(\mathrm{X}_{i_0} \cap \ldots \cap \mathrm{X}_{i_k})$$

where the sum extends over the set I_k of $(k+1)$-tuples $\mathbf{i} = (i_0, \ldots, i_k)$ such that $i_0 < \ldots < i_k$ and $\mathrm{X}_{i_0} \cap \ldots \cap \mathrm{X}_{i_k} \neq \varphi$. Let $u(\mathbf{j}, \omega) = (u(j_0, \omega), \ldots, u(j_k, \omega))$ and

$$\tau_{\mathbf{ji}}(\omega) = \begin{cases} 1 \ (\text{or} - 1) & \text{if } \mathrm{X}_{j_0}, \ldots, \mathrm{X}_{j_k} \subset \mathrm{V}_{\omega} \text{ and } u(\mathbf{j}, \omega) \text{ is an even} \\ & \text{(or odd) permutation of } \mathbf{i}, \\ 0 & \text{otherwise.} \end{cases}$$

We write then

$$(\mathcal{M}_k \Phi)_J(x) = \Sigma_I \int \mu(d\omega)\ \tau_{JI}(\omega)\ \varphi_\omega(x)\ \Phi_I(\psi_\omega x).$$

Let now

$$\bigoplus_{(a_0, \dots, a_k)} C^\alpha(X(a_0) \cap \dots \cap X(a_k))$$

be the sum over those $(k+1)$-tuples of elements of $J^{(m)}$ such that $X(a_0) \cap \dots \cap X(a_k) \neq \varnothing$ and $p^m a_0 = (i_0), \dots, p^m a_k = (i_k)$, with $i_0 < \dots < i_k$. We define then

$$Q_k^{(m)} : \bigoplus_{(i_0, \dots, i_k)} C^\alpha(X_{i_0} \cap \dots \cap X_{i_k}) \to \bigoplus_{(a_0, \dots, a_k)} C^\alpha(X(a_0) \cap \dots \cap X(a_k))$$

so that $Q_k^{(m)}$ is the restriction from $X_{i_0} \cap \dots \cap X_{i_k}$ to $X(a_0) \cap \dots \cap X(a_k)$.
We also define

$$\mathcal{M}_k^{(m)} : \bigoplus_{(a_0, \dots, a_k)} C^\alpha(X(a_0) \cap \dots \cap X(a_k)) \to \bigoplus_{(i_0, \dots, i_k)} C^\alpha(X_{i_0} \cap \dots \cap X_{i_k})$$

by

$$(\mathcal{M}_k^{(m)} \Phi)_J(x) = \int \mu(d\omega_1) \dots \mu(d\omega_m)\ \varphi_{\omega_m}(x) \dots \varphi_{\omega_1}(\psi_{\omega_2} \dots \psi_{\omega_m} x)$$
$$(\varepsilon \Phi_{(a_0, \dots, a_k)}(\psi_{\omega_1} \dots \psi_{\omega_m} x)),$$

where ε, and $a_0, \dots, a_k \in J^{(m)}$ are determined as follows. If $p^m v(j_0, \bar\omega), \dots, p^m v(j_k, \bar\omega)$ are not all different, write $\varepsilon = 0$. Otherwise, let π be the permutation which arranges these indices in increasing order, and write $\varepsilon = \operatorname{sign} \pi$, $(a_0, \dots, a_k) = \pi(v(j_0, \bar\omega), \dots, v(j_k, \bar\omega))$.

Finally, we choose an arbitrary point* $x(a) \in X(a_0) \cap \dots \cap X(a_k)$ for every $(k+1)$-tuple $\mathbf{a} = (a_0, \dots, a_k)$ and define an operator $T_k^{(m)}$ on

$$\bigoplus_{(a_0, \dots, a_k)} C^\alpha(X(a_0) \cap \dots \cap X(a_k)) \quad \text{by} \quad (T_k^{(m)} \Phi)_{\mathbf{a}} = \Phi(x(\mathbf{a})).$$

With these definitions we have

$$\| Q_k^{(m)} \| \leqslant 1, \qquad \| T_k^{(m)} \| \leqslant 1$$
$$\mathcal{M}_k^{(m)} Q_k^{(m)} = \mathcal{M}_k^m.$$

Note that for $k = 0$ the operators $\mathcal{M}_k$, $Q_k^{(m)}$, $\mathcal{M}_k^{(m)}$ reduce to $\mathcal{M}$, $Q^{(m)}$, $\mathcal{M}^{(m)}$.

2.7. *Proposition. — a) The spectral radius of $\mathcal{M}_k$ is $\leqslant e^P$.*

b) Given $\varepsilon > 0$, we have

$$\| \mathcal{M}_k^{(m)} - \mathcal{M}_k^{(m)} T_k^{(m)} \| < \operatorname{const}(\theta'^\alpha e^{P+\varepsilon})^m$$

and therefore the essential spectral radius of $\mathcal{M}_k$ is $\leqslant \theta^\alpha e^P$.

The proof is essentially the same as that of Proposition 2.5.

2.8. *Lemma. — Suppose that $\psi_{\omega_1} \dots \psi_{\omega_m}$ has a fixed point $x(\bar\omega) \in$ support φ_{ω_m}. Then*

(2.5) $$\Sigma_k (-1)^k \Sigma_{I_0, \dots, I_{m-1} \in I_k} \tau_{I_0 I_{m-1}}(\omega_m) \dots \tau_{I_2 I_1}(\omega_2)\ \tau_{I_1 I_0}(\omega_1) = 1.$$

---

* When $k = 0$, take $x(a_0)$ to be the center of $X(a_0)$ as before.

Let $\mathrm{I}^* = \{j : \mathrm{X}_j \in \mathrm{V}_{\omega_m}\}$ and $\alpha : \mathrm{I}^* \to \mathrm{I}$ be the map such that there exist $i_1, \ldots, i_{m-1}$ for which

$$\tau_{ji_{m-1}}(\omega_m) \ldots \tau_{i_1 \alpha(j)}(\omega_1) = 1.$$

By assumption $\mathrm{I}^* \neq \varnothing$, and clearly $\alpha \mathrm{I}^* \subset \mathrm{I}^*$. Let $\hat{\mathrm{I}}$ be the set of all α-periodic points in I^*, and $\hat{\alpha}$ the restriction of α to $\hat{\mathrm{I}}$. Then $\hat{\mathrm{I}} \neq \varnothing$ and $\hat{\alpha}$ is a permutation of $\hat{\mathrm{I}}$. Let $\hat{\alpha}$ consist of c (disjoint) cycles. Then, the non-zero terms of the left-hand side of (2.5) are those for which $\mathbf{i}_0$ consists of the elements of ℓ cycles of $\hat{\alpha}$, with $\ell \geqslant 1$. The value of such a term is thus

$$(-1)^k (-1)^{k+1-\ell} = (-1)^{\ell+1}$$

and the sum is

$$\Sigma_{\ell \geqslant 1}(-1)^{\ell+1} \frac{(c-\ell)!\,\ell!}{c!} = 1.$$

2.9. *Corollary. — Write*

$$\zeta_{mk} = \Sigma_{\mathbf{i}_0, \ldots, \mathbf{i}_{m-1} \in \mathrm{I}_k} \int \mu(d\omega_1) \ldots \mu(d\omega_m) \tag{2.6}$$
$$(\tau_{\mathbf{i}_0 \mathbf{i}_{m-1}}(\omega_m)\, \varphi_{\omega_m}(x(\overline{\omega}))) \ldots (\tau_{\mathbf{i}_1 \mathbf{i}_0}(\omega_1)\, \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x(\overline{\omega}))).$$

Then

$$\zeta_m = \Sigma_k (-1)^k \zeta_{mk}.$$

2.10. *Proposition. — The power series*

$$d_k(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \zeta_{mk}$$

converges for $|z|\, \theta^\alpha e^{\mathrm{P}} < 1$, *and its zeros in this domain are the inverses of the eigenvalues of* $\mathscr{M}_k$, *with the same multiplicities.*

Before proving this result, we note the following consequence.

2.11. *Corollary. — The power series*

$$1/\zeta(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \zeta_m$$

converges for $|z|\, \theta^\alpha e^{\mathrm{P}} < 1$, *and its zeros in this domain are the inverses of the eigenvalues of* $\mathscr{K}$, *with the same multiplicities.*

Corollary 2.9 yields

$$1/\zeta(z) = \Pi_{k \geqslant 0} [d_k(z)]^{(-1)^k}.$$

Corollary 2.11 therefore results from Proposition 2.10 if we can prove that, for $|\lambda| > \theta^\alpha e^{\mathrm{P}}$,

$$\overline{m}(\lambda) = \Sigma_{k \geqslant 0} (-1)^k m_k(\lambda) \tag{2.7}$$

where $\bar{m}(\lambda)$ and $m_k(\lambda)$ are the multiplicities of λ as eigenvalues of $\mathscr{K}$ and $\mathscr{M}_k$ respectively. To derive this result, let

$$C_k = \bigoplus_{(i_0, \ldots, i_k)} C^\alpha(X_{i_0} \cap \ldots \cap X_{i_k})$$

and define coboundary operators $\alpha_k : C_k \to C_{k+1}$ in the usual manner (i.e., $(\alpha_k \Phi)_{(i_0, \ldots, i_{k+1})} = \sum_{\ell=0}^{k+1} (-1)^\ell \Phi_{(i_0, \ldots, \hat{i}_\ell, \ldots, i_{k+1})} \mid X_{i_\ell})$. The existence of a C^α partition of unity associated with the covering (X_i) ensures that the following is an exact sequence:

$$0 \to C^\alpha(X) \xrightarrow{\beta} C_0 \xrightarrow{\alpha_0} C_1 \to \ldots \to C_k \xrightarrow{\alpha_k} C_{k+1} \to \ldots,$$

where β is the natural injection and $C_k = 0$ for sufficiently large k. We also have

$$\beta\mathscr{K} = \mathscr{M}_0 \beta, \qquad \alpha_k \mathscr{M}_k = \mathscr{M}_{k+1} \alpha_k.$$

Let $P_\lambda = \dfrac{1}{2\pi i}\oint \dfrac{dz}{z - \mathscr{L}}$ $\left(\text{resp. } P_{\lambda k} = \dfrac{1}{2\pi i}\oint \dfrac{dz}{z - \mathscr{M}_k}\right)$ where the integral is over a small circle centered at λ. Then, P_λ (resp. $P_{\lambda k}$) is a linear projection of $C^\alpha(X)$ (resp. C_k) onto the generalized eigenspace of $\mathscr{L}$ (resp. $\mathscr{M}_k$) corresponding to λ. Furthermore

$$P_{\lambda 0}\, \beta = \beta P_\lambda, \qquad P_{\lambda k+1}\, \alpha_k = \alpha_k\, P_{\lambda k}.$$

We therefore have an exact sequence

$$0 \to \operatorname{im} P_\lambda \xrightarrow{\beta} \operatorname{im} P_{\lambda 0} \xrightarrow{\alpha_0} \operatorname{im} P_{\lambda 1} \to \ldots \to 0$$

so that

$$\dim \operatorname{im} P_\lambda = \Sigma_{k \geqslant 0} (-1)^k \dim \operatorname{im} P_{\lambda k}$$

which is precisely (2.7).

2.12. *Proof of Theorem* **1.1.** — Theorem 1.1 follows from Proposition 2.7 and Corollary 2.11. We are thus left with Proposition 2.10 to prove.

2.13. *Proof of Proposition* **2.10.** — There is a finite number of eigenvalues λ_j of $\mathscr{M}_k$ such that $|\lambda_j| > \theta'^\alpha e^P$. If m_j is the multiplicity of λ_j, we may write

$$\Sigma_j\, m_j(\lambda_j)^m = \Sigma_j\, \lambda_j^m\, \Sigma_\gamma\, \sigma_{j\gamma}(S_{j\gamma}) = \Sigma_{j\gamma}\, \sigma_{j\gamma}(\mathscr{M}_k^m\, S_{j\gamma})$$

where $(\sigma_{j\gamma})$ and $(S_{j\gamma})$ are dual bases of the generalized eigenspaces of $\mathscr{M}_k^*$ and $\mathscr{M}_k$ respectively for the eigenvalue λ_j. Therefore

$$\Sigma_j\, m_j(\lambda_j)^m = \Sigma_{j\gamma}\, \sigma_{j\gamma}((\mathscr{M}_k^{(m)} - \mathscr{M}_k^{(m)}\, T_k^{(m)})\, Q_k^{(m)}\, S_{j\gamma}) + \Sigma_{j\gamma}\, \sigma_{j\gamma}(\mathscr{M}_k^{(m)}\, C_{j\gamma}) \tag{2.8}$$

where $C_{j\gamma}$ has the constant value $S_{j\gamma}(x(\mathbf{a}))$ on $X(a_0) \cap \ldots \cap X(a_k)$.

Using Proposition 2.7 we have

$$|\, \Sigma_{j\gamma}\, \sigma_{j\gamma}((\mathscr{M}_k^{(m)} - \mathscr{M}_k^{(m)}\, T_k^{(m)})\, Q_k^{(m)}\, S_{j\gamma})\,| \leqslant \text{const}(\theta'^\alpha e^{P+\varepsilon})^m. \tag{2.9}$$

Let $\chi_{\mathbf{a}}$ be the characteristic function of $X(a_0) \cap \ldots \cap X(a_k)$ as an element of

$$\bigoplus_{(a_0, \ldots, a_k)} C^\alpha(X(a_0) \cap \ldots \cap X(a_k)).$$

Then

$$(2.10) \qquad \Sigma_{j\gamma}\, \sigma_{j\gamma}(\mathcal{M}_k^{(m)} C_{j\gamma}) = \Sigma_{j\gamma} \Sigma_{\mathbf{a}} S_{j\gamma}(x(\mathbf{a}))\, \sigma_{j\gamma}(\mathcal{M}_k^{(m)} \chi_{\mathbf{a}}) = \Sigma_{\mathbf{a}}((1-\mathcal{P})\, \mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(\mathbf{a}))$$

where $\mathcal{P}$ is the projection corresponding to the part of the spectrum of $\mathcal{M}_k$ in $\{\lambda : |\lambda| \leqslant \theta'^{\alpha} e^{P}\}$.

The right-hand side of (2.10) is the sum of two terms. The first can be written as

$$\begin{aligned}
&\Sigma_{\mathbf{a}}(\mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(\mathbf{a})) \\
&\quad = \Sigma_{(i_0, \ldots, i_k) \in I_k} \Sigma_{a_0 : p^m a_0 = i_0} \ldots \Sigma_{a_k : p^m a_k = i_k} \int \mu(d\omega_1) \ldots \mu(d\omega_m) \\
&\qquad \Sigma_{\pi} (\operatorname{sign} \pi)\, \delta((a_0, \ldots, a_k), \pi(v((i_0), \bar\omega), \ldots, v((i_k), \bar\omega))) \\
&\qquad\qquad \varphi_{\omega_m}(x(\mathbf{a})) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x(\mathbf{a})) \\
&\quad = \Sigma_{\mathbf{i}_0, \ldots, \mathbf{i}_{m-1} \in I_k} \int \mu(d\omega_1) \ldots \mu(d\omega_m)\, \tau_{\mathbf{i}_0 \mathbf{i}_{m-1}}(\omega_m) \ldots \tau_{\mathbf{i}_2 \mathbf{i}_1}(\omega_2)\, \tau_{\mathbf{i}_1 \mathbf{i}_0}(\omega_1) \\
&\qquad\qquad \varphi_{\omega_m}(x(|v(\mathbf{i}_0, \bar\omega)|)) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_m} x(|v(\mathbf{i}_0, \bar\omega)|))
\end{aligned}$$

where $|v(\mathbf{i}, \bar\omega)|$ is the permutation of $(v(i_0, \bar\omega), \ldots, v(i_k, \bar\omega))$ such that $p^m |v(\mathbf{i}, \bar\omega)| = \mathbf{i}$. If we replace in the right-hand side $x(|v(\mathbf{i}_0, \bar\omega)|)$ by the fixed point $x(\bar\omega)$ of $\psi_{\omega_1} \ldots \psi_{\omega_m}$, the error is bounded by $\mathrm{const}(\theta'^{\alpha} e^{P+\varepsilon})^m$ (using the same sort of estimates as in the proof of Proposition 2.5). Therefore, by the definition (2.6) of ζ_{mk}, we have

$$(2.11) \qquad |\Sigma_{\mathbf{a}}(\mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(\mathbf{a})) - \zeta_{mk}| \leqslant \mathrm{const}(\theta'^{\alpha} e^{P+\varepsilon})^m.$$

We are left with the study of

$$\Sigma_{\mathbf{a}}(\mathcal{P}\mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(\mathbf{a})).$$

Remember that the sum is over the set $J_k^{(m)}$ of those $\mathbf{a} = (a_0, \ldots, a_k) \in (J^{(m)})^{k+1}$ such that $p^m \mathbf{a} = (i_0, \ldots, i_k)$ with $i_0 < \ldots < i_k$. Note that, if $0 \leqslant \ell \leqslant m$, we may write

$$\Sigma_{\mathbf{a} \in J_k^{(m)}}(\mathcal{P}\mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(p^{m-\ell} \mathbf{a})) = \Sigma_{\mathbf{b} \in J_k^{(\ell)}}(\mathcal{P}\mathcal{M}_k^{m-\ell} \mathcal{M}_k^{(\ell)} \chi_{\mathbf{b}})\, (x(\mathbf{b}))$$

(lump together those $\mathbf{a}$ such that $p^{m-\ell} \mathbf{a} = \mathbf{b}$). Therefore

$$\begin{aligned}
&\Sigma_{\mathbf{a}}(\mathcal{P}\mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(\mathbf{a})) - \Sigma_{\mathbf{i} \in I_k}(\mathcal{P}\mathcal{M}_k^{m} \chi_{\mathbf{i}})\, (x(\mathbf{i})) \\
&\quad = \sum_{\ell=1}^{m} \Sigma_{\mathbf{a}}((\mathcal{P}\mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(p^{m-\ell} \mathbf{a})) - (\mathcal{P}\mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(p^{m-\ell+1} \mathbf{a}))) \\
&\quad = \sum_{\ell=1}^{m} \Sigma_{\mathbf{b} \in J_k^{(\ell)}}((\mathcal{P}\mathcal{M}_k^{m-\ell} \mathcal{M}_k^{(\ell)} \chi_{\mathbf{b}})\, (x(\mathbf{b})) - (\mathcal{P}\mathcal{M}_k^{m-\ell} \mathcal{M}_k^{(\ell)} \chi_{\mathbf{b}})\, (x(p\mathbf{b}))).
\end{aligned}$$

From this we get, using (2.13) below,

$$\begin{aligned}
(2.12) \qquad |\Sigma_{\mathbf{a}}(\mathcal{P}\mathcal{M}_k^{(m)} \chi_{\mathbf{a}})\, (x(\mathbf{a}))| &\leqslant \mathrm{const}\, \|\mathcal{P}\mathcal{M}_k^{m}\| \\
&\quad + \mathrm{const} \sum_{\ell=1}^{m} \|\mathcal{P}\mathcal{M}_k^{m-\ell}\| . \Sigma_{\mathbf{b} \in J_k^{(\ell)}} \|\mathcal{M}_k^{(\ell)} \chi_{\mathbf{b}}\| . d(x(\mathbf{b}), x(p\mathbf{b}))^{\alpha} \\
&\leqslant \mathrm{const}\, [(\theta'^{\alpha} e^{P+\varepsilon})^m + \sum_{\ell=1}^{m} (\theta'^{\alpha} e^{P+\varepsilon})^{m-\ell} (e^{P+\varepsilon})^{\ell}\, \theta'^{\ell\alpha}] \\
&\leqslant \mathrm{const}\, m(\theta'^{\alpha} e^{P+\varepsilon})^m.
\end{aligned}$$

Putting together (2.8), (2.9), (2.10), (2.11), (2.12) we obtain

$$| \Sigma_j \, m_j(\lambda_j)^m - \zeta_{mk} | < \text{const}\, m(\theta'^{\alpha} e^{P+\varepsilon})^m$$

and therefore

$$\log(d_k(z)/\Pi_j(1-\lambda_j z)^{m_j} = \sum_{m=1}^{\infty} \frac{z^m}{m} (\Sigma_j \, m_j(\lambda_j)^m - \zeta_{mk})$$

converges for $| z | \theta'^{\alpha} e^{P+\varepsilon} < 1$, proving Proposition 2.10.

In deriving (2.12) we have used the inequality

$$\Sigma_{\mathbf{b} \in J_k^{(\ell)}} || \mathscr{M}_k^{(\ell)} \chi_{\mathbf{b}} || \leqslant \text{const}(e^{P+\varepsilon})^{\ell} \tag{2.13}$$

which we shall now prove.

Given $\beta > 0$ we set $\varphi_{\omega\beta} = | \varphi_\omega | + \beta || \varphi_\omega ||$. In the definition—Section 2.6—of $\mathscr{M}_k^{(\ell)}$ if we replace φ_ω by $\varphi_{\omega\beta}$ and suppress the factor $\varepsilon = \pm 1$ we obtain an opetaror $M_{k\beta}^{(\ell)}$:

$$(M_{k\beta}^{(\ell)} \Phi)_{\mathbf{j}}\,(x) = \int \mu(d\omega_1) \ldots \mu(d\omega_\ell)$$
$$\varphi_{\omega_\ell \beta}(x) \ldots \varphi_{\omega_1\beta}(\varphi_{\omega_2} \ldots \psi_{\omega_\ell}\, x)\, \Phi_{(a_0, \ldots, a_k)}(\psi_{\omega_1} \ldots \psi_{\omega_\ell}\, x)$$

where $(a_0, \ldots, a_k)$ is a permutation of $(v(j_0, \bar\omega), \ldots, v(j_k, \bar\omega))$. In particular

$$|| \mathscr{M}_k^{(\ell)} \chi_{\mathbf{b}} ||_0 \leqslant || M_{k\beta}^{(\ell)} \chi_{\mathbf{b}} ||_0.$$

If $x, y \in X_{j_0} \cap \ldots \cap X_{j_k}$, we also have

$$| (\mathscr{M}_k^{(\ell)} \chi_{\mathbf{b}})_{\mathbf{j}}\,(x) - (\mathscr{M}_k^{(\ell)} \chi_{\mathbf{b}})_{\mathbf{j}}\,(y) | \leqslant \int \mu(d\omega_1) \ldots \mu(d\omega_\ell)$$
$$| \varphi_{\omega_\ell}(x) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_\ell}\, x) - \varphi_{\omega_\ell}(y) \ldots \varphi_{\omega_1}(\psi_{\omega_2} \ldots \psi_{\omega_\ell}\, y) |$$
$$\leqslant \sum_{i=1}^{\ell} \int \mu(d\omega_1) \ldots \mu(d\omega_k)\, \varphi_{\omega_\ell \beta}(x) \ldots \varphi_{\omega_{i+1}\beta}(\psi_{\omega_{i+2}} \ldots \psi_{\omega_\ell}\, x)$$
$$| \varphi_{\omega_i}(\psi_{\omega_{i+1}} \ldots \psi_{\omega_\ell}\, x) - \varphi_{\omega_i}(\psi_{\omega_{i+1}} \ldots \psi_{\omega_\ell}\, y) |$$
$$\varphi_{\omega_{i-1}\beta}(\psi_{\omega_i} \ldots \psi_{\omega_\ell}\, y) \ldots \varphi_{\omega_1\beta}(\psi_{\omega_2} \ldots \psi_{\omega_\ell}\, y).$$

where the integrals are restricted to those $(\omega_1, \ldots, \omega_\ell)$ for which $\mathbf{b}$ is a permutation of $(v(j_0, \bar\omega), \ldots, v(j_k, \bar\omega))$. We may write

$$| \varphi_{\omega_i}(\psi_{\omega_{i+1}} \ldots \psi_{\omega_\ell}\, x) - \varphi_{\omega_i}(\psi_{\omega_{i+1}} \ldots \psi_{\omega_\ell}\, y) |$$
$$\leqslant || \varphi_{\omega_i} || \, (\theta^{\ell - i} d(x,y))^{\alpha} \leqslant \text{const}\, \varphi_{\omega_i\beta}(\psi_{\omega_{i+1}} \ldots \psi_{\omega_\ell}\, x)\, \theta^{\alpha(\ell - i)} d(x,y)^{\alpha}$$

and similarly

$$\varphi_{\omega_r \beta}(\psi_{\omega_{r+1}} \ldots \psi_{\omega_\ell}\, y) \leqslant \varphi_{\omega_r \beta}(\psi_{\omega_{r+1}} \ldots \psi_{\omega_\ell}\, x)\, (1 + \text{const}\, \theta^{\alpha(\ell - r)}). \tag{2.14}$$

Therefore

$$\frac{| (\mathscr{M}_k^{(\ell)} \chi_{\mathbf{b}})_{\mathbf{j}}\,(x) - (\mathscr{M}_k^{(\ell)} \chi_{\mathbf{b}})_{\mathbf{j}}\,(y) |}{d(x,y)^{\alpha}} \leqslant \text{const}\, \ell \, || M_{k\beta}^{(\ell)} \chi_{\mathbf{b}} ||_0$$

hence

$$|| \mathscr{M}_k^{(\ell)} \chi_{\mathbf{b}} || \leqslant C(\beta) . \ell \, || M_{k\beta}^{(\ell)} \chi_{\mathbf{b}} ||_0. \tag{2.15}$$

From (2.14) we also obtain

$$(M_{k\beta}^{(\ell)}\chi_b)_j(y) \leqslant C'(\beta)\,(M_{k\beta}^{(\ell)}\chi_b)_j(x)$$

where $C'(\beta)$ does not depend on ℓ. Therefore

$$\Sigma_{b\in J_k^{(\ell)}} \|M_{k\beta}^{(\ell)}\chi_b\|_0 \leqslant C'(\beta)\,\Sigma_j \sup_x |(M_{k\beta}^{(\ell)}1)_j(x)| \leqslant C''(\beta)\,\|M_{k\beta}^{\ell}1\|_0$$

and with (2.15) this gives

$$\Sigma_{b\in J_k^{(\ell)}} \|\mathscr{M}_k^{(\ell)}\chi_b\| \leqslant C(\beta)\,C''(\beta)\,\ell\,\|M_{k\beta}^{(\ell)}1\|_0 \leqslant C'''(\beta)\,(e^{P(\beta)+\varepsilon/2})^{\ell}$$

where $e^{P(\beta)}$ is the spectral radius with respect to the $\|\ \|_0$ norm of the operator $\mathscr{M}_\beta$ obtained if we replace φ_ω by $\varphi_{\omega\beta}$ in the definition of $\mathscr{M}$. Note that $\mathscr{M}_\beta$ is close to $|\mathscr{M}|$ for β small:

$$\|\mathscr{M}_\beta - |\mathscr{M}|\,\|_0 \leqslant \beta \int \mu(d\omega)\,\|\varphi_\omega\|.$$

Using the upper semicontinuity of the spectral radius we may thus choose β such that

$$\Sigma_{b\in J_k^{(\ell)}} \|\mathscr{M}_k^{(\ell)}\chi_b\| \leqslant C'''(\beta)\,(e^{P+\varepsilon})^{\ell}$$

i.e., (2.13) holds as announced.

3. Proof of Theorem 1.3

3.1. *The essential spectral radius of $\mathscr{K}$.* — We shall follow the proof of Theorem 1.1 in Section 2, and note what changes have to be performed to deal with the differentiable situation.

First of all, we make a choice of charts for the balls X_i, which will thus be identified in what follows with subsets of Euclidean space. We may assume that the balls X_i have small radii and that the Riemann metric is closely approximated by the Euclidean metric. Confusion between the two metrics is then inconsequential. The linear structures which we have chosen will allow us to define Taylor expansions.

Replacing C^α by C^r everywhere, we define $\mathscr{M}$, $\mathscr{M}^{(m)}$, $Q^{(m)}$, $\mathscr{M}_k$, $\mathscr{M}_k^{(m)}$, $Q_k^{(m)}$ as before. The operator $T^{(m)}$ on $\bigoplus_{a\in J^{(m)}} C^r(X(a))$ is now defined by

$$(T^{(m)}\Phi)_a = \text{Taylor expansion of order } r \text{ of } \Phi \text{ at } x(a)$$

and similarly for $T_k^{(m)}$. We have then

$$\|\Phi - T_k^{(m)}\Phi\|_0 \leqslant \text{const}\,\|\Phi\|\,\theta'^{m|r|}.$$

Following the arguments of Sections 2.5, 2.6, 2.7 with obvious changes, we get

$$\|\mathscr{M}_k^{(m)} - \mathscr{M}_k^{(m)}T_k^{(m)}\| \leqslant \text{const}(\theta'^{|r|}e^{P+\varepsilon})^m \tag{3.1}$$

and therefore the essential spectral radius of $\mathscr{M}_k$ is $\leqslant \theta^{|r|}e^P$. In particular, the same estimate holds for the essential spectral radii of $\mathscr{M}$ and $\mathscr{K}$.

3.2. *Proposition. — Define*

$$\operatorname{Tr}\mathcal{M}_k^m = \int \mu(d\omega_1)\,\ldots\,\mu(d\omega_m)\,(\det(1 - D_{x(\bar\omega)}\,\psi_{\omega_1}\ldots\psi_{\omega_m}))^{-1}$$
$$(\tau_{i_0 i_{m-1}}(\omega_m)\,\varphi_{\omega_m}(x(\bar\omega)))\,\ldots\,(\tau_{i_1 i_0}(\omega_1)\,\varphi_{\omega_1}(\psi_{\omega_2}\ldots\psi_{\omega_m}x(\bar\omega))).$$

Then, the power series

$$d_k^{(0)}(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \operatorname{Tr}\mathcal{M}_k^m$$

converges for $|\,z\,|\,\theta^{|r|}\,e^{\mathrm{P}} < 1$, *and its zeros in this domain are the inverses of the eigenvalues of* $\mathcal{M}_k$, *with the same multiplicities.*

Before proving this result, which corresponds to Proposition 2.10, we note that it allows us complete the demonstration of Theorem 1.3. We have indeed

$$d(z) = \Pi_{k\geq 0}\,(d_k^{(0)}(z))^{(-1)^k}$$

by Lemma 2.8. The proof of Corollary 2.11 again applies, and yields that the zeros of $d(z)$ in (1.5) are precisely the inverses of the eigenvalues of $\mathcal{K}$, with the same multiplicities.

3.3. *Remark.* — Before embarking in the demonstration of Proposition 3.2, we prove a necessary estimate. Let $n = (n_1, \ldots, n_{\dim \mathrm{X}})$ be a multi-index, $\dfrac{\partial^n}{\partial x^n}$ the corresponding derivative, and $n! = n_1!\ldots n_{\dim \mathrm{X}}!$. We assume that $|\,n\,| = n_1 + \ldots + n_{\dim \mathrm{X}} \leq r$. Define then

$$\mathrm{E}^{(n)}_{\xi_1\xi_2} = \Sigma_{i_0,\ldots,i_{m-1}}\int \mu(d\omega_1)\,\ldots\,\mu(d\omega_m)\,\tau_{i_0 i_{m-1}}(\omega_m)\,\ldots\,\tau_{i_1 i_0}(\omega_1)$$
$$\frac{\partial^n}{\partial x^n}(\varphi_{\omega_m}(x)\,\ldots\,\varphi_{\omega_1}(\psi_{\omega_2}\ldots\psi_{\omega_m}x)\,(\psi_{\omega_1}\ldots\psi_{\omega_m}x - \xi_1(i_0,\bar\omega))^n)\big|_{x=\xi_2(i_0,\bar\omega)}$$

and assume that $\xi_k \in \mathrm{X}(v(i_0,\bar\omega))$ for $k = 1, 2, 3$. Replace in $\mathrm{E}^{(n)}_{\xi_1\xi_2}$ the expression $\dfrac{\partial^n}{\partial x^n}(\ldots)$ by its Taylor expansion around $\xi_3(i_0,\bar\omega)$, keeping derivatives of total order up to r, and then put $x = \xi(i_0,\bar\omega)$. The error thus made is bounded by

$$\mathrm{const}(\theta'^m)^{|r|-|n|}.(e^{\mathrm{P}+\varepsilon})^m\,(\theta'^m)^{|n|} = \mathrm{const}(e^{\mathrm{P}+\varepsilon}\,\theta'^{|r|})^m.$$

Define now

$$\mathrm{E}_\xi = \Sigma_{n:|n|\leq r}\,\frac{1}{n!}\,\mathrm{E}^{(n)}_{\xi\xi}\quad\text{i.e.,}$$

(3.2)
$$\mathrm{E}_\xi = \Sigma_{n:|n|\leq r}\,\Sigma_{i_0,\ldots,i_{m-1}}\int \mu(d\omega_1)\,\ldots\,\mu(d\omega_m)\,\tau_{i_0 i_{m-1}}(\omega_m)\,\ldots\,\tau_{i_1 i_0}(\omega_1)$$
$$\frac{1}{n!}\frac{\partial^n}{\partial x^n}(\varphi_{\omega_m}(x)\,\ldots\,\varphi_{\omega_1}(\psi_{\omega_2}\ldots\psi_{\omega_m}x)\,(\psi_{\omega_1}\ldots\psi_{\omega_m}x - \xi(i_0,\bar\omega))^n)\big|_{x=\xi(i_0,\bar\omega)}.$$

Introducing limited Taylor expansions as explained above in each term of E_{ξ_1}, we simply obtain E_{ξ_2}. Therefore

$$|E_{\xi_1} - E_{\xi_2}| \leqslant \text{const}(e^{P+\varepsilon}\theta'^{|r|})^m. \tag{3.3}$$

3.4. *Proof of Proposition* **3.2.** — We shall prove the proposition for $\mathscr{M}(=\mathscr{M}_0)$ rather than $\mathscr{M}_k$. (This simplifies notation, and the general case is easily recovered by reference to Section 2.13.)

There is now a finite number of eigenvalues λ_j of $\mathscr{M}$ such that $|\lambda_j| > \theta'^{|r|}e^P$. Let m_j be the multiplicity of λ_j, and $(\sigma_{j\gamma})$, $(S_{j\gamma})$ be dual bases of the generalized eigenspaces of $\mathscr{M}^*$ and $\mathscr{M}$ respectively for the eigenvalue λ_j. Then

$$\begin{aligned}\Sigma_j m_j(\lambda_j)^m &= \Sigma_{j\gamma}\,\sigma_{j\gamma}(\mathscr{M}^m S_{j\gamma}) \\ &= \Sigma_{j\gamma}\,\sigma_{j\gamma}((\mathscr{M}^{(m)} - \mathscr{M}^{(m)}T^{(m)})\,Q^{(m)}S_{j\gamma}) + \Sigma_{j\gamma}\,\sigma_{j\gamma}(\mathscr{M}^{(m)}C_{j\gamma})\end{aligned} \tag{3.4}$$

where $C_{j\gamma}\,|\,X(a)$ is the Taylor expansion to order r of $S_{j\gamma}$ at $x(a)$. Note that (3.1) gives

$$|\Sigma_{j\gamma}\,\sigma_{j\gamma}((\mathscr{M}^{(m)} - \mathscr{M}^{(m)}T^{(m)})\,Q^{(m)}S_{j\gamma}) \leqslant \text{const}(\theta'^{|r|}e^{P+\varepsilon})^m. \tag{3.5}$$

Let χ_a denote the characteristic function of $X(a)$ and write ∂_ξ^n for the derivative of order $n = (n_1, \ldots, n_{\dim X})$ evaluated at ξ. We have

$$\begin{aligned}\Sigma_{j\gamma}\,\sigma_{j\gamma}(\mathscr{M}^{(m)}C_{j\gamma}) &= \Sigma_{j\gamma}\,\Sigma_a\,\Sigma_{n:|n|\leqslant r}\frac{1}{n!}\,\partial^n_{x(a)}S_{j\gamma}\,\sigma_{j\gamma}(\mathscr{M}^{(m)}((\cdot - x(a))^n\chi_a)) \\ &= \Sigma_a\,\Sigma_{n:|n|\leqslant r}\frac{1}{n!}\,\partial^n_{x(a)}((1-\mathscr{P})\,\mathscr{M}^{(m)}((\cdot - x(a))^n\chi_a))\end{aligned} \tag{3.6}$$

where $\mathscr{P}$ is the projection corresponding to the part of the spectrum of $\mathscr{M}$ in $\{\lambda : |\lambda| \leqslant \theta'^{|r|}e^P\}$. Further,

$$\begin{aligned}&\Sigma_a\,\Sigma_n\frac{1}{n!}\,\partial^n_{x(a)}\mathscr{M}^{(m)}((\cdot - x(a))^n\chi_a) \\ &\qquad= \Sigma_a\,\Sigma_n\frac{1}{n!}\int\mu(d\omega_1)\ldots\mu(d\omega_m)\,\delta(a, v(p^m a, \bar\omega)) \\ &\frac{\partial}{\partial x^n}(\varphi_{\omega_m}(x)\ldots\varphi_{\omega_1}(\psi_{\omega_2}\ldots\psi_{\omega_m}x)\,(\psi_{\omega_1}\ldots\psi_{\omega_m}x - x(a))^n)\big|_{x=x(a)} \\ &\qquad= \Sigma_{i_0,\ldots,i_{m-1}}\int\mu(d\omega_1)\ldots\mu(d\omega_m)\,\tau_{i_0 i_{m-1}}(\omega_m)\ldots\tau_{i_1 i_0}(\omega_1) \\ &\qquad\qquad\Sigma_n\frac{1}{n!}\frac{\partial}{\partial x^n}(\varphi_{\omega_m}(x)\ldots\varphi_{\omega_1}(\psi_{\omega_2}\ldots\psi_{\omega_m}x) \\ &\qquad\qquad(\psi_{\omega_1}\ldots\psi_{\omega_m}x - x(v(i_0, \bar\omega)))^n)\big|_{x=x(v(i_0,\bar\omega))} \\ &\qquad= E_{\xi_1}\end{aligned} \tag{3.7}$$

where we have used the notation (3.2) with $\xi_1(i_0, \bar\omega) = x(v(i_0, \bar\omega))$. If we choose $\xi_3(i_0, \bar\omega) = x(\bar\omega)$, we get

$$|E_{\xi_1} - E_{\xi_3}| \leqslant \text{const}(\theta'^{|r|}e^{P+\varepsilon})^m \tag{3.8}$$

in view of (3.3). Furthermore, since $x(\bar{\omega})$ is a fixed point of $\psi_{\omega_1} \dots \psi_{\omega_m}$,

$$E_{\xi_3} = \Sigma_{i_0, \dots, i_{m-1}} \int \mu(d\omega_1) \dots \mu(d\omega_m)\, \tau_{i_0 i_{m-1}}(\omega_m) \dots \tau_{i_1 i_0}(\omega_1)$$
$$\varphi_{\omega_m}(x(\bar{\omega})) \dots \varphi_{\omega_1}(\psi_{\omega_2} \dots \psi_{\omega_m} x(\bar{\omega}))\, \Sigma_{n:|n| \leqslant r} \frac{1}{n!} \Psi_n.$$

where Ψ_n is a polynomial of order $|n|$ in the elements of the matrix $D_{x(\bar{\omega})}(\psi_{\omega_1} \dots \psi_{\omega_m})$ of derivatives at $x(\bar{\omega})$, and $\Sigma \Psi_n/n!$ is invariant under linear changes of coordinates. It is easily recognized that $\Sigma \Psi_n/n!$ is the development of $(\det(1 - D_{x(\bar{\omega})} \psi_{\omega_1} \dots \psi_{\omega_m}))^{-1}$ to order r (take D to be in Jordan normal form). Therefore

$$| E_{\xi_3} - \operatorname{tr} \mathscr{M}^m | \leqslant \operatorname{const}(\theta^{r+1} e^{P+\varepsilon})^m. \tag{3.9}$$

From (3.7), (3.8), (3.9) we get

$$\left| \Sigma_a \Sigma_n \frac{1}{n!} \partial^n_{x(a)} \mathscr{M}^{(m)}((\cdot - x(a))^n \chi_a) - \operatorname{tr} \mathscr{M}^m \right| \leqslant \operatorname{const}(\theta'^{|r|} e^{P+\varepsilon})^m. \tag{3.10}$$

There remains to estimate

$$\Sigma_a \Sigma_{n:|n| \leqslant r} \frac{1}{n!} \partial^n_{x(a)} \mathscr{P}\mathscr{M}^{(m)}((\cdot - x(a))^n \chi_a).$$

Note that, if $0 \leqslant \ell \leqslant m$, we may write

$$\Sigma_{a \in J^{(m)}} \Sigma_n \frac{1}{n!} \partial^n_{x(p^{m-\ell} a)} \mathscr{P}\mathscr{M}^{(m)}((\cdot - x(p^{m-\ell} a))^n \chi_a)$$
$$= \Sigma_{b \in J^{(\ell)}} \Sigma_n \frac{1}{n!} \partial^n_{x(b)} \mathscr{P}\mathscr{M}^{m-\ell} \mathscr{M}^{(\ell)}((\cdot - x(b))^n \chi_b).$$

Thus

$$\Sigma_{a \in J^{(m)}} \Sigma_n \frac{1}{n!} \partial^n_{x(a)} \mathscr{P}\mathscr{M}^{(m)}((\cdot - x(a))^n \chi_a) - \Sigma_{i \in I} \Sigma_n \frac{1}{n!} \partial^n_{x_i} \mathscr{P}\mathscr{M}^m((\cdot - x_i)^n \chi_i)$$
$$= \sum_{\ell=1}^m \Sigma_{b \in J^{(\ell)}} \Sigma_n \frac{1}{n!} (\partial^n_{x(b)} \mathscr{P}\mathscr{M}^{m-\ell} \mathscr{M}^{(\ell)}((\cdot - x(b))^n \chi_b)$$
$$- \partial^n_{x(pb)} \mathscr{P}\mathscr{M}^{m-\ell} \mathscr{M}^{(\ell)}((\cdot - x(pb))^n \chi_b)).$$

The absolute value of the right-hand side can be estimated in terms of Taylor expansions (as in Remark 3.3). Using also (3.12) below, we get a bound

$$\operatorname{const} \sum_{\ell=1}^m \Sigma_{b \in J^{(\ell)}} \Sigma_n \frac{1}{n!} d(x(b), x(pb))^{|r|-|n|}$$
$$\| \mathscr{P}\mathscr{M}^{m-\ell} \| . \| \mathscr{M}^{(\ell)}((\cdot - x(b))^n \chi_b) \|$$
$$\leqslant \operatorname{const} \sum_{\ell=1}^m \Sigma_n \frac{1}{n!} (\theta'^{\ell})^{|r|-|n|} . (\theta'^{|r|} e^{P+\varepsilon})^{m-\ell} . (e^{P+\varepsilon})^\ell (\theta'^{\ell})^{|n|}$$
$$= \operatorname{const} m(\theta'^{|r|} e^{P+\varepsilon})^m.$$

Therefore

$$\left| \Sigma_a \Sigma_n \frac{1}{n!} \partial^n_{x(a)} \mathscr{P}\mathscr{M}^{(m)}((\cdot - x(a))^n \chi_a) \right| \leqslant \text{const}\, m(\theta'^{|r|} e^{P+\varepsilon})^m. \tag{3.11}$$

From (3.4), (3.5), (3.6), (3.10), (3.11) we conclude that

$$| \Sigma_j m_j(\lambda_j)^m - \operatorname{tr} \mathscr{M}^m | \leqslant \text{const}\, m(\theta'^{|r|} e^{P+\varepsilon})^m.$$

Therefore

$$\log(d_0^{(0)}(z)/\Pi_j(1-\lambda_j z)^{m_j}) = \sum_{m=1}^{\infty} \frac{z^m}{m} (\Sigma m_j(\lambda_j)^m - \operatorname{tr} \mathscr{M}^m)$$

converges for $|z| \theta'^{|r|} e^{P+\varepsilon} < 1$, proving Proposition 3.2.

We have used the inequality

$$\Sigma_{b \in \mathfrak{J}^{(\ell)}} || \mathscr{M}^{(\ell)}((\cdot - x(b))^n \chi_b) || \leqslant \text{const}(e^{P+\varepsilon})^{\ell} (\theta'^{\ell})^{|n|} \tag{3.12}$$

which is proved like (2.13).

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Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette

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