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Inventiones mathematicae

Sharp determinants

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Abstract. We introduce a sharp trace $\mathrm{Tr}^{\#}\mathcal{M}$ and a sharp determinant $\mathrm{Det}^{\#}(1 - z\mathcal{M})$ for an algebra of operators $\mathcal{M}$ acting on functions of bounded variation on the real line. We show that the zeroes of the sharp determinant describe the discrete spectrum of $\mathcal{M}$. The relationship with weighted zeta functions of interval maps and Milnor–Thurston kneading determinants is explained. This yields a result on convergence of the discrete spectrum of approximated operators.

1. Introduction

In the present paper we discuss a special case of the general problem of defining Fredholm-like determinants $\mathrm{Det}(1 - z\mathcal{M})$ where the operator $\Phi \to \mathcal{M}\Phi$ acts on a Banach space $\mathcal{B}$ of functions $\Phi : X \to \mathbb{C}$. (More generally, Φ may be a section of a vector bundle over X.)

We assume that $\mathcal{M}$ is a finite or countable linear combination

$$\mathcal{M} = \sum_{\omega} \mathcal{L}_{\omega}$$

of simple operators of the form

$$\mathcal{L}\Phi(x) = g(x) \cdot \Phi(\psi x) \,. \tag{1.1}$$

Under suitable conditions on the $g : X \to \mathbb{C}$ and $\psi : X \to X$, the operators of type $\mathcal{L}$ form a semi-group and the operators of type $\mathcal{M}$ form an algebra $\mathcal{A}$.

We may use the natural formula

$$\mathrm{Det}(1 - z\mathcal{M}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{Tr}\, \mathcal{M}^m$$

to define the determinant in terms of a trace Tr on $\mathcal{A}$. A successful definition should be such that $\mathrm{Det}(1 - z\mathcal{M})$ has a nontrivial radius of convergence in z, and that its zeroes λ^{-1} correspond to eigenvalues λ of $\mathcal{M}$.

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Note that the operators $\mathcal{M}$ need not be of trace class, and that the choice of a definition of Tr will in general depend on the fact that $\mathcal{M}$ is a functional operator. If X is a smooth finite dimensional manifold, and the graph of ψ is transversal to the diagonal in $X \times X$, a natural definition is that of the *flat trace* (see Atiyah and Bott [1967, 1968])

$$\mathrm{Tr}^\flat \mathcal{L} = \sum_{x \in \mathrm{Fix}\, \psi} \frac{g(x)}{|\det(1 - D_x\psi)|} .$$

This leads to a satisfactory definition of *flat determinants* if the ψ are contracting and the g, ψ are smooth or analytic. Under the same contraction assumption, when X is only a metric space and the degree of smoothness is only Hölder, the flat trace is replaced by the *counting trace*

$$\mathrm{Tr}\, \mathcal{L} = \sum_{x \in \mathrm{Fix}\, \psi} g(x) .$$

(See Ruelle [1976, 1990] and Fried [1993] for a discussion of these cases and further references.)

Here we do *not* assume that the ψ are contracting, and we shall use a different trace, which we may call *sharp trace*:

$$\mathrm{Tr}^\# \mathcal{L} = \sum_{x \in \mathrm{Fix}\, \psi} L(x, \psi) g(x) , \tag{1.2}$$

where $L(x, \psi)$ is a Lefschetz index which takes the values $0, \pm 1$ (and the definition will be modified to accommodate situations where $\mathrm{Fix}\, \psi$ is not finite, the sum in (1.2) being replaced by an integral). Taking X to be $\mathbb{R}$, and functions Φ of bounded variation, we shall obtain a *sharp determinant* closely related to the *kneading determinant* of Milnor and Thurston [1988] (see also Baladi–Ruelle [1994], Ruelle [1993], Baladi [1995]) and the *Fredholm determinant* of Mori ([1990, 1992]).

The specific functional theoretic situation in which we place ourselves in the present paper is described in Sect. 2. Our main result is that the sharp determinant $\mathrm{Det}^\#(1 - z\mathcal{M})$ can be expressed (by resummation of power series) as a more ordinary functional determinant (in fact, a mildly regularized Fredholm determinant)

$$\mathrm{Det}^\#(1 - z\mathcal{M}) = \mathrm{Det}_\star(1 + \widehat{\mathcal{D}}(z)) ,$$

where the *kneading operator* $\widehat{\mathcal{D}} = \widehat{\mathcal{D}}(z)$ is almost of trace class. Using this, one shows that $\mathrm{Det}^\#(1 - z\mathcal{M})$ is analytic in a disc where its zeroes are precisely the inverses λ^{-1} of the discrete eigenvalues λ of the quasicompact operator $\mathcal{M}$ (with the same multiplicity). This is the content of Sect. 3.

Using (1.2), we may write

$$\begin{aligned}\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{\omega_1, \dots, \omega_m} \sum_{x \in \mathrm{Fix}\, \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1}} \\ L(x, \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1}) g_{\omega_1}(x) g_{\omega_2}(\psi_{\omega_1} x) \cdots g_{\omega_m}(\psi_{\omega_{m-1}} \circ \cdots \circ \psi_{\omega_1} x) \\ = \frac{1}{\mathrm{Det}^\#(1 - z\mathcal{M})} ,\end{aligned}$$

where $\zeta(z)$ is a *dynamical zeta function*. This formula relates the present paper and the above mentioned work on kneading determinants. In particular the kneading operators introduced here correspond to the kneading matrices of Milnor and Thurston, and to the matrices introduced later by Baladi and Ruelle. This relationship will be further discussed in the Appendix.

2. Definitions, background, sharp trace

Let $g : \mathbb{R} \to \mathbb{C}$ be continuous, of bounded variation, and with compact support. (It would be sufficient to assume that g tends to zero at infinity, using a homeomorphism $\mathbb{R} \to (-1, 1)$, to revert to the compact support situation.) Note that allowing discontinuities in g is an interesting generalisation. (This can be handled by inserting intervals in $\mathbb{R}$ at the location of the discontinuities and their images by the ψ_ω or ψ_ω^{-1} and extending the g_ω continuously in the inserted intervals. We shall however not explore this approach in the present paper.) Let ψ be a homeomorphism of an interval J, containing the support of g, to an interval of $\mathbb{R}$. We define $\mathscr{L}$ by (1.1). (Note that since ψ can be extended arbitrarily outside of the support of g without modifying $\mathscr{L}$, we may assume that $J = \mathbb{R}$.) We write

$$\operatorname{Tr}^{\#} \mathscr{L} = \int d(g(x))\, \tfrac{1}{2} \operatorname{sgn}(\psi(x) - x)\,, \tag{2.1}$$

where

$$\operatorname{sgn}(\xi) = \begin{cases} +1 & \text{if } \xi > 0 \\ 0 & \text{if } \xi = 0 \\ -1 & \text{if } \xi < 0\,, \end{cases}$$

and $d(g(x))$ is by assumption a finite (signed) nonatomic measure with compact support.

Let now $\mathscr{A}$ be the algebra of operators $\mathscr{M}$ acting on the Banach space $\mathscr{B}$ of functions of bounded variation on $\mathbb{R}$, such that

$$\mathscr{M}\Phi(x) = \sum_{\omega} g_\omega(x)\Phi(\psi_\omega x)\,, \tag{2.2}$$

where ω varies over a countable set, and $\sum_\omega \operatorname{Var} g_\omega < \infty$ (Var denotes the total variation on $\mathbb{R}$). We define $\|\mathscr{M}\|_{\mathscr{A}}$ to be the infimum of the $\sum_\omega \operatorname{Var} g_\omega$ for all representations (2.2) of $\mathscr{M}$. It is easily seen that $\mathscr{A}$ is a Banach algebra with respect to the norm $\|\cdot\|_{\mathscr{A}}$, and that $|\operatorname{Tr}^{\#}\mathscr{L}| \leqq \|\mathscr{L}\|_{\mathscr{A}}$. We provisionally define $\operatorname{Tr}^{\#}\mathscr{M} = \sum_\omega \operatorname{Tr}^{\#}\mathscr{L}_\omega$. We will see that the value of $\operatorname{Tr}^{\#}\mathscr{M}$ does not depend on the representation (2.2) of $\mathscr{M}$ used, but first we introduce a dual operator $\widehat{\mathscr{M}}$ to $\mathscr{M}$.

Given the families (g_ω), (ψ_ω) used for the definition of $\mathscr{M}$ in (2.2), we let $\varepsilon_\omega = \pm 1$ depending on whether ψ_ω is increasing or decreasing. We can then introduce new (dual) families $(\hat{g}_\omega)$, $(\hat{\psi}_\omega)$ such that

$$\hat{g}_\omega = \varepsilon_\omega \cdot g_\omega \circ \psi_\omega^{-1}\,, \quad \hat{\psi}_\omega = \psi_\omega^{-1}\,,$$

and define $\widehat{\mathscr{M}}$ such that

$$\widehat{\mathscr{M}}\,\Phi(x) = \sum_{\omega} \varepsilon_\omega \cdot g_\omega(\psi_\omega^{-1}x)\Phi(\psi_\omega^{-1}x)\,. \tag{2.3}$$

Note that the operation $\hat{}$ is an involution and that

$$(\mathscr{M}_1\mathscr{M}_2)\hat{} = \widehat{\mathscr{M}}_2\widehat{\mathscr{M}}_1\,.$$

Let χ_y be the characteristic function of $\{y\}$. We associate with $\mathscr{M}$ the functions $F^{\pm}$ on $\mathbb{R}\times\mathbb{R}$ such that

$$F^+(x,y) = \sum_{\omega:\varepsilon_\omega=+1} g_\omega(x)\chi_y(\psi_\omega x)\,,$$
$$F^-(x,y) = \sum_{\omega:\varepsilon_\omega=-1} g_\omega(x)\chi_y(\psi_\omega x)\,.$$

If $\widehat{F}^{\pm}$ are similarly associated with $\widehat{\mathscr{M}}$, we have

$$\widehat{F}^+(x,y) = F^+(y,x), \quad \widehat{F}^-(x,y) = -F^-(y,x)\,. \tag{2.4}$$

Since χ_y is of bounded variation, the sum $F^+ + F^-$ is uniquely determined by the operator $\mathscr{M}$ (independently of the particular choice of the representation (2.2)). Let us show that both F^+ and F^- are determined by $\mathscr{M}$, i.e., $F^+ + F^- = 0$ implies $F^+ = F^- = 0$. Indeed, if $F^+ = -F^-$, then $F^+(x,y) \neq 0$ implies that there exist ω and ω' so that $y = \psi_\omega x$ with $\varepsilon_\omega = +1$, and $y = \psi_{\omega'}x$ with $\varepsilon_{\omega'} = -1$, hence $\{(x,y) : F^+(x,y) \neq 0\}$ is at most countable. But since the g_ω are continuous, $\{x : \exists y \text{ with } F^+(x,y) \neq 0\}$ is open, proving our contention.

Lemma 2.1. *The adjoint* $\widehat{\mathscr{M}}$ *and the function* $\mathrm{Tr}^\#\mathscr{M}$ *are uniquely determined by the operator* $\mathscr{M}$, *independently of the choice of the representation* (2.2)

Proof. For $\widehat{\mathscr{M}}$ this results from (2.4).

Let us now write

$$\mathrm{Tr}^\#\mathscr{M} = \int m(dx\,dy)\tfrac{1}{2}\operatorname{sgn}(y-x)\,,$$

where the bounded measure m on $\mathbb{R}\times\mathbb{R}$ is defined by

$$m(dx\,dy) = \sum_{\omega} d(g_\omega(x))\delta(y-\psi_\omega(x))\,dy\,.$$

If the functions Φ, Ψ are of bounded variation, continuous, and of compact support, we have

$$\begin{aligned}\int m(dx\,dy)\Psi(x)\Phi(y) &= \sum_{\omega}\int \Psi(x)d(g_\omega(x))\Phi(\psi_\omega x)\\ &= -\int d(\Psi(x))\sum_{\omega} g_\omega(x)\Phi(\psi_\omega x)\\ &\quad -\int\sum_{\omega}\varepsilon_\omega g_\omega(\psi_\omega^{-1}y)\Psi(\psi_\omega^{-1}y)d\Phi(y)\\ &= -\int d(\Psi(x))(\mathscr{M}\Phi)(x) - \int(\widehat{\mathscr{M}}\Psi)(y)d(\Phi(y))\,.\end{aligned}$$

By the theorem of Stone-Weierstrass, the linear combinations of products $\Psi(x)\Phi(y)$ are dense in the continuous functions vanishing at ∞ on $\mathbb{R} \times \mathbb{R}$. Therefore the knowledge of $\mathscr{M}$, $\widehat{\mathscr{M}}$ determines uniquely $m(dx\,dy)$ hence $\mathrm{Tr}^{\#}\mathscr{M}$. □

Note that an operator $\mathscr{M} \in \mathscr{A}$ also has an operator norm $\|\mathscr{M}\|_{\mathscr{B}}$ with respect to the Var norm on $\mathscr{B}$, and that $\|\mathscr{M}\|_{\mathscr{B}} \leqq \|\mathscr{M}\|_{\mathscr{A}}$, and $|\mathrm{Tr}^{\#}\mathscr{M}| \leqq \|\mathscr{M}\|_{\mathscr{A}}$. Note also that if

$$\mathscr{M}_1\Phi(x) = \sum_{\omega_1} g_{\omega_1}(x)\Phi(\psi_{\omega_1}x)$$

$$\mathscr{M}_2\Phi(x) = \sum_{\omega_2} g_{\omega_2}(x)\Phi(\psi_{\omega_2}x)\,,$$

the product $\mathscr{M}_1 \cdot \mathscr{M}_2$ is given by

$$\mathscr{M}_1\mathscr{M}_2\Phi(x) = \sum_{\omega_1}\sum_{\omega_2} g_{\omega_1}(x)g_{\omega_2}(\psi_{\omega_1}x)\Phi(\psi_{\omega_2}\psi_{\omega_1}(x))\,.$$

Lemma 2.2. $\mathrm{Tr}^{\#}$ *is a continuous trace on* $\mathscr{A}$.

Proof. It suffices to check the trace property $\mathrm{Tr}^{\#}(\mathscr{L}_1\mathscr{L}_2) = \mathrm{Tr}^{\#}(\mathscr{L}_2\mathscr{L}_1)$.

First assume that $\psi_2\psi_1$ is increasing, and let $\varepsilon = \pm 1$ depending on whether ψ_1 and ψ_2 are increasing or decreasing. Since ψ_1 and ψ_2 are continuous, the set $\{x : \psi_2\psi_1 x \neq x\}$ is the union of at most countably many open intervals (a_i, b_i). Correspondingly, $\{y : \psi_1\psi_2 y \neq y\}$ is the union of intervals (a_i', b_i') where

$$a_i' = \psi_1 a_i = \psi_2^{-1}a_i, \quad b_i' = \psi_1 b_i = \psi_2^{-1}b_i\,,$$

if $\varepsilon = 1$ and

$$a_i' = \psi_1 b_i = \psi_2^{-1}b_i, \quad b_i' = \psi_1 a_i = \psi_2^{-1}a_i\,,$$

if $\varepsilon = -1$. If σ_i is the sign of $\psi_2\psi_1 x - x$ on (a_i, b_i), then $\sigma_i' = \varepsilon\sigma_i$ is the signof $\psi_1\psi_2 y - y$ on (a_i', b_i'). We have

$$\begin{aligned}
\mathrm{Tr}^{\#}\mathscr{L}_1\mathscr{L}_2 &= \int d(g_1(x)g_2(\psi_1(x))\tfrac{1}{2}\operatorname{sgn}(\psi_2\psi_1 x - x) \\
&= \tfrac{1}{2}\sum_i \int_{a_i}^{b_i} d(g_1(x)g_2(\psi_1(x))\sigma_i \\
&= \tfrac{1}{2}\sum_i \sigma_i[g_1(b_i)g_2(\psi_1 b_i) - g_1(a_i)g_2(\psi_1 a_i)] \\
&= \tfrac{1}{2}\sum_i \sigma_i\varepsilon[g_1(\psi_2 b_i')g_2(b_i') - g_1(\psi_2 a_i')g_2(a_i')] \\
&= \tfrac{1}{2}\sum_i \sigma_i'[g_2(b_i')g_1(\psi_2 b_i') - g_2(a_i')g_1(\psi_2 a_i')] \\
&= \int d(g_2(y)g_1(\psi_2(y))\tfrac{1}{2}\operatorname{sgn}(\psi_1\psi_2 y - y) \\
&= \mathrm{Tr}^{\#}\mathscr{L}_2\mathscr{L}_1\,.
\end{aligned}$$

If $\psi_2\psi_1$ is decreasing, either it has no fixed point and $\psi_1\psi_2$ has no fixed point either, or it has a unique fixed point a and

$$a' = \psi_1 a = \psi_2^{-1} a$$

is the unique fixed point of $\psi_1\psi_2$. Then

$$\begin{aligned}\mathrm{Tr}^{\#}\mathscr{L}_1\mathscr{L}_2 &= g_1(a)g_2(\psi_1 a)\\ &= g_2(a')g_1(\psi_2 a')\\ &= \mathrm{Tr}^{\#}\mathscr{L}_2\mathscr{L}_1\end{aligned}$$

concluding the proof. □

Remark. From the proof of Lemma 2.2, one sees that whenever there are finitely many fixed points, the sharp trace takes the form presented in (1.2). In particular, one easily checks that if the ψ_ω are the finitely many contracting inverse branches of a piecewise monotone interval map, one obtains an expression of the type (1.2) where the Lefschetz numbers $L(x,\psi)$ are all equal to $+1$, thus recovering the usual formula for the dynamical zeta function.

The formulae (2.2), (2.3) define $\mathscr{M}, \widehat{\mathscr{M}}$ also as bounded operators on the space of bounded functions on $\mathbb{R}$, with the uniform norm $\|\cdot\|_0$ (instead of $\mathscr{B}$ with the norm Var); we denote the corresponding norms of $\mathscr{M}, \widehat{\mathscr{M}}$ by $\|\mathscr{M}\|_0$, $\|\widehat{\mathscr{M}}\|_0$ and define

$$\begin{aligned}R &= \lim_{m\to\infty} (\|\mathscr{M}^m\|_0)^{1/m}\\ \widehat{R} &= \lim_{m\to\infty} (\|\widehat{\mathscr{M}}^m\|_0)^{1/m} .\end{aligned}$$

Theorem 2.3.

a) *The spectral radius of $\mathscr{M}$ acting on $\mathscr{B}$ is $\leqq \max(R,\widehat{R})$ and $\geqq \widehat{R}$.*

b) *The essential spectral radius of $\mathscr{M}$ acting on $\mathscr{B}$ is $\leqq \widehat{R}$.*

c) *If $g_\omega \geqq 0$ for all ω, the spectral radius of $\mathscr{M}$ acting on $\mathscr{B}$ is $\geqq R$. If furthermore $\widehat{R} < R$, then R is an eigenvalue of $\mathscr{M}$ and there is a corresponding eigenfunction $\Phi_R \geqq 0$.*

Proof. This is Theorem B.1 of Ruelle [1993] (in the special case where the g_ω are continuous). □

Proposition 2.4. *We have identically*

$$\mathrm{Tr}^{\#}\mathscr{M} + \mathrm{Tr}^{\#}\widehat{\mathscr{M}} = 0 . \tag{2.5}$$

Proof. Indeed

$$\begin{aligned}\mathrm{Tr}^{\#}\widehat{\mathscr{L}} &= \int \varepsilon d(g\circ\psi^{-1}(x))\tfrac{1}{2}\mathrm{sgn}\,(\psi^{-1}(x)-x)\\ &= \int d(g(y))\tfrac{1}{2}\mathrm{sgn}\,(y-\psi(y))\\ &= -\int d(g(y))\tfrac{1}{2}\mathrm{sgn}\,(\psi(y)-y)\\ &= -\mathrm{Tr}^{\#}\mathscr{L} ,\end{aligned}$$

which proves the proposition. (We have not used the compact support property or the continuity of g.) $\square$

Note the duality between the pairs $(\mathscr{M}, R)$ and $(\widehat{\mathscr{M}}, \widehat{R})$. This duality can be formalised by introducing the bilinear form $\langle \mathscr{M}_1 : \mathscr{M}_2 \rangle = \mathrm{Tr}^{\#}(\widehat{\mathscr{M}}_1 \mathscr{M}_2)$, which is antisymmetric (i.e. $\langle \mathscr{M}_1 : \mathscr{M}_2 \rangle = -\langle \mathscr{M}_2 : \mathscr{M}_1 \rangle$), and for which

$$\langle \mathscr{M}_1 : \mathscr{M}\mathscr{M}_2 \rangle = \langle \widehat{\mathscr{M}}\mathscr{M}_1 : \mathscr{M}_2 \rangle .$$

(We shall not need to use this bilinear form.)

Remark. We shall use later the derivation property

$$d(g_1 \cdot g_2) = (dg_1) \cdot g_2 + g_1(dg_2) \tag{2.6}$$

which holds if g_1,g_2 are of bounded variation and at least one of the g_i is continuous.

The property (2.6) remains true if g_1, g_2 have only regular discontinuities (i.e. $g(x+) + g(x-) = 2g(x)$). However, functions with regular discontinuities do not form an algebra. This is why we assume that the *weights* g_i in the definition of $\mathscr{M}$ are continuous. This assumption was avoided in the papers of Baladi and Ruelle [1994] and Ruelle [1993], by making use of different Lefschetz numbers, but (among other things) (2.5), and its consequence (3.3), were replaced by a more complicated *functional equation* in Ruelle [1993]. In Baladi [1995], where the case of the finitely many inverse branches of a single map was considered, the weights g were only assumed to be continuous at the periodic points of the dynamical system, but a strong assumption of constancy on homtervals was also needed.

3. Sharp determinants, kneading operator

The sharp determinants of $\mathscr{M}, \widehat{\mathscr{M}}$ are defined by the following formal power series in z:

$$\Delta(z) = \mathrm{Det}^{\#}(1 - z\mathscr{M}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{Tr}^{\#} \mathscr{M}^m \tag{3.1}$$

$$\widehat{\Delta}(z) = \mathrm{Det}^{\#}(1 - z\widehat{\mathscr{M}}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{Tr}^{\#} \widehat{\mathscr{M}}^m . \tag{3.2}$$

In view of Proposition 2.4 we have thus the following *functional equation*

$$\widehat{\Delta}(z) = \Delta^{-1}(z) , \tag{3.3}$$

where we have used

$$\widehat{\mathscr{M}}^m = \widehat{\mathscr{M}^m} .$$

It is natural to define

$$\zeta(z) = \frac{1}{\Delta(z)} , \quad \hat{\zeta}(z) = \frac{1}{\widehat{\Delta}(z)} .$$

Note that (3.3) implies that $\zeta(z) = \widehat{\Delta}(z) = 1/\hat{\zeta}(z)$.

We shall now define the *kneading operators* $\mathscr{D} = \mathscr{D}(z)$, $\widehat{\mathscr{D}} = \widehat{\mathscr{D}}(z)$ as operators on $L^2(\mu)$ where the bounded nonatomic measure μ on $\mathbb{R}$ is defined by

$$\mu(dx) = \sum_\omega |d(g_\omega x)| + \sum_\omega |d(g_\omega \circ \psi_\omega^{-1} x)| \, .$$

The Radon–Nikodym derivative of $dg_\omega(x)$ with respect to $\mu(dx)$ is a bounded function which we shall denote by $g'_\omega(x)$, i.e. $dg_\omega(x) = g'_\omega(x)\mu(dx)$. Similarly for $d(g_\omega \circ \psi_\omega^{-1} x)$. In fact the measure μ is an auxiliary device and will mostly be omitted from the notation. Assuming $|z| < R^{-1}$, we introduce the bounded operator $\mathscr{D} = \mathscr{D}(z) : L^2(\mu) \to L^2(\mu)$ by

$$\begin{aligned}(\mathscr{D}\varphi)(y) &= \sum_\omega \int \varphi(x) d(z g_\omega x)[(1 - z\mathscr{M})^{-1} \tfrac{1}{2}\operatorname{sgn}(\cdot - y)](\psi_\omega x) \\ &= \int \mu(dx) \mathscr{D}_{xy} \varphi(x) \, , \end{aligned} \tag{3.4}$$

with the kernel

$$\begin{aligned}\mathscr{D}_{xy} = \sum_{k=1}^\infty z^k \sum_{\omega_1,\ldots,\omega_k} &(g'_{\omega_1}(x)) g_{\omega_2}(\psi_{\omega_1} x) \cdots g_{\omega_k}(\psi_{\omega_{k-1}} \cdots \psi_{\omega_1} x) \\ &\times \tfrac{1}{2}\operatorname{sgn}(\psi_{\omega_k} \cdots \psi_{\omega_1} x - y) \, . \end{aligned} \tag{3.5}$$

Replacing g_ω, ψ_ω, $\mathscr{M}$, and R by $\varepsilon_\omega g_\omega \circ \psi_\omega^{-1}$, ψ_ω^{-1}, $\widehat{\mathscr{M}}$, and $\widehat{R}$, we obtain an operator $\widehat{\mathscr{D}} = \widehat{\mathscr{D}}(z)$ with kernel $\widehat{\mathscr{D}}_{xy}$. The kernels $\mathscr{D}_{xy}$, $\widehat{\mathscr{D}}_{xy}$ are in $L^2(\mu \times \mu)$ hence these operators are Hilbert–Schmidt. We define a determinant

$$\begin{aligned}\operatorname{Det}_\star(1 + \mathscr{D}) &= 1 + \sum_{m=1}^\infty \frac{1}{m!} \int \mu(dx_1) \cdots \int \mu(dx_m)\, \Delta_m(x_1, \ldots, x_m) \\ &= (\exp \int \mu(dx)\, \mathscr{D}_{xx}) \cdot \operatorname{Det}_2(1 + \mathscr{D}) \, , \end{aligned}$$

(and similarly for $\widehat{\mathscr{D}}$) where Δ_n is the determinant of the $n \times n$ matrix with elements $\mathscr{D}_{x_i x_j}$ $(i, j = 1, \ldots, n)$; the integral $\int \mu(dx) \mathscr{D}_{xx}$ is well-defined and plays the role of a trace even though $\mathscr{D}$ is not of trace class; Det_2 is a *regularized determinant* defined by the power series

$$\operatorname{Det}_2(1 + \mathscr{D}) = \exp \sum_{m=2}^\infty \frac{(-1)^{m-1}}{m} \operatorname{Tr} \mathscr{D}^m$$

(see Simon [1979]).

Remark. In an earlier version of this paper it was stated that $\mathscr{D}$ is a trace class operator on $L^1(\mu)$, so that its Fredholm determinant $\operatorname{Det}(1 + \mathscr{D})$ exists in the sense of Grothendieck [1956]. This is incorrect but it is the case that $\operatorname{Det}_\star(1 + \mathscr{D})$ has nearly the same properties as a Fredholm determinant.

Proposition 3.1. *We have identically*

$$\Delta(z) = \operatorname{Det}^\#(1 - z\mathscr{M}) = \operatorname{Det}_\star(1 + \widehat{\mathscr{D}}(z)) \tag{3.6}$$

$$\widehat{\Delta}(z) = \operatorname{Det}^\#(1 - z\widehat{\mathscr{M}}) = \operatorname{Det}_\star(1 + \mathscr{D}(z)) \, . \tag{3.7}$$

Proof. It suffices to prove one of these dual formulas. We shall check (3.7), considered as an identity between formal power series. For this we shall use the expression

$$\mathrm{Det}_\star(1+\mathscr{D}) = \exp - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \mathrm{Tr}_\star \mathscr{D}^m ,$$

with

$$\mathrm{Tr}_\star \mathscr{D}^m = \int \mu(dx_1) \cdots \mu(dx_m) \mathscr{D}_{x_1x_2} \cdots \mathscr{D}_{x_{m-1}x_m} \mathscr{D}_{x_mx_1} .$$

Taking the logarithmic derivative of (3.7), it suffices to prove that

$$\sum_{n=1}^{\infty} z^{n-1} \mathrm{Tr}^\# \widehat{\mathscr{M}}^n = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \mathrm{Tr}_\star \left(\frac{d}{dz}\right) (\mathscr{D}(z))^m = \sum_{m=1}^{\infty} (-1)^m \mathrm{Tr}_\star [\mathscr{D}^\cdot \mathscr{D}^{m-1}] , \tag{3.8}$$

where $\mathscr{D}^\cdot(z) = (d/dz)\mathscr{D}(z)$ has the kernel

$$\mathscr{D}^\cdot_{xy} = \sum_{k=1}^{\infty} kz^{k-1} \sum_{\omega_1 \cdots \omega_k} (g'_{\omega_1}(x)) g_{\omega_2}(\psi_{\omega_1}x) \cdots g_{\omega_k}(\psi_{\omega_{k-1}} \cdots \psi_{\omega_1}x) \, \tfrac{1}{2} \, \mathrm{sgn}\, (\psi_{\omega_k} \cdots \psi_{\omega_1} x - y) . \tag{3.9}$$

Using (3.5) and (3.9) we expand the right-hand-side of (3.8) in powers of z:

$$\sum_{m=1}^{\infty} (-1)^m \mathrm{Tr}_\star \left[\mathscr{D}^\cdot \mathscr{D}^{m-1}\right] = \sum_{n=1}^{\infty} z^{n-1} \sum_{\omega_1,\ldots,\omega_n} \sum_{m=1}^{n} \int_{x_1,\ldots,x_m} \sum^{*} \prod_{[n,m,\omega_1,\ldots,\omega_n]} , \tag{3.10}$$

where a $-$ sign is affected to each factor g'_ω in $\mathscr{D}^\cdot$ or $\mathscr{D}$. Each product $\prod_{n,m,\omega_i}$ begins with one of the factors $-k \cdot g'_{\omega_1} \cdots g_{\omega_k}$ of (3.9); and is followed by $m-1$ strings, containing each exactly a $-g'_{\omega_i}$ followed by a product of g_{ω_i} and ending with a $\frac{1}{2}\mathrm{sgn}$. We view the $\frac{1}{2}\mathrm{sgn}$ as "markers" separating the strings (the sum $\sum^*$ is over the different possibilities of constructing the $m-1$ strings, i.e., of placing the markers). We shall see that it is more convenient to write the very last marker at the very beginning of the product. One integrates each $\prod_{n,m,\omega_i}$ over the variables $x_1, \ldots, x_m$ with respect to $\mu(dx_j)$ (replacing therefore each $-g'_{\omega_i}$ by $-dg_{\omega_i}$).

The case $n=1$ being trivial, we consider $n \geqq 2$. To make the book-keeping more systematic, we perform a preliminary operation, expressing the multiplicity k (coming from the initial factor $-k \cdot g'_{\omega_1} \cdots g_{\omega_k}$) as a sum over the first k cyclic permutations on $\{\omega_1, \ldots \omega_n\}$, and redistributing the permutated expressions in the corresponding term of the sum over the ordered ω_i (renumbering the variables x_j accordingly). After this operation, at the i^{th} position (not counting markers) of each new product $\prod'_{n,m,\omega_1,\ldots,\omega_n}$ there may be either a $-dg_{\omega_i}$ (preceded by a marker) or a g_{ω_i}. Summing over $\sum_m$ and $\sum^*$, we obtain for each fixed $\omega_1, \ldots, \omega_n$ exactly $2^n - 1$ possibilities (because there is at least one factor $-dg_{\omega_i}$ since $m \geqq 1$). We shall perform the sum over these $2^n - 1$ terms in $n-1$ steps. All the terms which differ only in the first l factors, and have at least one $-dg_{\omega_i}$ factor will be lumped together at the $(\ell - 1)$-th step. We shall see that the first ℓ factors are replaced by $-d$(product of ℓ factors g_{ω_i}).

Writing $\sigma = \frac{1}{2}\mathrm{sgn}$ and g_i, ψ_i, ε_i, instead of g_{ω_i}, ψ_{ω_i}, ε_{ω_i}, we first do the case $\ell = n = 2$, using an easy integration by parts explained below to get the first equality:

$$\begin{aligned}
&\int_{x_1,x_2} \sigma(\psi_2 x_2 - x_1) dg_1(x_1)\sigma(\psi_1 x_1 - x_2) dg_2(x_2) \\
&\quad - \int_{x_1} \sigma(\psi_2\psi_1 x_1 - x_1) dg_1(x_1) g_2(\psi_1 x_1) - \int_{x_2} g_1(\psi_2 x_2)\sigma(\psi_1\psi_2 x_2 - x_2) dg_2(x_2) \\
&= \int_{x_2} g_1(\psi_2 x_2)\sigma(\psi_1\psi_2 x_2 - x_2) dg_2(x_2) - \int_{y} \sigma(\psi_2\psi_1 y - y) g_1(y) dg_2(\psi_1 y) \\
&\quad - \int_{x_1} \sigma(\psi_2\psi_1 x_1 - x_1) dg_1(x_1) g_2(\psi_1 x_1) - \int_{x_2} g_1(\psi_2 x_2)\sigma(\psi_1\psi_2 x_2 - x_2) dg_2(x_2) \\
&= \int_{x} \sigma(\psi_2\psi_1 x - x)(-d(g_1(x) g_2(\psi_1 x)))\,. \qquad (3.11)
\end{aligned}$$

Since σ only has regular discontinuities and g is continuous, we can apply (2.6) and integrate by parts:

$$\int_{x_1} \sigma(u - x_1) dg_1(x_1)\sigma(\psi_1 x_1 - x_2) = g_1(u)\sigma(\psi_1 u - x_2) - \varepsilon_1\sigma(u - \psi_1^{-1} x_2) g_1(\psi_1^{-1} x_2)\,,$$

the change of variable $y = \psi_1^{-1} x_2$ then yields the first equality of (3.11).

We use now a similar calculation to treat the case $\ell = 2$ and $n \geqq 3$. The second factor is either followed by a marker or a factor $g_3(\cdot)$, we only consider the first situation (the other being similar):

$$\begin{aligned}
&\int_{x_1,x_2} \sigma(u - x_1) dg_1(x_1)\sigma(\psi_1 x_1 - x_2) dg_2(x_2)\sigma(\psi_2 x_2 - x_3) \\
&\quad - \int_{x_1} \sigma(u - x_1) dg_1(x_1) g_2(\psi_1 x_1)\sigma(\psi_2\psi_1 x_1 - x_3) \\
&\quad - \int_{x_2} g_1(u)\sigma(\psi_1 u - x_2) dg_2(x_2)\sigma(\psi_2 x_2 - x_3) \\
&= \int_{x_2} g_1(u)\sigma(\psi_1 u - x_2) dg_2(x_2)\sigma(\psi_2 x_2 - x_3) \\
&\quad - \int_{y} \sigma(u - y) g_1(y) dg_2(\psi_1 y)\sigma(\psi_2\psi_1 y - x_3) \\
&\quad - \int_{x_1} \sigma(u - x_1) dg_1(x_1) g_2(\psi_1 x_1)\sigma(\psi_2\psi_1 x_1 - x_3) \\
&\quad - \int_{x_2} g_1(u)\sigma(\psi_1 u - x_2) dg_2(x_2)\sigma(\psi_2 x_2 - x_3) \\
&= \int_{x_1} \sigma(u - x_1)(-d(g_1(x_1) g_2(\psi_1 x_1)))\sigma(\psi_2\psi_1 x_1 - x_3)\,. \qquad (3.12)
\end{aligned}$$

The reason why the same expression u (which only depends on ω_i for $i \geqq 3$, and on some x_r for $r \geqq 3$) appears in all lines of (3.12) is because the factors coincide after the ℓth position. (When the second term is followed by a $g_3(\cdot)$ factor, the situation is not exactly the same, but analogous.)

We continue in this way. Before the ℓ-th step we regroup terms as follows

$$\int \sigma d(g_1 \cdots g_{\ell-1})\sigma dg_l - \int \sigma d(g_1 \cdots g_{\ell-1})g_l - \int g_1 \cdots g_{\ell-1}\sigma dg_\ell \,,$$

and a calculation like (3.12) yields

$$= -\int \sigma d(g_1 \cdots g_\ell)\,.$$

The final step is like (3.11) and we obtain:

$$\begin{aligned}
&\sum_{m=1}^{\infty} (-1)^m \mathrm{Tr}_\star[\mathscr{D}^{\cdot}\mathscr{D}^{m-1}] \\
&\quad = \sum_{n=1}^{\infty} z^{n-1} \sum_{\omega_1\cdots\omega_n} \int -d(g_1(x)\cdots g_n(\psi_{n-1}\cdots\psi_1 x))\sigma(\psi_n\cdots\psi_1 x - x) \\
&\quad = \sum_{n=1}^{\infty} z^{n-1}(-\mathrm{Tr}^{\#}\mathscr{M}^n) \\
&\quad = \sum_{n=1}^{\infty} z^{n-1}\mathrm{Tr}^{\#}\widehat{\mathscr{M}}^{\,n}\,,
\end{aligned}$$

which proves (3.8) and therefore the proposition.

Note that the compact support assumption on g is essential when applying integration by parts to get rid of the boundary term. □

Corollary 3.2. *The function $\widehat{\Delta}(z) = \zeta(z)$ is holomorphic for $|z| < R^{-1}$. If $\widehat{R} < R$, $\widehat{\Delta}(z)$ extends to a function holomorphic in $|z| < \widehat{R}^{-1}$ except maybe for isolated singularities at points $z = \lambda^{-1}$ where λ is an eigenvalue of $\mathscr{M}$ acting on $\mathscr{B}$. If λ is a simple eigenvalue, $\widehat{\Delta}(z)$ has at most a simple pole at λ^{-1}.*

Proof. If $|z| < R^{-1}$, we use the fact that $1 - z\mathscr{M}$ is invertible on bounded functions. The inverse can hence be applied to $\mathrm{sgn}(\cdot - y)$. Therefore $\mathscr{D}(z)$ and $\widehat{\Delta}(z) = \mathrm{Det}_\star(1 + \mathscr{D}(z))$ depend holomorphically on z.

To proceed, it is convenient to regularize the kernel

$$\mathscr{D}_{xy}(z) = \sum_\omega z g'_\omega(x)[(1 - z\mathscr{M})^{-1}\tfrac{1}{2}\mathrm{sgn}(\cdot - y)](\psi_\omega x)$$

by convolution (to the right) with $\chi_n(\cdot) = n\chi(n\cdot)$ where χ is smooth, positive, and $\int \chi(y)\,dy = 1$. This amounts to replacing sgn by a smooth function $\mathrm{sgn} * \chi_n$ tending pointwise to sgn, and $\mathscr{D}(z)$ by a new operator $\mathscr{D}_{*n}(z)$ which we may assume to be of trace class on $L^2(\mu)$. Now $\mathrm{Det}_\star(1 + \mathscr{D}_{*n}(z)) = \mathrm{Det}(1 + \mathscr{D}_{*n}(z))$ is a true Fredholm determinant (see e.g. Simon [1979]) when $z \notin [\text{spectrum } \mathscr{M}]^{-1}$, analytic there in z with poles corresponding to the eigenvalues of $\mathscr{M}$. If λ is a simple eigenvalue of $\mathscr{M}$ it yields a contribution $A_n \cdot (1 - \lambda z)^{-1}$ to $\mathscr{D}_{*n}(z)$, where A_n is at most of rank 1, and therefore $\mathrm{Det}(1 + \mathscr{D}_{*n}(z))$ has at most a simple pole at λ^{-1}.

The kernels $(\mathscr{D}_{*n})_{xy}(z)$ and $\mathscr{D}_{xy}(z)$ are uniformly bounded functions of x, y when z is in a compact set K disjoint from $[\text{spectrum } \mathscr{M}]^{-1}$. It follows that the sequence $\mathrm{Det}_\star(1 + \mathscr{D}_{*n}(z))$ is uniformly bounded for $z \in K$. Furthermore, for $|z| < R^{-1}$, the functions $(\mathscr{D}_{*n})_{xy}(z)$ tend pointwise to $\mathscr{D}_{xy}(z)$ when $n \to \infty$, so that $\mathrm{Det}_\star(1+\mathscr{D}_{*n}(z)) \to \mathrm{Det}_\star(1+\mathscr{D}(z))$ in this disc. In conclusion $\mathrm{Det}(1+\mathscr{D}_{*n}(z))$ tends to $\mathrm{Det}_\star(1+\mathscr{D}(z))$ when $z \notin [\text{spectrum } \mathscr{M}]^{-1}$, uniformly on compact sets. When λ is an eigenvalue of $\mathscr{M}$ with $|\lambda| > \widehat{R}$ then λ^{-1} is an isolated singularity of $\mathrm{Det}_\star(1+\mathscr{D}(z))$ or a regular value, and if λ is a simple eigenvalue of $\mathscr{M}$, then λ^{-1} is at most a simple pole of $\mathrm{Det}_\star(1+\mathscr{D}(z))$. □

The following result will be useful below:

Lemma 3.3. *If $\varphi \in \mathscr{B}$ satisfies $\mathscr{M}\varphi = \lambda\varphi$ for $\widehat{R} < |\lambda| \leqq R$ then φ tends to zero at infinity and is continuous.*

Proof. The first assertion is true because each g_ω tends to zero at infinity. Let $\tilde{\varphi}(x) = \lim_{y\downarrow x}\varphi(y) - \lim_{y\uparrow x}\varphi(y)$ for all x. Then

$$\sum_\omega \varepsilon_\omega g_\omega(x)\tilde{\varphi}(\psi_\omega x) = \lambda\tilde{\varphi}(x)\ .$$

For bounded Φ we may define

$$(\Phi, \tilde{\varphi}) = \sum_x \Phi(x)\tilde{\varphi}(x)\ .$$

Writing

$$\mathscr{M}_\varepsilon\tilde{\varphi} = \sum_\omega \varepsilon_\omega g_\omega(x)\tilde{\varphi}(\psi_\omega x)$$

we have

$$(\Phi, \mathscr{M}_\varepsilon\tilde{\varphi}) = (\widehat{\mathscr{M}}\,\Phi, \tilde{\varphi})\ ,$$

and we find

$$(\Phi, \tilde{\varphi}) = \lambda^{-1}(\widehat{\mathscr{M}}\,\Phi, \tilde{\varphi})\ .$$

Therefore

$$(\Phi, \tilde{\varphi}) = \lambda^{-n}(\widehat{\mathscr{M}}^{\,n}\Phi, \tilde{\varphi})\ .$$

Since $|\lambda| > \widehat{R}$ we have $\lambda^{-n}\|\widehat{\mathscr{M}}^{\,n}\|_0 \to 0$ for $n \to \infty$, hence $(\Phi, \tilde{\varphi}) = 0$ for all Φ so that $\tilde{\varphi} = 0$, i.e., φ is continuous. □

Corollary 3.4. *If $\widehat{R} > R$ and $\hat{\lambda}$ is an eigenvalue of $\widehat{\mathscr{M}}$ acting on $\mathscr{B}$ (with $R < |\hat{\lambda}| \leqq \widehat{R}$) then $\hat{\lambda}^{-1}$ is a zero of $\widehat{\Delta}(z) = \mathrm{Det}^{\#}(1 - z\widehat{\mathscr{M}})$.*

Proof. If $\varphi \in \mathscr{B} \subset L^2(\mu)$ tends to zero at infinity and only has regular discontinuities, we have

$$\begin{aligned}((1+\mathscr{D})\varphi)(y) = \int -\,d\varphi(x)\{&\tfrac{1}{2}\operatorname{sgn}(x-y)\\ &+ \sum_\omega z g_\omega(x)[(1-z\mathscr{M})^{-1}\tfrac{1}{2}\operatorname{sgn}(\,\cdot\,-y)](\psi_\omega x)\}\\ &+ \int\sum_\omega d(zg_\omega(x)\varphi(x))[(1-z\mathscr{M})^{-1}\tfrac{1}{2}\operatorname{sgn}(\,\cdot\,-y)](\psi_\omega x)\end{aligned}$$

$$\begin{aligned}
&= \int - d\varphi(x)[(1 + z\mathscr{M}(1 - z\mathscr{M})^{-1})\, \tfrac{1}{2}\mathrm{sgn}\,(\,\cdot\, - y)](x) \\
&\quad + \int d\left(z \sum_{\omega} \varepsilon_\omega g_\omega(\psi_\omega^{-1}x)\varphi(\psi_\omega^{-1}x)\right) \\
&\quad \times [(1 - z\mathscr{M})^{-1}\, \tfrac{1}{2}\mathrm{sgn}\,(\,\cdot\, - y)](x) \\
&= -\int d(\varphi(x) - (z\widehat{\mathscr{M}}\,\varphi)(x))[(1 - z\mathscr{M})^{-1}\, \tfrac{1}{2}\mathrm{sgn}\,(\,\cdot\, - y)](x)\,.
\end{aligned}$$

If we take $z = \hat{\lambda}^{-1}$ and assume $\widehat{\mathscr{M}}\,\varphi = \hat{\lambda}\varphi$ (by the dual Lemma 3.3 φ is continuous and tends to zero at infinity), the right-hand-side vanishes, hence $(1 + \mathscr{D})\varphi = 0$, i.e., -1 is an eigenvalue of $\mathscr{D}(z)$ hence the regularized determinant $\mathrm{Det}_2(1 + \mathscr{D}(z))$ vanishes (see Simon [1979]), and therefore also $\widehat{\Delta}(z) = \mathrm{Det}_\bigstar(1 + \mathscr{D}(z)) = 0$. □

Theorem 3.5. *The determinant $\Delta(z) = \mathrm{Det}^\#(1 - z\mathscr{M})$ is holomorphic for $|z| < \widehat{R}^{-1}$, vanishes only at points λ^{-1} where $\widehat{R} < |\lambda| \leqq R$, and λ is an eigenvalue of $\mathscr{M}$ acting on $\mathscr{B}$. The multiplicity of λ^{-1} as a zero of $\Delta(z)$ is the multiplicity of λ as an eigenvalue of $\mathscr{M}$.*

Proof. The dual of Corollary 3.2 shows that $\Delta(z)$ is holomorphic for $|z| < \widehat{R}^{-1}$. In this region, Corollary 3.2 shows that the inverse $\Delta^{-1}(z) = \widehat{\Delta}(z)$ either is everywhere holomorphic (if $\widehat{R} > R$) or holomorphic except for isolated singularities occuring only at points λ^{-1} where $\widehat{R} < |\lambda| \leqq R$, and λ is an eigenvalue of $\mathscr{M}$ (if $\widehat{R} < R$).

In fact, if λ is an eigenvalue of $\mathscr{M}$, $\Delta(z)$ vanishes at λ^{-1} by the dual of Corollary 3.4. If λ is a simple eigenvalue of $\mathscr{M}$, then λ^{-1} is a simple zero of $\Delta(z)$ because (by Corollary 3.2) it is at most a simple pole of $\widehat{\Delta}(z)$.

Finally, if λ is an eigenvalue of multiplicity k of $\mathscr{M}$, a small perturbation of $\mathscr{M}$ (with respect to $\|\cdot\|_{\mathscr{A}}$) will replace λ by k simple eigenvalues, and therefore a small perturbation of $\Delta(z)$ will have k simple zeroes. Therefore $\Delta(z)$ itself has a zero of order k at λ^{-1}, so that $\widehat{\Delta}(z) = \Delta^{-1}(z)$ has in fact a pole of order k at λ^{-1}. □

Remark. In Theorem 3.5 we recover in particular, by the means of a totally different proof, a previous result on weighted zeta functions of positively expansive piecewise monotone maps (Baladi–Keller [1990]), but only in the special case when the weight is continuous and vanishes at the endpoints of the intervals of monotonicity. See also Mori [1992]. See Ruelle [1994] for more general results.

Corollary 3.6. *The function*

$$\mathscr{M} \to \mathrm{Det}^\#(1 - \mathscr{M})$$

is holomorphic in

$$\{\mathscr{M} \in \mathscr{A} \,:\, \widehat{R} < 1\}\,,$$

and meromorphic without zero in

$$\{\mathcal{M} \in \mathcal{A} \,:\, R < 1\} .$$

Proof. This is because, for $\widehat{R} < 1$ the maps

$$\mathcal{M} \mapsto \widehat{\mathcal{M}} \mapsto \widehat{\mathcal{D}}(1) \mapsto \mathrm{Det}_\star(1 + \widehat{\mathcal{D}}(1)) = \mathrm{Det}^\#(1 - \mathcal{M})$$

are holomorphic, while for $R < 1$ the maps

$$\mathcal{M} \mapsto \mathcal{D}(1) \mapsto \mathrm{Det}_\star(1 + \mathcal{D}(1)) = \frac{1}{\mathrm{Det}^\#(1 - \mathcal{M})}$$

are holomorphic. Here we consider $\mathcal{D}(1)$ as an element of the Banach space of bounded Kernels $\mathcal{D}_{xy}$ (with the uniform norm), and similarly for $\widehat{\mathcal{D}}(1)$. □

Appendix

In this appendix we see how the sharp determinant $\mathrm{Det}^\#(1 - z\mathcal{M})$ of a fixed operator $\mathcal{M}$ of the form (2.2) can be obtained as a limit of determinants of finite kneading matrices (Milnor–Thurston [1988], Baladi-Ruelle [1994], Ruelle [1993]). From this, we obtain convergence of the discrete spectrum of approximations of $\mathcal{M}$.

We use the assumptions and notations of Sect. 2, considering only the case where the set of indices ω is finite (the countable case can be treated by considering finite approximations, see e.g. Ruelle [1993]). The idea is to approach each g_ω by a sequence of finite linear combinations g_ω^n of functions

$$\Upsilon_\omega^x = \tfrac{1}{2}\left(-\chi_{(u_\omega, x)} + \chi_{(x, v_\omega)}\right) ,$$

where $J_\omega = (u_\omega, v_\omega)$ contains the support of g_ω and $\chi_{(a,b)}$ is the characteristic function of (a, b). More precisely, at the n^{th} step we decompose the interval $[u_\omega, v_\omega]$ into a finite number of intervals $[t_{k-1}, t_k] = [t^n_{\omega,k-1}, t^n_{\omega,k}]$ (with $\lim_{n\to\infty} \max_{\omega,k} |t^n_{\omega,k} - t^n_{\omega,k-1}| = 0$), and place the mass $\mathcal{G}^n_{\omega,k} = \int_{t_{k-1}}^{t_k} dg_\omega(x) = g(t_k) - g(t_{k-1})$ at $X_k = X^n_{\omega,k} \in (t_k, t_{k-1})$. This amounts to taking $g^n_\omega = \sum_k \mathcal{G}^n_{\omega,k} \Upsilon_\omega^{X_k}$, and produces an approximation of $dg_\omega(x)$ by $dg^n_\omega(x)$ in the weak sense (in the dual of the space of continuous functions). Also, the g^n_ω tend uniformly to g_ω, because g_ω is of bounded variation and continuous, and vanishes at the endpoints of J_ω:

$$\int dg_\omega(x)\, \Upsilon_\omega^x(y) = -\int dg_\omega(x)\, \Upsilon_\omega^y(x) = \frac{1}{2}\left[\lim_{x\uparrow y} g_\omega(x) + \lim_{x\downarrow y} g_\omega(x)\right] = g_\omega(y) .$$

We may consider the following operators, acting on $\mathcal{B}$:

$$\begin{aligned} \mathcal{M}_n \Phi(x) &= \sum_\omega g^n_\omega(x) \Phi(\psi_\omega x) \\ \widehat{\mathcal{M}}_n \Phi(x) &= \sum_\omega \varepsilon_\omega g^n_\omega(\psi_\omega^{-1} x) \Phi(\psi_\omega^{-1} x) . \end{aligned} \tag{A.1}$$

Using the atomic measures dg^n_ω, we may define the sharp traces of $\mathscr{M}_n$, $\widehat{\mathscr{M}}_n$ and their powers by (2.1) and linearity, and Proposition 2.4 still holds. (We do *not* claim that all results of Sects. 2 and 3 hold in this discontinuous setting.)

Consider an approximation $g^n_\omega = \sum_k \mathscr{G}^n_{\omega,k} \Upsilon^{X_k}_\omega$ as described above, and construct a corresponding sequence $\mathscr{F}^n$ of families, indexed by $\eta = (\omega, k, \pm)$:

$$J^n_{\omega,k,-} = (u_\omega, X^n_{\omega,k}) \,, \quad J^n_{\omega,k,+} = (X^n_{\omega,k}, v_\omega) \,,$$

$$\psi^n_{\omega,k,\pm} = \psi_\omega|_{(J^n_{\omega,k,\pm})} \,, \quad \varepsilon^n_{\omega,k,\pm} = \varepsilon_\omega \,, \quad G^n_{\omega,k,\pm} = \pm\tfrac{1}{2}\mathscr{G}^n_{\omega,k} \,.$$

Since each G^n_η is constant, we are in the setting considered by Ruelle [1993]. The operators defined by (A.1) can also be written:

$$\mathscr{M}_n \Phi(x) = \sum_\eta G^n_\eta \chi_{J^n_\eta}(x) \Phi(\psi^n_\eta x)$$

(similarly for $\widehat{\mathscr{M}}_n$). Note that the sharp trace of $\mathscr{M}_n$ ($\widehat{\mathscr{M}}_n$) can also be computed from the above decomposition.

Using for $m \geqq 1$ the notation $\underset{\sim}{\eta} = (\eta_1, \ldots, \eta_m)$, and $|\underset{\sim}{\eta}| = m$, we define

$$J^n_{\underset{\sim}{\eta}} = J^n_{\eta_1} \cap (\psi^n_{\eta_1})^{-1}(J^n_{\eta_2} \cap (\psi^n_{\eta_2})^{-1}(\cdots (\psi^n_{\eta_{m-1}})^{-1} J^n_{\eta_m}))) \,,$$

and $\psi^n_{\underset{\sim}{\eta}} : J^n_{\underset{\sim}{\eta}} \to \mathbb{R}$ by $\psi^n_{\underset{\sim}{\eta}} = \psi^n_{\eta_m} \circ \cdots \circ \psi^n_{\eta_1}$. If $J^n_{\underset{\sim}{\eta}} \neq \emptyset$ we also write $J^n_{\underset{\sim}{\eta}} = (u^n_{\underset{\sim}{\eta}}, v^n_{\underset{\sim}{\eta}})$. Finally, we set

$$G^n(\underset{\sim}{\eta}) = \prod_{i=1}^{|\underset{\sim}{\eta}|} G^n_{\eta_i} \,, \quad \varepsilon(\underset{\sim}{\eta}) = \prod_{i=1}^{|\underset{\sim}{\eta}|} \varepsilon_{\eta_i} \,.$$

The first important observation is that, for all m, n,

$$\mathrm{Tr}^{\#} \mathscr{M}^m_n = \sum_{|\underset{\sim}{\eta}|=m} L_1(\psi^n_{\underset{\sim}{\eta}}) G^n(\underset{\sim}{\eta}) \,,$$

$$\mathrm{Tr}^{\#} \widehat{\mathscr{M}}^m_n = \sum_{|\underset{\sim}{\eta}|=m} \varepsilon(\underset{\sim}{\eta}) L_1((\psi^n_{\underset{\sim}{\eta}})^{-1}) G^n(\underset{\sim}{\eta}) \,, \tag{A.2}$$

where the Lefschetz numbers $L_1(\psi)$, for $\psi : (a,b) \to \mathbb{R}$ are as defined in Ruelle [1993]:

$$L_1(\psi) = \tfrac{1}{2}[\mathrm{sgn}\,(\bar\psi(a) - a) - \mathrm{sgn}\,(\bar\psi(b) - b)] \,,$$

with $\bar\psi$ the extension of ψ to $[a,b]$ by continuity. (To see this, we apply as we may the definition of the trace to $\mathscr{M}^m_n$ and $\widehat{\mathscr{M}}^m_n$ written as sums over $\underset{\sim}{\eta}$.)

The zeta function associated with the families $\mathscr{F}^n$ (Ruelle [1993]) is

$$\zeta^n(z) = \exp \sum_{\underset{\sim}{\eta}} \frac{z^{|\underset{\sim}{\eta}|}}{|\underset{\sim}{\eta}|} L(\psi^n_{\underset{\sim}{\eta}}) G^n(\underset{\sim}{\eta}) \tag{A.3}$$

with Lefschetz numbers $L(\psi) = L_0(\psi) + L_1(\psi)$, where, writing $\varepsilon = +1$ if ψ is increasing, $\varepsilon = -1$ otherwise,

$$L_0(\psi) = \frac{\varepsilon}{2}[\mathrm{del}\,(\bar{\psi}(a) - a) + \mathrm{del}\,(\bar{\psi}(b) - b)]\,,$$

and

$$\mathrm{del}\,(\xi) = \begin{cases} +1 & \text{if } \xi = 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Considering $\mathscr{M}_n$ acting on bounded functions we set $R_n = \lim_{m\to\infty} (\|\mathscr{M}_n^m\|_0)^{1/m}$, and similarly $\widehat{R}_n$. Ruelle [1993] proved that the spectral radius of $\mathscr{M}_n$ acting on $\mathscr{B}$ is not bigger than $\max(R_n, \widehat{R}_n)$, that its essential spectral radius is not bigger than $\widehat{R}_n$ (using the duality between $\mathscr{M}_n$ and $\widehat{\mathscr{M}}_n$ we may assume that $\widehat{R}_n \leqq R_n$), and that the zeta function $\zeta^n(z)$ is holomorphic in the disc of radius R_n^{-1}, and coincides in this disc with the kneading determinant which we now define. Consider the set $\{a_1 < \cdots < a_{L_n}\}$ of all endpoints of the intervals J_η^n. The $L_n \times L_n$ kneading matrix is:

$$D_{ij}^n(z) = \delta_{ij} + \sum_{m=1}^{\infty} z^m \left[D_{ij}^{(m)+} - D_{ij}^{(m)-}\right],$$

where

$$D_{ij}^{(m)+} = \lim_{x\downarrow a_i} \sum_{\eta: u_\eta^n = a_i} G_\eta^n[\mathscr{M}_n^{m-1}\left(\tfrac{1}{2}\mathrm{sgn}\,(\,\cdot\, - a_j)\right)]\left[\psi_\eta(x)\right]$$

$$D_{ij}^{(m)-} = \lim_{x\uparrow a_i} \sum_{\eta: v_\eta^n = a_i} G_\eta^n[\mathscr{M}_n^{m-1}\left(\tfrac{1}{2}\mathrm{sgn}\,(\,\cdot\, - a_j)\right)]\left[\psi_\eta(x)\right].$$

The kneading determinant is $\det\left[D_{ij}^n(z)\right] = \zeta^n(z)$. Ruelle also proved that the zeta function $\zeta^n(z)$ admits a meromorphic extension to the disc of radius $1/\widehat{R}_n$, and that in the disc of radius $\min(1/\widehat{R}_n, 1/\widehat{R}_{n,\varepsilon})$, where $\widehat{R}_{n,\varepsilon}$ is obtained by considering the operator

$$\widehat{\mathscr{M}}_{n,\varepsilon}\Phi(x) = \sum_\omega g_\omega^n(\psi_\omega^{-1}x)\Phi(\psi_\omega^{-1}x)\,,$$

the poles of $\zeta^n(z)$ coincide (including multiplicities) with the inverses of the eigenvalues λ of $\mathscr{M}_n$ with $\max(\widehat{R}_n, \widehat{R}_{n,\varepsilon}) < |\lambda| \leqq R_n$.

Replacing the families $\mathscr{F}_n$ by the dual families, we may define a dual zeta function $\hat{\zeta}^n(z)$ ($L_0(\psi) + L_1(\psi)$ being replaced by $\varepsilon L_0(\psi) - L_1(\psi)$ in (A.3) and a dual kneading matrix $\widehat{D}_{ij}^n(z)$. We have $\hat{\zeta}^n(z) = \det[\widehat{D}_{ij}^n(z)]$, and the function $\hat{\zeta}^n(z)$ is analytic in the disc of radius $1/\widehat{R}_n$. In general, $\zeta^n \cdot \hat{\zeta}^n \neq 1$, but Ruelle proves that in the disc of radius $\min(1/\widehat{R}_n, 1/\widehat{R}_{n,\varepsilon})$ the zeroes of $\hat{\zeta}^n(z)$ coincide with the inverses of eigenvalues of $\mathscr{M}_n$ (including multiplicities). We have:

Theorem. *The approximations g_ω^n of the g_ω may be chosen in such a way that the holomorphic functions $\hat{\zeta}^n(z)$ converge to the holomorphic function* $\mathrm{Det}^\#(1 - z\mathscr{M})$ *in the disc of radius $1/\widehat{R}$. In particular, if $\widehat{R} < R$, the set of eigenvalues of $\mathscr{M}_n$ acting on $\mathscr{B}$* (*counted with multiplicities*) *in $\widehat{R} < |\lambda| \leqq R$ converges to the set of eigenvalues of $\mathscr{M}$ acting on $\mathscr{B}$* (*counted with multiplicities*) *in the same annulus.*

Proof. Defining $\widehat{R}_\varepsilon$ by analogy with $\widehat{R}_{n,\varepsilon}$, we first remark that $\widehat{R}_\varepsilon = \widehat{R}$. Indeed, for any fixed m, let the function Φ with $\|\Phi\|_0 = 1$ satisfy $\|\widehat{\mathscr{M}}^m \Phi\|_0 = \|\widehat{\mathscr{M}}^m\|_0$, and consider the set $Z_m = \{x : \psi_{\underset{\sim}{\omega}}^{-1}(x) = \psi_{\underset{\sim}{\omega}'}^{-1}(x) \text{ with } |\underset{\sim}{\omega}| = |\underset{\sim}{\omega}'| = m, \varepsilon(\underset{\sim}{\omega}) = -\varepsilon(\underset{\sim}{\omega}')\}$ (the complement of Z_m is open). If the set of points x_0 such that $|\widehat{\mathscr{M}}^m \Phi(x_0)| = \|\widehat{\mathscr{M}}^m \Phi\|_0$ is a subset of Z_m, we modify the function Φ to ensure that $\Phi(\psi_{\underset{\sim}{\omega}}^{-1}(x_0)\pm) = \Phi(\psi_{\underset{\sim}{\omega}}^{-1}(x_0))$, for one of the points x_0 and all $|\underset{\sim}{\omega}| = m$ (without changing the uniform norm or the value $\Phi(\psi_{\underset{\sim}{\omega}}^{-1}x_0)$). Since the g_ω are continuous, for each $\gamma > 0$, we may thus find $x \notin Z_m$ with $|\widehat{\mathscr{M}}^m \Phi(x)| \geqq \|\widehat{\mathscr{M}}^m \Phi\|_0 - \gamma$, and construct Φ_ε with $\|\Phi_\varepsilon\|_0 = 1$ and $\Phi_\varepsilon(\psi_{\underset{\sim}{\omega}}^{-1} y) = \varepsilon(\underset{\sim}{\omega})\Phi(\psi_{\underset{\sim}{\omega}}^{-1} y)$ for $|\underset{\sim}{\omega}| = m$ and all y in a small neighborhood of x. Then $\|\widehat{\mathscr{M}}_\varepsilon^m \Phi_\varepsilon\|_0 \geqq \|\widehat{\mathscr{M}}^m \Phi\|_0 - \gamma$ so that $\|\widehat{\mathscr{M}}_\varepsilon^m\|_0 \geqq \|\widehat{\mathscr{M}}^m\|_0$. By symmetry we obtain the other inequality.

Using the submultiplicativity of the sequence $\|\widehat{\mathscr{M}}_n^m\|_0$ for $m \geqq 0$, and the uniform convergence of the g_ω^n to the g_ω, one obtains $\limsup_{n\to\infty} \widehat{R}_n \leqq \widehat{R}$, and $\limsup_{n\to\infty} \widehat{R}_{n,\varepsilon} \leqq \widehat{R}_\varepsilon = \widehat{R}$, in particular for any $\gamma > 0$ $\hat{\zeta}^n$ is holomorphic in the disc of radius $(\widehat{R} + \gamma)^{-1}$ for large enough n.

By the definitions and the observation (A.2), to prove the convergence of $\hat{\zeta}^n(z)$ to $\mathrm{Det}^\#(1 - z\mathscr{M})$ it suffices to show that for all m,

$$\lim_{n\to\infty} \mathrm{Tr}^\# \mathscr{M}_n^m = \mathrm{Tr}^\# \mathscr{M}^m\,, \tag{A.4}$$

and

$$\lim_{n\to\infty} \sum_{|\underset{\sim}{\eta}|=m} \varepsilon(\underset{\sim}{\eta}) L_0(\psi_{\underset{\sim}{\eta}}^n) G^n(\underset{\sim}{\eta}) = 0\,, \tag{A.5}$$

together with a uniform bound on $|\zeta^n(z)|$ for all $|z| \leqq (\widehat{R} + \gamma)^{-1}$. Convergence of the discrete spectrum then follows from the above-mentioned results of Ruelle [1993].

It will suffice to prove that for every fixed M_0, the approximations g_ω^n of the g_ω may be chosen such that (A.4) and (A.5) hold for all $m \leqq M_0$. We first consider (A.4). We must see that, for all $m \leqq M_0$,

$$\sum_{\omega_1,\dots,\omega_m} \int d(g_{\omega_1}^n(x) g_{\omega_2}^n(\psi_1 x) \cdots g_{\omega_m}^n(\psi_{\omega_{m-1}} \circ \cdots \circ \psi_{\omega_1} x))$$

$$\times \tfrac{1}{2} \mathrm{sgn}\, \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1}(x) - x)$$

converges to

$$\sum_{\omega_1,\dots,\omega_m} \int d(g_{\omega_1}(x) g_{\omega_2}(\psi_1 x) \cdots g_{\omega_m}(\psi_{\omega_{m-1}} \circ \cdots \circ \psi_{\omega_1} x))$$

$$\times \tfrac{1}{2} \mathrm{sgn}\, (\psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1}(x) - x)$$

when $n \to \infty$. We fix a sequence $\underset{\sim}{\omega} = (\omega_1, \dots, \omega_m)$, and write g_j, g_j^n, and ψ_j, for g_{ω_j}, $g_{\omega_j}^n$, and ψ_{ω_j}. The main observation is that (2.6) holds for $m = 2$

because the g_j^n are regular, and that for $m \geqq 3$ the correction is of the form

$$d(g_1^n(x)g_2^n(\psi_1 x)\cdots g_m^n(\psi_{m-1}\circ\cdots\circ\psi_1 x))$$

$$-\sum_{i=1}^{m} d(g_i^n(\psi_{i-1}\circ\cdots\circ\psi_1 x))\left[\prod_{\ell\neq i} g_l^n(\psi_{\ell-1}\circ\cdots\circ\psi_1 x)\right]$$

$$=\sum_{i=1}^{m-2}\left[\prod_{j=1}^{i-1} g_j^n(\psi_{j-1}\circ\cdots\circ\psi_1 x)\right] d(g_i^n(\psi_{i-1}\circ\cdots\circ\psi_1 x))$$

$$\times\left\{\left[\prod_{l=i+1}^{m} g_\ell^n(\psi_{l-1}\circ\cdots\circ\psi_1\ \cdot\ )\right]_{\mathrm{reg}}(x)-\left[\prod_{l=i+1}^{m} g_\ell^n(\psi_{\ell-1}\circ\cdots\circ\psi_1 x)\right]\right\},$$

where $\varphi_{\mathrm{reg}}(x)=\frac{1}{2}(\varphi(x+)-\varphi(x-))$. (Because for general φ_1, φ_2 of bounded variation $d(\varphi_1\varphi_2)=(\varphi_1)_{\mathrm{reg}}\,d(\varphi_2)+(\varphi_2)_{\mathrm{reg}}\,d(\varphi_1)$.) Since the g_i are continuous,

$$c_n=\sup_{i;\omega_1,\ldots,\omega_i}\sum_{\omega_{i+1},\ldots,\omega_m}\left|\left[\prod_{\ell=i+1}^{m} g_{\omega_\ell}^n(\psi_{\omega_{\ell-1}}\circ\cdots\circ\psi_{\omega_1}\ \cdot\ )\right]_{\mathrm{reg}}(x)\right.$$
$$\left.-\prod_{l=i+1}^{m} g_{\omega_\ell}^n(\psi_{\omega_{\ell-1}}\circ\cdots\circ\psi_{\omega_1} x)\right|$$

tends to zero as $n\to\infty$, so that the sum of all corrections

$$\sum_{\omega_1,\ldots,\omega_m}\sum_{i=1}^{m-2}\left|\int d(g_{\omega_i}^n(\psi_{\omega_{i-1}}\circ\cdots\circ\psi_{\omega_1}x))\left[\prod_{j=1}^{i-1} g_{\omega_j}^n(\psi_{\omega_{j-1}}\circ\cdots\circ\psi_{\omega_1}x)\right]\right.$$
$$\cdot\left\{\left[\prod_{l=i+1}^{m} g_{\omega_l}^n(\psi_{\omega_{l-1}}\circ\cdots\circ\psi_{\omega_1}\ \cdot\ )\right]_{\mathrm{reg}}(x)\right.$$
$$\left.\left.-\left[\prod_{l=i+1}^{m} g_{\omega_l}^n(\psi_{\omega_{l-1}}\circ\cdots\circ\psi_{\omega_1}x)\right]\right\}\cdot\frac{1}{2}\mathrm{sgn}\ (\psi_{\underset{\sim}{\omega}}(x)-x)\right|$$
$$\leqq\frac{c_n}{2}\sum_{i=1}^{m-2}\sum_{\omega_1,\ldots,\omega_i}\left[\prod_{j=1}^{i-1}\sup|g_{\omega_j}^n|\right]\int|d(g_{\omega_i}^n)|$$
$$\leqq\sum_{i=1}^{m-2}\frac{c_n}{2}\sum_{\omega_1,\ldots,\omega_i}\prod_{j=1}^{i}\mathrm{Var}\,g_{\omega_j}^n\ \leqq\ \sum_{i=1}^{m-2}\frac{c_n}{2}\left(\sum_{\omega}\mathrm{Var}\,g_\omega\right)^i$$

tends to zero as $n\to\infty$. Fixing again a sequence $\underset{\sim}{\omega}$, it thus suffices to verify that

$$L_{\underset{\sim}{\omega}}^n=\sum_{i=1}^{m}\int d(g_i^n(\psi_{i-1}\circ\cdots\circ\psi_1 x))\left[\prod_{l\neq i} g_l^n(\psi_{l-1}\circ\cdots\circ\psi_1 x)\right]\tfrac{1}{2}\mathrm{sgn}\,(\psi_{\underset{\sim}{\omega}}(x)-x)$$

converges to

$$L_{\underset{\sim}{\omega}} = \sum_{i=1}^{m} \int d(g_i(\psi_{i-1} \circ \cdots \circ \psi_1 x)) \left[\prod_{\ell \neq i} g_l(\psi_{l-1} \circ \cdots \circ \psi_1 x)\right] \frac{1}{2} \operatorname{sgn}(\psi_{\underset{\sim}{\omega}}(x) - x)$$

when $n \to \infty$. We introduce the set $S_{\underset{\sim}{\omega}} = \{\alpha : \psi_{\underset{\sim}{\omega}}\alpha = \alpha\}$. Since the g_i^n only have finitely many discontinuities and tend uniformly to the g_i, we obtain the desired convergence as follows: By making good choices of the division points t_k we may avoid intervals $[t_{k-1}, t_k]$ such that their images by compositions $\psi_1^{-1} \cdots \psi_{i-1}^{-1}$ intersect on $S_{\underset{\sim}{\omega}}$ only at an endpoint. In the intervals containing u_{ω_j} and v_{ω_j}, we may assume g_j and g_j^n to vanish. In the other intervals we arrange that if $\psi_1^{-1} \cdots \psi_{i-1}^{-1}[t_{k-1}, t_k]$ contains a point of $S_{\underset{\sim}{\omega}}$, then also $\psi_1^{-1} \cdots \psi_{i-1}^{-1} X_k \in S_{\underset{\sim}{\omega}}$. With such a choice, the contribution to $L_{\underset{\sim}{\omega}}^n$ coming from an open intervalwhere $\psi_{\underset{\sim}{\omega}}x > x$ or $\psi_{\underset{\sim}{\omega}}x < x$ tends to the corresponding contribution to $L_{\underset{\sim}{\omega}}$. (Note that $S_{\underset{\sim}{\omega}}$ is closed and $\{\alpha : \psi_{\underset{\sim}{\omega}}\alpha > \alpha\}$, $\{\alpha : \psi_{\underset{\sim}{\omega}}\alpha < \alpha\}$ are open. To ensure convergence of $L_{\underset{\sim}{\omega}}^n$ to $L_{\underset{\sim}{\omega}}$ for a finite family of $\underset{\sim}{\omega}$'s we have to ensure that in each small interval (t_{k-1}, t_k) a point X_k is chosen such that X_k belongs to each of a finite number of closed sets $\mathscr{T}_\lambda = \psi_{\omega(\lambda)_{i-1}} \cdots \psi_{\omega(\lambda)_1} S_{\underset{\sim}{\omega}(\lambda)}$ with $\mathscr{T}_\lambda \cap [t_{k-1}, t_k] \neq \emptyset$. This can be achieved by subdividing the intervals $[t_{k-1}, t_k]$. Indeed, for any two $\mathscr{T}_\lambda$, $\mathscr{T}_\mu$ so that $\mathscr{T}_\lambda \cap [t_{k-1}, t_k] \neq \emptyset$ and $\mathscr{T}_\mu \cap [t_{k-1}, t_k] \neq \emptyset$ either $\mathscr{T}_\lambda \cap \mathscr{T}_\mu \cap [t_{k-1}, t_k] \neq \emptyset$ or we can divide $[t_{k-1}, t_k]$ into finitely many subintervals, each of which intersects only one of the $\mathscr{T}_\lambda$ or $\mathscr{T}_\mu$. Repeating a similar construction we can arrange that if an interval $[t_{k-1}, t_k]$ intersects certain $\mathscr{T}_\lambda$'s, it contains a point of their common intersection, which we may take as X_k.)

Finally, when choosing the X_k, we may in fact remove from $S_{\underset{\sim}{\omega}}$ the points with an at most countable neighbourhood in $S_{\underset{\sim}{\omega}}$: At most countably many points are thus excluded, since the g_i are continuous, a countable set is negligible for $d(g_i(\psi_{i-1} \circ \cdots \circ \psi_1 x))$. Using this remark we see that we can require that $X_{\omega',k'}^n = X_{\omega,k}^n$ if and only if $k = k'$ and $\omega = \omega'$.

We now check (A.5). For $m = 1$, the fact that the g_ω^n vanish at u_ω and v_ω, and that the G_η^n have opposite signs on opposite sides of all other endpoints of the intervals J_η^n yields $\sum_\eta \varepsilon_\eta L_0(\psi_\eta^n) G^n(\eta) = 0$. For $m \geqq 2$, a composition $\psi_{\eta_j} \circ \cdots \circ \psi_{\eta_i}$ can send an endpoint of $J_{\eta_i}^n$ to an endpoint of $J_{\eta_{j+1}}^n$ and make the sum nonzero. We note that we may modify slightly each ψ_ω outside of the support of g_ω (which contains the support of g_ω^n for all n) to ensure that the sets $S_{\underset{\sim}{\omega}}$ for $|\underset{\sim}{\omega}| \leqq M_0$ do not contain any u_ω or v_ω (therefore, the endpoints of $J_{\underset{\sim}{\eta}}$ of the form u_ω or v_ω cannot contribute to the function L_0, we call them trivial endpoints). Fixing some $m \geqq 2$, we observe that for each cyclic permutation $\underset{\sim}{\eta}^*$ of $\underset{\sim}{\eta}$ we have

$$\varepsilon(\underset{\sim}{\eta}^*) L_0(\psi_{\underset{\sim}{\eta}^*}^n) G^n(\underset{\sim}{\eta}^*) = \varepsilon(\underset{\sim}{\eta}) L_0(\psi_{\underset{\sim}{\eta}}^n) G^n(\underset{\sim}{\eta}) .$$

We choose a representative $\underset{\sim}{\eta}^*$ of the equivalence class $[\underset{\sim}{\eta}]$ (for the equivalence relation generated by the cyclic permutations) such that if there exists a composition $\psi_{\eta_j} \circ \cdots \circ \psi_{\eta_i}$ sending the non trivial endpoint of $J^n_{\eta_i}$ to the non trivial endpoint of $J^n_{\eta_{j+1}}$ then $\underset{\sim}{\eta}^*$ is obtained from $\underset{\sim}{\eta}$ by applying the circular permutation sending $j+1$ to m (for one of the possible $i \leqq j$ satisfying the requirement). Then

$$\begin{aligned}
\left| \sum_{|\underset{\sim}{\eta}|=m} \varepsilon(\underset{\sim}{\eta}) L_0(\psi^n_{\underset{\sim}{\eta}}) G^n(\underset{\sim}{\eta}) \right| &\leqq m \cdot \left| \sum_{[\underset{\sim}{\eta}]:|\underset{\sim}{\eta}|=m} \varepsilon(\underset{\sim}{\eta}^*) L_0(\psi^n_{\underset{\sim}{\eta}^*}) G^n(\underset{\sim}{\eta}^*) \right| \\
&\leqq m \cdot \sum_{|\underset{\sim}{\eta}'|=m-1} |G^n(\underset{\sim}{\eta}')| \cdot \sup_\eta |G^n_\eta| \\
&\leqq m \cdot \sup_\eta |G^n_\eta| \cdot \left(\sum_\eta |G^n_\eta| \right)^{m-1} \\
&\leqq M_0 \cdot \sup_\eta |G^n_\eta| \cdot \left(\sum_\omega \operatorname{Var} g_\omega \right)^{M_0-1} \qquad \text{(A.6)}
\end{aligned}$$

Indeed, if $\underset{\sim}{\eta}^*$ is the representative of a class with a non-cancelled contribution, then for some $0 \leqq k$, the map $\psi^n_{\eta^*_{m-1}} \circ \cdots \circ \psi^n_{\eta^*_{m-1-k}}$ sends the non trivial endpoint of $J^n_{\eta^*_{m-1-k}}$ to the non trivial endpoint of $J^n_{\eta^*_m}$. Since the non-trivial endpoints are pairwise distinct, if the first $m-1$ components of η^* are specified, the last one is unambiguously defined. The right hand-side of (A.6) tends to zero when $n \to \infty$, because continuity of the g_ω implies that each G^n_η tends to zero.

We must still verify that the $\hat{\zeta}^n(z) = \det[\widehat{D}^n_{ij}(z)]$ are uniformly bounded when $|z| < (\widehat{R} + \gamma)^{-1}$. We shall check that the condition of the main lemma in Baladi [1995] is satisfied.

The index set $A_n = \{\hat{a}_1, \ldots, \hat{a}_{L_n}\}$ can be partitioned into two subsets $A_n = A^0_n \cup A^1_n$, where A^0_n is the set of endpoints of the original dual intervals $[\hat{u}_\omega, \hat{v}_\omega] = \psi_\omega J_\omega$.

For $\hat{a}_i \in A^0_n$ and all large enough n, we have for $1 \leqq j \leqq L_n$.

$$\begin{aligned}
\widehat{D}^{(m)+}_{ij} &= \lim_{x \downarrow \hat{a}_i} \sum_{\eta : \hat{u}_\eta = \hat{a}_i} \varepsilon_\eta G^n_\eta [\widehat{\mathscr{M}}^{m-1}_n (\tfrac{1}{2} \operatorname{sgn}(\,\cdot\, - \hat{a}_j))] \left[\psi^{-1}_\eta(x)\right] \\
&= \lim_{x \downarrow \hat{a}_i} \sum_{\omega : \hat{u}_\omega = \hat{a}_i} \varepsilon_\omega g^n_\omega (\psi_\omega)^{-1}(x) [\widehat{\mathscr{M}}^{m-1}_n (\tfrac{1}{2} \operatorname{sgn}(\,\cdot\, - \hat{a}_j))][\psi^{-1}_\omega(x)] = 0\,,
\end{aligned}$$

because the g^n_ω vanish near u_ω and v_ω. Similarly $\widehat{D}^{(m)-}_{ij} = 0$, so that for $\hat{a}_i \in A^0_n$ we have $\widehat{D}^n_{ij}(z) = \delta_{ij}$.

Each $\hat{a}_i \in A_n^1$ is of the form $\hat{a}_i = \psi_\omega(X^n_{\omega,k(\omega,i)})$, for ω in some set $\Omega(i)$ and uniquely defined $k(\omega,i)$s. Thus

$$\widehat{D}_{ij}^{(m)+} = \sum_{\omega\in\Omega(i)} + \varepsilon_\omega \frac{\mathscr{G}^n_{\omega,k(\omega,i)}}{2} \cdot \lim_{x\downarrow\hat{a}_i}[\widehat{\mathscr{M}}_n^{m-1}(\tfrac{1}{2}\operatorname{sgn}(\,\cdot\, - \hat{a}_j))]\,[\psi_\omega^{-1}(x)]$$

$$\widehat{D}_{ij}^{(m)-} = \sum_{\omega\in\Omega(i)} - \varepsilon_\omega \frac{\mathscr{G}^n_{\omega,k(\omega,i)}}{2} \cdot \lim_{x\uparrow\hat{a}_i}[\widehat{\mathscr{M}}_n^{m-1}(\tfrac{1}{2}\operatorname{sgn}(\,\cdot\, - \hat{a}_j))][\psi_\omega^{-1}(x)]\,.$$

Hence, for all large enough n and all $1 \leqq j \leqq L_n$

$$\begin{aligned} |\widehat{D}^n_{ij}(z) - \delta_{ij}| &\leqq \sum_{\omega\in\Omega(i)} |\mathscr{G}^n_{\omega,k(\omega,i)}| \sum_{m=1}^{\infty} |z|^m \cdot [|\lim_{x\downarrow\hat{a}_i}[\widehat{\mathscr{M}}_n^{m-1}\tfrac{1}{2}\operatorname{sgn}(\,\cdot\, - \hat{a}_j)]| \\ &\qquad + |\lim_{x\uparrow\hat{a}_i}[\widehat{\mathscr{M}}_n^{m-1}\tfrac{1}{2}\operatorname{sgn}(\,\cdot\, - \hat{a}_j)]]|(\psi_\omega^{-1}(x)) \\ &\leqq \sum_{\omega\in\Omega(i)} |\mathscr{G}^n_{\omega,k(\omega,i)}| \sum_{m=1}^{\infty} |z|^m \cdot \|\widehat{\mathscr{M}}_n^{m-1}\|_0 \\ &\leqq \sum_{\omega\in\Omega(i)} |\mathscr{G}^n_{\omega,k(\omega,i)}| \cdot \frac{|z|}{1 - |z| \cdot (\widehat{R} + \gamma)}\,. \end{aligned}$$

Since the sets $\Omega(i)$ are pairwise disjoint and the points $X^n_{\omega,k(\omega,i)}$ are all different, we have $\sum_{\hat{a}_i\in A_n^1} \sum_{\omega\in\Omega(i)} |\mathscr{G}_{\omega,k(\omega,i)}| \leqq \sum_\omega \operatorname{Var} g_\omega$, so that

$$\sum_i \sup_j |\widehat{D}^n_{ij}(z) - \delta_{ij}|$$

is uniformly bounded for $|z| \leqq (\widehat{R} + \gamma)^{-1}$, which concludes the proof of the theorem. □

Remark. We recover in particular the results in Baladi [1995] when each weight g_ω is continuous and vanishes at the endpoints of the intervals J_ω. However, our method really requires the nonfunctional setting of Ruelle [1993]. In Baladi [1995], the limitations of the functional setting imposed a certain choice of the subdivision points (for which the contribution of the L_0 remained constant) and an assumption of constancy of the weight on homtervals. See Mori [1994] for related results.

Note. In *Sharp determinants and kneading operators for holomorphic maps* by V. Baladi, A. Kitaev, D. Ruelle and S. Semmes (IHES preprint, 1995) a variation on the theme of the present paper is discussed, where homeomorphisms of intervals of $\mathbb{R}$ are replaced by holomorphic homeomorphisms of domains of $\mathbb{C}$; the results obtained there are less complete however.

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References

1. M. Atiyah, R. Bott: A Lefschetz fixed point formula for elliptic complexes: I. Ann. Math. **86**, 374–407 (1967)
2. M. Atiyah, R. Bott: A Lefschetz fixed point formula for elliptic complexes: II. Applications, Ann. Math. **88**, 451–491 (1968)
3. V. Baladi: Infinite kneading matrices and weighted zeta functions of interval maps. J. Funct. Anal. **128**, 226–244 (1995)
4. V. Baladi, G. Keller: Zeta functions and transfer operators for piecewise monotone transformations. Commun. Math. Phys. **127**, 459–479 (1990)
5. V. Baladi, D. Ruelle: An extension of the theorem of Milnor and Thurston on the zeta functions of interval maps. Ergodic Theory Dynamical Systems **14**, 621–632 (1994)
6. D. Fried: The flat-trace asymptotics of a uniform system of contractions, preprint (1993)
7. A. Grothendieck: La théorie de Fredholm. Bull. Soc. Math. France **84**, 319–384 (1956)
8. J. Milnor, W. Thurston: Iterated maps of the interval, Dynamical Systems (Maryland 1986–1987) (J.C. Alexander (ed.)) Lecture Notes in Math. Vol. 1342, Springer, Berlin Heidelberg New York, 1988
9. M. Mori: Fredholm determinant for piecewise linear transformations. Osaka J. Math **27**, 81–116 (1990)
10. M. Mori: Fredholm determinant for piecewise monotonic transformations. Osaka J. Math. **29**, 497–529 (1992)
11. M. Mori: On the convergence of the spectrum of Perron–Frobenius operators (preprint 1994)
12. D. Ruelle: Zeta functions for expanding maps and Anosov flows. Invent. Math. **34**, 231–242 (1976)
13. D. Ruelle: An extension of the theory of Fredholm determinants. Publ. Math. I.H.E.S. **72**, 175–193 (1990)
14. D. Ruelle: Functional equation for dynamical zeta functions of Milnor–Thurston type. IHES (preprint 1993)
15. D. Ruelle: Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval. CRM Monograph Series, Vol. 4, Amer. Math. Soc., Providence, NJ, 1994
16. B. Simon: Trace Ideals and their Applications, Cambridge University Press, Cambridge, 1979