THE STATES OF CLASSICAL STATISTICAL MECHANICS

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Abstract. A state of an infinite system in classical statistical mechanics is usually described by its correlation functions. We discuss here other descriptions in particular as 1) a state on a B^*- algebra, 2) a collection of density distributions, 3) a field theory 4) a measure on a "space of configurations of infinitely many particles". We consider the relations between these various descriptions and prove under very general conditions an integral representation of a state as superposition of "extremal invariant states" corresponding to pure thermodynamical phases.

1. Introduction.

The idea of representing states of physical systems by states on B^*- algebras has been present for some time*). Actually the word "state", used to describe a normalized positive linear functional on a B^*- algebra, is borrowed from physics. Recently, a number of nontrivial results of general nature have been obtained**) concerning the use of B^*- algebras in physics. In the present paper we shall apply some of these results to classical statistical mechanics.

The use of states in statistical mechanics is not new. The well-known ensembles of Gibbs correspond to states both in classical and quantum statistical mechanics. They describe, however, only systems with - essentially - a finite number of degrees of freedom. If one takes the limit of an infinite system (the thermodynamic limit) another description is needed, and is given by the correlation functions or the reduced density matrices. The correlation functions in classical statistical mechanics and the reduced density matrices in quantum statistical mechanics may be considered as "states" on algebras of unbounded operators (see respectively [18] and [19]). Dealing with unbounded operators is, however, a serious mathematical drawback and we shall here make use of a B^*- algebra which may be thought of as generated by bounded functions of these unbounded operators.

We proceed by giving some motivation for the definitions and assumptions which we shall make below.

The systems considered in classical statistical mechanics are formed by a large number of "particles". These particles may be points in R^ν or on a lattice, or points in R^ν with a velocity vector, or more complicated objects like continuous mappings of the interval [0,1] into R. Furthermore, a system may be composed of several species of particles. There is thus a naturally defined one-particle space T. In general also there will be a natural group G acting on T. Typically G might be the euclidean or the

*) See the book of Segal [22] and references quoted there, and the paper of Haag and Kastler [11].

**) See [7], [21], [8], [12], [17], [13].

translation group in ν dimensions, or a lattice group.

Let $T^{\hat{n}}$ be the symmetrized product of n copies of T. The sum $\hat{\mathcal{C}}$ of the $T^{\hat{n}}$ is the space of configurations of an arbitrary finite number of particles. If we have a locally compact topology on T, we may now define states "with an essentially finite number of particles" as probability measures on $\hat{\mathcal{C}}$. It would be natural to represent the states of statistical mechanics, which have typically an infinite number of particles, by probability measures on a new space $\mathcal{X}$ of configurations of an infinite number of particles.

We shall consider, as states of classical statistical mechanics, states which are invariant under the action of G and have the property that their restriction to a compact region has an essentially finite number of particles. These states will be exhibited as states on a B^* - algebra $\mathcal{O}\!\!\!\mathcal{L}$, actually an algebra of functions on $\hat{\mathcal{C}}$. A large part of the paper will consist in obtaining equivalent characterizations of these states and in connecting them with the correlation functions. The space $\mathcal{X}$ mentioned above will appear as a subset of the set of pure states on $\mathcal{O}\!\!\!\mathcal{L}$ and this imbedding will yield a natural compactification of $\mathcal{X}$.

2. Assumptions.

It will be convenient, for reasons of conciseness and generality, to axiomatize that part of classical statistical mechanics (CSM) in which we shall be interested. We shall call CSM theory a triple (T,G,τ) satisfying the following properties

(T) · T is a locally compact space called one-particle space,

(G) · G is a topological group,

(τ) · τ is a homomorphism of G into the homeomorphisms of T such that the mapping $(g,x) \to \tau_g x$ of $G \times T$ onto T is continuous.

A number of the results which we shall derive do not depend upon the existence of a topology on G. Our assumptions are, however, not restrictive since, when G is given the discrete topology, the continuity of $\tau : G \times T \to T$ is insured by the fact that $\tau_g : T \to T$ is a homeomorphism. We shall for some results need a stronger assumption than (T), namely

(T') $\cdot$ T <u>is locally compact with countable basis</u>.

This means that the topology of T is generated by a countable family of open sets, and is equivalent to requiring that T is countable at infinity and that its compacts are metrizable (see [2] 8.3 and 8.19). Condition (T') is for instance satisfied for $T = R^{\nu}$.

The above axiomatic setup is, of course, more general than required by the applications, but seems to contain precisely those assumptions which we shall need later, so that a particularization would lead to no simplification of the arguments.

3. <u>Basic definitions</u>.

In what follows, we shall make constant use of the topological sum $\mathcal{T}$ of the powers T^n of T

$$\mathcal{T} = \sum_{n\geq 0} T^n \tag{3.1}$$

where T^0 is by definition reduced to a point. It would be natural to consider instead of T^n its quotient $\hat{T^n}$ by permutations and to define $\hat{\mathcal{T}} = \sum_{n\geq 0} \hat{T^n}$ as indicated in the introduction. It will, however, be more convenient to work with $\mathcal{T}$.

Let $\mathcal{K}(\mathcal{T})$ be the space of real continuous functions with compact support on $\mathcal{T}$, i.e. the direct sum of the spaces $\mathcal{K}(T^n)$ of real continuous functions with compact support on T^n. Let $f_1, f_2 \in \mathcal{K}(\mathcal{T})$ and $f_1 = (f_1{}^n)$, $f_2 = (f_2{}^n)$, we define $f_1 * f_2 \in \mathcal{K}(\mathcal{T})$ by

$$(f_1 * f_2)^n(x_1,\ldots,x_n) = \sum_{m=0}^{n} f_1{}^m(x_1,\ldots,x_m) f_2{}^{n-m}(x_{m+1},\ldots,x_n) \tag{3.2}$$

With respect to this multiplication $\mathcal{K}(\mathcal{T})$ becomes a non-commutative algebra which we note $\mathcal{K}_*$ to avoid confusion with the structure of algebra defined on $\mathcal{K}(\mathcal{T})$ by the usual multiplication of functions. We note $\mathbb{1}$ the identity in $\mathcal{K}_*$.

Let $f \in \mathcal{K}(T^n)$, $n > 0$ and let ω be a partition of the set $\{1,2,\ldots,n\}$ into r subsets

$S_1 = \{i_{11}, i_{12}, \ldots\}$, ..., $S_r = \{i_{r1}, i_{r2}, \ldots\}$. We may suppose that $i_{jk} < i_{jk'}$ if $k < k'$ and $i_{j1} < i_{j'1}$ if $j < j'$. For all $y = (y_1, \ldots, y_r) \in T^r$ let $x_i^{\,w}(y) = y_j$ if $i \in S_j$. We define $f_w \in \mathcal{K}(T^r)$ by

$$f_w(y_1, \ldots, y_r) = f_w(y) = f(x_1^{\,w}(y), \ldots, x_n^{\,w}(y)) \tag{3.3}$$

The sum of the f_w overall partitions of $\{1,2,\ldots,n\}$ is an element Δf of $\mathcal{K}_*$. For $f^\circ \in \mathcal{K}(T^\circ)$ we write $\Delta f^\circ = f^\circ$. Δ extends then to a linear mapping of $\mathcal{K}_*$ into itself, and one sees readily that this mapping has an inverse Δ^{-1}.

Let $\mathcal{C}(\mathcal{T})$ be the algebra of complex continuous functions on $\mathcal{T}$ (for the usual multiplication of functions), i.e. the product over $n \geq 0$ of the spaces $\mathcal{C}(T^n)$ of complex continuous functions on T_n. If $F = (F^n) \in \mathcal{C}(\mathcal{T})$ and $g \in G$ we define $\tau_g F \in \mathcal{C}(\mathcal{T})$ by

$$(\tau_g F)^n(x_1, \ldots, x_n) = F^n(\tau_{g^{-1}} x_1, \ldots, \tau_{g^{-1}} x_n) \tag{3.4}$$

In particular, it is seen that if $f_1, f_2 \in \mathcal{K}(\mathcal{T}) \subset \mathcal{C}(\mathcal{T})$, then $\tau_g f_1 \in \mathcal{K}(\mathcal{T})$, $\tau_g \ \Delta f_1 = \Delta \tau_g f_1$, $\tau_g(f_1 * f_2) = \tau_g f_1 * \tau_g f_2$.

If $f = (f^n) \in \mathcal{K}_*$ we define $Sf \in \mathcal{C}(\mathcal{T})$ by

$$(Sf)^\circ = f^\circ \tag{3.5}$$
$$(Sf)^n(x_1, \ldots, x_n) = f^\circ + \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} f^m(x_{i_1}, \ldots, x_{i_m}) \quad (n > 0)$$

One checks readily that S is a homomorphism of $\mathcal{K}_*$ into $\mathcal{C}(\mathcal{T})$ such that $S\tau_g f = \tau_g Sf$. Furthermore, it is easily seen that

$$(S\Delta^{-1} f)^n(x_1, \ldots, x_n) = f^\circ + \sum_{m=1}^{n} \ \sum_{i_1, \ldots, i_m}{}' \ f^m(x_{i_1}, \ldots, x_{i_m}) \tag{3.6}$$

where the summation Σ' extends over the $n!/(n-m)!$ sequences $(i_1, \ldots, i_m)$ of m different integers i, with $1 \leq i \leq n$.

We define $\mathfrak{A}$ to be the closure, with respect to the uniform norm, of the subalgebra of $\mathcal{C}(\mathcal{T})$ constituted by the elements of the form $\phi(Sf_1, \ldots, Sg_q)$ for all integers $q \geq 0$, $f_1, \ldots, f_q \in \mathcal{K}_*$ and ϕ bounded continuous complex function on R^q. With respect to complex conjugation and the uniform norm, $\mathfrak{A}$ is an abelian B^*- algebra with identity 1. We note

$E \subset \mathcal{O}\!\mathcal{l}'$ the set of all states on $\mathcal{O}\!\mathcal{l}$.

If $g \in G$, τ_g is an automorphism of $\mathcal{O}\!\mathcal{l}$. If $\rho \in E$ we define $\tau'_g\rho$ by $(\tau'_g\rho)(A) = \rho(\tau_{g^{-1}}A)$ for all $A \in \mathcal{O}\!\mathcal{l}$. Let $\mathcal{L}_G$ be the subspace of $\mathcal{O}\!\mathcal{l}$ generated by the elements of the form $A - \tau_g A$ and

$$\mathcal{L}_G^{\perp} = \{f \in \mathcal{O}\!\mathcal{l}' : A \in \mathcal{L}_G \Rightarrow f(A) = 0\} \tag{3.7}$$

Then, the set of τ'_g-invariant states on $\mathcal{O}\!\mathcal{l}$ is $E \cap \mathcal{L}_G^{\perp}$. The sets E and $E \cap \mathcal{L}_G^{\perp}$ are both convex and weakly compact.

4. The set $\mathcal{F}$ of states.

We shall concern ourselves in what follows with the set $\mathcal{F} \subset E$ consisting of those states ρ on $\mathcal{O}\!\mathcal{l}$ which satisfy the condition

($\mathcal{F}$). if $f_1, \ldots, f_q \in \mathcal{K}_*$ then the functional on the bounded continuous complex functions on R^q defined by

$$\phi \to \rho(\phi(Sf_1, \ldots, Sf_q)) \tag{4.1}$$

is a measure on R^q.

To understand ($\mathcal{F}$), we interpret $\rho(\phi\,(Sf_1, \ldots, Sf_q))$ as an expectation value of the function $\phi(Sf_1, \ldots, Sf_q)$ on $\mathcal{Z}$. If we restrict ϕ to tend to zero at infinity, the positive linear functional $\phi \to \rho(\phi(Sf_1, \ldots, Sf_q))$ defines a positive measure on R^q which represents the probability distribution of finite values of $Sf_1, \ldots, Sf_q$. Condition ($\mathcal{F}$) means that this distribution has total mass 1, in other words that the functions $Sf_1, \ldots, Sf_q$ take the value ∞ with probability 0.

As an example take $q = 1$, $f_1^o = 0$, $f_1^n = 0$ for $n > 1$ and $f_1^1 \geqslant 0$ not vanishing identically. Since f_1^1 has compact support and

$$(Sf_1)^n(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_1^1(x_i) \tag{4.2}$$

it is seen that for ($\mathcal{F}$) to hold it is necessary (in fact it is also sufficient) that the probability of finding simultaneously an infinite number of particles on a compact subset of T vanishes.

We shall now express condition ($\mathcal{F}$) in a manner better suited to later purposes (see Proposition 4.3.2. below). Let $0 \leq h \in \mathcal{K}_*$. We denote by $\mathcal{A}_{(h)}$ the sub - B* - algebra of $\mathcal{A}$ generated by the elements of the form $\phi(Sh, Sf_1,\ldots,Sf_q)$ where supp $f_1,\ldots,$ supp f_q are contained in $\{X \in \mathcal{Z}: h(X) > 0\}$. We denote by $\mathcal{J}_{(h)}$ the closed ideal of $\mathcal{A}_{(h)}$ generated by those elements for which ϕ tends to zero at infinity.

Proposition 4.1. If supp h is metrizable, then $\mathcal{J}_{(h)}$ is separable.

If ϕ tends to zero at infinity, we may approximate $\phi(Sh, Sf_1,\ldots,Sf_q)$ by $\tilde{\phi}(Sh, Sf_1,\ldots,Sf_q)$ where $\tilde{\phi}$ belongs to a countable family of functions with compact support on R^{q+1}. The space $\{f \in \mathcal{K}_* : \text{supp } f \subset \text{supp } h\}$ is separable (see [2] 13.27.) and therefore, given $\varepsilon > 0$, one may find $\tilde{f}_1,\ldots,\tilde{f}_q \in \mathcal{K}_*$ such that $|f_1 - \tilde{f}_1| < \varepsilon h,\ldots,$ $|f_q - \tilde{f}_q| < \varepsilon h$, where $\tilde{f}_1,\ldots,\tilde{f}_q$ belong to a countable family of elements of $\mathcal{K}_*$ with support in $\{X \in \mathcal{Z}: h(X) > 0\}$. Since $\tilde{\phi}$ has compact support one may approximate $\tilde{\phi}(Sh, Sf_1,\ldots,Sf_q)$ by $\tilde{\phi}(Sh,S\tilde{f}_1,\ldots,S\tilde{f}_q)$.

Proposition 4.2. Let $\rho \in E$ then $\rho \in \mathcal{F}$ if and only if, for all h, the restriction of ρ to $\mathcal{J}_{(h)}$ has norm 1.

If $\rho \in \mathcal{F}$ then, according to the comment following the statement of condition ($\mathcal{F}$), the restriction of ρ to $\mathcal{J}_{(h)}$ has norm 1.

Conversely let the restriction of $\rho \in E$ to $\mathcal{J}_{(h)}$ have norm 1. Given $\varepsilon > 0$, there exists then $\phi(Sh, Sf'_1,\ldots,Sf'_{q'}) \in \mathcal{J}_{(h)}$ such that ϕ has compact support, $0 \leq \phi \leq 1$ and $\rho(\phi(Sh,Sf'_1,\ldots,Sf'_{q'})) > 1 - \varepsilon$. One may then choose $\phi'(Sh) \in \mathcal{J}_{(h)}$ such that ϕ has compact support, $0 \leq \phi \leq 1$ and $\rho(\phi'(Sh)) > 1 - \varepsilon$. Given $f_1,\ldots,f_q \in \mathcal{K}_*$ let $h \in \mathcal{K}_*$ be such that $h \geq 0$ and supp $f_1,\ldots,$supp $f_q \subset \{X \in \mathcal{Z}: h(X) > 0\}$. One may choose $\phi''(Sf_1,\ldots,Sf_q)$ such that ϕ'' has compact support, $0 \leq \phi'' \leq 1$ and $\phi'(Sh)\,\phi''(Sf_1,\ldots,Sf_q) = \phi'(Sh)$, we have then

$$\begin{aligned}\rho\,(\phi''(Sf_1,\ldots,Sf_q)) &\geq \rho\,(\phi'(Sh)\,\phi''(Sf_1,\ldots,Sf_q))\\ &= \rho\,(\phi'(Sh)) \geq 1 - \varepsilon\end{aligned} \tag{4.3}$$

Proposition 4.3. Let $(h_\iota)_{\iota \in I}$ be a family such that

(i) for each $\iota \in I$: $0 \leq h_\iota \in \mathcal{K}_*$

(ii) for each compact $K \subset \mathcal{E}$ there exists $\iota \in I$ such that $K \subset \{X \in \mathcal{E} : h(X) > 0\}$. We shall then write $\mathfrak{A}_\iota = \mathfrak{A}_{(h_\iota)}$, $\mathcal{J}_\iota = \mathcal{J}_{(h_\iota)}$. We have

1. the union $\cup_\iota \mathfrak{A}_\iota$ is dense in $\mathfrak{A}$

2. let $\rho \in E$, then $\rho \in \mathcal{F}$ if and only if, for all $\iota \in I$, the restriction of ρ to $\mathcal{J}_\iota$ has norm 1.

3. if T is countable at infinity, then one can choose for (h_ι) a countable family.

1. follows from the fact that, by (ii), every $\phi(Sf_1,\ldots,Sf_q)$ belongs to some $\mathfrak{A}_\iota$.

The proof of 2. is identical to that of Proposition 4.2. except that everywhere one has to take h in the family (h_ι).

In 3. one can even choose for $(\mathfrak{A}_\iota)$ an increasing sequence (see [2] 8.19.)

Corollary 4.4. If T has a countable basis one can choose for (h_ι) a countable family and the $\mathcal{J}_\iota$ are separable.

This follows from Proposition 4.1. and Proposition 4.3.3.

5. G-systems of density distributions.

The discussion of $(\mathcal{F})$ in Section 4 suggests to describe a state $\rho \in \mathcal{F}$ by giving for each relatively compact open set $\Lambda \subset T$ and each integer n the probability of finding n particles in Λ and the probability distribution of their positions.

For every relatively compact open set $\Lambda \subset T$, and integer $n \geq 0$, let $\mu_\Lambda^n \geq 0$ be a measure on $\Lambda^n \subset T^n$. We assume that μ_Λ^n is invariant under permutation of the n factors of T^n. We shall say that the μ_Λ^n form a G-system of density distributions if they satisfy the following conditions

(D.1).
$$\mu_\emptyset^o(T^o) = 1 \tag{5.1}$$

(D.2). Let $\Lambda \subset \Lambda'$ and $\chi_{\Lambda'-\Lambda}$ be the characteristic function of $\Lambda' - \Lambda$.

If $f^n \in \mathcal{K}(\sum_{n \geq 0} \Lambda^n) \subset \mathcal{K}_*$, then

$$\mu_\Lambda^n(f^n) = \sum_{m=0}^{\infty} \frac{(n+m)!}{n!\, m!} \mu_{\Lambda'}^{n+m}(f^n \otimes \chi_{\Lambda'-\Lambda}^{\otimes m}) \tag{5.2}$$

(D.3). if $f^n \in \mathcal{K}(\Lambda^n)$ and $g \in G$, then

$$\mu_\Lambda^n(f^n) = \mu^n_{\tau_g\Lambda}(\tau_g f^n) \tag{5.3}$$

Notice that from (D.1), (D.2) we obtain

$$\sum_n \mu_\Lambda^n(\Lambda^n) = 1 \tag{5.4}$$

Theorem 5.1. Given $\rho \in \mathcal{F} \cap \mathcal{L}_G^1$, there exists a unique G-system (μ_Λ^n) of density distributions such that if $f_1, \ldots, f_q \in \mathcal{K}(\sum_{n\geq 0} \Lambda^n) \subset \mathcal{K}_*$ and ϕ is a bounded continuous complex function on R^q, then

$$\begin{aligned}&\rho(\phi(Sf_1, \ldots, Sf_q)) \\ &= \sum_{n\geq 0} \int d\,\mu_\Lambda^n(x_1, \ldots, x_n)\, \phi(Sf_1(x_1, \ldots, x_n), \ldots, Sf_q(x_1, \ldots, x_n))\end{aligned} \tag{5.5}$$

The mapping $\rho \to (\mu_\Lambda^n)$ is one-to-one from $\mathcal{F} \cap \mathcal{L}_G^1$ onto the G-systems of density distributions.

We prove here only that (5.5) determines a unique mapping of the G-systems of density distributions into $\mathcal{F} \cap \mathcal{L}_G^1$.

Given a G-system of density distributions, (5.5) defines a linear functional $\tilde{\rho}$ on expressions of the form $\phi(Sf_1, \ldots, Sf_q)$ (because of (D.2) the definition does not depend on Λ). It follows from (5.4) that $\tilde{\rho}$ is continuous with respect to the uniform norm of functions on $\mathcal{C}$, and extends thus uniquely to a continuous linear functional ρ on $\mathcal{O}\!\mathcal{L}$. Clearly ρ is positive and, by (D.1), normalized, hence $\rho \in E$. By (D.3), $\rho \in \mathcal{L}_G^1$. On the other hand one checks easily from (5.4) and (5.5) that $(\mathcal{F})$ is satisfied. The other half of the theorem will be proved in Section 7.

6. G-field theories.

By definition we call G-field theory a quadruple $(\mathcal{H}, Q, U, \Omega)$ satisfying the following conditions.

$(\mathcal{H})$. $\mathcal{H}$ is a complex Hilbert space.

(Q). Q is a mapping of $\mathcal{K}_*$ to self-adjoint operators in $\mathcal{H}$ such that

(Q1) for all f_1, $f_2 \in \mathcal{K}_*$ the spectral projections of $Q(f_1)$, $Q(f_2)$ commute,

(Q2) Q is a homomorphism in the sense that for all f_1, $f_2 \in \mathcal{K}_*$ and $\lambda \in R$ we have

$$Q(\mathbb{1}) = 1 \tag{6.1}$$

$$Q(\lambda f_1) \supset \lambda Q(f_1) \tag{6.2}$$

$$Q(f_1+f_2) \supset Q(f_1) + Q(f_2) \tag{6.3}$$

$$Q(f_1 * f_2) \supset Q(f_1) \cdot Q(f_2) \tag{6.4}$$

(Q3) . if $0 \leqslant f \in \mathcal{K}_*$, then $Q(\Delta^{-1} f) \geqslant 0$.

(U) . U is a unitary representation of G in $\mathcal{H}$ such that for all $g \in G$, $f \in \mathcal{K}_*$

$$U(g)Q(f)U(g^{-1}) = Q(\tau_g f) \tag{6.5}$$

(Ω) . Ω is an element of $\mathcal{H}$ such that $\|\Omega\| = 1$ and

(Ω1) for all $g \in G$, $U(g)\Omega = \Omega$

(Ω2) Ω is cyclic with respect to Q in the sense that if $\tilde{\mathfrak{A}}$ is the C^*- algebra generated by the bounded continuous complex functions of the Q(f), then $\tilde{\mathfrak{A}}\Omega$ is dense in $\mathcal{H}$.

Theorem 6.1. Given $\rho \in \mathcal{F} \cap \mathcal{S}_G^1$ there exists a G-field theory ($\mathcal{H}$, Q, U, Ω) unique up to unitary equivalence, such that if $f_1, \ldots, f_q \in \mathcal{K}_*$ and if ϕ is a bounded continuous complex function on R^q, then

$$(\Omega, \phi(Q(f_1), \ldots, Q(f_q))\Omega) = \rho(\phi(Sf_1, \ldots, Sf_q)) \tag{6.6}$$

The mapping $\rho \to (\mathcal{H}, Q, U, \Omega)$ is one-to-one from $\mathcal{F} \cap \mathcal{S}_G^1$ onto the G-field theories defined up to unitary equivalence.

We prove here only that (6.6) determines a unique mapping from $\mathcal{F} \cap \mathcal{S}_G^1$ to G-field theories defined up to unitary equivalence.

Let $\rho \in \mathcal{F} \cap \mathcal{S}_G^1$. The Gel'fand-Segal construction yields a complex Hilbert space $\mathcal{H}$, a *-homomorphism π_ρ of $\mathfrak{A}$ into the bounded operators on $\mathcal{H}$, a unitary representation U of G in $\mathcal{H}$ and a vector $\Omega \in \mathcal{H}$ such that $\|\Omega\| = 1$ and the following conditions are satisfied for all $A \in \mathfrak{A}$, $g \in G$

$$(\Omega, \pi_\rho(A)\Omega) = \rho(A) \tag{6.7}$$

(U') $$U(g)\pi_\rho(A)U(g^{-1}) = \pi_\rho(\tau_g A) \tag{6.8}$$

(Ω1) $$U(g)\Omega = \Omega \tag{6.9}$$

(Ω2) $$\pi_\rho(\mathfrak{A})\Omega \text{ is dense in } \mathcal{H} \tag{6.10}$$

Furthermore, the Gel'fand-Segal construction is unique within unitary equivalence.

Given $f\in\mathscr{K}_*$, there exists according to ($\mathscr{F}$) and the Appendix of [21] a self-adjoint operator $Q(f)$ such that

$$\phi(Q(f)) = \pi_\rho(\phi(Sf)) \tag{6.11}$$

for all continuous complex functions ϕ tending to zero at infinity on R. It is immediate that (Q1) is satisfied.

Let $f_1,\ldots,f_q \in \mathscr{K}_*$. Using the simultaneous spectral decomposition of $Q(f_1),\ldots,Q(f_q)$ and (6.11) we see that

$$\psi(Q(f_1),\ldots,Q(f_q)) = \pi_\rho(\psi(Sf_1,\ldots,Sf_q)) \tag{6.12}$$

if ψ is a complex continuous function tending to zero at infinity on R^q. The properties (Q2) are seen to hold on vectors of the form

$$\psi(Q(f_1),\ldots,Q(f_q))\Psi \tag{6.13}$$

where $\Psi\in\mathscr{G}$ and ψ has compact support. This actually proves (Q2) because the operators involved are the closure of their restriction to such vectors.

Let P be a complex polynomial on R^q, then

$$\begin{aligned} &P(Q(f_1),\ldots,Q(f_q))\ \psi(Q(f_1),\ldots,Q(f_q))\Psi \\ &= \pi_\rho(P(Sf_1,\ldots,Sf_q)\ \psi(Sf_1,\ldots,Sf_q))\Psi \end{aligned} \tag{6.14}$$

If ϕ is a bounded continuous complex function on R^q, it can be approximated uniformly on compacts by polynomials so that P may be replaced by ϕ in the above equation, yielding

$$\phi(Q(f_1),\ldots,Q(f_q)) = \pi_\rho(\phi(Sf_1,\ldots,Sf_q)) \tag{6.15}$$

Using (6.15) and (6.8), (6.10) one checks readily (U) and (Q2). Property (Q3) follows from the fact that, if $f \geqslant 0$, then $S\Delta^{-1}f \geqslant 0$ by (3.6). This concludes the verification of the conditions defining a G-field theory.

Finally (6.6) follows from (6.7) and (6.15). Given $\rho\in\mathscr{F}$ we have thus proved that there exists a G-field theory satisfying (6.6), this theory is unique within unitary equivalence because of the uniqueness of the Gel'fand-Segal construction and of the uniqueness of the construction of $Q(f)$ when the $\varphi(Q(f))$ are given. The other half of the theorem will be proved in Section 7.

7. Proof of Theorems 5.1. and 6.1.

To conclude the proof of theorems 5.1. and 6.1. we have to show that, given a G-field theory $(\mathcal{H},Q,U,\Omega)$ there is a unique G-system of density distributions (μ^n_Λ) such that if $f_1,\ldots,f_q \in \mathcal{K}(\sum_{n\geq 0} \Lambda^n)$ and ϕ is a bounded continuous complex function on R^q, then

$$\sum_n \int d\mu^n_\Lambda(x_1,\ldots,x_n)\ \phi(Sf_1(x_1,\ldots,x_n),\ldots,Sf_q(x_1,\ldots,x_n))$$
$$= (\Omega, \phi(Q(f_1),\ldots,Q(f_q))\Omega) \tag{7.1}$$

Let $(\mathcal{H},Q,U,\Omega)$ be a G-field theory and Λ be an open relatively compact subset of T. Let $h \in \mathcal{K}(T^1) \subset \mathcal{K}_*$ be such that $h \geqslant 0$ and $h(x) = 1$ if $x \in \Lambda$. If $f^n \in \mathcal{K}(\Lambda^n)$, the reader will check that

$$\Delta\,[(\Delta^{-1}f^n) * h] = f^n * h + nf^n \tag{7.2}$$

We assume that $f^n \geqslant 0$, then

$$\Delta\,[(\Delta^{-1}f^n) * h] - nf^n = f^n * h \geqslant 0 \tag{7.3}$$

hence, by (Q2), (Q3),

$$Q(\Delta^{-1}f^n)\,[Q(h) - n] \geqslant 0 \tag{7.4}$$

For every integer $p > 0$, let $\alpha_p \geqslant 0$ be a continuous real function with support in the closed interval $[-1,p+1] \subset R$ and such that $\alpha_p(t) = 1$ if $0 \leqslant t \leqslant p$. We assume also that $\alpha_p \leqslant \alpha_{p+1}$. From (Q1), (Q3) and (7.4) we obtain

$$Q(\Delta^{-1}f^n)\ \alpha_p(Q(h)) = 0 \qquad \text{if } n > p \tag{7.5}$$

Furthermore, for all n, $Q(\Delta^{-1}f)\ \alpha_p(Q(h))$ is bounded because f^n is bounded by a multiple Ch^{*n} of h^{*n} and therefore

$$0 \leqslant Q(\Delta^{-1}f^n) \leqslant C\,Q(\Delta^{-1}h^{*n}) \leqslant C\,Q(h^{*n}) = CQ(h)^n \tag{7.6}$$

This shows that $(\Omega, Q(\Delta^{-1}f)\,\alpha_p(Q(h))\Omega)$ is a positive linear functional of $(f^0, f^1,\ldots, f^p)$. There exist thus bounded measures $\nu^n \geqslant 0$ on Λ^n for $n = 0,1,\ldots,p$ such that

$$(\Omega, Q(\Delta^{-1}f)\alpha_p(Q(h))\Omega) = \sum_{n=0}^{p} \nu^n(f^n) \tag{7.7}$$

and (Q1) implies that ν^n is symmetric in its n arguments.

If f^n is assumed to be symmetric in its n arguments for $n = 0,1,\ldots,p$, $(f^0,f^1,\ldots,f^p)$ is uniquely determined by the restrictions of $S\Delta^{-1}f$ to $\Lambda^0,\Lambda^1,\ldots,\Lambda^p$ of $S\Delta^{-1}f$, and the correspondence is such that there exist bounded measures $\mu^n_{(p)}$ on Λ^n for $n = 0,1,\ldots,p$, symmetric in their arguments and for which

$$(\Omega, Q(\Delta^{-1}f)\,\alpha_p(Q(h))\Omega) = \sum_{n=0}^{p} \mu^n_{(p)}((S\Delta^{-1}f)^n) \tag{7.8}$$

We define further $\mu^n_{(p)} = 0$ for $n \geqslant p$. We have then, writing f instead of $\Delta^{-1}f$

$$\begin{aligned}&(\Omega, Q(f)\,\alpha_p(Q(h))\Omega)\\ &= \sum_{n=0}^{\infty}\int d\mu^n_{(p)}(x_1,\ldots,x_n)\,Sf(x_1,\ldots,x_n) = \mu_{(p)}(Sf)\end{aligned} \tag{7.9}$$

This formula is valid for all $f \in \mathcal{K}(\sum_{n\geqslant 0}\Lambda^n)$. Let now $f_1,\ldots,f_q \in \mathcal{K}(\sum_{n\geqslant 0}\Lambda^n)$ and let P be a complex polynomial on R^q, we have

$$(\Omega, P(Q(f_1),\ldots,Q(f_q))\,\alpha_p(Q(h))\Omega) = \mu_{(p)}(P(Sf_1,\ldots,Sf_q)) \tag{7.10}$$

If P tends uniformly on the compacts to a bounded continuous complex function ϕ on R^q, this gives

$$(\Omega, \phi(Q(f_1),\ldots,Q(f_q))\,\alpha_p(Q(h))\Omega) = \mu_{(p)}(\phi(Sf_1,\ldots,Sf_q)) \tag{7.11}$$

We take $f^m \in \mathcal{K}(\Lambda^m)$ symmetric in its arguments, $k \in \mathcal{K}(\Lambda)$. We assume that k has values in [0,1] and that $k(x) = 1$ if $x \in K$, where K is a compact such that supp $f^m \subset K^m \subset \Lambda^m$. We have

$$(\Omega, Q(\Delta^{-1}f^m)\,\alpha_m(Q(k))\,\alpha_p(Q(h))\Omega) = \sum_{n=0}^{p} \mu^n_{(p)}((S\Delta^{-1}f^m)^n \cdot \alpha_m(Sk)^n) \tag{7.12}$$

Clearly $(S\Delta^{-1}f^m)^n\,\alpha_m(Sk)^n$ is zero if $n < m$ by (3.6), is $m!f^m$ if $n = m$ and vanishes on K^n if $n > m$. By taking K adequately large, the terms with $n > m$ are made arbitrarily small and we obtain

$$f^m \geqslant 0 \Rightarrow \mu^m_{(p)}(f^m) \geqslant 0 \tag{7.13}$$

$$f^m \geqslant 0 \Rightarrow \mu^m_{(p+1)}(f^m) - \mu^m_{(p)}(f^m) \geqslant 0 \tag{7.14}$$

(7.13) and (7.14) show that $\mu_{(p)} \geqslant 0$, $\mu_{(p+1)} \geqslant \mu_{(p)}$ and therefore

$$\lim_{p,p'\to\infty} \|\mu_{(p')} - \mu_{(p)}\| = \lim_{p,p'\to\infty} |\mu_{(p')}(1) - \mu_{(p)}(1)| = 0 \tag{7.15}$$

Let $\mu_\Lambda = \lim_{p\to\infty} \mu_{(p)}$, then (7.1) follows from (7.11), (D1) follows from (6.1), (Ω), and (D3) follows from (U), (Ω1). Let $\Lambda' \supset \Lambda$ and $h(x) = 1$ if $x \in \Lambda'$. We define $\mu_{(p)'}$ like $\mu_{(p)}$ except for the replacement of Λ by Λ', then if $f \in \mathcal{K}(\sum_{n\geq 0} \Lambda^n)$ we have

$$\sum_n \mu^n_{(p)}(Sf)^n = \sum_{n'} \mu^{n'}_{(p)'}(Sf)^{n'}$$

$$= \sum_{n'} \sum_{m=o}^{n'} \frac{n'!}{(n'-m)!m!} \mu^{n'}_{(p)'}(\chi_\Lambda^{\otimes(n'-m)}(Sf)^{n'-m} \otimes \chi^{\otimes m}_{\Lambda'-\Lambda})$$

$$= \sum_n \sum_m \frac{(n+m)!}{n!m!} \mu^{(n+m)}_{(p)'}(\chi_\Lambda^{\otimes n}(Sf)^n \otimes \chi^{\otimes m}_{\Lambda'-\Lambda}) \tag{7.16}$$

Taking successively $f = \Delta^{-1} f^n$ with $f^n \in \mathcal{K}(\Lambda^n)$ and $n = p,\dots,1,0$ yields

$$\mu^n_{(p)}(f^n) = \sum_m \frac{(n+m)!}{n!m!} \mu^{n+m}_{(p)'}(f^n \otimes \chi^{\otimes m}_{\Lambda'-\Lambda}) \tag{7.17}$$

hence (D2) when $p \to \infty$.

Finally we show that the system (μ^n_Λ) is uniquely determined by $(\mathcal{G}, Q, U, \Omega)$. Let indeed f^m and k be as above (7.12), then

$$(\Omega, Q(\Delta^{-1}f^m)\alpha_m(Q(k)\Omega) - m!\mu^m_\Lambda(f^m)$$
$$= \sum_{n>m} \mu^m_\Lambda((S\Delta^{-1}f^m)^n \alpha_m (Sk)^n) \tag{7.18}$$

and the absolute value of the right hand side can be made arbitrarily small by taking K adequately large so that, if $f^m \geq 0$,

$$\mu^m_\Lambda(f^m) = (m!)^{-1} \sup_k (\Omega, Q(\Delta^{-1}f^m)\alpha_m(Q(k))\Omega) \tag{7.19}$$

8. Correlation functions.

Let $(\mathcal{G}, Q, U, \Omega)$ be a G-field theory. If Ω is contained in the intersection of the domains of all the $Q(f)$, $f \in \mathcal{K}_*$, there exist positive measures $\bar\rho^n$ on T^n, $n \geq 0$, such that

$$(\Omega, Q(\Delta^{-1}f)\Omega) = \sum_{n=0}^\infty \bar\rho^n(f^n) = \sum_{n=0}^\infty \int d\bar\rho^n(x_1,\dots,x_n) f^n(x_1,\dots,x_n) \tag{8.1}$$

$\bar\rho^n$ is invariant under permutations of its arguments and is called the n-body correlation function *) of the G-field theory.

*) This terminology originates from the situation where $T = R^\nu$ and $\bar\rho^n$ is absolutely continuous with respect to the Lebesgue measure on T^n, in this case $\bar\rho^n$ is identified to a locally integrable function. If this function is bounded by a constant C^n, where C is independent of n, then Proposition 8.1. applies.

The correlation functions, when they exist, may or may not determine the G-field theory (up to unitary equivalence). It is of interest to know when they do because the information about a CSM system is usually given in terms of its correlation functions. The following criterion may be useful for this purpose.

Proposition 8.1. Suppose that for every $f^1 \in \mathcal{K}(T)$, $f^1 \geq 0$, there exists $C > 0$ such that

$$\int d\bar{\rho}^n(x_1,\ldots,x_n) f^1(x_1)\ldots f^1(x_n) \leq C^n \tag{8.2}$$

Then, the G-field theory is determined up to unitary equivalence by $(\bar{\rho}^n)$.

We have assumed that

$$(\Omega, Q(\Delta^{-1}(f^1)^{*n})\Omega) \leq C^n \tag{8.3}$$

Since the number of partitions of $\{1,\ldots,n\}$ is $\leq n!$ (use the cycle representation of permutations), there exists $C' > 0$ such that

$$(\Omega, Q(f^1)^n\Omega) \leq n!C'^n \tag{8.4}$$

If $f \in \mathcal{K}_*$, we have thus for some $C'' > 0$

$$\begin{aligned}\frac{1}{n!}\ \|Q(f^1)^n Q(f)\Omega\| &= \frac{1}{n!}\ (Q(f^1)^{2n}\Omega,\ Q(f)^2\Omega)^{\frac{1}{2}} \\ &\leq \frac{1}{n!}\ (\Omega, Q(f^1)^{4n}\Omega)^{\frac{1}{4}}\ \|Q(f)^2\Omega\|^{\frac{1}{2}} \\ &\leq \frac{1}{n!}\ [(4n)!]^{\frac{1}{4}}\ C'^n\ \ \|Q(f)^2\Omega\|^{\frac{1}{2}} < C''^n\end{aligned} \tag{8.5}$$

which shows that $Q(f)\Omega$ is an analytic vector for $Q(f^1)$ (see Nelson [14]), hence that $Q(f^1)$ is essentially self-adjoint on the complex Hilbert space generated by $Q(\mathcal{K}_*)\Omega$ (see [14], Lemma 5.1., and for a similar application see [1]).

For arbitrary $\tilde{f} \in \mathcal{K}_*$ one may find $f^1 \in \mathcal{K}(T)$, $f^1 \geq 0$, such that $|S\tilde{f}|$ is bounded by a polynomial in Sf^1. In this case, the vectors $\alpha(Q(f^1))Q(Sf)\Omega$ where $f \in \mathcal{K}_*$ and α is complex continuous with compact support are analytic for $Q(S\tilde{f})$, and $Q(S\tilde{f})$ is thus essentially self-adjoint on these vectors.

Therefore, if the $\bar{\rho}^n$ are known, the $Q(f)$ are known as self-adjoint operators on the complex Hilbert space generated by $Q(\mathcal{K}_*)\Omega$, and it follows from the cyclicity (Ω2) of Ω that this Hilbert space coincides with $\mathfrak{H}$, which concludes the proof.

9. Properties of G-field theories.

Proposition 9.1. In a G-field theory the representation U of G is strongly continuous.

Let $f_1,\ldots,f_q \in \mathcal{K}_*$ and ϕ be a bounded continuous complex function on R^q, we write

$$\psi_g = [\tau_g \phi^*(Sf_1,\ldots,Sf_q)]\ \phi(Sf_1,\ldots,Sf_q) - |\phi(Sf_1,\ldots,Sf_q)|^2 \quad (9.1)$$

for $g \in G$. We have to prove that, if $\rho \in \mathcal{F}$, then

$$\lim_{g\to e} \rho(\psi_g) = 0 \quad (9.2)$$

where e is the identity of G.

We choose Λ open relatively compact in T such that $f_1,\ldots,f_q \in \mathcal{K}(\sum_{n\geqslant 0}\Lambda^n)$, then given $\varepsilon > 0$, there exists p such that

$$\sum_{n>p} \|\mu_\Lambda^n\| < (3\,\|\phi(Sf_1,\ldots,Sf_q)\|^2)^{-1}\,\varepsilon \quad (9.3)$$

Because of the continuity of $(g,x) \to \tau_g x$, there exists a neighbourhood $\mathcal{N}$ of e in G such that if $g \in \mathcal{N}$ then $\tau_g f_1,\ldots,\tau_g f_q \in \mathcal{K}(\sum_{n\geqslant 0}\Lambda^n)$ and

$$\max_{n\leqslant p}\ \max_{x_1,\ldots,x_n\in\Lambda} |\psi_g^n(x_1,\ldots,x_n)| < \frac{\varepsilon}{3} \quad (9.4)$$

If $g \in \mathcal{N}$, (9.3) and (9.4) give $|\rho(\psi_g)| < \varepsilon$ which proves (9.2) and the proposition.

Proposition 9.2. If the condition (T') is satisfied (the topology of T has a countable basis), then the Hilbert space $\mathfrak{H}$ of a G-field theory is separable.

Let ρ be the state on $\mathcal{O}\!\mathcal{C}$ corresponding to the G-field theory. Let $(\mathcal{O}_\iota)$, $(\mathcal{J}_\iota)$ be as in Proposition 4.3. According to Corollary 4.4. we may choose these families countable and the $\mathcal{J}_\iota$ are separable. Let now $\mathfrak{H}_\iota$ be the closed subspace of $\mathfrak{H}$ spanned by $\pi_\rho(\mathcal{O}_\iota)\Omega$. By the uniqueness of the Gel'fand-Segal construction, the representation of

$\mathfrak{A}_\iota$ in $\mathfrak{H}_\iota$ is unitarily equivalent to the representation π_{ρ_ι} constructed from the restriction ρ_ι of ρ to $\mathfrak{A}_\iota$. Since the restriction of ρ_ι to $\mathfrak{I}_\iota$ has norm 1 by 4.3.2., $\pi_\rho(\mathfrak{I}_\iota)\Omega$ is dense in $\mathfrak{H}_\iota$ (see [6] 2.11.7.), hence $\mathfrak{H}_\iota$ is separable. Since the $\mathfrak{H}_\iota$ form a countable family and span $\mathfrak{H}$ by 4.3.1., $\mathfrak{H}$ is separable.

10. Integral representations on $\mathcal{F} \cap \mathcal{L}_G^\perp$.

In this section we apply to the states in $\mathcal{F} \cap \mathcal{L}_G^\perp$ some recent general results*) which are summarized in the Appendix for the convenience of the reader. If $A \in \mathfrak{A}$, we denote by $\hat{A}$ the function $\rho \to \hat{A}(\rho) = \rho(A)$ on E. If K is a convex set in a topological vector space, $\mathcal{E}(K)$ will denote the set of its extremal points.

Theorem 10.1. Let (T') hold. Given $\rho \in \mathcal{F} \cap \mathcal{L}_G^\perp$, there exists a unique measure μ_ρ on $E \cap \mathcal{L}_G^\perp$ concentrated on $\mathcal{E}(E \cap \mathcal{L}_G^\perp)$ such that for all $A \in \mathfrak{A}$

$$\rho(A) = \mu_\rho(\hat{A}) \tag{10.1}$$

The mapping $\rho \to \mu_\rho$ is one-to-one from $\mathcal{F} \cap \mathcal{L}_G^\perp$ onto the positive measures of norm 1 on $E \cap \mathcal{L}_G^\perp$ which are concentrated on $\mathcal{F} \cap \mathcal{E}(E \cap \mathcal{L}_G^\perp)$.

This follows from Proposition A3.1. and Theorem A3.2. of the Appendix, using the fact (Theorem A2.3.) that since $\mathfrak{A}$ is abelian, it is G-abelian.

Proposition 10.2. Let $\rho \in \mathcal{F} \cap \mathcal{L}_G^\perp$, U be the unitary representation of G in the Hilbert space $\mathfrak{H}$ of the corresponding G-field theory. If P_ρ is the projection on the subspace of $\mathfrak{H}$ formed by the vectors invariant under U(G), then

(i) the measure μ_ρ introduced in Theorem 10.1. is determined by

$$\mu_\rho(\hat{A}_1 \ldots\ldots \hat{A}_\ell) = (\Omega, \pi_\rho(A_1)P_\rho \pi_\rho(A_2)P_\rho \ldots\ldots P_\rho \pi_\rho(A_\ell)\Omega) \tag{10.2}$$

(ii) $\rho \in \mathcal{E}(E \cap \mathcal{L}_G^\perp) \Leftrightarrow P_\rho$ is one-dimensional.

(i) and (ii) follow respectively from Theorem A2.2. and Proposition A2.4. of the Appendix.

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*) See [21], [8], [12], [13].

Let G now be a locally compact group. A directed set (χ_α) of functions on G will be called a $\mathfrak{M}$-directed set (see [12]) if

(i) $\chi_\alpha \geq 0$ (10.3)

(ii) $\int dg \quad \chi_\alpha(g) = 1$ (10.4)

(iii) $\lim\limits_{\alpha \to \infty} \int dg \ |\chi_\alpha(gg_1) - \chi_\alpha(g)| = 0$ (10.5)

where the integrations are with respect to the right Haar measure. The existence of $\mathfrak{M}$-directed sets is insured if G has an invariant mean (see A 4., Appendix). This is true in general for the groups of interest in classical statistical mechanics (the abelian groups, the euclidean group, etc.) and the $\mathfrak{M}$-directed sets may be taken to be sequences.

Proposition 10.3*). With the notations of 10.2., we have

(i) $\mu_\rho(\hat{A}_1 \cdot \ldots \cdot \hat{A}_\ell)$

$$= \lim_{\alpha_1, \ldots, \alpha_\ell \to \infty} \int dg_1 \ldots \int dg_\ell \, \chi_{\alpha_1}(g_1) \ldots \chi_{\alpha_\ell}(g_\ell) \, \rho(\tau_{g_1} A_1 \cdot \ldots \cdot \tau_{g_\ell} A_\ell) \qquad (10.6)$$

(ii) $\rho \in \mathcal{E}(E \cap \mathcal{S}_G^\perp) \Leftrightarrow \lim\limits_{\alpha \to \infty} \int dg \ \chi_\alpha(g) \ \rho(A_1 \cdot \tau_g A_2) = \rho(A_1)\ \rho(A_2)$

for all $A_1, A_2 \in \mathfrak{A}$

The representation U of G is strongly continuous by Proposition 9.1., therefore Theorem A4.1. of the Appendix yields

$$\lim_{\alpha \to \infty} \int dg \ \chi(g) \ U(g) = P_\rho \quad \text{strongly} \qquad (10.7)$$

In view of Proposition 10.2.(i), we prove (i) by showing that

$$\lim_{\alpha_1, \ldots, \alpha_\ell \to \infty} \Big\| \int dg_1 \ \chi_{\alpha_1}(g_1) \pi_\rho(\tau_{g_1} A_1) \ldots \int dg_\ell \chi_{\alpha_\ell}(g_\ell) \pi_\rho(\tau_{g_\ell} A_\ell) \Omega$$

$$- P_\rho \, \pi_\rho(A_1) P_\rho \cdot \ldots \cdot P_\rho \, \pi_\rho(A_\ell) \Omega \Big\| = 0 \qquad (10.8)$$

The norm to be evaluated is majorized by a sum of ℓ terms of the form

$\Big\| \int dg_1 \chi_{\alpha_1}(g_1) \pi_\rho(\tau_{g_1} A_1) \cdot \ldots \cdot \int dg_{m-1} \chi_{\alpha_{m-1}}(g_{m-1}) \pi_\rho(\tau_{g_{m-1}} A_{m-1}) \cdot$

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*) Cf. [12], [21].

$$\cdot [\int dg_m \chi_{\alpha_m}(g_m)\pi_\rho(\tau_{g_m}A_m) - P_\rho \pi_\rho(A_m)]\, P_\rho \pi_\rho(A_{m+1}) \ldots\ldots P_\rho \pi_\rho(A_\ell)\Omega \|$$

$$\leq (\prod_{i=1}^{m-1} \|A_i\|)\ \ \| [\int dg_m \chi_{\alpha_m}(g_m) U(g_m) - P_\rho]\, \pi_\rho(A_m) P_\rho \pi_\rho(A_{m+1}) \ldots\ldots P_\rho \pi_\rho(A_\ell)\Omega \| \tag{10.9}$$

which tend to zero in view of (10.7) when $\alpha_1, \ldots, \alpha_\ell \to \infty$.

(ii) results from (10.7) and Proposition 10.2.(ii).

Remark. The interpretation of the integral representation in Theorem 10.1 as a decomposition of a state into phases has been discussed in [18].

11. Pure states.

Let X be a function from T to the integers ≥ 0 such that for every compact $K \subset T$, X restricted to T vanishes except at a finite number of points. We call $\mathcal{X}$ the set of all such X. Given Λ, relatively compact open set in T, and $X \in \mathcal{X}$, there exist $n \geq 0$ and $(x_1, \ldots, x_n) \subset \Lambda^n$ such that, for all $x \in \Lambda$, $X(x)$ is the number of elements of $(x_1, \ldots, x_n)$ which are equal to x. If $f \in \mathcal{K}(\sum_{n \geq 0} \Lambda^n)$ we define

$$Sf(X) = f^0 + \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \ldots \sum_{i_m=1}^{n} f^m(x_{i_1}, \ldots, x_{i_m}) \tag{11.1}$$

Proposition 11.1. If $\sigma \in \mathcal{F} \cap \mathcal{E}(E)$, there exists a unique $X_\sigma \in \mathcal{X}$ such that if $f_1, \ldots, f_q \in \mathcal{K}_*$ and ϕ is a bounded continuous complex function on R^q, then

$$\sigma(\phi(Sf_1, \ldots, Sf_q)) = \phi(Sf_1(X_\sigma), \ldots, Sf_q(X_\sigma)) \tag{11.2}$$

The mapping $\sigma \to X_\sigma$ is one-to-one from $\mathcal{F} \cap \mathcal{E}(E)$ onto $\mathcal{X}$.

Notice that $\mathcal{E}(E)$ is the set of extremal, i.e. pure states on the abelian B^*-algebra $\mathcal{O}\!\iota$, and is thus also the set of homomorphisms of $\mathcal{O}\!\iota$ onto the complex field. On the other hand we may write $\mathcal{F} = \mathcal{F} \cap \mathcal{L}_{G_o}^{\perp}$ where G_o is reduced to the identity.

Let Λ_o be a relatively compact open subset of T and $\mathcal{O}\!\iota(\Lambda_o)$ be the sub - B^*-algebra of $\mathcal{O}\!\iota$ generated by the bounded continuous complex functions of the Sf for $f \in \mathcal{K}(\sum_{n \geq 0} \Lambda^n)$. If $\sigma \in \mathcal{F} \cap \mathcal{E}(E)$ then the restriction of σ to $\mathcal{O}\!\iota(\Lambda_o)$ is a homomorphism onto the complex field. Therefore, if (μ_Λ^n) is the G_o-system of density distributions associated to σ, there exists

n_o such that $\mu^n_{\Lambda_o} = 0$ for $n \neq n_o$ and $\mu^{n_o}_{\Lambda_o}$ is obtained by symmetrizing the measure $\delta_{x_1} \otimes \ldots \otimes \delta_{x_{n_o}}$ on $\Lambda_o^{n_o}$ for some $x_1, \ldots, x_{n_o} \in \Lambda$. Let $X_{(\Lambda_o)}$ be the function on Λ_o defined by

$$X_{(\Lambda_o)}(x) = \sum_{i=1}^{n_o} f_i(x) \tag{11.3}$$

where $f_i(x_i) = 1$ and $f_i(x) = 0$ if $x \neq x_i$. We define X_σ to be such that its restriction to each Λ is $X_{(\Lambda)}$. It is then easily checked that (11.2) is satisfied, furthermore it is clear that (11.2) determines X_σ uniquely.

Let now $X \in \mathcal{X}$, the state σ_X on $\mathcal{O}\!\mathcal{L}$ defined by

$$\sigma_X(\phi(Sf_1, \ldots, Sf_q)) = \phi(Sf_1(X), \ldots, Sf_q(X)) \tag{11.4}$$

is in $\mathcal{F}$ and is a homomorphism onto the complex numbers, hence $\sigma_X \in \mathcal{F} \cap \mathcal{E}(E)$ which shows that $\sigma \to X_\sigma$ is one-to-one onto, concluding the proof.

Since $\mathcal{E}(E)$ is identical to the spectrum of $\mathcal{O}\!\mathcal{L}$, the Gel'fand isomorphism associates to any $\rho \in E$ a measure $\nu_\rho \geq 0$ on $\mathcal{E}(E)$. This measure is actually the unique measure on $\mathcal{E}(E)$ with resultant ρ. If (T') holds *) and if $\rho \in \mathcal{F}$, ν_ρ is thus identical to the measure μ_ρ of Theorem 10.1. for the case $G = G_o$, in particular ν_ρ is concentrated on $\mathcal{F} \cap \mathcal{E}(E)$. The set $\mathcal{X}$, identified to $\mathcal{F} \cap \mathcal{E}(E)$ is the "space of configurations of an infinite number of particles" promised in the introduction, this interpretation being justified by Proposition 11.1.

<u>Proposition 11.2. Let (T') hold *) and $\rho \in \mathcal{F} \cap \mathcal{L}^1_G$.</u>

(i) <u>The mappings $\sigma \to \tau'_g \sigma$ define homeomorphisms of $\mathcal{X}$ which leave the measure ν_ρ invariant.</u>

(ii) <u>If $A \in \mathcal{O}\!\mathcal{L}$, let $\tilde{A}$ be the function on $\mathcal{X}$ defined by $\tilde{A}(X_\sigma) = \sigma(A)$ then $L^2(\nu_\rho)$ is identified to the Hilbert space $\mathcal{H}$ by $\tilde{A} \to \pi_\rho(A)\Omega$.</u>

(iii) <u>The group G acts ergodically on $(\mathcal{X}, \nu_\rho)$ if and only if $\rho \in \mathcal{E}(E \cap \mathcal{L}^1_G)$</u>

(i) is obvious, the identification (ii) goes via the Gel'fand isomorphism, and

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*) It is in fact sufficient to assume that T is countable at infinity.

$\rho \in \mathcal{E}(E \cap \mathcal{L}_G^{\perp})$ is equivalent to the fact that ν_ρ has no nontrivial decomposition into two invariant measures $\geqslant 0$, proving (iii).

12. Averages over translations.

In this section we consider the case where $G = R^\nu$ is the group of translations in ν dimensions, this will allow us to use a pointwise ergodic theorem.

Theorem 12.1. Let (T') hold, $G = R^\nu$ and let χ_α be the characteristic function of the cube

$$\Lambda_\alpha = \{g : 0 \leqslant g^i < \alpha \quad \text{for } i = 1,\dots,\nu\} \tag{12.1}$$

divided by α^ν. If $\rho \in \mathcal{E}(E \cap \mathcal{L}_G^{\perp}) \cap \mathcal{F}$, then ν_ρ is concentrated on those $X \in \mathcal{X}$ such that, for all $A \in \mathcal{O}\!\!\iota$,

$$\lim_{\alpha\to\infty} \int dg\, \chi_\alpha(g)\, \tau_g \tilde{A}(X) = \rho(A) \tag{12.2}$$

If a system $(\bar{\rho}^n)$ of correlation functions is associated with ρ, ν_ρ is concentrated on those $X \in \mathcal{X}$ such that, for all $f \in \mathcal{K}_*$

$$\lim_{\alpha\to\infty} \int dg\, \chi_\alpha(g)\, S\Delta^{-1}\tau_g f(X) = \sum_n \bar{\rho}^n(f^n) \tag{12.3}$$

(i) Using the identifications made in Section 11 (in particular, Proposition 11.2) one sees that the functions $X \to \tilde{A}(X)$ (with $A \in \mathcal{O}\!\!\iota$) and (if the $\bar{\rho}^n$ exist) also $X \to Sf(X)$ (with $f \in \mathcal{K}_*$) are in $L^1(\nu_\rho)$. The continuity and ergodicity of the representation U of G in $L^2(\nu_\rho)$ (see Proposition 9.1. and Proposition 11.2.) and a pointwise ergodic theorem (see [9] VIII.7.17.) imply then that ν_ρ is concentrated on those X such that (12.2) (resp. 12.3. if the $\bar{\rho}^n$ are defined) holds for a given $A \in \mathcal{O}\!\!\iota$ (resp. $f \in \mathcal{K}_*$).

(ii) Let $(\mathcal{O}\!\!\iota_\iota)$, $(\mathcal{J}_\iota)$ be countable families as in Proposition 4.3. We let $(A_{\iota\lambda})$ be a countable dense set in $\mathcal{J}_\iota$ and $(U_{\iota\lambda})$ be a countable increasing approximate identity in $\mathcal{J}_\iota$ (see [6] 1.7.). Let $\mathcal{U}$ be the set of those $X \in \mathcal{X}$ such that (12.2) holds for $A = A_{\iota\lambda}$ and $A = U_{\iota\lambda}$, then ν_ρ is concentrated on $\mathcal{U}$. Since $(A_{\iota\lambda})$ is dense in $\mathcal{J}_\iota$, (12.2) holds for $X \in \mathcal{U}$ and all $A \in \mathcal{J}_\iota$. We have $\lim_{\lambda\to\infty} \rho(1-U_{\iota\lambda}) = 0$, every $A \in \mathcal{O}\!\!\iota$ may be written as $AU_{\iota\lambda} + A(1 -$

$U_{\iota\lambda}$) where $AU_{\iota\lambda} \subset \mathcal{T}_\lambda$ and $|\tilde{A}(1-\tilde{U}_{\iota\lambda})| \leq \|A\| \, (1-\tilde{U}_{\iota\lambda})$ so that (12.2) holds for $X \in \mathcal{U}$ and all $A \in \mathcal{O}\!\iota_\iota$. Finally, since $\cup_\iota \mathcal{O}\!\iota_\iota$ is dense in $\mathcal{O}\!\iota$, ν_ρ is concentrated on those $X \in \mathcal{X}$ such that (12.2) holds for all $A \in \mathcal{O}\!\iota$.

(iii) Let (f_λ) be a countable family of elements of $\mathcal{K}_*$ such that any $f \in \mathcal{K}_*$ may be approximated uniformly on some compact by elements of (f_λ) (such a family exists because T is countable at infinity and its compacts are metrizable). Let (h_ι) be a countable family as in Proposition 4.3. If $\mathcal{U}$ is the set of those $X \in \mathcal{X}$ such that (12.3) holds for $f = f_\lambda$ and $f = h_\iota$, then ν_ρ is concentrated on $\mathcal{U}$. Given $f \in \mathcal{K}_*$ and $\varepsilon > 0$ there exist λ, ι such that $|f - f_\lambda| < \varepsilon\, h_\iota$. Therefore $|\sum_n \bar{\rho}^n(f^n - f^n_\lambda)| < \varepsilon \sum_n \bar{\rho}^n(h_\iota)$ may be chosen arbitrarily small and (12.3) holds for all $X \in \mathcal{U}$, $f \in \mathcal{K}_*$, concluding the proof.

Remark. Theorem 12.1. shows in particular that if (T') holds and $\rho \in \mathcal{E}(E \cap \mathcal{S}^\perp_G) \cap \mathcal{F}$, there exists $X \in \mathcal{X}$ such that, for all $f_1,\ldots,f_q \in \mathcal{K}_*$ and ϕ a bounded continuous complex function on R^q,

$$\rho(\phi(Sf_1,\ldots,Sf_q)) = \lim_{\alpha\to\infty} \int dg\, \chi_\alpha(g)\, \phi(S\tau_g f_1(X),\ldots,S\tau_g f_q(X)) \tag{12.4}$$

This is precisely the statement made in [20] except for the replacement of the condition $\rho \in \mathcal{F} \cap \mathcal{S}^\perp_G$ by the more stringent condition $\rho \in \mathcal{E}(E \cap \mathcal{S}^\perp_G) \cap \mathcal{F}$. The proof alluded to in [20] is rather different from that given here.

13. Remarks and questions.

(i) Hardcores. Let $T = R^\nu$, one often imposes the condition that, for some $a > 0$, the euclidean distance between two particles be always $\geqslant a$. One sees easily that to impose such a condition on a state ρ on $\mathcal{O}\!\iota$ is equivalent to requiring that ρ vanishes on a family of positive elements of $\mathcal{O}\!\iota$. If $E \cap \mathcal{V}$ is the set of states satisfying this condition, then $E \cap \mathcal{V} \subset \mathcal{F}$ and $E \cap \mathcal{S}^\perp_G \cap \mathcal{V}$ is a simplex (see part A5 of the Appendix). If $\rho \in E \cap \mathcal{V}$, the correlation functions of $\mathcal{V}$ are defined, and describe ρ completely.

Similar remarks hold for the case where T is a lattice and two particles are forbidden to occupy the same lattice point.

(ii) Example: state of a language. A language with N symbols may be idealized as a state of classical statistical mechanics with $G = \mathbb{Z}$ (the additive group of integers) and T : N copies of Z. A "hard core" type condition must be introduced to avoid the occupation of a site by more than one symbol. The symbols may be letters, the corresponding correlation functions are well known in cryptography, or they may be words.

(iii) Entropy per unit volume. Let $T = R^\nu$ and $G = R^\nu$, then an entropy per unit volume $s(\rho)$ can be defined for $\rho \in \mathcal{F} \cap \mathcal{L}_G^\perp$ along the lines indicated in [18]. It has been proved by the author (unpublished) that s is an affine upper semi-continuous function on $\mathcal{F} \cap \mathcal{L}_G^\perp$. It would be interesting to have a simple and more general proof of this fact, and to prove the equivalence of various definitions of the entropy per unit volume (for another definition see [19]). One should be able to prove that the equilibrium state of statistical mechanics is the solution of a variational problem (involving s) under more general conditions than those given in [18]. One should be able to prove the Gibbs phase rule (for almost every interaction, temperature, chemical potential, the equilibrium state is in $\mathcal{E}(E \cap \mathcal{L}_G^\perp)$).

(iv) The problems of evolution. We may describe the positions and momenta of point particles by taking $T = R^\nu \times (R^\nu)^\cdot$ where $(R^\nu)^\cdot$ is the one-point compactification of R^ν. The first factor is the one-particle position space, the second factor the one-particle momentum space (the use of $(R^\nu)^\cdot$ corresponds to the fact that we want to restrict the number of particles to be finite on compacts of position space).

No non-trivial existence theorem seems to be known for the evolution of a realistic system of infinitely many particles. Probably the evolution of states can be discussed for suitable interactions (Cf. the stability conditions in [10] and references quoted there) and suitable states (states having finite energy per unit volume with respect to the interaction). In particular an equilibrium state would be a fixed point for the evolution of states.

It is unclear to the author whether the evolution of an infinite system should increase its entropy per unit volume. Another possibility is that, when the time tends to $+\infty$, a state has a limit with strictly larger entropy.

(v) The situation described in Sections 11 and 12: a group G acting on a space $\mathcal{X}$ with an invariant measure ν_ρ, is the natural set up for ergodic theory, we have used only the mean and pointwise ergodic theorems, but much more could probably be done.

In quantum statistical mechanics, problems similar to those considered in this paper arise. For instance a decomposition theorem analogous to Theorem 10.1. can be proved (see part A6 of the Appendix).

Appendix

A1. Integral representations on convex compact sets. (see [3] Ch. 4. § 7., [4] and [15]). Let K be a convex compact set in a locally convex topological vector space. We denote by $\mathscr{C}(K)$ the space of complex continuous functions on K and by $\mathscr{E}(K)$ the set of extremal points of K.

An order $\prec$ is defined on the set $\mathscr{M}_+$ of positive measures on K by

$\mu_1 \prec \mu_2 \Leftrightarrow \mu_1(\phi) \leq \mu_2(\phi)$ for all convex $\phi \in \mathscr{C}(K)$. If $\mu_1 \prec \mu_2$, then μ_1 and μ_2 have the same norm and (if this norm is 1) the same resultant. If δ_ρ is the unit mass at $\rho \in K$, $\delta_\rho \prec \mu$ means that ρ is the resultant of μ.

A measure $\mu \in \mathscr{M}_+$ will be called maximal if it is maximal for the order $\prec$. For every $\mu \in \mathscr{M}_+$ there exists a maximal μ' such that $\mu \prec \mu'$.

If $\mu \in \mathscr{M}_+$ is concentrated on $\mathscr{E}(K)$, then μ is maximal. Conversely, if K is metrizable and μ is maximal, then μ is concentrated on $\mathscr{E}(K)$.

The set K is called a simplex if for every $\rho \in K$ there is a unique maximal measure $\mu_\rho \succ \delta_\rho$.

In particular if K is metrizable and a simplex, there is a unique mapping $\rho \to \mu_\rho$ of K to the probability measures concentrated on $\mathscr{E}(K)$ such that $\delta_\rho \prec \mu_\rho$. This mapping is one-to-one onto and μ_ρ may be considered as an integral representation of ρ on $\mathscr{E}(K)$.

A2. G-abelian B^*- algebras. Let $\mathfrak{A}$ be a B^*- algebra with identity, aut $(\mathfrak{A})$ the group of its *- automorphisms, G a group and τ a (group -)homomorphism $G \to$ aut $(\mathfrak{A})$. If $g \in G$ we denote by $\tau_g : A \to \tau_g A$ the corresponding automorphism. Let $\mathscr{L}_G$ be the subspace of $\mathfrak{A}$ generated by the elements of the form $A - \tau_g A$ with $g \in G$, $A \in \mathfrak{A}$ and let

$$\mathscr{L}_G^\perp = \left\{ f \in \mathfrak{A}' : A \in \mathscr{L}_G \Rightarrow f(A) = 0 \right\}$$

$\mathscr{L}_G^\perp$ is thus the space of continuous linear forms on $\mathfrak{A}$ which are invariant under the action of G. If E is the set of states on $\mathfrak{A}$, $E \cap \mathscr{L}_G^\perp$ is the set of G-invariant states on $\mathfrak{A}$.

For $\rho \in$ E, the Gel'fand-Segal construction yields a complex Hilbert space $\mathfrak{H}_\rho$, a representation π_ρ of $\mathfrak{A}$ in $\mathfrak{H}_\rho$ and a normalized vector $\Omega_\rho \in \mathfrak{H}_\rho$, cyclic with respect to

$\pi_\rho(\mathfrak{A})$ and such that for all $A \in \mathfrak{A}$

$$\rho(A) = (\Omega_\rho, \pi_\rho(A)\Omega_\rho)$$

If $\rho \in E \cap \mathcal{S}_G^\perp$, there is a unique unitary representation U_ρ of G in $\mathfrak{H}_\rho$ such that for all $g \in G$, $A \in \mathfrak{A}$

$$U_\rho(g)\,\Omega_\rho = \Omega_\rho, \quad U_\rho(g)\pi_\rho(A)U_\rho(g^{-1}) = \pi_\rho(\tau_g A)$$

Let P_ρ be the projection on the subspace of $\mathfrak{H}_\rho$ formed by the vectors invariant under $U_\rho(G)$.

Definition A2.1. $\mathfrak{A}$ is said to be G-abelian if, for all $\rho \in E \cap \mathcal{S}_G^\perp$, the von Neumann algebra generated by $P_\rho\ \pi_\rho(\mathfrak{A})P_\rho$ is abelian (in other words, if $A_1, A_2 \in \mathfrak{A}$, then $[P_\rho\ \pi_\rho(A_1)P_\rho,\ P_\rho\ \pi_\rho(A_2)P_\rho] = 0$)

Theorem A2.2. If $\mathfrak{A}$ has an identity and is G-abelian, then $E \cap \mathcal{S}_G^\perp$ is a simplex and the unique maximal measure μ_ρ with resultant $\rho \in E \cap \mathcal{S}_G^\perp$ is determined by

$$\mu_\rho(\hat{A}_1 \cdot \ldots \cdot \hat{A}_\ell) = (\Omega_\rho,\ \pi_\rho(A_1)P_\rho\ \pi_\rho(A_2)P_\rho \cdot \ldots \cdot P_\rho\ \pi_\rho(A_\ell)\Omega_\rho)$$

Theorem A2.3.*) If, for each $\rho \in E \cap \mathcal{S}_G^\perp$, there exists a filter $\mathcal{F}$ on G such that for all $A_1, A_2 \in \mathfrak{A}$

$$\lim_{\mathcal{F}}\ \rho([A_1, \tau_g A_2]) = 0$$

then $\mathfrak{A}$ is G-abelian. This is true in particular

(i) if $E \cap \mathcal{S}_G^\perp$ is empty

(ii) if $\mathfrak{A}$ is abelian

(iii) if $\mathfrak{A}$ is asymptotically abelian**) i.e. G is locally compact non-compact and for all $A_1, A_2 \in \mathfrak{A}$

$$\lim_{g \to \infty} \|[A_1, \tau_g A_2]\| = 0$$

Proposition A2.4. Let $\mathfrak{A}$ be G-abelian and $\rho \in E \cap \mathcal{S}_G^\perp$, then $\rho \in \mathcal{E}(E \cap \mathcal{S}_G^\perp) \Leftrightarrow P_\rho$ is one-dimensional.

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*) A good characterization of G-abelian algebras is given in [13].

**) This terminology was introduced by Doplicher, Kastler and Robinson [8].

A.3. Integral representations of G-invariant states.

Proposition A3.1. Let $\mathcal{A}$ be a B^*- algebra with identity and $(\mathcal{B}_\alpha)$ a countable family of self-adjoint subalgebras of $\mathcal{A}$, define

$$\mathcal{F} = \{\sigma \in E : \text{the restriction of } \sigma \text{ to } \mathcal{B}_\alpha \text{ has norm 1 for all } \alpha\}$$

If μ is a positive measure of norm 1 with resultant ρ on $\mathcal{A}$, then

$$\rho \in \mathcal{F} \iff \mu \text{ is concentrated on } \mathcal{F}$$

The proof is essentially that of part 4 of the theorem in [21].

Theorem A3.2. Let $\mathcal{A}$ be a B^*- algebra and $(\mathcal{A}_\alpha)$ a countable family of sub-B^*- algebras of $\mathcal{A}$ such that $\cup_\alpha \mathcal{A}_\alpha$ is dense in $\mathcal{A}$. Let $\mathcal{J}_\alpha$ be a separable closed two-sided ideal for each α and define

$$\mathcal{F} = \{\sigma \in E : \text{the restriction of } \sigma \text{ to } \mathcal{J}_\alpha \text{ has norm 1 for all } \alpha\}$$

(i) If $\rho \in \mathcal{F}$, the Hilbert space $\mathfrak{h}_\rho$ of the Gel'fand-Segal construction is separable

(ii) If $E \cap \mathcal{S}_G^\perp$ is a simplex (in particular if $\mathcal{A}$ has an identity and is G-abelian) and if the positive measure μ of norm 1 on $E \cap \mathcal{S}_G^\perp$ has resultant $\rho \in \mathcal{F}$, then

$$\mu \text{ maximal on } E \cap \mathcal{S}_G^\perp \iff \mu \text{ concentrated on } \mathcal{E}(E \cap \mathcal{S}_G^\perp)$$

The proof of (i) is essentially that of our Proposition 9.2. and the proof of (ii) is essentially that of part 5 of the theorem in [21].

A4. Groups with an invariant mean*). Let G be a locally compact group and $\mathcal{C}_B(G)$ be the abelian B^*- algebra of bounded continuous complex functions on G. If $f \in \mathcal{C}_B(G)$ we denote by f_g the right translate of f by $g \in G$. A state $\mathcal{M}$ on $\mathcal{C}_B(G)$ is called a right-invariant mean if, for all $g \in G$, $f \in \mathcal{C}_B(G)$,

$$\mathcal{M}(f_g) = \mathcal{M}(f)$$

If there exists a right-invariant mean on G, there also exists a left-invariant mean and a two-sided invariant mean, one says then that G is a group with an invariant mean.

G is a group with an invariant mean if it is abelian, or compact, or admits a

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*) Information about groups with an invariant mean is conveniently collected in [16] which we have used as a source for the indications given here.

composition series consisting of such groups.

One can prove that G has an invariant mean if and only if for every $\varepsilon > 0$ and compact $K \subset G$ there exists a function χ on G such that

(i) $\chi \geq 0$

(ii) $\int dg\, \chi(g) = 1$

(iii) $\int dg \left| \chi(gg_1) - \chi(g) \right| < \varepsilon \qquad \text{if } g_1 \in K$

where the integrations are with respect to the right Haar measure. In that case let $\chi_{(K,\varepsilon)}$ be such a function, the family $(\chi_{(K,\varepsilon)})$ is a $\mathfrak{M}$-directed set (see Section 10) for the order.

$$(K,\varepsilon) \leq (K',\varepsilon') \Leftrightarrow K \subset K', \quad \varepsilon \geq \varepsilon'$$

of the indices. If the topology of G has a countable basis, there is a subsequence of $(\chi_{(K,\varepsilon)})$ which is a $\mathfrak{M}$-directed set.

Theorem A4.1. Let (χ_α) be a $\mathfrak{M}$-directed set on G, U a strongly continuous unitary representation of G in a complex Hilbert space $\mathfrak{H}$, and P the projection on the subspace of $\mathfrak{H}$ formed by the vectors invariant under U, then

$$\lim_{\alpha \to \infty} \int dg\, \chi_\alpha(g) U(g) = P \quad \text{strongly}$$

This is a mean ergodic theorem (see [12] for a proof in the case $G = R^\nu$).

A5. States vanishing on positive elements. Let $\mathfrak{A}$ be a B^*- algebra with an identity $\mathfrak{A}_1$, a sub-B^*- algebra of $\mathfrak{A}$. A state ρ on $\mathfrak{A}$ vanishes on $\mathfrak{A}_1$, if and only if it vanishes on the positive elements of $\mathfrak{A}_1$. Let (A_λ) be a family of positive elements in $\mathfrak{A}$ and let

$$\mathcal{V} = \left\{ f \in \mathfrak{A}' : f(A_\lambda) = 0 \quad \text{for all } \lambda \right\}$$

If $\rho \in E$ and μ is a positive measure on E such that $\mu \succ \delta_\rho$, then

$$\rho \in \mathcal{V} \Leftrightarrow \operatorname{supp} \mu \subset \mathcal{V}$$

In particular, if $\mathfrak{A}$ is G-abelian, $E \cap \mathcal{L}_G^{\perp} \cap \mathcal{V}$ is a simplex.

A6. States of Quantum Statistical Mechanics. For each Lebesgue-measurable set $\Lambda \subset R^\nu$ let $\mathcal{H}(\Lambda)$ be the Fock space of the canonical commutation relations constructed with the real square integrable functions on Λ as test functions. If $\Lambda_1 \cap \Lambda_2 = \emptyset$ one may in a natural manner identify $\mathcal{H}(\Lambda)$ to $\mathcal{H}(\Lambda_1) \bar{\otimes} \mathcal{H}(\Lambda_2)$. It is natural (see [19], [5]) to identify the states occuring in Quantum Statistical Mechanics to collections $(\rho(\Lambda))$ where $\rho(\Lambda)$ is a density matrix on $\mathcal{H}(\Lambda)$ and, if $\Lambda_1 \cap \Lambda_2 = \emptyset$,

$$\rho(\Lambda_1) = \mathrm{Tr}_{\mathcal{H}(\Lambda_2)}\, \rho(\Lambda_1 \cup \Lambda_2)$$

We will furthermore require the invariance of $\mathcal{H}(\Lambda)$ under the group G of translations (or the euclidean group) of R^{ν}.

Let now Λ_n be the sphere of radius n around the origin, $\mathcal{O}\!\mathcal{C}_n$ the algebra of all bounded operators on $\mathcal{H}(\Lambda_n)$ identified to a subalgebra of $\mathcal{H}(R^{\nu})$, $\mathcal{J}_n$ the ideal of $\mathcal{O}\!\mathcal{C}_n$ consisting of the compact operators. Let $\mathcal{O}\!\mathcal{C}$ be the C^*- algebra on $\mathcal{H}(R^{\nu})$ generated by the $\mathcal{O}\!\mathcal{C}_n$. There is a one-to-one correspondence between families $(\rho(\Lambda))$ and the set $\mathcal{F}$ of states on $\mathcal{O}\!\mathcal{C}$ with restrictions of norm 1 to each $\mathcal{J}_n$. By Theorem A2.3. (iii), $\mathcal{O}\!\mathcal{C}$ is G-abelian and therefore by Theorem A3.2. the states in $\mathcal{F} \cap \mathcal{L}_G^{\perp}$ have a unique integral representation on $\mathcal{E}(E \cap \mathcal{L}_G^{\perp}) \cap \mathcal{F}$. Furthermore the Hilbert space of their Gel'fand-Segal construction is separable and one can see that the corresponding unitary representation of G is strongly continuous.

Bibliography.

[1] H. J. Borchers and W. Zimmermann. On the Self-Adjointness of Field Operators. Nuovo Cim. 31, 1047-1059, 1964.

[2] N. Bourbaki. Topologie Générale. Fascicule de Résultats. Hermann, Paris, 1953.

[3] N. Bourbaki. Intégration. Ch. 1-4, 2^e éd. Hermann, Paris, 1965.

[4] G. Choquet et P.-A. Meyer. Existence et unicité des représentations intégrales dans les convexes compacts quelconques. Ann. Inst. Fourier 13, 139-154, 1963.

[5] G.-F. Dell'Antonio, S. Doplicher and D. Ruelle. A Theorem on Canonical Commutation and Anticommutation Relations. Commun. Math. Phys. 2, 223-230, 1966.

[6] J. Dixmier. Les C^*- algèbres et leurs représentations. Gauthier-Villars, Paris, 1964.

[7] S. Doplicher. An algebraic Spectrum Condition. Commun. Math. Phys. 1, 1-5, 1965.

[8] S. Doplicher, D. Kastler and D. Robinson. Covariance Algebras in Field Theory and Statistical Mechanics. Commun. Math. Phys. To appear.

[9] N. Dunford and J. Schwartz. Linear Operators. Part I. Interscience, New York, 1958.

[10] M. E. Fisher and D. Ruelle. The Stability of Many-Paricle Systems. J. Math. Phys. 7, 260-270, 1966.

[11] R. Haag and D. Kastler. An Algebraic Approach to Quantum Field Theory. J. Math. Phys. 5, 848-861, 1964.

[12] D. Kastler and D. Robinson. Invariant States in Statistical Mechanics. Commun. Math. Phys. To appear.

[13] O. Lanford and D. Ruelle. Integral Representations of Invariant States on B^*-Algebras. Preprint.

[14] E. Nelson. Analytic Vectors. Ann. of Math. 70, 572-615, 1959.

[15] R. Phelps. Lectures on Choquet's Theorem. Van Nostrand Mathematical Studies # 7. D. Van Nostrand, Princeton, 1966.

[16] J.-P. Pier. Sur une classe de groupes localement compacts remarquables du point de

vue de l'Analyse harmonique. Thèse 3e cycle. Nancy, 1965. (Unpublished).

[17] D. Robinson and D. Ruelle. Extremal Invariant States. Preprint.

[18] D. Ruelle. Correlation Functionals. J. Math. Phys. 6, 201-220, 1965.

[19] D. Ruelle. Quantum Statistical Mechanics and Canonical Commutation Relations. Lecture notes of the Summer School of Theoretical Physics. Cargise, Corsica, July, 1965.

[20] D. Ruelle. A Field Theory Like Axiom System. Endicott House Conference. September, 1965.

[21] D. Ruelle. States of Physical Systems. Commun. Math. Phys. To appear.

[22] I. E. Segal. Mathematical Problems of Relativistic Physics. Am. Math. Soc., Providence, Rhode Island, 1963.