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On the Interpretation of $1/f$ Noise

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Abstract. We propose a model of $1/f$ noise based on a random walk in a random potential. Numerical support for the model is given, and physical applicability discussed.

1. Introduction

The frequency analysis of fluctuations in a number of physical phenomena exhibits a remarkable feature. It is found that the experimental power spectrum (i.e., essentially, the frequency distribution of the squared amplitude) behaves like f^{-1} at low frequency f. (This explains the name of $1/f$ noise.) The prime example of this type of behaviour is seen in voltage fluctuations across a conductor carrying electric current (see Hooge et al. [1], Dutta and Horn [2] for reviews). One also observes $1/f$ noise (sometimes called *flicker* noise) in such diverse questions as fluctuations of marine currents, or the temporal distribution of loudness in a musical recording (see for instance Press [3]). The $1/f$ law sometimes extends over many decades of frequency, implying the existence of correlations over surprisingly long times for the systems considered. (Since the integrated spectrum would diverge logarithmically, one expects that there is a low frequency cutoff.)

It is easy to obtain a power spectrum $\sim$constant (independently distributed events) or $\sim 1/f^2$ (independently distributed increments). The $1/f$ law is more difficult to explain, especially that its universality requires an interpretation of general applicability. There is no natural time scale associated with pure $1/f$ noise. Considerations of self-similarity therefore come naturally to mind (see Mandelbrot [4]) but something more specific is needed. Explanations based on spatial diffusion depend on special geometric assumptions (see Voss and Clarke [5] and Omnes [6])[1]. A guide towards understanding flicker noise in conductors is provided by the experimental fact that they are due to *equilibrium fluctuations* of the resistance (see [5, 1], and more detailed results about so-called α noise in [1]).

1 Explanations using the deterministic noise associated with low-dimensional strange attractors are also in doubt (see Arrechi and Lisi [7], Beasley et al. [8]). See however remark (b) below

Here we shall obtain $1/f$ noise from the study of random walks in random potentials in $\mathbb{R}^N$ under natural self-similarity conditions. More precisely we obtain a power spectrum $\sim|\log f|^k/f$ by an argument which is not rigorous, but is confirmed by numerical experiments. We do not claim that our specific model is of general applicability, but variations on the general theme of walks in random potentials may be adequate to describe the various $1/f$ noises seen in nature or heard in music [see remark (c) below].

2. A Model for $1/f$ Noise

We consider a system represented by a point $\mathbf{x}$ of a finite dimensional state space $\mathbb{R}^N$. A potential V is defined on this state space; we think of V as a random function with statistical properties discussed below. The time evolution of the system is given by a random walk $t \to \mathbf{x}(t)$ corresponding to a diffusion in the presence of the potential V. We choose the diffusion equation to be

$$\frac{\partial c}{\partial t} = k\nabla\cdot\mathbf{J}\,, \qquad \mathbf{J} = \nabla c + c\nabla V\,. \tag{1}$$

This is the continuous limit of a random walk on a lattice with the nearest neighbor transition $i \to j$ proportional to $\exp\frac{1}{2}[V_i - V_j]$.

The diffusion (1) is chosen such that it has an equilibrium distribution $c \sim \exp[-V]$ in a bounded box[2], corresponding to thermal equilibrium if $V = E/kT$. In the landscape created by the potential V in $\mathbb{R}^N$, one expects that $\mathbf{x}(t)$ will occasionally go through a "mountain pass" and then rapidly relax to equilibrium in the intermediate valleys. Mountain passes will thus dominate the time evolution. Other things being similar, the flux through a mountain pass is proportional to the density c at the pass (the profile $\nabla c/c$ and the gradient ∇V are taken to be the same). An approximate value of c at the pass is given by the equilibrium distribution $\exp[-V]$ normalized to already occupied valleys.

The above considerations permit an estimate of the long time behavior of the random walk if a scaling assumption is made on the potential V at large distances. We assume that V belongs to an ensemble which is invariant (at least for the large distance behavior) under the transformation $V \to V^*$, where

$$V^*(\lambda\mathbf{x}) - V^*(0) = \lambda^\alpha(V(\mathbf{x}) - V(0))\,. \tag{2}$$

For instance, if $N=1$, potentials with independent increments correspond to $\alpha = 1/2$. (Examples with $N>1$ and $\alpha \neq 1/2$ are discussed in Mandelbrot [4, Chap. 28].) Since only potential differences are important, (2) expresses scaling in terms of such differences.

When distances are multiplied by λ, the height of a mountain pass is multiplied by λ^α. The flux through the pass then changes from $g(1)\exp[-V]$ to $g(\lambda)\exp[-\lambda^\alpha V]$, where $g(\lambda)$ is a geometric coefficient, polynomial in λ, including a factor λ^{-N} for volume normalization and another factor for the width of the pass. The time scale is correspondingly multiplied by $g(1)/g(\lambda)\cdot\exp[(\lambda^\alpha - 1)V]$.

2 More generally one could take $J_\alpha = A_{\alpha\beta}(\nabla_\beta c + c\nabla_\beta V)$ with a constant matrix $(A_{\alpha\beta})$

Conversely, multiplication of the time by τ corresponds to multiplication of distances by a factor $\lambda(\tau)$ and, for large τ,

$$\begin{aligned}\lambda(\tau) &= [V^{-1}\log\tau + 1 + 0(\log\log\tau)]^{1/\alpha} \\ &\cong [V^{-1}(\log\tau + \log\tau_0^{-1})]^{1/\alpha} \\ &\sim \left(\log\frac{\tau}{\tau_0}\right)^{1/\alpha}.\end{aligned}$$

In particular, for a random walk on a one-dimensional lattice in a potential with independent increments we recover $\lambda(\tau) \sim |\log\tau|^2$ in agreement with the rigorous study of Sinai [9].

We define the power spectrum by

$$P(f) = \lim_{T\to\infty} \frac{1}{T}\left|\int_0^T e^{ift}x(t)dt\right|^2,$$

where $x(t)$ is a linear component of the vector $\mathbf{x}(t)$ (or a sum over components is taken). By scaling

$$\begin{aligned}P(f/\tau) &= \lim_{T\to\infty} \frac{1}{\tau T}\left|\int_0^{\tau T} e^{ift/\tau}x(t)dt\right|^2 \\ &= \tau \lim_{T\to\infty} \frac{1}{T}\left|\int_0^T e^{ift'}x(\tau t')dt'\right|^2 \\ &= \tau\lambda(\tau)^2 P(f),\end{aligned}$$

so that, replacing f by 1 and τ by f^{-1}, we obtain for small f

$$P(f) = P(1)f^{-1}\lambda(f^{-1})^2 \sim \frac{|\log f/f_0|^{2/\alpha}}{f}. \tag{3}$$

For the random walk on a one-dimensional lattice in a potential with independent increments we then find

$$P(f) \sim \frac{|\log f|^4}{f} \quad \text{for small } f.$$

A numerical confirmation of this prediction is presented below.

3. Remarks

(a) Physical experiments showing $1/f$ noise presumably cannot detect the factor $|\log f|^{2/\alpha}$ in (3). In particular the scaling assumption need not be verified very precisely since a variation of α with frequency would not be visible. Also, scaling concerns only valleys and passes, the mountain peaks are rarely visited and their behaviour is indifferent.

(b) A deterministic (rather than random) walk in a random potential would have the same scaling behavior, if ergodicity arguments can be applied, and would therefore again lead to $1/f$ noise.

(c) Instead of $\mathbb{R}^N$ one may consider a discrete space like Z^N or the product of N copies of the two-element set $\{0, 1\}$ for N large. The state space of an extended physical system (like a conductor) presumably has a large dimension N. Relaxation to equilibrium in potential valleys may then no longer be realistic, because the random walk $x(t)$ wanders without coming back to visited points. In this situation a new analysis would be necessary.

We may however guess the answer as follows. Suppose that the system can be approximated as the incoherent sum of N/n "small" systems with noise c/f. Then the full system has noise $c'N/f$. The factor N is indeed experimentally present in the flicker noise of conductors (see [1] or [2]).

(d) The range of distances over which approximate scaling should hold is small (logarithmic) compared to the range of times (or frequencies) over which the $1/f$ law will be verified.

(e) If the potential V is multiplied by β and the time t by τ, distances are multiplied by

$$\lambda(\tau, \beta) \sim \left(\beta^{-1} \log \frac{\tau}{\tau_0}\right)^{1/\alpha},$$

and therefore

$$P_{\beta V}(f) \sim \beta^{-2/\alpha} \frac{|\log f/f_0|^{2/\alpha}}{f}.$$

If $\alpha = 1/2$, the noise is thus proportional to the 4^{th} power of the temperature, but for larger α a weaker dependence on the temperature is obtained.

If $N = 1$ and $\alpha = 1/2$, let

$$\langle \Delta V^2 \rangle = \lim_{x \to \infty} \frac{1}{x} |V(x) - V(0)|^2,$$

then there should exist a universal constant K such that

$$P(f) \approx K \langle \Delta V^2 \rangle^{-2} \frac{|\log f/f_0|^4}{f}.$$

4. Numerical Simulations: The One Dimensional Case

In Sect. 2 we argued that random walk models where the average distance behaves for large t as $(\log t)^p$ are likely to have a power spectrum of the form $(\log f)^q f^{-1}$. In the next two sections we will describe our numerical simulations of such a model in one dimension, and some possible extensions to two dimensions.

The class of models we are going to examine can be seen as a "randomization" of the standard random walk. By this we mean that the hopping probabilities are randomly distributed over the points. Let us consider a site n belonging to a d-dimensional cubic lattice, and $n = (n_1, n_2, \ldots, n_d) \in \Lambda$. We will assume we are working in the infinite volume limit (i.e. $\Lambda = Z^d$). A traveller is moving randomly on the lattice. At each time t its probability to move in direction μ depends on its position, and is nonzero only for moving to nearest neighbour sites. These hopping probabilities $\pi_{\varepsilon\mu}(n)$ ($\varepsilon = \pm$, $\mu = 1, 2, \ldots, d$) are randomly distributed [according to

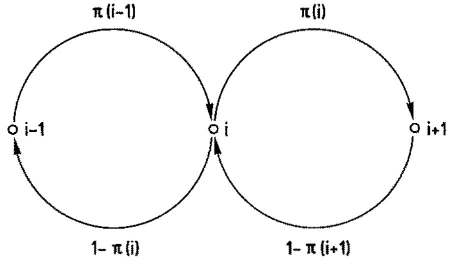

Fig. 1. Transition probabilities from and to the site i (for $d=1$)

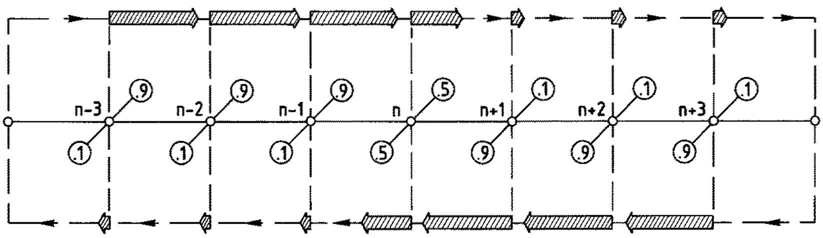

Fig. 2. A typical trapping configuration. On the lower strip the left transition probabilities: on the upper one the right ones

some probability measure $\varrho(\pi)$] and, in all the cases we will consider, the total probability to leave the site is equal to one:

$$\sum_{\varepsilon=\pm} \sum_{\mu=1}^{d} \Pi_{\varepsilon\mu}(n) = 1, \qquad \forall n, \tag{4}$$

where $\Pi_{\varepsilon\mu}(n)$ is the probability to jump from the site n in the $\varepsilon\mu$ direction. Let us remark that this class of models falls in the "nonsymmetric category": $\Pi_{\varepsilon\mu}(n)$ is a priori different from $\Pi_{-\varepsilon\mu}(n+\varepsilon\mu)$. We choose them to be uncorrelated (see Fig. 1 for the $d=1$ case).

If the probability distribution $\varrho(\pi(n))$ is such that

$$\int d\pi(n)\varrho(\pi(n)) \ln\left(\frac{\pi(n)}{1-\pi(n)}\right) = 0 \tag{5}$$

in $d=1$, we get the model analyzed by Sinai [9]. He proved that, with probability one,

$$\lim_{t\to\infty} x(t) \propto (\ln t)^2 . \tag{6}$$

For models in which the constraint (5) is not satisfied (see [11]), and references quoted therein. We will describe here our numerical studies of systems satisfying the Sinai constraint: we will be concerned both with the analysis of the temporal behavior and of its power spectrum.

We consider an infinite regular chain, and assign to each site a right transition probability $\pi(n)$, uniformly distributed in the interval $(0,1)$. The left transition

Table 1. Average distance $\bar{d}$ for the one dimensional walk. Average is over 30,000 walks, composed of 4096 time steps. The error is defined to be $((\overline{d^2}-\bar{d}^2)/(N-1))^{1/2}$, where N is the number of walks taken in account

t	$\bar{d}$
2	1.003 ± 0.006
4	1.484 ± 0.007
8	2.10 ± 0.010
16	2.89 ± 0.010
32	3.86 ± 0.030
64	5.01 ± 0.030
128	6.42 ± 0.030
256	8.06 ± 0.040
512	10.05 ± 0.050
1024	12.21 ± 0.060
2048	14.67 ± 0.080
4096	17.30 ± 0.100

Table 2. Same as Table 1, but for time from 2^{13} to 2^{19}, averaged over 100 walks

t	$\bar{d}$
8192	22.6 ± 1.8
16384	27.0 ± 2.4
32768	29.1 ± 2.6
65536	32.3 ± 2.9
131072	35.3 ± 3.2
262144	39.2 ± 3.3
524288	46.4 ± 4.6

probability will be $1-\pi(n)$. We will use discrete time (for all our simulations). We will denote with brackets $\langle \cdot \rangle$ the average over the walks (in a *given* configuration of transition probabilities), and with a long bar $\overline{\cdot}$ the average over the probability distribution $\varrho(\pi)$.

At this point it is useful to underscore that Sinai shows that, with probability one, $x(t) \sim \log^2 t$ in the asymptotic regime. This logarithmic behaviour is essentially produced by the presence of trapping configurations (see Fig. 2): the traveller spends most of his time travelling through barriers. On the time scales we are able to analyze numerically (t of the order of $10^4 \div 10^6$) $\langle d(t) \rangle$, where $d(t)$ is the distance covered in time t by the traveller, will be strongly influenced by the particular set of transition probabilities which has been chosen: for example in our model at a time of order 10^9 (that we cannot reach) the traveller will be at an average distance of, let us say, 100 steps from his starting point. It is clear that the details of each randomly chosen set of probabilities will influence strongly $\langle d(t=10^9) \rangle$. The question that has to be answered now is: if for a large but finite time T we observe $\overline{d(T)}$ instead of $\langle d(T) \rangle$, will we find the same kind of behavior? It is not trivial a priori that an average over $\varrho(\pi)$ will not change the behaviour of the considered expectation values (see [12] for a detailed discussion of this phenomenon). Let us

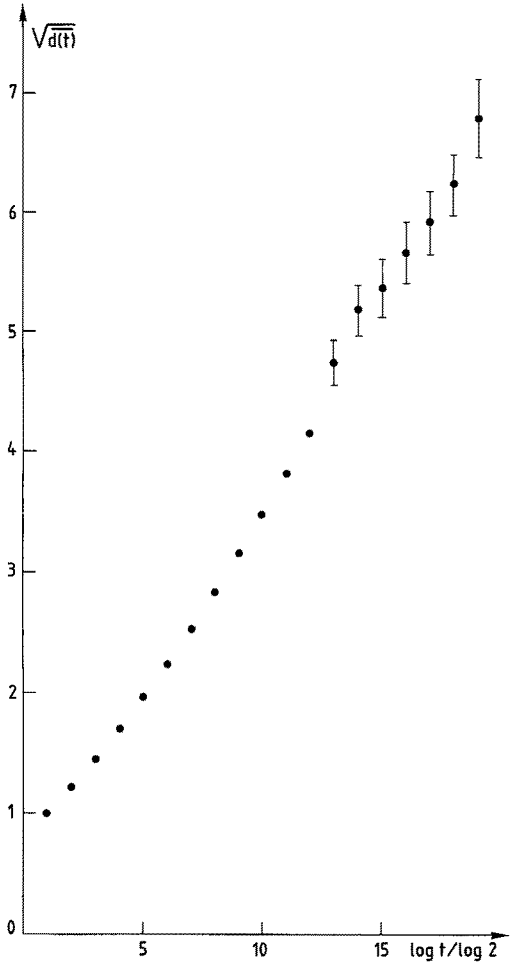

Fig. 3. Square root of the average distance (see Tables 1 and 2) vs. $\ln t/\ln 2$. The error is defined to be $\Delta((\bar{d})^{1/2})=\Delta(\bar{d})/2(\bar{d})^{1/2}$. For the high statistics data ($\ln t/\ln 2$ ranging from 1 to 12) the error is contained in the drawing of the points

consider an example: we assume that for any site n the right transition probability $\pi(n)$ can take the values ε or $1-\varepsilon$, with probability 1/2. A realization with all the $\pi(n)$ equal to $1-\varepsilon$, up to the time T, will have a probability

$$\exp\{-T\ln 2\}, \tag{7}$$

and for this particular realization

$$\langle d(T)\rangle \sim T. \tag{8}$$

This "exceptional" configuration will than add to the leading behaviour $(\ln T)^2$ a term Te^{-T}, which becomes irrelevant also for $T\simeq 0$ (10). It should be noticed that this conclusion does not hold, for example, for $\langle e^{d(t)}\rangle$. More precisely, we always measured expectation values integrated over the probability distribution: for

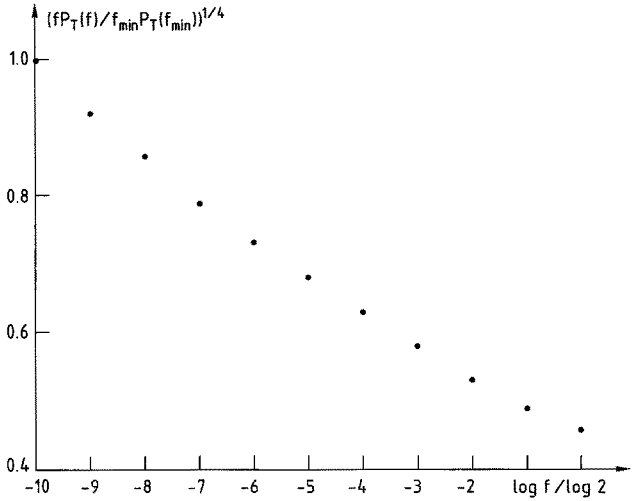

Fig. 4. $(fP_T(f)/f_{\min}P_T(f_{\min}))^{1/4}$ vs. $\ln f/\ln 2$

every walk contributing to our expectation values a different random set of transition probabilities was used. This is, moreover, the numerical approach adequate to the physical picture we have in mind: we are thinking about random processes evolving in random environments, where the macroscopic effect is given by the exploration of different realizations of the transition probabilities.

For the simulations we always used a chain long enough so that the traveller never hits the boundary. For that sake only modest memory requirements are needed in $d=1$. We analyzed $3\cdot 10^4$ walks of 2^{12} steps, and 10^2 walks of 2^{19} steps (see Tables 1 and 2). The asymptotic regime seems to be reached after $0\ (2^7 \div 2^9)$ time steps. In Fig. 3 we plot $(\overline{d(t)})^{1/2}$ versus $\ln t/\ln 2$. The behaviour is clearly linear.

We also computed

$$P_T(f)=\frac{1}{T}\left|\int_0^T e^{ift}x(t)dt\right|^2$$

for all the frequencies present in the walk of $T=2^{12}$ steps, and, for the same frequencies, for the $T=2^{19}$ steps walks. In Fig. 4 we plot the quantity $(fP_{T=2^{19}}(f))^{1/4}$ versus $\ln f$. Its behaviour is compatible with the linear behaviour guessed in Eq. (3). We should remark that observing numerically logarithmic corrections is a very delicate matter (and the same consideration holds for true experiments). First a log correction is hard to distinguish from a small power. Secondly we know that the $(\log f)^4 f^{-1}$ law will set in only in the asymptotic region. A contribution of small power corrections cannot be excluded: if we assumed $P_{T=2^{19}}(f)\sim f^{-(1+A)}$ we can just bound A by $A_{\mathrm{Max}}\sim 0.4$. A power correction for finite T is to be expected: we will have $A=A(T)$, and

$$\lim_{T\to\infty} A(T)=0\,. \tag{9}$$

Table 3. **a** $\overline{d(t)}$ vs. $\ln t/\ln 2$ for the two dimensional $k=1$ model. 500 walks are analyzed. **b** As in **a**, but $k=2$ and 2000 walks. **c** As in **a**, but $k=7$, and 3500 walks of 4096 time steps and 1000 walks of 8192 time steps. **d** As in **a**, but $k=20$ and 500 walks

a		**b**		**c**		**d**	
$\ln t/\ln 2$	$\overline{d(t)}$	$\ln t/\ln 2$	$\overline{d(t)}$	$\ln t/\ln 2$	$\overline{d(t)}$	$\ln t/\ln 2$	$\overline{d(t)}$
1	1.17 ± 0.03	1	1.22 ± 0.01	1	1.20 ± 0.01	1	1.16 ± 0.03
2	1.73 ± 0.04	2	1.73 ± 0.02	2	1.68 ± 0.02	2	1.61 ± 0.05
3	2.51 ± 0.06	3	2.47 ± 0.03	3	2.24 ± 0.03	3	2.12 ± 0.07
4	3.56 ± 0.08	4	3.47 ± 0.03	4	2.95 ± 0.03	4	2.49 ± 0.09
5	5.21 ± 0.11	5	4.85 ± 0.05	5	3.77 ± 0.03	5	2.83 ± 0.10
6	7.25 ± 0.16	6	6.79 ± 0.08	6	4.74 ± 0.05	6	3.26 ± 0.11
7	10.0 ± 0.20	7	9.45 ± 0.11	7	5.98 ± 0.06	7	3.48 ± 0.12
8	14.6 ± 0.30	8	13.45 ± 0.15	8	7.52 ± 0.08	8	3.76 ± 0.13
9	20.7 ± 0.50	9	18.65 ± 0.21	9	9.34 ± 0.09	9	4.03 ± 0.14
10	29.0 ± 0.70	10	26.40 ± 0.3	10	11.63 ± 0.12	10	4.29 ± 0.15
11	40.5 ± 0.90	11	36.30 ± 0.5	11	14.43 ± 0.15	11	4.67 ± 0.16
12	57.4 ± 1.40	12	51.40 ± 0.6	12	18.13 ± 0.17	12	5.12 ± 0.17
		13	73.40 ± 0.9	13	22.6 ± 0.5		

5. The Two Dimensional Case

The straightforward extension to $d=2$ of the model we analyzed in the previous section can be discussed in this way: to each site we assign $2d=4$ numbers Q_μ ($\mu=1,2,3,4$), uniformly distributed in $(0,1)$, and define the transition probability to each of the $2d$ neighbouring sites by

$$\pi_i = Q_i Z^{-1}, \tag{10}$$

where

$$Z = \sum_{i=1}^{2d} Q_i. \tag{11}$$

In the class of models we studied

$$\pi_i(k) = Q_i^k Z(k)^{-1}, \tag{12}$$

where

$$Z(k) = \sum_{i=1}^{2d} Q_i^k, \tag{13}$$

and k is an integer power ranging between 0 and $+\infty$. When $k=0$ it reduces to the normal random walk. In the limit $k \to \infty$ we get a deterministic model where the moving point is localized on a closed loop: for every site we will get that for a random $\mu=\tilde{\mu}$

$$\pi_{\tilde{\mu}} = 1, \tag{14}$$

and for all the other μ

$$\pi_{\mu \neq \tilde{\mu}} = 0. \tag{15}$$

Table 4. $\tilde{x}(t) = \dfrac{\sqrt{4096}}{d(t=4096)} \cdot \overline{d(t)}$ versus $\sqrt{t}$

$\sqrt{t}$	$\tilde{x}(t)$, $K=1$	$\tilde{x}(t)$, $K=2$
4.00	3.95	4.32
5.66	5.78	6.04
8.00	8.08	8.45
11.31	11.15	11.77
16.00	16.28	16.75
22.63	23.08	23.22
32.00	32.33	32.87
45.25	45.16	45.20
64.00	64.00	64.00
90.51	–	91.39

Then as soon as the traveller comes back to a site he already visited he starts repeating indefinitely the same path. For $k=1$ we recover the naive extension we just described.

We analyzed this model for several values of k: we produced 500 walks of 4096 time steps for $k=1$, 2000 walks of 8192 time steps for $k=2$, 3500 walks of 4096 time steps and 1000 of 8192 for $k=7$ and 500 walks of 4096 time steps for $k=20$. In Tables 3a–d we list $\overline{d(t)}$ for these four cases.

For the $d=2$ case we needed quite a big lattice to keep the moving point away from the boundary: for smaller k a bigger lattice is of course needed. For the cases of $k=7$ and $k=20$ we needed a 160×160 sites lattice, for $k=2$ a 400×400 lattice, while for $k=1$ we used dynamical allocation of the memory (when the traveller was reaching far away regions new parts of lattice were created in this given direction).

For $k=1$ and $k=2$ our results clearly indicate

$$\overline{d(t)} \propto t^{\alpha}, \tag{16}$$

$\alpha \simeq 0.5$, for the time regions we explored. In Table 4 we give the quantity $(\tilde{t}^{1/2}/\overline{d(\tilde{t})}) \cdot \overline{d(t)}$, where $\tilde{t}=4096$. In the case $k=1$ the power seems to be exactly a square root, while for $k=2$ a slightly lower power seems to be preferable.

Another possible way of analyzing the numerical results consists in defining the parameter

$$\zeta(2^{\tau}) = \frac{\overline{d(2^{\tau+1})} - \overline{d(2^{\tau})}}{\overline{d(2^{\tau})} - \overline{d(2^{\tau-1})}}. \tag{17}$$

If $d(t) \sim \alpha + \beta(\ln t^2)$ we get $\zeta(2^{\tau}) \sim \dfrac{2\tau+1}{2\tau-1}$, while if $d(t) \sim A + Bt^{\alpha}$ we get $\zeta(2^{\tau}) = 2^{\alpha}$ $=$ constant. This gives us

$$\alpha(k=1) = 0.50 \pm 0.13, \tag{18}$$

$$\alpha(k=2) = 0.50 \pm 0.08. \tag{19}$$

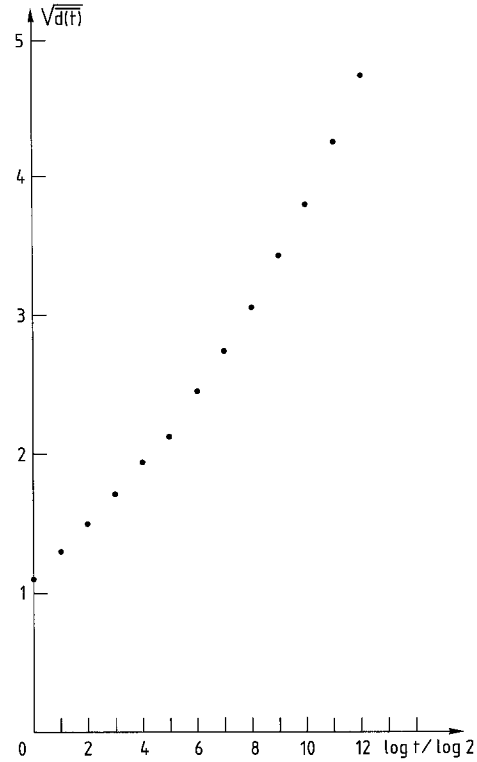

Fig. 5. $\sqrt{d}(t)$ vs. $\ln t/\ln 2$ for the $K=7$ model

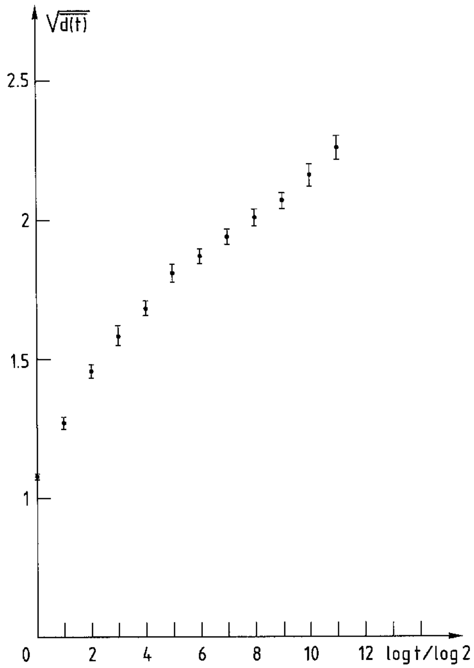

Fig. 6. As in Fig. 5, but $K=20$

For $k=7$ and $k=20$ we plot in Figs. 5 and 6 $\overline{(d(t))}^{1/2}$ versus $\ln t/\ln 2$. The $k=7$ case does not seem to show a linear behaviour of $\overline{(d(t))}^{1/2}$ in log scale. The ζ seems to be constant enough to suggest a power behaviour: under this assumption we get

$$\alpha(k=7)=0.30 \pm 0.09\,. \tag{20}$$

but in this case we have again to face the outstanding problem of distinguishing between a log behaviour and a small power one. The $k=20$ walk has, in the precision of our errors bars, a behaviour completely compatible with $(\log t)^2$.

All these numerical computations ($d=1$ and 2) required the equivalent of ~ 5 CPU h of CDC 7600.

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