Repellers for real analytic maps

DAVID RUELLE

Institut des Hautes Études Scientifiques, 35, Route de Chartres,
91440 Bures-sur-Yvette, France

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Abstract. The purpose of this note is to prove a conjecture of D. Sullivan† that
when the Julia set $J$ of a rational function $f$ is hyperbolic, the Hausdorff dimension
of $J$ depends real analytically on $f$. We shall obtain this as corollary of a general
result on repellers of real analytic maps (see corollary 5).

Let $M$ be a real analytic manifold of finite dimension $N$, $J$ a compact subset of
$M$, and $V$ an open neighbourhood of $J$ in $M$. We say that $J$ is a (mixing) repeller
for the real analytic map $f : V \to M$ if the following conditions are satisfied

(a) there exist $C > 0, \alpha > 1$ such that

$$\| (T_n f^n) u \| \geq C \alpha^n \| u \|$$

(1)

for all $u \in J$, $u \in T_n M$, $n \geq 1$ (and some Riemann metric on $TM$),

(b) $J = \{ x \in V : f^n x \in V$ for all $n > 0 \}$,

(c) $f$ is topologically mixing on $J$, i.e. for every non-empty open set $O$ intersecting

$J$ there is an $n > 0$ such that $f^n O \supset J$.

From (b) and (c) it follows that $f^2 J = J$. Our results would extend easily to the
case where $J$ is topologically + transitive instead of topologically mixing (see [12]).

1. PROPOSITION. Let $J$ be a mixing repeller for the real analytic map $f : V \to M$, and

let $\phi : V \to \mathbb{R}$ be a real analytic function. Then the series

$$\zeta(u) = \exp \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{x \in \text{Fix} f^n} \exp \sum_{k=0}^{n-1} \phi(f^k x)$$

has non-vanishing convergence radius and extends to a meromorphic function of $u$, again noted $\zeta(u)$. This function has a simple pole at $\exp P(\phi) > 0$, and every other
zero or pole of $\zeta$ has modulus $> \exp P(\phi)$. The function $\phi \mapsto P(\phi)$ is convex. There
is a unique Radon measure $\rho$ on $J$ such that

$$P(\phi + \psi) - P(\phi) > \rho(\psi)$$

(2)

for all $\psi$, and $\rho$ is an $f$-invariant probability measure (Gibbs measure).

To see this, one observes that expanding maps have Markov partitions.† Markov
partitions permit a study of the periodic points of $f$. Assuming only that $\phi$ is Hölder

† Formulated at the conference on dynamical systems in Rio de Janeiro, 1981, see [15].

‡ Markov partitions have been introduced by Sinai [13] for Anosov diffeomorphisms. Their existence
for expanding maps is implicit in Bowen [1]. For an explicit discussion see Ruelle [12]. One may
choose an 'adapted' metric on $M$ such that $C-1$ in (1). Characterizations of expanding maps as
needed for the existence of Markov partitions are analysed in [8].
continuous one shows, by methods of statistical mechanics, that \( \zeta \) extends to a circle of radius \( > \exp P \) in which it has no zero and only a simple pole at \( \exp P \). One obtains then \( \rho \) satisfying (2) for all Hölder continuous functions \( \phi, \psi : \mathcal{F} \to \mathbb{R} \).

The real analyticity of \( f \) and \( \phi \) is needed to prove the meromorphy of \( \zeta \) in \( \mathbb{C} \). Using the Markov partition and complex extensions of \( f \) and \( \phi \), one expresses \( \zeta \) in the terms of Fredholm determinants in the form

\[
\zeta(u) = \prod_{k=0}^{N} \left[ \det \left( 1 - u \mathcal{L}_{\lambda}^{k} \right) \right]^{-1} \cdot \chi^{-1}
\]

where the \( \mathcal{L}_{\lambda} \) have continuous kernels on compact sets, depending analytically on \( f \) and \( \phi \) (see Ruelle [11, theorem 1], the application considered here is much the same as that of theorem 2 of [11]; the Fredholm theory used is based on Grothendieck [6]). In particular, if \( f \) and \( \phi \) depend analytically on parameters, then \( \zeta \) will depend analytically on the same parameters.† We now formulate this result more precisely.

2. PROPOSITION. With the notation of proposition 1, let \( f \) and \( \phi \) (now noted \( f_{\lambda}, \phi_{\lambda} \)) depend on a parameter \( \lambda \in U \subseteq \mathbb{R}^{n} \) such that \( (\lambda, x) \mapsto f_{\lambda}, \phi(x) \) are analytic, and \( f_{\lambda} \) has a repeller \( J_{\lambda} \) depending continuously on \( \lambda \). We may take \( U \) open by \( \Omega \) stability. Under these conditions \( \zeta = d_{1}/d_{2} \) where \( d_{1}, d_{2} \) are entire holomorphic in \( u \) and real analytic in \( \lambda \in U \).

3. COROLLARY. The function \( \lambda \mapsto P \) is real analytic and \( \lambda \mapsto \rho \) is real analytic in the sense that \( \lambda \mapsto \rho(\psi) \) is analytic for real analytic \( \psi : V \to \mathbb{R} \). If \( \phi_{\lambda} < 0 \) on \( J_{\lambda} \), the function \( \lambda \mapsto \psi \) is analytic, where \( \psi \) is defined by \( P(\psi_{\lambda}) = 0 \).

The analyticity of \( \lambda \mapsto e^{\psi} \) (and thus \( \lambda \mapsto P \)) results from the implicit function theorem applied to the function \( (\lambda, u) \mapsto 1/\zeta \). We consider now two applications of the analyticity of \( \lambda \mapsto P \), where \( \lambda \) is replaced by \( (t, \lambda) \), \( t \in \mathbb{R} \).

If \( \psi : V \to \mathbb{R} \) is real analytic, we see that \( (t, \lambda) \mapsto P(\phi_{\lambda} + t\psi) \) is real analytic, and therefore also

\[
\lambda \mapsto \left. \frac{d}{dt} P(\phi_{\lambda} + t\psi) \right|_{t=0} = \rho(\phi).
\]

This proves the real analyticity of \( \lambda \mapsto \rho \) as announced.

Similarly \( (t, \lambda) \mapsto P(\phi_{\lambda}) \) is real analytic. We also have the variational principle††

\[
P(\phi_{\lambda}) = \max \{ h(\sigma) + \int \sigma(\phi_{\lambda}) : \sigma \text{ invariant probability measure} \}
\]

where \( h \) is the measure-theoretic entropy. Therefore if \( \phi_{\lambda} < 0 \) on \( J_{\lambda} \), the function \( t \mapsto P(\phi_{\lambda}) \) has derivative \( < 0 \) and goes from positive to negative values.‡‡ Its unique zero is a real analytic function of \( \lambda \) by the implicit function theorem.

† See Ruelle [10] or [12], Mayer [8]. For related \( \zeta \)-functions see Chen & Manning [4].

‡ One could also deduce this from the fact that the periodic points of \( f \) depend analytically on the parameters, and that one has control over their positions when the parameters become complex (see lemma 1 in [11]). Therefore the coefficients of \( \zeta \) depend holomorphically on the parameters, and the same is true of \( \zeta \).

†† In its general form, this is due to Walters [16], see also Misiurewicz [9], Bowen [1], Ruelle [12].

‡‡ The existence of the Markov partition gives an explicit upper bound on \( h \).
4. PROPOSITION. Let $J$ be a repeller for a map $f: V \rightarrow M$. We assume that $f$ is conformal with respect to some continuous Riemann metric, and of class $C^{1+\varepsilon}$ ($\varepsilon > 0$). If we write

$$\phi(x) = -\log \|Tf(x)\|$$

the Hausdorff dimension $t$ of $J$ is defined by Bowen’s formula $P(t\phi) = 0$. Furthermore the $t$-Hausdorff measure $v$ on $J$ is equivalent to the Gibbs measure $\rho$ corresponding to $t\phi$.

In the formulation of this proposition we have allowed $f$ to be $C^{1+\varepsilon}$ rather than real analytic as in our earlier definitions. Apart from this, the proposition is due to Bowen [2] (who worked with groups of fractional linear transformations of the Riemann sphere). For the convenience of the reader, appendix 1 reproduces a proof of proposition 4. See Sullivan [15] for an analogous determination of $t$.

Actually the results of Bowen and Sullivan allow the map $f$ to be discontinuous, as we shall indicate below.

5. COROLLARY. Let $J_\lambda$ be a repeller for a real analytic conformal map $f_\lambda$, depending real analytically on $\lambda$. (Thus $(\lambda, x) \rightarrow f_\lambda(x)$ is an analytic $U \times V \rightarrow M$ and the linear maps $Df_\lambda$ are of the form: scalar $\times$ isometry.) Then the Hausdorff dimension of $J_\lambda$ is a real analytic function of $\lambda$.

This follows from proposition 4 and corollary 3.

6. COROLLARY. If the Julia set $J$ of a rational function $f$ is hyperbolic, the Hausdorff dimension of $J$ depends real analytically on $f$.

We let $f = P/Q$ where $P, Q$ are polynomials of fixed degrees, so that $f$ can be parametrized by a family of coefficients varying over $\mathbb{R}^n$. Hyperbolicity means that condition (a) in the definition of a repeller is satisfied. Conditions (b) and (c) in the definition of a repeller are satisfied for general Julia sets (see Brolin [3, theorems 4.2 and 4.3]). It follows therefore that the Hausdorff dimension of $J$ depends analytically on $f$.

The polynomial map $z \mapsto z^q$, with $q \geq 2$, has the unit circle

$$\{z \in \mathbb{C} : |z| = 1\}$$

as hyperbolic Julia set. Corollary 6 applies therefore to the maps

$$z \mapsto z^q + \lambda$$

for small complex $\lambda$. A formal calculation (see appendix 2) gives

$$t = 1 + \frac{|\lambda|^2}{4 \log q} + \text{higher order terms in } |\lambda|.$$ 

The case $q = 2$ has been particularly studied (see Brolin [3] and references quoted there, and Mandelbrot [7] which also contains beautiful pictures of the corresponding $J_\lambda$). A computer calculation of $t$ as a function of $\lambda$ (real) for $z \mapsto z^2 + \lambda$ was performed by L. Garnett (unpublished) and prompted Sullivan’s conjecture that
λ → t is analytic.† Sullivan [15] proved that t > 1 when λ ≠ 0 (and |λ| is sufficiently small).

7. Generalization
As mentioned above, Bowen originally established the formula \( P(t \cdot \phi) = 0 \) for the Hausdorff dimension of a repeller \( J \) in the context of groups of fractional linear transformations of the Riemann sphere. (The Hausdorff dimension results were extended by Sullivan to more general groups of conformal maps [14].) In Bowen’s study, \( J \) is the quasi-circle associated with a quasi-Fuchsian group \( G \), and there is a Markov partition \( \{ S_\alpha \} \) of \( J \) such that \( f \) is a different fractional linear transformation on each \( S_\alpha \), and thus discontinuous. Arguments similar to those given above show in this case that the Hausdorff dimension of the quasi-circle depends real analytically on \( G \) or, equivalently, on pairs of points in Teichmüller space.

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Appendix 1: Proof of proposition 4
The pressure (function \( P \)) and Gibbs state \( \rho \) occurring in proposition 4 translate to similar concepts for the symbolic dynamical system associated with a Markov partition of \( J \). A Markov partition \( \{ S_\alpha \} \) is a finite collection of closed non-empty subsets of \( J \) such that \( \bigcup S_\alpha = J \) and \( \text{int} S_\alpha \) is dense in \( S_\alpha \) (int denotes the interior in \( J \)). Furthermore,

(i) \( \text{int} S_\alpha \cap \text{int} S_\beta = \emptyset \) if \( \alpha \neq \beta \),

(ii) each \( S_\alpha \) is a union of sets \( \tilde{S}_\alpha \).

For a study of symbolic dynamics, the reader must be referred to Bowen [2] or Ruelle [12].

Let \( \{ S_\alpha \} \) be a Markov partition of \( J \) into small subsets. We call \( K \) the maximum number of \( \tilde{S}_\alpha \) which intersect any \( S_\beta \):

\[
K = \max_{\alpha} \text{card} \{ \tilde{S}_\beta : S_\alpha \cap \tilde{S}_\beta \neq \emptyset \}.
\]

Let \( \tilde{S}_\alpha \) be a small open neighbourhood of \( S_\alpha \) in \( V \), for each \( \alpha \), such that

\[
\tilde{S}_\alpha \cap \tilde{S}_\beta = \emptyset \quad \text{whenever} \quad S_\alpha \cap S_\beta = \emptyset.
\]

We assume that for all \( \alpha \) the diameter of \( \tilde{S}_\alpha \) is <\( \Delta \), and that \( \tilde{S}_\alpha \) contains the \( \delta \)-neighbourhood of \( S_\alpha \) (0 <\( \delta < \Delta \)). If \( \xi_0, \xi_1, \ldots, \xi_n \) is an admissible sequence of elements of the Markov partition, i.e., \( f_j^{\xi_{j-1}} \supset \xi_j \) for \( j = 1, \ldots, n \), we define

\[
E(\xi_0, \ldots, \xi_n) = \prod_{j=0}^{n} f_j^{\xi_j}
\]

\[
\tilde{E}(\xi_0, \ldots, \xi_n) = \prod_{j=0}^{n} f_j^{\xi_j}.
\]

† The results of the calculation suggest \( r = 1 + C|\lambda|^2 \) and are compatible with \( r = 1 + |\lambda|^2/(4 \log 2) \).
The sets $\tilde{E}(\xi_0, \ldots, \xi_n)$ which intersect a given $\tilde{E}(\xi_0^*, \ldots, \xi_n^*)$ are determined successively as follows:

(a) choose $\xi_*, \xi_*$ such that $\xi_0 \cap \xi_*^* = \emptyset$,

(b) $\xi_*$ is uniquely determined for $k = n - 1, \ldots, 1, 0$ by

$$\left[ \bigcap_{i=0}^{n} f^{-i+k-1}\xi_i \right] \cap \left[ \bigcap_{i=k}^{n} f^{-i+k}\xi_i^* \right] \neq \emptyset.$$

In particular the sets $\tilde{E}(\xi_0, \ldots, \xi_n)$ which intersect $\tilde{E}(\xi_0^*, \ldots, \xi_n^*)$ correspond precisely to the sets $E(\xi_0, \ldots, \xi_n)$ which intersect $E(\xi_0^*, \ldots, \xi_n^*)$, and there are at most $K$ of those. We also see that, if $\Delta$ has been taken sufficiently small, there are $\beta \in (0, 1)$ and $G > 0$ (and $G$ independent of $n$, $\xi_0^*, \ldots, \xi_n^*$) such that

$$\text{dist} (\xi, \xi^*) \leq G\beta^n \quad \text{if} \quad \xi \in \tilde{E}(\xi_0, \ldots, \xi_n) \text{ and } \xi^* \in \tilde{E}(\xi_0^*, \ldots, \xi_n^*).$$

(A.1)

(Use part (a) of the definition of a repeller. In particular, $\text{diam} \tilde{E}(\xi_0^*, \ldots, \xi_n^*) \leq G\beta^n$.

Let

$$F_\xi : \tilde{E}(\xi_0, \ldots, \xi_n) \to \tilde{E}(\xi_0, \ldots, \xi_n)$$

be the inverse of the restriction of $f^*$ to $\tilde{E}(\xi_0, \ldots, \xi_n)$. If $x \in \tilde{E}$ we have, since $f$ is conformal,

$$\log \|F_{\xi_0, \ldots, \xi_n}(x)\| = \sum_{k=0}^{n-1} \log \|(f^{-1})'(F_{\xi_{k+1}, \ldots, \xi_n}(x))\| = -\sum_{k=0}^{n-1} \log \|f(F_{\xi_{k+1}, \ldots, \xi_n}(x))\|$$

$$\quad = \sum_{k=0}^{n-1} \phi(F_{\xi_{k+1}, \ldots, \xi_n}(x))$$

(A.2)

where we have denoted the tangent map by a dash. If $\tilde{E}(\xi_0, \ldots, \xi_n) \cap \tilde{E}(\xi_0^*, \ldots, \xi_n^*) \neq \emptyset$ and $x \in \tilde{E}*, x* \in \tilde{E}^*$ we have thus, using (A.1),

$$|\log \|F_{\xi_0, \ldots, \xi_n}(x)\| - \log \|F_{\xi_0^*, \ldots, \xi_n^*}(x^*)\| | \leq C_* \sum_{k=0}^{n-1} (G\beta^n)^{k} < C_* G^n \frac{1}{1-\beta^n} = D$$

(A.3)

where $C_*$ is the $\epsilon$-Hölder norm of $\phi$. In particular, if $x^* \in \xi_*$, the ball of radius $e^{-D\delta}\|F_{\xi_0^*, \ldots, \xi_n^*}(x^*)\|$ centred at $F_{\xi_0^*, \ldots, \xi_n^*}(x^*)$ is entirely contained in $\tilde{E}(\xi_0^*, \ldots, \xi_n^*)$.†

† We assume here for simplicity that $\phi < 0.$
The Gibbs measure $\rho$ corresponding to $t\phi$ is determined (since $P(t\phi) = 0$) by the fact that there is a constant $\gamma$ such that
\[(\log \rho(E(\xi_0, \ldots, \xi_n)) - \sum_{k=0}^{n-1} t\phi(F_{\xi_k}, \ldots, \xi_{k+1})) < \gamma \tag{A.4}\]
where $\gamma$ is independent of $n, E(\xi_0, \ldots, \xi_n)$, and $x \in \xi_n$. Using (A.2) and (A.4) we have, for each $E(\xi_0, \ldots, \xi_n)$, the following estimate of the $i$-Hausdorff measure $\nu$:
\[
\nu(E(\xi_0, \ldots, \xi_n)) \leq \lim_{p \to \infty} \frac{1}{2} \text{diam}(\bar{E}(\xi_0, \ldots, \xi_{n+p}))^{1/p} \\
\leq \lim_{p \to \infty} \frac{1}{2} \text{diam}(\bar{E}(\xi_0, \ldots, \xi_{n+p}))^{1/p} \\
= (2\Delta e^{D})^p \lim_{p \to \infty} \frac{1}{2} \text{diam}(\bar{E}(\xi_0, \ldots, \xi_{n+p}))^{1/p} \\
\leq (2\Delta e^{D})^p \epsilon \rho(E(\xi_0, \ldots, \xi_n)).
\]
This shows that $\nu$ is absolutely continuous with respect to $\rho$.

On the other hand $\nu(E(\xi_0, \ldots, \xi_n))$ is the infimum of
\[
\sum_{j=1}^{\infty} (\text{diam } U_j)^j
\]
for an open cover $\{U_j\}$ of $E(\xi_0, \ldots, \xi_n)$ when $\text{diam } U_j \to 0$. For each $j$ take
\[y_j \in E(\xi_0, \ldots, \xi_n) \cap U_j\]
and notice that $E(\xi_0, \ldots, \xi_n)$ is covered by the balls
\[B_{y_j}(\text{diam } U_j).
\]
For each $j$ let $n_j$ be the smallest integer such that if
\[y_j \in E(\xi^*_0, \ldots, \xi^*_{n_j})
\]
then
\[e^{-D} \epsilon \delta \|F_{\xi^*_0} \cdots (f_{n_j} y_j)\| \leq \text{diam } U_j. \tag{A.5}\]
(We may assume that $\text{diam } U_j$ is small, and therefore
\[n_j > n, \quad \xi^*_0 = \xi_0, \ldots, \xi^*_{n_j} = \xi_n,
\]
the further $\xi^*_k$ depend on $j$.) By assumption
\[e^{-D} \epsilon \delta \|F_{\xi^*_0} \cdots (f_{n_j} y_j)\| > \text{diam } U_j.
\]
Therefore, the set $E(\xi_0, \ldots, \xi_n)$ is covered by the $\bar{E}(\xi^*_0, \ldots, \xi^*_{n_j})$ and, using (A.5) and (A.2) we see that
\[
\sum_{j=1}^{\infty} (\text{diam } U_j)^j \geq e^{-D} \epsilon \delta \sum_{j=1}^{\infty} \exp \sum_{k=0}^{n_j-1} \phi(F_{\xi^*_0} \cdots (f_{n_j} y_j)) \\
\geq e^{-D} \epsilon \delta \sum_{j=1}^{\infty} \exp \sum_{k=0}^{n_j-1} \phi(F_{\xi^*_0} \cdots (f_{n_j} y_j)).
\]
\[\text{See Bowen [2] or Ruelle [6].}\]
where $E$ is an upper bound to $|\phi(x)|$. We recall that each $\tilde{E}(\xi_0^k, \ldots, \xi_n^k)$ intersects at most $K$ sets $E(\xi_0, \ldots, \xi_n)$. Redistributing the contribution of the index $j$ among those, and using (A.2) and (A.3) we find

$$\sum_{j=1}^{\infty} \delta_j \leq K^{-1} e^{-2D_1 - E_1} \delta_1 \sum_{j=1}^{\infty} \exp \left( \sum_{k=0}^{n_k-1} \phi(F_{\xi_0^k} \ldots, \xi_n^k, x_k) \right)$$

where the $E(\xi_0^k, \ldots, \xi_n^k)$ cover $E(\xi_0, \ldots, \xi_n)$. So, finally, using (A.4), we obtain

$$\nu(\tilde{E}(\xi_0^k, \ldots, \xi_n^k)) \leq K^{-1} e^{-2D_1 - E_1} \delta_1 e^{-\gamma} \rho(\tilde{E}(\xi_0, \ldots, \xi_n)).$$

This shows that $\rho$ is absolutely continuous with respect to $\nu$, completing the proof of the proposition. \hfill \Box

**Appendix 2: Hausdorff dimension of the Julia set $J$ of the map $f: z \mapsto z^q - p$.**

We shall formally show that the Hausdorff dimension of $J$ is

$$t = 1 + \frac{|p|^2}{4 \log q} + \text{terms of order} > 2 \text{ in } p.$$ 

For small $|p|$, $f$ has a fixed point $\alpha$ close to 1, so that

$$\alpha + p = \alpha^q \quad \text{and} \quad \alpha = 1 + \frac{p}{q-1} + \cdots.$$ 

Write $\gamma = \exp \frac{2i \pi}{q}$. With $e_1 = 0, 1, \ldots, q-1$ we define

$$\zeta(e_1, \ldots, e_n) = \gamma^{e_1}(p + \gamma^{e_1}(p + \gamma^{e_1} \cdots (p + \gamma^{e_1} \alpha)^{e_1})^{e_1} / q^{e_1} q^{e_1})$$

$$= \exp \left[ Q(e_1, \ldots, e_n) 2i \pi + r(e_1, \ldots, e_n) \right]$$

where

$$Q(e_1, \ldots, e_n) = \frac{e_1}{q} + \frac{e_{n-1}}{q} + \cdots + \frac{e_1}{q^n}$$

$$r(e_1, \ldots, e_n) = \frac{1}{q} r(e_1, \ldots, e_{n-1}) + \frac{1}{q} \log \left( 1 + p/\zeta(e_1, \ldots, e_n) \right)$$

$$\approx \frac{1}{q} r(e_1, \ldots, e_{n-1}) + \frac{1}{q} p/\zeta(e_1, \ldots, e_n)$$

$$= \frac{1}{q} r(e_1, \ldots, e_{n-1}) + \frac{1}{q} p \exp \left( -Q(e_1, \ldots, e_n) \cdot 2i \pi \right)$$

to first order in $p$. Therefore, if $u = \exp(-Q(e_1, \ldots, e_n) \cdot 2i \pi)$,

$$r(e_1, \ldots, e_n) = p \left[ \frac{1}{q} u + \frac{1}{q^2} u^q + \frac{1}{q^3} u^{q^2} + \cdots + \frac{1}{q^{n-1}} u^{q^{n-2}} + \frac{1}{q^n} \right]$$

$$= p \sum_{q=0}^{\infty} \frac{1}{q^k} u^{q^k}.$$ 

Writing

$$\phi(z) = -\log |f'(z)| = -\log q |z|^{q-1}$$
we have
\[ \phi(\zeta(e_1, \ldots, e_n)) = -\log q - \Re (q-1)r(e_1, \ldots, e_{n-1}), \]
hence
\[ \sum_{k=1}^{n} \phi(\zeta(e_1, \ldots, e_k)) = -n \log q - \Re (q-1) \sum_{k=1}^{n} r(e_1, \ldots, e_{k-1}). \]

We have, to first order in \( p \),
\[ \Re (q-1) \sum_{k=1}^{n} r(e_1, \ldots, e_{k-1}) = \Re p \Phi_n(u) \]
where
\[ \Phi_n(u) = \left(1 - \frac{1}{q}\right)u + \left(1 - \frac{1}{q^2}\right)u^q + \cdots + \left(1 - \frac{1}{q^n}\right)u^{q^{n-1}} + \frac{q-q^n}{q-1}. \]

To second order in \( p \) we have, using the induction formula,
\[ \sum_{e_1, \ldots, e_n} r(e_1, \ldots, e_n) = \frac{1}{q} \sum_{e_1, \ldots, e_{n-1}} \left[ r(e_1, \ldots, e_{n-1})(1 - pu) + pu - \frac{1}{2} p^2 u^2 \right] \]
\[ = \sum_{e_1, \ldots, e_{n-1}} r(e_1, \ldots, e_{n-1}) \]
so that, for large \( n \),
\[ \sum_{e_1, \ldots, e_{n=1}} r(e_1, \ldots, e_{n-1}) = O(q^n). \]

The Hausdorff dimension \( \tau = 1 + \beta \) of the Julia set \( J \) of \( z \mapsto z^q - p \) is determined by
\[ \sum_{e_1, \ldots, e_n} \exp(1 + \beta) \sum_{k=1}^{n} \phi(\zeta(e_1, \ldots, e_k)) = O(1) \]
for large \( n \) or, to second order in \( p \),
\[ O(1) \approx \sum_{e_1, \ldots, e_n} q^{-n(1+\beta)} \exp \left[ -\Re (q-1) \sum_{k=1}^{n} r(e_1, \ldots, e_{k-1}) \right] \]
\[ = \sum_{e_1, \ldots, e_n} q^{-n(1+\beta)} \left[ 1 - \Re (q-1) \sum_{k=1}^{n} r(e_1, \ldots, e_{k-1}) + \frac{1}{2} (\Re p \Phi_n(u))^2 \right] \]
\[ = q^{-n} + O(q^{-d})|p| + O(q^{-n})|p|^2 \]
\[ + q^{-n(1+\beta)} \left[ \frac{q^n}{2} |p|^2 \left( \left(1 - \frac{1}{q}\right)^2 + \left(1 - \frac{1}{q^2}\right)^2 + \cdots + \left(1 - \frac{1}{q^{n-1}}\right)^2 \right) \right] + |p|^2 o(n) . \]

We have used
\[ \sum_{e_1, \ldots, e_n} (\Re pu^{\sigma-1})(\Re pu^{\sigma-1}) = 0 \quad \text{if } 1 \leq r < s \leq n + 1, \]
\[ \sum_{e_1, \ldots, e_n} (\Re pu^{\sigma-1})^2 = \sum_{e_1, \ldots, e_n} \frac{1}{2} (|p|^2 + \Re p^2 u^{2\sigma-1}) \left[ \frac{q^n}{2} |p|^2 \left( \left(1 - \frac{1}{q}\right)^2 + \left(1 - \frac{1}{q^2}\right)^2 + \cdots + \left(1 - \frac{1}{q^{n-1}}\right)^2 \right) \right] \]
\[ \quad \text{if } r < n, \]
\[ \leq q^n |p|^2 \quad \text{if } r = n \text{ or } n + 1. \]
Thus, omitting negligible terms

$$O(1) = q^{-n} \left(1 + \frac{|p|^2}{4} n \right) = \exp \left(\frac{|p|^2}{4} - \beta \log q \right)$$

giving

$$\beta = \frac{|p|^2}{4 \log q} + \cdots, \text{ or } l = 1 + \frac{|p|^2}{4 \log q} + \cdots.$$ 

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