Interaction Graphs: Graphings

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Abstract

In two previous papers [Sei12b, Sei12a], we exposed a combinatorial approach to the program of Geometry of Interaction, a program initiated by Jean-Yves Girard [Gir89b]. The strength of our approach lies in the fact that we interpret proofs by simpler structures — graphs — than Girard's constructions, while generalizing the latter since they can be recovered as special cases of our setting. This third paper extends this approach by considering a generalization of graphs named *graphings*, which is in some way a *geometric realization* of a graph. This very general framework leads to a number of new models of multiplicative-additive linear logic which generalize Girard's geometry of interaction models and opens several new lines of research. As an example, we exhibit a family of such models which account for second-order quantification without suffering the same limitations as Girard's models.

1. Introduction

1.1. Context

Geometry of Interaction. This research program was introduced by Girard [Gir87, Gir89b] after his discovery of linear logic [Gir95a]. In a first approximation, it aims at defining a semantics of proofs that accounts for the dynamics of cut-elimination. Namely, the geometry of interaction models differ from usual (denotational) semantics in that the interpretation of a proof π and its normal form ρ are not equal, but one has a way of computing the interpretation of the normal form ρ from the interpretation of the proof π (illustrated in Figure 1). As a consequence, a geometry of interaction models not only proofs — programs — but also their normalization — their execution. This semantical counterpart to the cut-elimination procedure was called the *execution formula* by Girard in his first papers about geometry of interaction [Gir89a, Gir88, Gir95b], and it is a way of computing the solution to the so-called *feedback equation*. This equation turned out to have a more general solution [Gir06], which lead Girard to the definition of a geometry of interaction in the hyperfinite factor [Gir11].

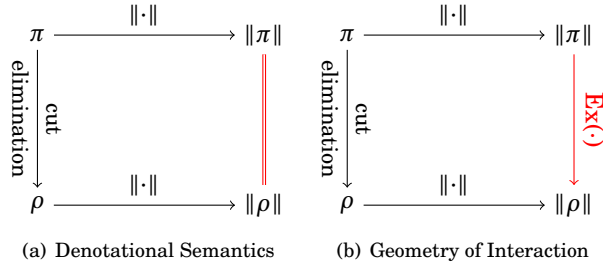(a) Denotational Semantics   (b) Geometry of Interaction

Figure 1: Denotational Semantics vs Geometry of Interaction

Geometry of Interaction, however, is not only about the interpretation of proofs and their dynamics, but also about reconstructing logic around this semantical counterpart to the cut-elimination procedure. This means that logic arises from the dynamics and interactions of proofs — programs —, as a syntactical description of the possible behaviors of proofs — programs. This aspect of the geometry of interaction program has been less studied than the proof interpretation part.

We must also point out that geometry of interaction has been successful in providing tools for the study of computational complexity. The fact that it models the execution of programs explains that it is well suited for the study of complexity classes in time [BP01, Lag09], as well as in space [AS13, AS12]. It was also used to explain [GAL92] Lamping's optimal reduction of lambda-calculus [Lam90].

Interaction Graphs. They were first introduced [Sei12b] to define a combinatorial approach to Girard's geometry of interaction in the hyperfinite factor [Gir11]. The main idea was that the execution formula — the counterpart of the cut-elimination procedure — can be computed as the set of alternating paths between graphs, and that the measurement of interaction defined by Girard using the Fuglede-Kadison determinant [FK52] can be computed as a measurement of a set of cycles.

The setting was then extended to deal with additive connectives [Sei12a], showing by the way that the constructions were a combinatorial approach not only to Girard's hyperfinite GoI construction but also to all the earlier constructions [Gir87, Gir89a, Gir88, Gir95b]. This result could be obtained by unveiling a single geometrical property, which we called the *trefoil property*, upon which all the constructions of geometry of interaction introduced by Girard are founded. This property, which can be understood as a sort of associativity, suggests that computation — as modeled by geometry of interaction — is closely related to algebraic topology.

This paper takes another direction though: based on ideas that appeared in the author's phd thesis [Sei12c], it extends the setting of graphs by considering a generalization of graphs named *graphings*, which is in some way a *geometric realization* of a graph. This very general framework leads to a number of new

models of multiplicative-additive linear logic which generalize Girard's geometry of interaction models and opens several new lines of research. As an example, we exhibit a family of such models which account for second-order quantification without suffering the same limitations as Girard's models.

1.2. Outline

We introduce in this paper a family of models which generalizes and axiomatizes the notion a GoI model. This systematic approach is obtained by extending previous work [Sei12b, Sei12a] in which the objects under study were directed weighted graphs. We will consider here a generalization of such graphs named *graphings*, and show how to define a very rich hierarchy of models parametrized by two monoids: the *weight monoid* and the so-called *microcosm*.

A weight monoid is nothing more than a monoid which elements will be used to give weights to edges of the graphs. A microcosm, on the other hand, is a monoid of measurable maps, i.e. it is a subset of $\mathscr{M}(X)$, the set of non-singular Borel-preserving transformations from a measured space $(X, \mathscr{B}, \mu)$ to itself, which is closed under composition and contains the identity transformation.

Once chosen a weight monoid Ω and a microcosm $\mathfrak{m}$, we define the notion of Ω-*weighted graphing in* $\mathfrak{m}$. A Ω-weighted graphing is simply a directed graph F whose edges are weighted by elements in Ω, whose vertices are measurable subsets of the measurable space $(X, \mathscr{B})$, and whose edges are *realized* by elements of $\mathfrak{m}$, i.e. for each edge e there exists an element ϕ_e in $\mathfrak{m}$ such that $\phi_e(s(e)) = t(e)$, where s, t denote the source and target maps. For convenience, we use in the following a different — thus less "graph-like", but equivalent definition of graphings.

The main result of this paper is then expressed as follows.

Theorem 1. *Let Ω be a monoid and $\mathfrak{m}$ a microcosm. There exists a family of GoI model of multiplicative-additive linear logic (MALL) whose objects are Ω-weighted graphings in $\mathfrak{m}$.*

The proof of this theorem is decomposed in three steps. Indeed, using previous results [Sei12a], we only need, for any two graphings F, G, to define:

- the *execution $F :_{\mathfrak{m}} G$* between F and G, which is associative;
- the measurement $[\![F, G]\!]_m$ of the interaction between F and G;

and show that these two notions satisfy the so-called *trefoil property* which ensures that the constructions described in our previous paper define a satisfying model of MALL. This property is recalled in Section 2.

Section 3 is therefore concerned with the definition of the execution and the proof that it is indeed associative (Theorem 22). We then define in Section 4 abstractly a notion of measurement and prove, under certain conditions, that the trefoil property holds (Theorem 35). These conditions being quite involved, we then show in Section 5 that in most reasonable cases, one can define whole families of measurement satisfying them (Theorem 48).

We then give some examples of applications of this result, showing how all of Girard's frameworks, either based on operator algebras [Gir89a, Gir88, Gir11]

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or on the more syntactical "unification algebra" [Gir95b, Gir13a, Gir13b], can be understood as special cases of our construction.

Section 6, finally, studies a simple example of GoI models obtained from our construction. We show how this framework allows to interpret multiplicative-linear logic with second-order quantification. This result actually solves an important issue, due to locativity, that arose in Girard's hyperfinite geometry of interaction [Gir11].

1.3. Motivations and Perspectives

This result is very technical, and we believe that the work needed to obtain it should be motivated. The importance of this result lies in its great generality and the new lines of research it opens. We believe that the notion of graphing is an excellent mathematical abstraction of the notion of program. First, any pure lambda-term can be represented as a graphing since Girard's GoI model [Gir88] is a particular case of our constructions. But one can represent a lot more. In following work, we will show how to represent quantum computation with the same objects. Results of Mazza [Maz05] and de Falco [dF08] leads us to believe that concurrent models of computations, such as pi-calculus can be represented as graphings. In the long-term, we would also like to obtain such a representation for more mathematical frameworks, such as cellular automata.

We therefore consider that this framework offers a perfect mathematical abstraction of the notion of program, one that is machine-independent, which captures all the different computational paradigms, and which allows for a fine control on computational principles. Indeed, while linear logic introduced a syntax in which one could talk about resource-usage, the approach taken here goes a step further and introduces a framework where subtle distinctions on the computational principles allowed in the model. These distinctions are made by considering the hierarchies of weight monoid and microcosms.

This will lead to some interesting results in computational complexity on one hand, and to interesting models of quantum computation on the other. We now detail these motivating perspectives.

1.3.1. Fine-Grained Implicit Computational Complexity

Firstly, the results obtained in previous work with C. Aubert characterizing the classes **L** and **coNL** can be improved, extended and linked with logical constructions in this setting. The following summarizes a work in preparation [Sei14b].

In the previously described models of multiplicative-additive linear logic, one can define the type of binary lists $\mathbf{Nat_2}$ in a quite natural fashion (the representation of lists is thoroughly explained in previous papers on complexity [AS12, AS13]). Moreover, in a number of cases one can define exponential connectives in the model and therefore consider the type $!\mathbf{Nat_2} \multimap \mathbf{Bool}$. Elements of this type, when applied to a element $!N_n$ of $!\mathbf{Nat_2}$, yields one of the distinguished elements `true` or `false`. Such an element F thus *recognize* the language

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$\mathscr{L}(F) = \{n \in \mathbf{N} \mid F :: !N_n = \texttt{true}\}$. As a consequence, we can study the set of languages recognized by all elements in the type $!\mathbf{Nat_2} \multimap \mathbf{Bool}$. By modifying the microcosm, one then modifies the expressivity of the model.

The intuition is that a microcosm $\mathfrak{m}$ represents the set of computational principles available to write programs in the model. Considering a bigger microcosm $\mathfrak{n} \supsetneq \mathfrak{m}$ thus corresponds to extending the set of principles at disposal, consequently increasing expressivity. The set of languages characterized by the type $!\mathbf{Nat_2} \multimap \mathbf{Bool}$ becomes larger and larger as we consider extensions of the microcosms. We can then work on this remark, and use intuitions gained from earlier work [AS12, AS13]. This leads to a perfect correspondance between a hierarchy of monoids on the measured space $\mathbf{Z} \times [0,1]^{\mathbf{N}}$ and a hierarchy of classes of languages in between regular languages and logarithmic space predicates (both included) [Sei14b].

1.3.2. Quantum Computing and Unitary Bases

The second interest in the hierarchy of models thus obtained concerns quantum computation. Indeed, quantum computation can be represented in a very natural way as graphings. Using the weight monoid to include complex coefficients, one can represent qbits in a natural way as graphings corresponding to their density matrix. One can then represent unitary operators in a similar way and show that the execution as graphings corresponds to the conjugation of the density matrix by the unitary operator. Using these techniques, and showing that one can construct semantically a tensor product of qbits that allows for entanglement, we can obtain a representation of quantum computation that allows for entanglement of functions.

The construction just described is interesting in itself, and describes work in preparation. However, it opens a particularly interesting line of research if one considers the fact that the obtained model lives in the hierarchy of models described by the microcosms. Indeed, contrarily to theoretical quantum computation which allows the use of unitary gates for any unitary operator, a real, physical, quantum computer would allow only for a finite number of already implemented unitary gates. The set of such gates is called a *basis of unitary operators* and satisfied the property that the span, under composition, of this set of operators is dense in the set of all unitaries. In other words, any unitary operator can be approximated by composites of elements of the basis.

Such a restriction is necessary, but there are a number of different choices for the basis. To compare two different bases, one can ask a physicist which unitary gates would be easier to create. This is how bases are compared nowadays: the one which is easiest to construct is considered as better. However, the choice of the basis may have important consequences from a computational and/or logical point of view. Our construction provides an adequate framework to tackle this issue. Indeed, one can restrict the microcosm to allow only unitary gates in a given basis and study the obtained model computation.

2. Interaction Graphs: Execution and the Trefoil Property

We defined in earlier work [Sei12b, Sei12a] a graph-theoretical construction where proofs — or more precisely paraproofs, that is generalized proofs — are interpreted by finite objects[2]. The graphs we considered were directed and weighted, with the weights taken in a monoid $(\Omega, \cdot)$. We briefly expose the main results obtained in these previous works.

Definition 2. A *directed weighted graph* is a tuple G, where V^G is the set of vertices, E^G is the set of edges, s^G and t^G are two functions from E^G to V^G, the *source* and *target* functions, and ω^G is a function $E^G \to \Omega$.

The construction is centered around the notion of alternating paths. Given two graphs F and G, an alternating path is a path $e_1 \ldots e_n$ such that $e_i \in E^F$ if and only if $e_{i+1} \in E^G$. The set of alternating paths will be used to define the interpretation of cut-elimination in the framework, i.e. the graph $F::G$ — the *execution of F and G* — is defined as the graph of alternating paths between F and G whose source and target are in the symmetric difference $V^F \Delta V^G$. The weight of a path is naturally defined as the product of the weights of the edges it contains. One easily verifies that this operation is associative: as long as the three graphs F, G, H satisfy $V^F \cap V^G \cap V^H = \emptyset$, we have:

$$(F::G)::H = F::(G::H)$$

As it is usual in mathematics, this notion of paths cannot be considered without the associated notion of cycle: an *alternating cycle* between two graphs F and G is a cycle which is an alternating path $e_1 e_2 \ldots e_n$ such that $e_1 \in V^F$ if and only if $e_n \in V^G$. For technical reasons, we actually consider the related notion of 1-circuit, which is a cycle satisfying some technical property.

Definition 3. We define the following notions of cycles:
- a *cycle* in a graph F is a sequence $\pi = e_0 \ldots e_n$ of edges such that for all $i > n$ the source of the edge e_{i+1} coincides with the target of the edge e_i;
- a 1-*cycle* in a graph F is a cycle π such that there are no cycle ρ and integer $k > 1$ with $\pi = \rho^k$, where ρ^k denotes the concatenation of k copies of ρ;
- a *circuit* is an equivalence class of cycles for the equivalence relation defined by $e_0 \ldots e_n \sim f_0 \ldots f_n$ if and only if there exists an integer k such that for all i, $e_i = f_j$ with $j = k + i[n+1]$.
- a 1-*circuit* ρ is a circuit which is not a proper power of a smaller circuit, i.e. is the equivalence class of a 1-cycle.

We will denote by $\mathscr{C}(F, G)$ the set of 1-circuits. It can be shown that these notions of paths and cycles satisfy a property we call the *trefoil property* which

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[2]Even though the graphs we consider can have an infinite set of edges, linear logic proofs are represented by finite graphs (disjoint unions of transpositions).
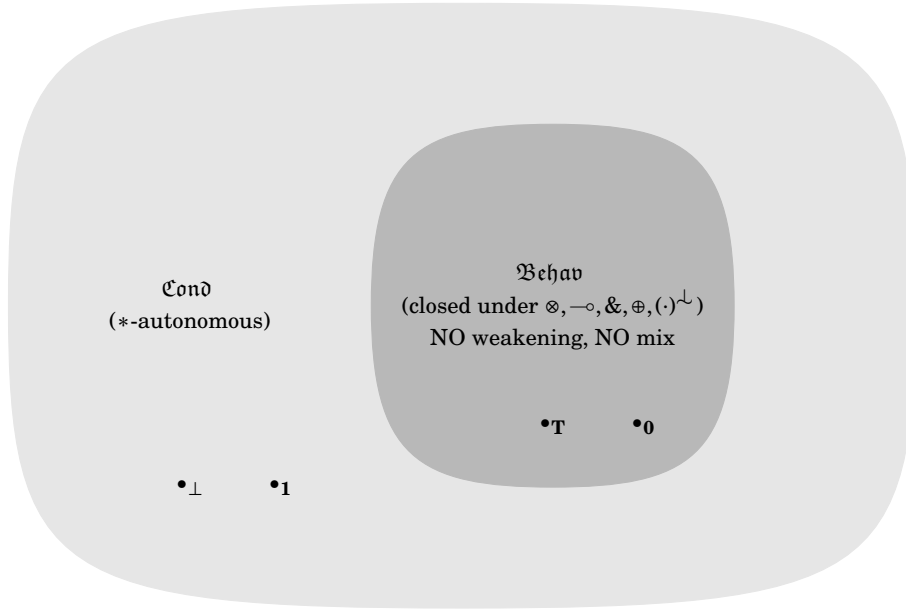
Figure 2: Structure of the categorical models

turns out to be fundamental for constructing models of linear logic. This property states the existence of weight-preserving bijections between sets of 1-circuits:

$$\mathscr{C}(F::G,H)\cup\mathscr{C}(F,G)\cong\mathscr{C}(G::H,F)\cup\mathscr{C}(G,H)\cong\mathscr{C}(H::F,G)\cup\mathscr{C}(H,F)$$

In this setting, one can define the multiplicative and additive connectives of Linear Logic. This construction is parametrized by a map from the set Ω to $\mathbb{R}_{\geqslant 0}\cup\{\infty\}$. We thus obtain not only one but a whole family of models. This parameter is introduced to define the notion of orthogonality in our setting and is used to measure the sets of 1-circuits. Indeed, given a map m and two graphs F,G we define $[\![F,G]\!]_m$ as the sum $\sum_{\pi\in\mathscr{C}(F,G)}m(\omega(\pi))$, where $\omega(\pi)$ is the weight of the cycle π.

From any of these constructions, one can obtain a $*$-autonomous category $\mathfrak{Graph}_{MLL}$ with $\mathfrak{N}\not\cong\otimes$ and $1\not\cong\bot$, i.e. a non-degenerate denotational semantics for Multiplicative Linear Logic (MLL). A consequence of the trefoil property is that this category can be quotiented by an observational equivalence while conserving its structure of $*$-autonomous category. The categorical model obtained in this way has two layers (see Figure 2). The first layer consists in a non-degenerate (i.e. $\otimes\neq\mathfrak{N}$ and $1\neq\bot$) $*$-autonomous category $\mathfrak{Cond}$ obtained as a quotient, hence a denotational model for MLL with units. The second layer is a full subcategory $\mathfrak{Behav}$ which does not contain the multiplicative units but is a non-degenerate model (i.e. $\otimes\neq\mathfrak{N}$, $\oplus\neq\&$ and $\mathbf{0}\neq\top$) of MALL with additive units that does not satisfy the mix and weakening rules.

3. Graphings, Paths and Execution

We define in this section the notion of graphing. This notion was considered by Levitt [Lev95], and later by Gaboriau [Gab00] in order to study measurable group actions. It generalizes in the setting of measure theory the topological notion of pseudo-group [KN96] which was introduced by E. Cartan [Car04, Car09].

We will here consider graphings as a generalization of graph or, more acutely, as a *realization* of a graph. We will therefore define the notion of path and built upon it a generalization of the notion of execution mentioned in the previous section. To stay consistent with the spirit of measure-theory, we will however need first to examine the notion of almost-everywhere equality between graphings.

3.1. First Definitions

The idea is that a graphing is a sort of "geometric realization" of a graph: the vertices correspond to measurable subsets of a measured space, and edges correspond to measurable maps[3] from the source subset onto the target subset. Some difficulties arise when one wants to define a tractable notion of graphing. Indeed, a new phenomenon appears when vertices are measurable sets: what should one do when two vertices are neither disjoint or equal, i.e. when two vertices are not equal but their intersection is not of null measure? One solution would be to define graphings where vertices are disjoint subsets (i.e. their intersection is of null measure), but this makes the definition of execution extremely complex.

Let us consider for instance two graphings with a single edge each, and whose plugging is represented in Figure 3. To represent the set of alternating paths whose source and target are subsets of the symmetric difference of the carriers — the execution of the two graphs — we would need to decompose each of the measurable sets into a disjoint union of sets, each one corresponding to the source and/or target of a path. In the particular case we show in the figure, this operation is not that complicated: it is sufficient to consider the sets $(\phi\psi)^{-k}(U_t - V_s) \cap (U_s - V_t)$. However, the operation quickly becomes much more complicated as we add new edges and create cycles. Figure 4 represents the case of two graphings with two edges each. Defining the decomposition of the set of vertices induced by the execution is — already in this case — quite difficult. In particular, since the sets of vertices considered can be infinite (but countable), the number of cycles can be infinite, and the operation is then of an extreme complexity.

As a consequence, we have chosen to work with a different presentation of graphings, where two distinct vertices can have a intersection of strictly positive measure — they can even be equal. We will now define the notion of graphing taking into account these remarks. The terminology is borrowed from works of Levitt [Lev95] and Gaboriau [Gab00], in which the underlying notion of graphing (forgetting about the weights) is defined.
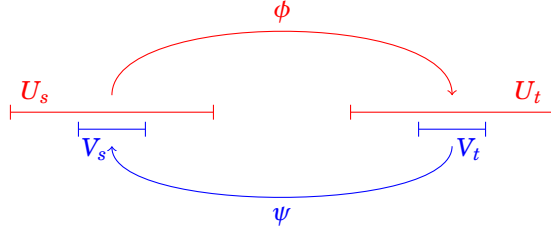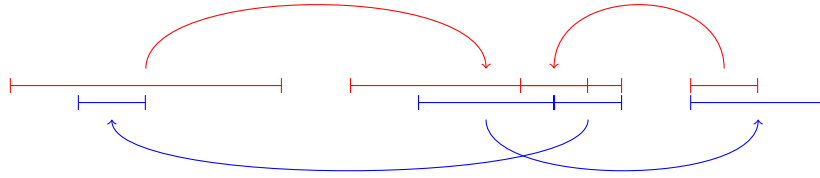
Figure 3: Example of a plugging between graphings

Figure 4: Example of a plugging between graphings

Definition 4. Let $(X, \mathcal{B}, \lambda)$ be a measured space. We denote by $\mathcal{M}(X)$ the set of non-singular Borel-preserving transformations[4] $X \to X$. A *microcosm* of the measured space X is a subset $\mathfrak{m}$ of $\mathcal{M}(X)$ which is closed under composition and contains the identity.

In the following, we will consider a notion of graphing depending on a *weight-monoid* Ω, i.e. a monoid $(\Omega, \cdot, 1)$ which contains the possible weights of the edges.

Definition 5 (Graphings)**.** Let $\mathfrak{m}$ be a microcosm of a measured space $(X, \mathcal{B}, \lambda)$ and V^F a measurable subset of X. A Ω-*weighted graphing in* $\mathfrak{m}$ of carrier V^F is a countable family $F = \{(\omega_e^F, \phi_e^F : S_e^F \to T_e^F)\}_{e \in E^F}$, where, for all $e \in E^F$ (the set of *edges*):

- ω_e^F is an element of Ω, the *weight* of the edge e;
- $S_e^F \subset V^F$ is a measurable set, the *source* of the edge e;
- $T_e^F = \phi_e^F(S_e^F) \subset V^F$ is a measurable set, the *target* of the edge e;
- ϕ_e^F is the restriction of an element of $\mathfrak{m}$ to S_e^F, the *realization* of the edge e.

3.2. Almost-Everywhere Equality

For the remaining of this section, we consider that we fixed once and for all the weight monoid Ω and the microcosm $\mathfrak{m}$. We will therefore refer to Ω-weighted graphings in $\mathfrak{m}$ simply as *graphings*.

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[3]To be exact, we will consider graphings whose edges are taken in a *microcosm*, that is a subset of all measurable maps which is closed under composition and contains the identity.

[4]A non-singular transformation $f : X \to X$ is a measurable map which preserves the sets of null measure, i.e. $\lambda(f(A)) = 0$ if and only if $\lambda(A) = 0$. A map $f : X \to X$ is Borel-preserving if it maps every Borel set to a Borel set.

It is usual, when doing measure theory, to work modulo sets of null measure. Similarly, we will work with graphings modulo almost everywhere equality, a notion that we need to define first. Before giving the definition, we will define the useful notion of empty graphing. An empty graphing will be almost everywhere equal to the graphing without edges.

Definition 6 (Empty graphings). A graphing F is said to be *empty* if its effective carrier is of null measure.

Definition 7 (Almost Everywhere Equality). Two graphings F, G are *almost everywhere equal* if there exists two empty graphings $0_F, 0_G$ and a bijection $\theta : E^F \uplus E^{0_F} \to E^G \uplus E^{0_G}$ such that:
- for all $e \in E^F \uplus E^{0_F}$, $\omega_e^{F \cup 0_F} = \omega_{\theta(e)}^{G \cup 0_G}$;
- for all $e \in E^F \uplus E^{0_F}$, $S_e^{F \cup 0_F} \Delta S_{\theta(e)}^{G \cup 0_G}$ is of null measure;
- for all $e \in E^F \uplus E^{0_F}$, $T_e^{F \cup 0_F} \Delta T_{\theta(e)}^{G \cup 0_G}$ is of null measure;
- for all $e \in E^F \uplus E^{0_F}$, $\phi_{\theta(e)}^{G \uplus 0_G}$ and $\phi_e^{F \uplus 0_F}$ are equal almost everywhere on $S_{\theta(e)}^{G \uplus 0_G} \cap S_e^{F \uplus 0_F}$;

Proposition 8. *We define the relation $\sim_{a.e.}$ between graphings:*

$$F \sim_{a.e.} G \text{ if and only if } F \text{ and } G \text{ are almost everywhere equal}$$

This relation is an equivalence relation.

Proof. It is obvious that this relation is reflexive and symmetric (it suffices to take the bijection θ^{-1}). We therefore only need to show transitivity. Let F, G, H be three graphings such that $F \sim_{a.e.} G$ and $G \sim_{a.e.} H$. Therefore there exists four empty graphings $0_F, 0_{G^F}, 0_{G^H}, 0_H$ and two bijections $\theta_{F,G} : E^F \uplus E^{0_F} \to E^G \uplus E^{0_{G^F}}$ and $\theta_{G,H} : E^G \uplus E^{0_{G^H}} \to E^H \uplus E^{0_H}$ that satisfy the properties listed in the preceding definition. We notice that $0_F \uplus 0_{G^H}$ and $0_{G^F} \uplus 0_H$ are empty graphings. One can then define $\theta_{F,H} = (\theta_{G,H} \uplus \mathrm{Id}_{E^{0_{G^F}}}) \circ (\mathrm{Id}_{E^G} \uplus \tau) \circ (\theta_{F,G} \uplus \mathrm{Id}_{E^{0_{G^H}}})$, where τ represents the symmetry $E^{0_{G^F}} \uplus E^{0_{G^H}} \to E^{0_{G^H}} \uplus E_{0_{G^F}}$;

$$
\begin{array}{ccc}
E^F \uplus E^{0_F} \uplus E^{0_{G^H}} & \xrightarrow{\theta_{F,G} \uplus \mathrm{Id}_{E^{0_{G^H}}}} & E^G \uplus E^{0_{G^F}} \uplus E^{0_{G^H}} \\
\Big\downarrow{\scriptstyle\theta_{F,H}} & & \Big\downarrow{\scriptstyle\mathrm{Id}_{E^G} \uplus \tau} \\
E^H \uplus E^{0_H} \uplus E^{0_{G^F}} & \xleftarrow{\theta_{G,H} \uplus \mathrm{Id}_{E^{0_{G^F}}}} & E^G \uplus E^{0_{G^H}} \uplus E^{0_{G^F}}
\end{array}
$$

It is then easy to verify that the three first properties of almost everywhere equality are satisfied. We will only detail the proof that the fourth property also holds.

We will forget about the superscripts in order to simplify notations. We will moreover denote by $\tilde{\theta}_{F,G}$ (resp. $\tilde{\tau}$, resp. $\tilde{\theta}_{G,H}$) the function $\theta_{F,G} \uplus \mathrm{Id}_{E^{GH}}$ (resp. $\mathrm{Id}_{E^G} \uplus \tau$, resp. $\theta_{G,H} \uplus \mathrm{Id}_{E^{GF}}$).

Chose $e \in E^F \uplus E^{0_F} \uplus E^{0_{GH}}$:

- if $e \in E^{0_{GH}}$, then $\tilde{\theta}_{F,G}(e) = e$, and $\phi_{\tilde{\theta}(e)} = \phi(e)$;
- if $e \in E^F \uplus E^{0_F}$ then, by the definition of $\theta_{F,G}$, $\phi_{\theta_{\tilde{(e)}}}$ is almost everywhere equal to ϕ_e on $S_e \cap S_{\theta_{\tilde{(e)}}}$.

Thus $\phi_{\tilde{\theta}(e)}$ and ϕ_e are equal almost everywhere on $S_e \cap S_{\tilde{\theta}(e)}$ in all cases. A similar reasoning shows that for all $f \in E^G \uplus E^{0_H} \uplus E^{G^F}$, the functions $\phi_{\theta_{G,H}(f)}$ and ϕ_f are almost everywhere equal on $S_{\theta_{G,H}(f)} \cap S_f$.

Moreover, $\phi_{\tilde{\theta}_{F,G}(e)}$ and $\phi_{\tilde{\tau}(\tilde{\theta}_{F,G}(e))}$ are equal and have the same domain $S_{\tilde{\theta}_{F,G}(e)} = S_{\tilde{\tau}(\tilde{\theta}_{F,G}(e))}$. Thus $\phi_{\tilde{\tau}(\tilde{\theta}_{F,G}(e))}$ and ϕ_e are almost everywhere equal on the intersection $S_{\tilde{\tau}(\tilde{\theta}(e))} \cap S_e$. Moreover, $\phi_{\tilde{\tau}(\tilde{\theta}_{F,G}(e))}$ and $\phi_{\tilde{\theta}_{G,H}(\tilde{\tau}(\tilde{\theta}_{F,G}(e)))}$ are almost everywhere equal on the intersection $S_{\tilde{\tau}(\tilde{\theta}_{F,G}(e))} \cap S_{\tilde{\theta}_{G,H}(\tilde{\tau}(\tilde{\theta}_{F,G}(e)))}$. We deduce from this that the functions ϕ_e and $\phi_{\theta_{F,H}(e)}$ are almost everywhere equal on

$$S_e \cap S_{\theta_{F,H}(e)} \cap S_{\tilde{\tau}(\tilde{\theta}_{F,G}(e))} = S_e \cap S_{\theta_{F,H}(e)} \cap S_{\tilde{\theta}_{F,G}(e)}$$

We denote by Z the set of null measure on which they differ. Since $S_e \Delta S_{\tilde{\theta}_{F,G}(e)}$ is of null measure, there exists two sets X, Y of null measure such that $S_e \cup X = S_{\tilde{\theta}_{F,G}(e)} \cup Y$. We can deduce[5] that $S_{\tilde{\theta}_{F,G}(e)} = S_e \cup X - Y$. Thus

$$
\begin{aligned}
& S_e \cap S_{\theta_{F,H}(e)} \cap S_{\tilde{\theta}_{F,G}(e)} \\
= {} & S_e \cap S_{\tilde{\theta}_{F,H}(e)} \cap (S_e \cup X - Y) \\
= {} & S_e \cap S_{\tilde{\theta}_{F,H}(e)} \cap S_e - Y \\
= {} & S_e \cap S_{\tilde{\theta}_{F,H}(e)} - Y
\end{aligned}
$$

We then conclude that the functions ϕ_e and $\phi_{\tilde{\theta}_{F,H}(e)}$, restricted to $S_e \cap S_{\tilde{\theta}_{F,H}(e)}$, are equal outside of $Y \cup Z$ which is a set of null measure. $\qquad\square$

3.3. Paths and Execution

We now need to define what is a path, since we won't be able to work with the usual notion of a path in a graph. Obviously, a path will be a finite sequence of edges. We will replace the condition that the source of an edge be equal to the target of the preceding edge by the condition that the intersection of these source and target sets be of non-null measure.

Definition 9 (Plugging). Being given two graphings F, G, we define their plugging $F \tilde{\square} G$ as the graphing $F \uplus G$ endowed with the coloring function $\delta : E^{F \uplus G} \to \{0, 1\}$ such that $\delta(e) = 1$ if and only if $e \in E^G$.

---

[5] One can chose X in such a way so that $S_{\tilde{\theta}_{F,G}(e)} \cap Y = \emptyset$.

Definition 10 (Alternating Paths). A path in a graphing F is a finite sequence $\{e_i\}_{i=0}^n$ of elements of E^F such that for all $0 \le i \le n-1$, $T_{e_i}^F \cap S_{e_{i+1}}^F$ is of strictly positive measure.

An *alternating path* between two graphings F, G is a path $\{e_i\}_{i=0}^n$ in the graphing $F \tilde{\square} G$ such that for all $0 \le i \le n-1$, $\delta(e_i) \ne \delta(e_{i+1})$. We will denote by $\mathrm{Ch}^m(F, G)$ the set of alternating paths in $F \tilde{\square} G$.

We also define the *weight* of a path $\pi = \{e_i\}_{i=0}^n$ in the graphing F as the scalar $\omega_\pi^F = \prod_{i=0}^n \omega_{e_i}^F$.

Given a path $\{e_i\}_{i=0}^n$ in a graphing F, one can define a function ϕ_π^F as the partial transformation:

$$\phi_\pi^F = \phi_{e_n}^F \circ \chi_{T_{e_n}^F \cap S_{e_{n-1}}^F} \circ \phi_{e_{n-1}}^F \circ \chi_{T_{e_{n-1}}^F \cap S_{e_{n-2}}^F} \circ \cdots \circ \chi_{T_{e_1}^F \cap S_{e_0}^F} \circ \phi_{e_0}^F$$

where for all measurable set A, the function χ_A is the partial identity $A \to A$.

We denote by S_π and T_π respectively the domain and codomain of this partial transformation $S_{e_0}^F \to T_{e_n}^F$. It is then clear that the transformation $\phi_\pi^F : S_\pi \to T_\pi$ is measurable. Moreover, if all ϕ_{e_i} are in a microcosm $\mathfrak{m}$, the transformation ϕ_π is itself in the microcosm $\mathfrak{m}$.

We now introduce the notion of *carving* of a graphing along a measurable set C. This operation will consists in replacing an edge by four disjoint edges whose source and target are either subsets of C or subsets of the complementary set of C.

Definition 11 (Carvings). Let $\phi : S \to T$ be a measurable transformation, C a measurable set and C^c its complementary set. We define the measurable transformations:

$$
\begin{aligned}
[\phi]_i^i &= \phi\restriction_{C \cap \phi^{-1}(C)} : A \cap C \cap \phi^{-1}(C) \to B \cap \phi(C) \cap C \\
[\phi]_i^o &= \phi\restriction_{C \cap \phi^{-1}(C^c)} : A \cap C \cap \phi^{-1}(C^c) \to B \cap \phi(C) \cap C^c \\
[\phi]_o^i &= \phi\restriction_{C^c \cap \phi^{-1}(C)} : A \cap C^c \cap \phi^{-1}(C) \to B \cap \phi(C^c) \cap C \\
[\phi]_o^o &= \phi\restriction_{C^c \cap \phi^{-1}(C^c)} : A \cap C^c \cap \phi^{-1}(C^c) \to B \cap \phi(C^c) \cap C^c
\end{aligned}
$$

We will denote by $[S]_a^b, [T]_a^b$ ($a, b \in \{i, o\}$) the domain and codomain of $[\phi]_a^b$.

If F is a graphing in a microcosm $\mathfrak{m}$, define the *carving of F along C* as the graphing $F^{\curlyvee C} = \{(\omega_e^F, [\phi_e^F]_a^b) \mid e \in E^F, a, b \in \{i, o\}\}$ which is a graphing in the microcosm $\mathfrak{m}$.

In some cases, the carving a graphing G along a measurable set C is almost the same as G. Indeed, if each edge have its source and target (up to a null-measure set) either in C or in its complementary set, the graphing obtained from the carving operation is almost everywhere equal to G.

Definition 12. Let A, B be two measurable sets. We say that A intersects B trivially if $\lambda(A \cap B) = 0$ or $\lambda(A \cap B^c) = 0$.

If F is a graphing and $e \in E^F$, S_e^F and T_e^F intersect C trivially, then F will be said to be C-tough.

Lemma 13. *Let F be a graphing and C be a measurable set. If F is C-tough, then $F^{\backslash\!\backslash C} \sim_{a.e.} F$.*

Proof. Chose $e \in E^F$. Since F is C-tough, we are in one of the four following cases:

- $S_e^F \cap C$ and $T_e^F \cap C$ are of null measure;
- $S_e^F \cap C$ and $T_e^F \cap C^c$ are of null measure;
- $S_e^F \cap C^c$ and $T_e^F \cap C$ are of null measure;
- $S_e^F \cap C^c$ and $T_e^F \cap C^c$ are of null measure;

These four cases are treated in a similar way. Indeed, among the functions $[\phi_e^F]_a^b$, $a, b \in \{i, o\}$, only one is of domain (and thus of codomain) a set of strictly positive measure. We thus define an empty graphing 0_F, with $E^{0_F} = E^F \times \{1, 2, 3\}$, and a bijection $E^{F^{\backslash\!\backslash C}} \to E^F \uplus E^{0_F}$ which associates to the element (e, a, b) ($e \in E^F$, $a, b \in \{i, o\}$) the element $e \in E^F$ if the domain of $[\phi_e^F]_a^b$ is of strictly positive measure, and one of the elements $(e, i) \in E^{0_F}$ otherwise. One can then easily show that this bijection satisfies all the necessary properties to conclude that $F \sim_{a.e.} F^{\backslash\!\backslash C}$. $\square$

Thanks to the carving operation, we are now able to define the execution of two graphings F and G: we consider the set of alternating paths between F and G, and we then keep the part of each path which is external to the intersection C — the location of the *cut* — of the carriers of F and G. The execution for graphings is therefore the natural generalization of the execution we defined earlier on graphs.

Definition 14 (Execution). Let F, G be two graphings in a microcosm $\mathfrak{m}$ of respective carriers V^F, V^G and let $C = V^F \cap V^G$. We define the execution of F and G, denoted by $F :_\mathfrak{m} G$, as the graphing in the microcosm $\mathfrak{m}$ defined as follows:

$$\{(\omega_\pi^{F \tilde{\square} G}, \phi_\pi^{F \tilde{\square} G} : [S_\pi]_o^o \to [T_\pi]_o^o) \mid \pi \in \mathrm{Ch}^m(F, G), \lambda([S_\pi]_o^o) \neq 0\}$$

3.4. Cycles and Circuits

Definition 15 (Alternating Cycles). A cycle in a graphing F is a path $\{e_i\}_{i=0}^n$ in F such that $S_{e_0}^F \cap T_{e_n}^F$ is of strictly positive measure.

An alternating cycle between two graphings F, G is a cycle $\{e_i\}_{i=0}^n$ in $F \tilde{\square} G$ which is an alternating path and such that $\delta(e_0) \neq \delta(e_n)$. We will denote by $\mathrm{Cy}^m(F, G)$ the set of alternating cycles between F and G.

We now want to define the functions representing the circuits between graphings. This is where things get a little bit more complicated: if π_1 and π_2 are two cycles representing the same circuit (i.e. π_1 is a cyclic permutation of π_2) the functions ϕ_{π_1} and ϕ_{π_2} are not equal in general! We will for now skip this complication by considering such a set of function for each choice of representative of circuits. We will however need to take this non-uniformity later, when defining the notion of circuit-quantifying maps (in the cases — which are those if interest— where these maps depend on the functions $\phi_{\pi_1}, \phi_{\pi_2}$ associated to the representatives of circuits).

Definition 16. Let F, G be two graphings. We denote by $\mathrm{Cy}^m(F,G)$ the set of alternating paths between F and G. A *choice of representatives of circuits* is a set $\mathrm{Rep}(F,G)$ such that for all ρ in $\mathrm{Cy}^m(F,G)$ there exists a unique element π in $\mathrm{Rep}(F,G)$ such that $\bar{\rho} = \bar{\pi}$ ($\bar{\pi}$ denotes the equivalence class of π modulo the action of cyclic permutations, see Theorem 3).

Definition 17 (Circuits and 1-circuits)**.** If F and G are graphings and $\mathrm{Rep}(F,G)$ is a choice of representatives of circuits between F and G, we define:

$$\mathrm{Circ}^m(F,G) \quad = \quad \{[\phi_\pi]_i^i \mid \pi \in \mathrm{Rep}(F,G)\}$$

Proposition 18. *Let F, F', G be graphings such that $F \sim_{a.e.} F'$. Then there exists a bijection*

$$\theta : \mathrm{Ch}^m(F,G) \to \mathrm{Ch}^m(F',G)$$

such that $\phi_\pi = \phi_{\theta(\pi)}$ for all path π.

Proof. By definition, there exists two empty graphings $0_F, 0_{F'}$ and a bijection $\theta : E^{F'} \uplus E^{0_{F'}} \to E^F \uplus E^{0_F}$ such that:

- for all $e \in E^{F'} \uplus E^{0_{F'}}$, $\omega_e^{F' \cup 0_F} = \omega_{\theta(e)}^{F \cup 0_F}$;
- for all $e \in E^{F'} \uplus E^{0_{F'}}$, $S_e^{F' \cup 0_{F'}} \Delta S_{\theta(e)}^{F \cup 0_F}$ is of null measure;
- for all $e \in E^{F'} \uplus E^{0_{F'}}$, $T_e^{F \cup 0_F} \Delta T_{\theta(e)}^{F \cup 0_F}$ is of null measure;
- for all $e \in E^{F'} \uplus E^{0_{F'}}$, $\phi_{\theta(e)}^{F \uplus 0_F}$ et $\phi_e^{F' \uplus 0_{F'}}$ are almost everywhere equal;

Let π be an alternating path in $F\tilde{\Box}G$. We treat the case $\pi = f_0 g_0 \dots f_n g_n$ as an example, the other cases are dealt with in a similar way. We can define a path $\theta{-}1(\pi)$ in $F'\tilde{\Box}G$ by $\pi' = \theta^{-1}(f_0) g_0 \theta^{-1}(f_1) \dots \theta^{-1}(f_n) g_n$. Indeed, since the sets $S_{f_{i+1}} \cap T_{g_i}$ (resp. $S_{g_i} \cap T_{f_i}$) are of strictly positive measure, then $\theta^{-1}(f_{i+1})$ (resp. $\theta^{-1}(f_i)$) is of strictly positive measure (hence an element of $E^{F'}$) and moreover satisfies that $S_{\theta^{-1}(f_{i+1})} \cap T_{g_i}$ (resp. $S_{g_i} \cap T_{\theta^{-1}(f_i)}$) is of strictly positive measure.

Conversely, a path $\pi' = e_0 g_0 \dots e_n g_n$ in $F'\tilde{\Box}G$ allows one to define a path $\theta(\pi') = \theta(e_0) g_0 \theta(e_1) \dots \theta(e_n) g_n$. It is clear that $\theta(\theta^{-1}(\pi)) = \pi$ (resp. $\theta^{-1}(\theta(\pi')) = \pi'$) for all path π (resp. π') in $F\tilde{\Box}G$ (resp. $F'\tilde{\Box}G$). $\qquad\square$

We also need to check that the operation of execution is compatible with the notion of almost everywhere equality. Indeed, since we want to work with graphings considered up to almost everywhere equality, the result of the execution should not depend on the representative of the equivalence class considered.

Corollary 18.1. *Let F, F', G be graphings such that $F \sim_{a.e.} F'$. Then $F:_m:G \sim_{a.e.} F':_m:G$.*

Proof. Let θ be the bijection defined in the statement of the preceding proposition. We notice that $\omega_\pi = \omega_{\theta^{-1}(\pi)}$, and that ϕ_π and $\phi_{\theta^{-1}(\pi)}$ are almost everywhere equal as compositions of pairwise almost everywhere equal maps. In particular, their domain and codomain are equal up to a set of null measure. We can then conclude that $\theta : E^{F':m:G} \to E^{F:m:G}$ satisfies all the necessary properties: $F':_m:G$ and $F:_m:G$ are almost everywhere equal. $\qquad\square$

Corollary 18.2. *Let F, F', G be graphings such that $F \sim_{a.e.} F'$. Then $\mathrm{Cy}^m(F,G) \cong \mathrm{Cy}^m(F',G)$.*

Proof. Let θ be the bijection defined in the proof of Theorem 18. The functions ϕ_π and $\phi_{\theta(\pi)}$ are almost everywhere equal and their domains and codomains are equal up to a set of null measure. We can deduce from this that $[\phi_\pi]_i^i$ and $[\phi_{\theta^{-1}(\pi)}]_i^i$ are almost everywhere equal, and their domains and codomains are equal up to a set of null measure. $\square$

3.5. Carvings, Cycles and Execution

We now show a technical result that will be useful later, and which gives better insights on the operation of execution between two graphings. The execution of the graphings F and G is defined as a restriction of the set of alternating paths between F and G. One could have also considered the carvings of F and G along the intersection C of the carriers of F and G, and then define the execution as the set of alternating paths whose source and target lie outside of the set C. The technical lemma we now state and prove shows that these two operations are equivalent.

Let F, G be two graphings and $C = V^F \cap V^G$. One can notice that there should be a bijective correspondence between the edges of $F \!:\!\mathrm{m}\!:\! G$ and those of $F^{\vec{\curlyvee} C} \!:\!\mathrm{m}\!:\! G^{\vec{\curlyvee} C}$. Indeed, for two edges e, f to follow each other in a path, one should have that $S_e \cap T_f$ is of strictly positive measure. But since $S_g \cap T_f$ and $S^f \cap S_g$ are subsets of C, the following expressions are equal:

$$\chi_{S_g \cap T_f} \circ \phi_f^F \circ \chi_{S^f \cap S_g} \quad \text{and} \quad \chi_{S_g \cap T_f \cap C \cap \phi_f^F(C)} \circ (\phi_f^F)_{\restriction_{C \cap (\phi_f^F)^{-1}(C)}} \circ \chi_{S_f \cap S_{g_i} \cap C \cap (\phi_f^F)^{-1}(C)}$$

One can deduce from this the following equality:

$$\chi_{S_g \cap T_f} \circ \phi_f^F \circ \chi_{S^f \cap S_g} = \chi_{S_g \cap [T_f]_i^i} \circ [\phi_f^F]_i^i \circ \chi_{[S^f]_i^i \cap S_g}$$

One can obtain the following equalities in a similar manner:

$$\chi_{C^c} \circ \phi_f \circ \chi_{S_f \cap T_g} \quad = \quad \chi_{C^c} \circ [\phi_f]_i^o \circ \chi_{[S_f]_o^i \cap T_g}$$

$$\chi_{S_g \cap T_f} \circ \phi_f \circ \chi_{C^c} \quad = \quad \chi_{S_g \cap [T_f]_o^i} \circ [\phi_f]_o^i \circ \chi_{C^c}$$

$$\chi_{C^c} \circ \phi_f \circ \chi_{C^c} \quad = \quad \chi_{C^c} \circ [\phi_f]_o^o \circ \chi_{C^c}$$

Lemma 19. *Let F, G be two graphings, V^F, V^G their carrier and $C = V^F \cap V^G$. Then:*

$$F \!:\!\mathrm{m}\!:\! G = F^{\vec{\curlyvee} C} \!:\!\mathrm{m}\!:\! G$$

Proof. By definition, the execution $F \!:\!\mathrm{m}\!:\! G$ is the graphing:

$$\{(\omega_\pi^{F \bar{\square} G}, \phi_\pi^{F \bar{\square} G} : [S_\pi]_o^o \to [T_\pi]_o^o) \mid \pi \in \mathrm{Ch}^m(F,G), \lambda([S_\pi]_o^o) \neq 0\}$$

Similarly, the execution $F^{\vec{\curlyvee} C} \!:\!\mathrm{m}\!:\! G$ is the graphing:

$$\{(\omega_\pi^{F^{\vec{\curlyvee} C} \bar{\square} G}, \phi_\pi^{F^{\vec{\curlyvee} C} \bar{\square} G} : [S_\pi]_o^o \to [T_\pi]_o^o) \mid \pi \in \mathrm{Ch}^m(F^{\vec{\curlyvee} C}, G), \lambda([S_\pi]_o^o) \neq 0\}$$

Let π be an alternating path in $F\tilde{\Box}G$. Then π is an alternating sequence of elements in E^F and elements in E^G. Suppose for instance $\pi = f_0g_0f_1\ldots f_kg_kf_{k+1}$, and let us define $\tilde{\pi} = [f_0]_i^o g_0[f_1]_i^i\ldots[f_k]_i^i g_k[f_{k+1}]_i^o$. The function $[\phi_\pi^{F\tilde{\Box}G}]_o^o$ is equal to:

$$\chi_{C^c\cap\phi_\pi^{F\tilde{\Box}G}(C^c)}\circ\phi_{f_{k+1}}^F\circ\chi_{S_{f_{k+1}}\cap T_{g_k}}\circ\phi_{g_k}^G\circ\ldots$$
$$\ldots\circ\phi_{g_{i+1}}^G\circ\chi_{S_{g_{i+1}}\cap T_{f_{i+1}}}\circ\phi_{f_{i+1}}^F\circ\chi_{S_{f_{i+1}}\cap T_{g_i}}\circ\phi_{g_i}^G\circ\ldots$$
$$\ldots\circ\phi_{g_0}^G\circ\chi_{S_{g_{i+1}}\cap T_{f_{i+1}}}\circ\phi_{f_{i+1}}^F\circ\chi_{C^c\cap(\phi_\pi^{F\tilde{\Box}G})^{-1}(C^c)}$$

From the remarks preceding the statement of the lemma, one can conclude that $[\phi_\pi^{F\tilde{\Box}G}]_o^o$ is equal to:

$$\chi_{C^c\cap\phi_{\tilde{\pi}}^{F^{\tilde{Y}C}\tilde{\Box}G}(C^c)}\circ[\phi_{f_{k+1}}^F]_i^o\circ\chi_{[S_{f_{k+1}}]_i^o\cap T_{g_k}}\circ\phi_{g_k}^G\circ\ldots$$
$$\ldots\circ\phi_{g_{i+1}}^G\circ\chi_{S_{g_{i+1}}\cap[T_{f_{i+1}}]_i^i}\circ[\phi_{f_{i+1}}^F]_i^i\circ\chi_{[S_{f_{i+1}}]_i^i\cap T_{g_i}}\circ\phi_{g_i}^G\circ\ldots$$
$$\ldots\circ\phi_{g_0}^G\circ\chi_{S_{g_{i+1}}\cap[T_{f_{i+1}}]_o^i}\circ[\phi_{f_{i+1}}^F]_o^i\circ\chi_{C^c\cap(\phi_{\tilde{\pi}}^{F^{\tilde{Y}C}\tilde{\Box}G})^{-1}(C^c)}$$

We therefore obtain that $[\phi_\pi^{F\tilde{\Box}G}]_o^o = [\phi_{\tilde{\pi}}^{F^{\tilde{Y}C}\tilde{\Box}G}]_o^o$. Conversely, each alternating path in $F^{\tilde{Y}C}\tilde{\Box}G$ whose first and last edges are elements of $F^{\tilde{Y}C}$ is necessarily of the form $[f_0]_i^o g_0[f_1]_i^i\ldots[f_k]_i^i g_k[f_{k+1}]_o^i$ where the path $f_0g_0f_1\ldots f_kg_kf_{k+1}$ is an alternating path in $F\tilde{\Box}G$.

The other cases are treated in a similar way. $\qquad\square$

Corollary 19.1. *Let F,G be two graphings, V^F,V^G their carrier and $C = V^F\cap V^G$. Then:*
$$F :_m: G = F^{\tilde{Y}C} :_m: G^{\tilde{Y}C}$$

We can also show the sets of cycles are equal.

Lemma 20. *Let F,G be two graphings, V^F,V^G their carrier, and $C = V^F\cap V^G$. Then:*
$$\mathrm{Cy}^m(F,G) = \mathrm{Cy}^m(F^{\tilde{Y}C},G)$$

Proof. The argument is close to the one used in the preceding proof. Indeed, if $\pi = e_0\ldots e_n$ is an alternating cycle between F and G, then we can associate it to the cycle $[\pi] = [e_0]_i^i[e_1]_i^i\ldots[e_n]_i^i$. Conversely, if π' is a cycle in $F^{\tilde{Y}C}\tilde{\Box}G$, then each edge in π' necessarily is of the form $[e]_i^i$ for an element e in F. Moreover, the associated functions are equal, i.e. $[\phi_\pi]_i^i = \phi_{[\pi]}$. $\qquad\square$

Notice that the following lemma is the only place where the fact that our transformations are non-singular is used. It is however fundamental, as it is the key to obtain the associativity of execution.

Lemma 21. *Let F be a graphing, and $\pi = e_0\ldots e_n$ a path in F such that S_π is of strictly positive measure. We define, for all couple of integers $i < j$, $\rho_{i,j}$ the path $e_ie_{i+1}\ldots e_j$. Then:*

- *for all $0 < i < j \leqslant n$, $S_{\rho_{i,j}} \cap T_{e_{i-1}}$ is of strictly positive measure;*
- *for all $0 \leqslant i < j < n$, $T_{\rho_{i,j}} \cap S_{e_{j+1}}$ is of strictly positive measure.*

Proof. Let us fix i, j. We suppose that $S_{\rho_{i,j}} \cap T_{e_{i-1}}$ is of null measure. Then for all $x \in S_{pi}$, $\phi_{e_0 \ldots e_{i-1}}$ is defined at x, and such that $\phi_{e_i \ldots e_n}$ is defined at $\phi_{e_0 \ldots e_{i-1}}(x)$. In particular, $\phi_{e_i \ldots e_j}$ is defined at $\phi_{e_0 \ldots e_{i-1}}(x)$, i.e. $\phi_{e_0 \ldots e_{i-1}}(x)$ is an element in $S_{\rho_{i,j}}$. Moreover, by the definition, $\phi_{e_0 \ldots e_{i-1}}(x)$ is an element of $T_{e_{i-1}}$. Thus $\phi_{e_0 \ldots e_{i-1}}(S_\pi) \subset S_{\rho_{i,j}} \cap T_{e_{i-1}}$. Since $\phi_{e_0 \ldots e_{i-1}}$ is a non-singular transformation which is defined at all $x \in S_\pi$, we deduce that $\lambda(S_\pi) = 0 \Leftrightarrow \lambda(S_{\rho_{i,j}} \cap T_{e_{i-1}}) = 0$. This lead us to a contradiction since this implies that $\lambda(S_\pi) = 0$.

A similar argument shows that $T_{\rho_{i,j}} \cap S_{e_{j+1}}$ is of strictly positive measure. $\square$

As this was the case with the execution between graphs in earlier constructions [Sei12b, Sei12a], we can show the associativity of execution under the hypothesis that the intersection of the carriers is of null measure.

Theorem 22 (Associativity of Execution)**.** *Let F, G, H be three graphings such that $\lambda(V^F \cap V^G \cap V^H) = 0$. Then:*

$$F :_m: (G :_m: H) = (F :_m: G) :_m: H$$

Proof. We can first suppose that F (resp. G, resp. H) is $C_F = V^F \cap (V^G \cup V^H)$-tough (resp. $C_G = V^G \cap (V^F \cup V^H)$-tough, resp. $C_H = V^H \cap (V^F \cup V^G)$-tough). Indeed, if this was not the case, we can always consider the carving along the set C_F (resp. C_G, resp. C_H). This simplifies the following argument since it allows us to consider paths instead of restrictions of paths. The proof then follows the proof of the associativity of the execution for directed weighted graphs obtained in our previous papers [Sei12b, Sei12a].

We can define the simultaneous plugging of the three graphings F, G, H as the graphing $F \uplus G \uplus H$ endowed with a coloring map δ defined by $\delta(e) = 0$ when $e \in E^F$, $\delta(e) = 1$ when $e \in E^G$ and $\delta(e) = 2$ when $e \in E^H$. We can then define the set of 3-alternating paths between F, G, H as the paths $e_0 e_1 \ldots e_n$ such that $\delta(e_i) \neq \delta(e_{i+1})$.

If $e_0 f_0 e_1 \ldots f_{k-1} e_k f_k$ is an alternating path in $F :_m: (G :_m: H)$, where every e_i is an alternating path $e_i = g_0^i h_0^i \ldots g_{n_i}^i h_{n_i}^i$, then the sequence of edges obtained by replacing each e_i by the associated sequence (and forgetting about parentheses) is a path. Indeed, we know that, for instance, $S_{e_i} \cap T_{f_{i-1}}$ is of strictly positive measure, and $S_{e_i} \subset S_{g_0}$, thus $S_{g_0} \cap T_{f_{i-1}}$ is of strictly positive measure. We therefore defined a 3-alternating path between F, G and H. The two paths define the same measurable partial transformation, and have the same domains dans codomains.

Conversely, if $e_0 e_1 \ldots e_n$ is a 3-alternating path between F, G and H, then we can see it as an alternating sequence of edges in F and alternating sequences between G and H. Let $\pi = g_0 h_0 \ldots g_k h_k$ be the path defined by such a sequence appearing in the path $e_0 \ldots e_n$. We can use the preceding lemma to insure that $S_\pi \cap T_{e_j}$ is of strictly positive measure. Similarly, $T_\pi \cap S_{e_k}$ is of strictly positive measure. We thus showed that we had an edge in $F :_m: (G :_m: H)$. The two paths define the same partial measurable transformation, and have the same domains and codomains. $\square$
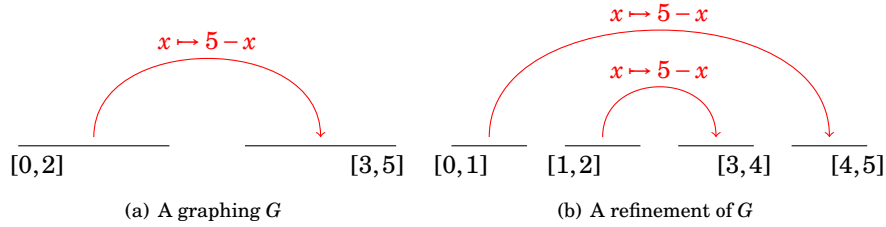
(a) A graphing G (b) A refinement of G

Figure 5: Illustration of refinement

4. Cycles and The Trefoil Property

In this section, we go a bit further into the theory of graphings. Indeed, one of the main motivations behind the use of continuous sets as vertices of a graph instead of usual discrete sets lies in the idea that edges of a graphing may be *split* into sub-edges. In order to formalize this idea (illustrated in Figure 5), we define the notion of refinement of a graphing. Once this notion defined, we will want to consider graphings *up to refinement*, meaning that a graphing and one of it refinements should intuitively represent the same computation. We therefore define a equivalence relation on the set of graphings by saying that two graphings are equivalent if they possess a common refinement. We show that this equivalence relation is compatible with the execution defined in the previous section.

We then explore the notion of cycles between graphings. If a cycle might be defined in the obvious and natural way, we want to define a measurement which is compatible with the equivalence relation based on refinements. This means that the measurement should be "refinement-invariant", something that involves lots of complex combinatorics. Moreover, this measurement should satisfy the trefoil property with respect to the execution defined earlier. We therefore introduce the notion of "circuit-quantifying map" which satisfies two abstract properties. We then show that any map satisfying these properties defines a measurement which is both refinement-invariant and satisfies the trefoil property.

4.1. Refinements

We now define the notion of *refinement* of a graphing. This a very natural operation to consider. A simple example of refinement is to consider a graphing F and one of its edges $e \in E^F$: one can obtain a refinement of F by replacing e with two edges f, f' such that $S_f \cup S_{f'} = S_e$ and $S_f \cap S_{f'}$ is of null measure (one should then define $T_f = \phi_e(S_f)$ and $T_{f'} = \phi_e(S_{f'})$ accordingly). This is illustrated in Figure 5.

Definition 23 (Refinements)**.** Let F, G be two graphings. We will say that F is a *refinement* of G — denoted by $F \leqslant G$ — if there exists a function $\theta : E^F \to E^G$ such that:

- for all $e, e' \in E^F$ such that $\theta(e) = \theta(e')$ and $e \neq e'$, $S_e^F \cap S_{e'}^F$ and $T_e^F \cap T_{e'}^F$ are of null measure;
- for all $e \in E^F$, $\omega_{\theta(e)}^G = \omega_e^F$;
- for all $f \in E^G$, S_f^G and $\cup_{e \in \theta^{-1}(f)} S_e^F$ are equal up to a set of null measure;
- for all $f \in E^G$, T_f^G and $\cup_{e \in \theta^{-1}(f)} T_e^F$ are equal up to a set of null measure;
- for all $e \in E^F$, $\phi_{\theta(e)}^G$ and ϕ_e^F are equal almost everywhere $S_{\theta(e)}^G \cap S_e^F$.

We will say that F is a *refinement of G along $g \in E^G$* if there exists a set D of elements of E^F such that:

- $\theta^{-1}(g) = D$;
- $\theta_{\restriction_{E^F - D}} : E^F - D \to E^G - \{g\}$ is bijective.

If D contains only two elements, we will say that F is a *simple refinement along g*. The refinements will sometimes be written (F, θ) in order to precise the function θ.

Proposition 24. *We define the relation $\sim_{\leqslant}$ on the set of graphings as follows:*

$$F \sim_{\leqslant} G \Leftrightarrow \exists H, (H \leqslant F) \wedge (H \leqslant G)$$

This is an equivalence relation.

Proof. Reflexivity and symmetry are straightforward. We are therefore left with transitivity: let F, G, H be three weighted graphs such that $F \sim_{\leqslant} G$ and $G \sim_{\leqslant} H$. We will denote by $(P_{F,G}, \theta)$ (resp. $(P_{G,H}, \rho)$) a common refinement of F and G (resp. of G and H). We will now define a graphing P such that $P \leqslant P_{F,G}$ and $P \leqslant P_{G,H}$. Let us define:

- $E^P = \{(e, f) \in E^{P_{F,G}} \times E^{P_{G,H}} \mid \theta(e) = \rho(f)\}$;
- $S_{(e,f)}^P = S_e^{P_{F,G}} \cap S_f^{P_{G,H}}$ when $\theta(e) = \rho(f)$;
- $T_{(e,f)}^P = T_e^{P_{F,G}} \cap T_f^{P_{G,H}}$ when $\theta(e) = \rho(f)$;
- $\omega_{(e,f)}^P = \omega_e^{P_{F,G}} = \omega_f^{P_{G,H}}$;
- $\phi_{(e,f)}^P$ is the restriction of $\phi_e^{P_{F,G}}$ to $S_{(e,f)}^P$;
- $\mu_{F,G} : (e, f) \mapsto e$ and $\mu_{G,H} : (e, f) \mapsto f$.

It is then easy to check that $(P, \mu_{F,G})$ (resp. $(P, \mu_{G,H})$) is a refinement of $P_{F,G}$ (resp. of $P_{G,H}$).

Since $(P_{F,G}, \theta)$ is a refinement of F and $(P, \mu_{F,G})$ is a refinement of $P_{F,G}$, it is clear that $(P, \theta \circ \mu_{F,G})$ is a refinement of F. In a similar way, $(P, \theta \circ \mu_{G,H})$ is a refinement of H. Finally, $P \leqslant F$ and $P \leqslant H$, which shows that $F \sim_{\leqslant} H$. $\qquad\square$

Proposition 25. *The relation $\sim_{\leqslant}$ contains the relation $\sim_{a.e.}$.*

Proof. Let F, G be two graphings such that $F \sim_{a.e.} G$. We will show that $F \sim_{\leqslant} G$. We will use the notations of Theorem 7: $0_F, 0_G$ for the empty graphings and θ for the bijection between the sets of vertices. First, we notice that $F \leqslant F \uplus 0_F$ and $G \leqslant G \uplus 0_G$. As a consequence, $F \sim_{\leqslant} F \uplus 0_F$ and $G \sim_{\leqslant} 0_G$. Moreover, the bijection $\theta : E^F \uplus E^{0_F} \to E^G \uplus E^{0_G}$ clearly satisfies the necessary conditions for $(F \uplus 0_F, \theta)$ to be a refinement of $G \uplus 0_G$, which implies that $F \uplus 0_F \sim_{\leqslant} G \uplus 0_G$. Using the transitivity of $\sim_{\leqslant}$, we can now conclude that $F \sim_{\leqslant} G$. $\qquad\square$

Of course, the carving of a graphing G along a measurable set C defines a refinement of G where each edge is replaced by exactly four disjoint edges. This will be of use to simplify some conditions later: a measurement that is invariant under refinements will be invariant under carvings too.

Proposition 26. *Let G be a graphing and C a measurable set. The graphing $G^{\curlyvee C}$ is a refinement of G.*

Proof. It is sufficient to verify that the function $\theta : E^{G^{\curlyvee C}} \to E^G$, $(e,a,b) \to e$ satisfies all the necessary conditions. Firstly, using the definition, the weights $\omega^{G^{\curlyvee C}}_{(e,a,b)}$ and ω^G_e are equal. Then, the sets $[S^G_e]^b_a$, $a,b \in \{i,o\}$ (resp. $[T^G_e]^b_a$) define a partition of S^G_e (resp. T^G_e). Finally, using the definition again, $[\phi^G_e]^b_a$ is equal to ϕ^G_e on its domain. $\square$

Lemma 27. *Let F,G be two graphings, $e \in E^F$ and $F^{(e)}$ be a simple refinement of F along e. Then $F^{(e)} :_m: G$ is a refinement of $F :_m: G$.*

Proof. By definition,

$$F :_m: G = \{(\omega^{F \tilde{\square} G}_\pi, \phi^{F \tilde{\square} G}_\pi : [S_\pi]^o_o \to [T_\pi]^o_o) \mid \pi \in \mathrm{Ch}^m(F,G), \lambda([S_\pi]^o_o) \neq 0\}$$

Since $F^{(e)}$ is a simple refinement of F along e, there exists a partition of S^F_e in two sets S_1,S_2, and a partition of T^F_e in two sets T_1,T_2 such that $\phi^F_e(S_i) = T_i$. We can suppose, without loss of generality, that $S_1 \cap S_2 = \emptyset$ since there exists a graphing which is almost everywhere equal to $F^{(e)}$ and satisfies this additional condition, and since execution is compatible with almost everywhere equality. This additional assumption implies in particular that $T_1 \cap T_2$ is of null measure. We denote by f_1,f_2 the two elements of $E^{F^{(e)}}$ whose image is e by θ.

To any element $\pi = \{e_i\}^n_{i=0}$ of $F^{(e)} :_m: G$, we associate the path $\theta(\pi) = \{\theta(a_i)\}^n_{i=0}$. We now need to check that this is indeed a refinement. Let π_1, π_2 be two distinct paths such that $\theta(\pi_1) = \theta(\pi_2)$. We want to show that $S_{\pi_1} \cap S_{\pi_2}$ is of null measure. Since $\pi_1 = \{p_i\}^{n_1}_{i=0}$ and $\pi_2 = \{q_i\}^{n_2}_{i=0}$ are distinct, they differ at least on one edge. Let k be the smallest integer such that $p_k \neq q_k$. We can suppose without loss of generality that $p_k = f_1$ and $q_k = f_2$. If $x \in S_{\pi_1}$, then $x \in \phi^{-1}_{p_0 \ldots p_{k-1}}(S_1)$. Similarly, if $x \in S_{\pi_2}$, then $x \in \phi^{-1}_{q_0 \ldots q_{k-1}}(S_2) = \phi^{-1}_{p_0 \ldots p_{k-1}}(S_2)$. Since we supposed that $S_1 \cap S_2 = \emptyset$, we deduce that $S_{\pi_1} \cap S_{\pi_2} = \emptyset$.

By definition, the weight of a path π is equal to the weight of every path π' such that $\theta(\pi') = \pi$. Moreover, the functions $\phi_{\pi'}$ and $\phi_{\theta(\pi')}$ are by definition almost everywhere equal on the intersection of their domain since every ϕ_e is almost everywhere equal to $\phi_{\theta(e)}$.

We are now left to show $S_\pi = \cup_{\pi' \in \theta^{-1}(\pi)} S_{\pi'}$ (the result concerning T_π is then obvious). It is clear that $S_{\pi'} \subset S_\pi$ when $\theta(\pi') = \pi$, and it is therefore enough to show one inclusion: that for all $x \in S_\pi$ there exists a π' with $\theta(\pi') = \pi$ such that $x \in S_{\pi'}$. Let $\pi = \pi_0 e_0 \pi_1 e_1 \ldots \pi_n e_n \pi_{n+1}$ where for all i, $e_i = e$, and π_i is a path (that could be empty if $i = 0$ or $i = n + 1$). Now chose $x \in S_\pi$. Then for all $i = 0, \ldots, n$, $\phi_{\pi_0 e_0 \ldots \pi_i}(x) \in S_{e_i} = S_e$, thus $\phi_{\pi_0 e_0 \ldots \pi_i}(x)$ is either in S_1 or in S_2. We obtain in this way a sequence $a_0, \ldots, a_n$ in $\{1,2\}^n$. It is then easy to see that $x \in S_{\pi'}$ where $\pi' = \pi_0 f_{a_0} \pi_1 f_{a_1} \ldots \pi_n f_{a_n} \pi_{n+1}$. $\square$

Lemma 28. *Let F, G be two graphings, $e \in E^F$ and (F', θ) a refinement of F along e. Then $F' :\!m\!: G$ is a refinement of $F :\!m\!: G$.*

Proof. This is a simple adaptation of the proof of the preceding lemma. Let D be the set of elements such that $\theta^{-1}(e) = D$; we can suppose, modulo considering an almost everywhere equal graphing, that the sets S_d $(d \in D)$ are pairwise disjoint. To every path $\pi = (f_i)_{i=0}^n$ in $F' :\!m\!: G$, we associate $\tilde{\theta}(\pi) = \{\theta(f_i)\}_{i=0}^n$. Conversely, a path $\pi = (g_i)_{i=0}^n$ in $F :\!m\!: G$ defines a countable set of paths:

$$C_\pi = \{(f_i)_{i=0}^n \mid \theta(f_i) = g_i\}$$

We are left with the task of checking that $\tilde{\theta} : \pi \mapsto \tilde{\theta}(\pi)$ is a refinement. For this, we consider two paths π_1 and π_2 such that $\tilde{\theta}(\pi_1) = \tilde{\theta}(\pi_2)$. Using the same argument as in the preceding proof, we show that $S_{\pi_1} \cap S_{\pi_2}$ is of null measure. The verification concerning the weights is straightforward, as is the fact that the functions are almost everywhere equal on the intersection of their domains. The last thing left to show is that $S_\pi = \cup_{\pi' \in C_\pi} S_{\pi'}$. Here, the argument is again the same as in the preceding proof: an element $x \in S_\pi$ is in the domain of one and only one $S_{\pi'}$ for $\pi' \in C_\pi$. $\qquad\square$

Theorem 29. *Let F, G be graphings and (F', θ) be a refinement of F. Then $F' :\!m\!: G$ is a refinement of $F :\!m\!: G$.*

Proof. If π is an alternating path $f_0 g_0 f_1 \dots f_n g_n$ between F' and G, we define $\theta(\pi) = \theta(f_0) g_0 \dots \theta(f_1) \dots g_n \theta(f_n)$. This clearly defines a path, since $S_{f_i} \subset S_{\theta(f_i)}$ (resp. $T_{f_i} \subset T_{\theta(f_i)}$) and π is itself a path.

Let us denote by $f_0, \dots f_n, \dots$ the edges of F. We define the graphings F^n as the following restrictions of F: $\{(\omega_{f_i}^F, \phi_{f_i}^F)\}_{i=0}^n$. We define the corresponding restrictions of F' as the graphings $(F')^n = \{(\omega_e^{F'}, \phi_e^{F'}) \mid e \in \theta^{-1}(f_i)\}_{i=0}^n$. By an iterated use of the preceding lemma, we obtain that $((F')^n :\!m\!: G, \theta)$ is a refinement of $F^n :\!m\!: G$ for every integer n. It is then easy to see that $(F' :\!m\!: G, \theta) = (\cup_{n \geqslant 0} (F')^n :\!m\!: G, \theta)$ is a refinement of $\cup_{n \geqslant 0} F^n :\!m\!: G$, i.e. of $F :\!m\!: G$. $\qquad\square$

4.2. Measurement of circuits

We would like to define a measurement of circuits between two graphings F and G in such a way that if (F', θ) is a refinement of F, the measurements $[\![F, G]\!]_m$ and $[\![F', G]\!]_m$ are equal. Firstly, one should be aware that to define the notion of circuit-quantifying maps one should take into account the fact that if π_1, π_2 are two representatives of a given circuit, the functions ϕ_{π_1} and ϕ_{π_2} are not equal in general.

Secondly, suppose that we obtained such a map q (which does not depend on the choice of representatives), that π is an alternating cycle between F and G and that (F', θ) is a refinement of F. We will try to understand what the set of circuits induced by π in $F' \tilde{\square} G$ looks like. We first notice that the cycle π corresponds to a family E_π of alternating cycles between F' and G. If for instance $\pi = f_0 g_0 \dots f_n g_n$, one should consider the set of sequences $\{f'_0 g_0 \dots f'_n g_n \mid \forall i, f'_i \in \theta^{-1}(f_i)\}$. However, each of these sequences does not necessarily define a path: it is possible that

(a) A cycle doubling its length
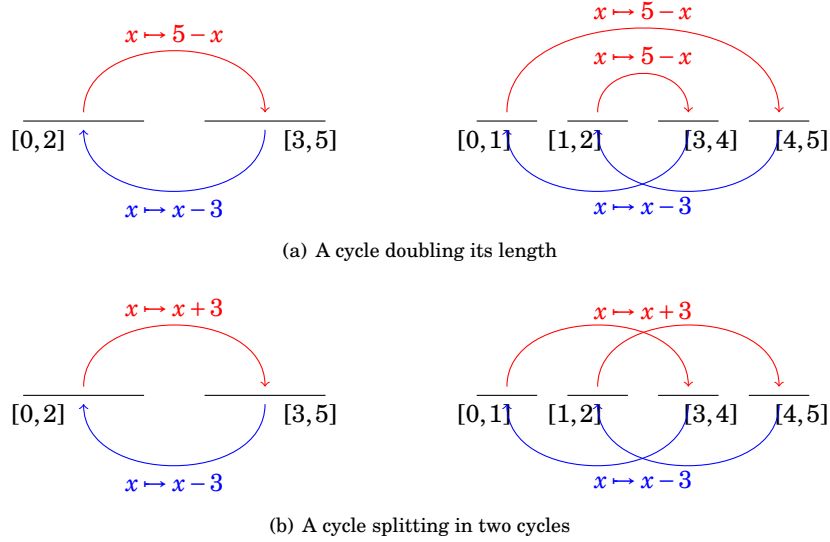

(b) A cycle splitting in two cycles

Figure 6: Examples of the evolution of a cycle when performing a refinement

$S_{g_i} \cap T_{f'_i}$ (or $S_{f'_{i+1}} \cap T_{g_i}$) is of null measure. It is even possible that such a sequence will be a path without being a cycle, and that a cycle of length l, once decomposed along the refinement, becomes a cycle of length $m \times l$, where m is an arbitrary integer. Figure 6 shows how a cycle of length 2 can induce either a cycle of length 4 or a set of two cycles of length 2 after a refinement. However, a cycle of length 4 could very well be induced by the cycle π^2 if the latter is an element of $\mathrm{Cy}^m(F,G)$. The following definition takes all these remarks into account.

Definition 30. Let π be a cycle between two graphings F, G, et $\pi^\omega = \{\pi^k \mid k \in \mathbf{N}\} \cap \mathrm{Cy}^m(F,G)$. Let (F', θ) be a refinement of F. We fix $\mathrm{Rep}(F', G)$ a choice of representatives of circuits, and we write

$$E_\pi^{(F',\theta)} = \{\rho = f'_0 g_0 f'_1 g_1 \ldots f'_n g_n \in \mathrm{Rep}(F',G) \mid \exists k \in \mathbf{N}, \theta(\rho) = \pi^k\}$$

A function q from the set[6] of cycles into $\mathbf{R}_{\geqslant 0} \cup \{\infty\}$ is *refinement-invariant* if for all graphings F, G and simple refinement (F', θ) of F, the following equality holds:

$$\sum_{\rho \in \pi^\omega} q(\rho) = \sum_{\rho \in E_\pi^{(F',\theta)}} q(\rho)$$

---

[6] As in the graph setting, we work modulo a the renaming of edges. As a consequence, if the class of cycles is not a priori a set, the function q will not depend on the name of edges, but only on the weight and the transformation associated to the cycle. We can therefore define q as a function on the set of equivalence classes of cycles modulo renamings of edges.

This is the most general notion one could state and it allows one to define what it means to be invariant under refinement in the general setting of circuits. In the particular case we will be interested in, i.e. the case where one considers only the set of 1-circuits, we can notice that the definition becomes much simpler. Indeed, the set π^ω is reduced to the singleton $\{\pi\}$, and the equality that should be verified becomes

$$q(\pi) = \sum_{\rho \in E_\pi^{(F', \theta)}} q(\rho)$$

Definition 31 (Circuit-Quantifying Maps). Let $\mathfrak{m}$ denote a microcosm. A map q from the set of $\mathfrak{m}$-cycles[7] into $\mathbf{R}_{\geqslant 0} \cup \{\infty\}$ is a ($\mathfrak{m}$-)circuit-quantifying map if:
1. for all representatives π_1, π_2 of a circuit π, $q(\pi_1) = q(\pi_2)$;
2. q is refinement-invariant.

A circuit-quantifying map should therefore meet quite complex conditions and it is a very natural question to ask wether such maps exist. We will define in the next section a family of circuit-quantifying maps for the set of 1-circuits, answering this question positively.

We will now define the measurement associated to a circuit-quantifying map. If the formal definition depends on the choice of a family of representatives of circuits, the result $[\![F, G]\!]_m$ is obviously independent of it. This is a consequence of the first condition in the definition of circuit-quantifying maps.

Definition 32 (Measure). Let q be a circuit-quantifying map. We define the associated measure of interaction as the function $[\![\cdot, \cdot]\!]_m$ which associates, to all couple of graphings F, G, the quantity:

$$[\![F, G]\!]_m = \sum_{\pi \in \mathrm{Circ}^m(F, G)} q(\pi)$$

Where $\mathrm{Circ}^m(F, G)$ depends on a choice of a set of representatives of circuits.

Lemma 33. *Let F, G be two graphings, $e \in E^F$, and $F^{(e)}$ a simple refinement along e of F. Then:*

$$[\![F, G]\!]_m = [\![F^{(e)}, G]\!]_m$$

Proof. We write $\theta : E^{F^{(e)}} \to E^F$ and $\{f, f'\} = \theta^{-1}(e)$. We will use the notations introduced in Theorem 30.

We will also denote by $O(\{e\}, G)$ the set of 1-circuits in $\mathrm{Cy}^m(\{e\}, G)$. Then the family $\{\pi^\omega\}_{\pi \in O(\{e\}, G)}$ is a partition of $\mathrm{Cy}^m(\{e\}, G)$: it is clear that if π, π' are two distinct elements of $O(\{e\}, G)$, π^ω and $(\pi')^\omega$ are disjoint, and it is equally obvious that $\mathrm{Cy}^m(\{e\}, G) = \cup_{\pi \in O(\{e\}, G)} \pi^\omega$ since the fact that $\pi^k \in \mathrm{Cy}^m(F, G)$ implies that $\pi \in \mathrm{Cy}^m(F, G)$.

---

[7]This is the set of cycles realized by maps in the microcosm $\mathfrak{m}$.

By definition, one has:

$$\begin{aligned}
\llbracket F^{(e)},G \rrbracket_m &= \sum_{\pi \in \mathrm{Cy}^m(F^{(e)},G)} q(\pi) \\
&= \sum_{\pi \in \mathrm{Cy}^m(F^{(e)}-\{f,f'\},G)} q(\pi) + \sum_{\pi \in \mathrm{Cy}^m(\{f,f'\},G)} q(\pi) \\
&= \sum_{\pi \in \mathrm{Cy}^m(F-\{e\},G)} q(\pi) + \sum_{\pi \in O(\{e\},G)} \sum_{\rho \in E_\pi^{(F^{(e)},\theta)}} q(\rho) \\
&= \sum_{\pi \in \mathrm{Cy}^m(F-\{e\},G)} q(\pi) + \sum_{\pi \in O(\{e\},G)} \sum_{\rho \in \pi^\omega} q(\rho) \\
&= \sum_{\pi \in \mathrm{Cy}^m(F-\{e\},G)} q(\pi) + \sum_{\pi \in \mathrm{Cy}^m(\{e\},G)} q(\pi) \\
&= \sum_{\pi \in \mathrm{Cy}^m(F,G)} q(\pi)
\end{aligned}$$

Which shows that $\llbracket F^{(e)},G \rrbracket_m = \llbracket F,G \rrbracket_m$. $\qquad\square$

Theorem 34. *Let F,G be graphings and (F',θ) a refinement of F. Then:*

$$\llbracket F,G \rrbracket_m = \llbracket F',G \rrbracket_m$$

Proof. The argument is now usual. We first enumerate the edges of F, and denote them by $f_0,\ldots,f_n,\ldots$. We then define:

$$\begin{aligned}
F^n &= \{(\omega_{f_i}^F,\phi_{f_i}^F)\}_{i=0}^n \\
(F')^n &= \{(\omega_e^{F'},\phi_e^{F'}) \mid \theta(e) = f_i\}_{i=0}^n
\end{aligned}$$

Then $((F')^n,\theta)$ is a refinement of F^n, and an iterated use of the preceding lemma shows that:

$$\llbracket (F')^n,G \rrbracket_m = \llbracket F^n,G \rrbracket_m$$

Then:

$$\begin{aligned}
\llbracket F',G \rrbracket_m &= \sum_{\pi \in \mathrm{Cy}^m(F',G)} q(\pi) \\
&= \lim_{n\to\infty} \sum_{\pi \in \mathrm{Cy}^m((F')^n,G)} q(\pi) \\
&= \lim_{n\to\infty} \llbracket (F')^n,G \rrbracket_m \\
&= \lim_{n\to\infty} \llbracket F^n,G \rrbracket_m \\
&= \lim_{n\to\infty} \sum_{\pi \in \mathrm{Cy}^m(F^n,G)} q(\pi) \\
&= \sum_{\pi \in \mathrm{Cy}^m(F,G)} q(\pi)
\end{aligned}$$

Finally, we showed that $\llbracket F',G \rrbracket_m = \llbracket F,G \rrbracket_m$. $\qquad\square$

Theorem 35 (Trefoil Property)**.** *Let F,G,H be graphings satisfying the condition* $\lambda(V^F \cap V^G \cap V^H) = 0$. *Then:*

$$[\![F,G:_m:H]\!]_m + [\![G,H]\!]_m = [\![H:_m:F,G]\!]_m + [\![H,F]\!]_m$$

Proof. We consider the expression $[\![F,G:_\mathrm{m}:H]\!]_m + [\![G,H]\!]_m$. We can suppose without loss of generality that F (resp. G, resp H) is $V^F \cap (V^G \cup V^H)$-tough (resp. $V^G \cap (V^F \cup V^H)$-tough, resp. $V^H \cap (V^F \cup V^G)$-tough). Indeed, if this is not the case the preceding proposition allows us to replace F,G,H by the adequate carvings without changing the measure of interaction.

The end of the proof is now very similar to the proof of the trefoil property for graphs.

Let π be an element in $\mathrm{Circ}^m(F,G:_\mathrm{m}:H)$. Then π is an alternating path between F and $G:_\mathrm{m}:H$, for instance $\pi = f_0\rho_0 f_1 \ldots f_n \rho_n$. Now, each ρ_i is an alternating path between G and H. Either each ρ_i is an element of G in which case π is an alternating path between F and G, and therefore corresponds to an element in $\mathrm{Circ}^m(F,G)$, either at least one of the ρ_i contains an edge of H. In this second case, it is clear that π is an element of $\mathrm{Cy}^m(F:_\mathrm{m}:G,H)$ (we use Theorem 21 to insure that alternating paths between F and G that appear as part of π have a domain —hence a codomain — of strictly positive measure). Similarly, an element of $\mathrm{Cy}^m(G,H)$ is an element of $\mathrm{Cy}^m(F:_\mathrm{m}:G,H)$. $\qquad\square$

We now will work with equivalence classes of graphings for the equivalence relation $\sim_{\leqslant}$. Theorem 34 and Theorem 29 insure us that the operations of plugging and execution are well defined in this setting. As we showed, each circuit-quantifying map gives rise to a measurement on graphings that satisfies the trefoil property. One question remains unanswered at this point: do such functions exists? The existence of such maps is not clear from their definition, which is quite involved. The next section will give explicit constructions of such maps exists in a very general setting.

5. Models of Multiplicative-Additive Linear Logic

We have now shown how to define execution between graphings and, given a map m, a measurement between graphings. We have shown that the execution is associative and that if m is a *circuit-quantifying map*, that is m satisfies a number of conditions, the trefoil property holds. We have therefore almost finished the proof that for any microcosm $\mathfrak{m}$ and monoid Ω one can construct a GoI model based on Ω-weighted graphings in $\mathfrak{m}$. The last step is to show the existence of circuit-quantifying maps, i.e. exhibit at least one measurement which satisfies the trefoil property. In fact, we will show how to define, given a measured space X satisfying some reasonable properties, one can define a whole family of such circuit-quantifying maps, a family parametrized by the choice of a measurable map $m : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$.

We first define the notion of *trefoil space*, and prove some basic but essential properties on them. We then define a family of functions and prove that they are circuit-quantifying maps.

5.1. Trefoil Spaces

We now introduce the notion of trefoil space. One of the conditions for a space X to be a trefoil space is that it should be second-countable. This is justified by the fact that we will need the measurability of the fixed-point set of a measurable transformation on X. We will indeed use the following theorems.

Proposition 36 (Dravecký [Dra75]). *Let $(Y,\mathcal{T})$ be a measurable space. The following statements are equivalent:*
- *$(Y,\mathcal{T})$ has a measurable diagonal;*
- *For every measurable space $(X,\mathcal{S})$ and every measurable mapping $f : X \to Y$, the graph of f is measurable.*

Proposition 37 (Dravecký [Dra75]). *Let $(Y,\mathcal{G})$ be a topological space and $\mathcal{T}$ a σ-algebra generated by $\mathcal{G}$. Then $(Y,\mathcal{T})$ has a measurable diagonal if and only if there is a topology $\mathcal{H} \subset \mathcal{G}$ such that $(Y,\mathcal{H})$ is a second-countable T_0 space.*

Now, we will therefore ask our spaces to be second-countable[8] in order to obtain the measurability of the fixed point sets of measurable transformations. Moreover, our measurements will be defined using integrals, and we thus need a space in which one can define a reasonable notion of integral. In particular, we will ask our space to be Hausdorff[9], and endowed with its Borel σ-algebra and a σ-additive Radon measure.

Definition 38. Let $(X,\mathcal{T})$ be a second-countable Hausdorff space, and $(X,\mathcal{B},\mu)$ be a measured space where $\mathcal{B}$ is the Borel σ-algebra and μ a σ-additive Radon measure on $(X,\mathcal{B})$. Such a measured space will be referred to as a *trefoil space*.

Proposition 39. *Let X be a trefoil space. For all measurable map $\phi : X \to X$, the fixed point set $\mathscr{F}(\phi) = \{x \in X \mid \phi(x) = x\}$ is measurable.*

Proof. The point is that the diagonal $\Delta = \{(x,x) \mid x \in X\} \subset X \times X$ is a measurable set for the product σ-algebra. This is true because of second countability. Then, we have that $\mathscr{F}(\phi) = F^{-1}(\Delta)$ for $F(x) = (x,f(x))$ measurable from X to $X \times X$. Hence $\mathscr{F}(\phi)$ is measurable. $\qquad\square$

Corollary 39.1. *Let X be a trefoil space and $\phi : X \to X$ be a measurable map. Then the following map is measurable:*

$$
\rho_\phi : \begin{cases} X & \to & \mathbf{N} \cup \{\infty\} \\ x & \mapsto & \inf\{n \in \mathbf{N} \mid \phi^n(x) = x\} \quad \textit{if } \{n \in \mathbf{N} \mid \phi^n(x) = x\} \neq \emptyset \\ x & \mapsto & \infty \qquad\qquad\qquad\quad \textit{otherwise} \end{cases}
$$

Proof. We define $X_i = \rho_\phi^{-1}(i)$ for all integer $i \in \mathbf{N}$. Then it is clear that X_i is equal to the fixed point set $\mathscr{F}(\phi^i)$ of ϕ^i. Applying Theorem 39, we deduce that X_i is measurable. Finally, the set $X_\infty = X - \cup_{i \in \mathbf{N}} X_i$ is also measurable. $\qquad\square$

---

[8]And satisfying the first axiom of separation T_0, but this will be strengthened.

[9]In fact, this restriction could be weakened, as integration with respect to Radon measures can be defined in non-Hausdorff spaces. However it is not clear that the weakening this condition would be of interest. We therefore consider the case of Hausdorff spaces, keeping in mind that this is not an essential condition.

5.2. Circuit-Quantifying Maps for Measure-Inflating Transformations

We have chosen to explain the construction of circuit-quantifying maps on the microcosm of measure-inflating transformations first. Indeed, this particular case allows for a simpler definition of the maps which should be more intuitive for the reader. We will then built on the results of this section to define circuit-quantifying maps in the general setting.

Definition 40. Let $\phi : X \to X$ be a non-singular transformation. We define the measurable set:

$$\{\phi\} = \bigcap_{n \in \mathbf{N}} \phi^n(X) \cap \phi^{-n}(X)$$

Definition 41. Let π be a cycle in the weighted graphing F. Then the map ϕ_π restricted to $X = \{\phi_\pi\}$ is a non-singular transformation $X \to X$. We can then define the map ρ_{ϕ_π} on X. We define the *support* $\mathrm{supp}(\pi)$ of π as the set $\rho_{\phi_\pi}^{-1}(\mathbf{N})$.

Remark. In the author's PhD, a similar work was presented, only restricted to the particular case of the microcosm of measure-preserving maps on the real line. We showed in this case the existence of a family of circuit-quantifying maps. Indeed, for any measurable map $m : \Omega \to \mathbf{R} \cup \{\infty\}$, we defined q_m as the function:

$$q_m : \pi \mapsto \int_{\mathrm{supp}(\pi)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda(x)$$

This function was a circuit-quantifying map for the above mentioned microcosm. As it turns out, this approach can be generalized to the microcosm of measure-preserving maps on any trefoil space by using the very same formula.

We now want to extend these circuit-quantifying maps to the microcosm of measure-inflating maps[10] on any trefoil space (i.e. transporting μ to a scalar multiple of μ). One easy way to do so is by considering an extension using push-forward measures. Recall that if μ is a measure on X, and $f : X \to Y$ a measurable map, then the push-forward measure $\mu_* f$ is defined by $\mu_* f(A) = \mu(f^{-1}(A))$ and satisfies, for all g measurable such that $g \circ f$ is integrable (this is equivalent to saying that g is $\mu_* f$ integrable):

$$\int_Y g(y) d\mu_* f(y) = \int_X g(f(x)) d\mu(x)$$

Definition 42. Let $(X, \mathscr{B}, \mu)$ be a trefoil space, and $m : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$ be a measurable map. We define the function:

$$q_m : \pi = e_0 \dots e_n \mapsto \frac{1}{n+1} \sum_{i=0}^{n} \int_{\mathrm{supp}(\pi)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_i(x)$$

---

[10] We use this terminology for maps that transport the measure μ onto a scalar multiple of μ.

Corollary 42.1. *Let $(X, \mathcal{B}, \mu)$ be a trefoil space, $\mathfrak{m}$ the microcosm of measure-preserving maps, and $m : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$ be a measurable map. The map q_m can be expressed as:*

$$q_m : \pi \mapsto \int_{supp(\pi)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda(x)$$

Proof. If all ϕ_k are measure-preserving maps, then we have, for all integer j, the following equality:

$$\int_{\text{supp}(\pi)_j} \frac{m(\omega(\pi)^j)}{j} d\lambda_* \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_i(x) = \int_{\text{supp}(\pi)_j} \frac{m(\omega(\pi)^j)}{j} d\lambda(x)$$

We can then compute:

$$
\begin{aligned}
q_m(\pi) &= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \int_{\text{supp}(\pi)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_i(x) \\
&= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \sum_{j \in \mathbf{N}} \int_{\text{supp}(\pi)_j} \frac{m(\omega(\pi)^j)}{j} d\lambda_* \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_i(x) \\
&= \sum_{j \in \mathbf{N}} \sum_{i=0}^{\lg(\pi)-1} \int_{\text{supp}(\pi)_j} \frac{1}{\lg(\pi)} \frac{m(\omega(\pi)^j)}{j} d\lambda_* \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_i(x) \\
&= \sum_{j \in \mathbf{N}} \sum_{i=0}^{\lg(\pi)-1} \int_{\text{supp}(\pi)_j} \frac{1}{\lg(\pi)} \frac{m(\omega(\pi)^j)}{j} d\lambda(x) \\
&= \sum_{j \in \mathbf{N}} \int_{\text{supp}(\pi)_j} \frac{m(\omega(\pi)^j)}{j} d\lambda_* \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_i(x) \\
&= \int_{\text{supp}(\pi)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda(x)
\end{aligned}
$$

$\square$

Lemma 43. *For all measurable map m, the function q_m has a constant value on the equivalence classes of cycles modulo the action of cyclic permutations.*

Proof. Let $\pi = e_0 e_1 \ldots e_n$ be a cycle, $\text{supp}(\pi)$ its support. For all $i \in \mathbf{N}$, we write $(\text{supp}(\pi))_i = \rho_\pi^{-1}(i)$. Consider now $\pi^1 = e_1 e_2 \ldots e_n e_0$, and $\text{supp}(\pi^1)$ its support. We define $(\text{supp}(\pi^1))_i = \rho_{\pi^1}^{-1}(i)$. We will first show that $(\text{supp}(\pi^1))_i = \phi_{e_0}((\text{supp}(\pi))_i)$ for all integer i.

Let us now pick $x \in (\text{supp}(\pi^1))_i$, which means that $x \in \text{supp}(\pi^1)$ and $\phi_{\pi^1}^i(x) = x$. Since $\phi_{\pi^1}(x) = \phi_{e_0}(\phi_{e_1 \ldots e_n}(x))$, we have $x = \phi_{e_0}(\phi_{e_1 \ldots e_n} \phi_{\pi^1}^{i-1}(x))$. We now define $y = \phi_{e_1 \ldots e_n} \phi_{\pi^1}^{i-1}(x))$ and we will show that $y \in (\text{supp}(\pi))_i$. Since $\phi_{e_0}(y) \in \text{supp}(\pi^1)$,

we have $\phi_{e_0} \in S_{e_1 \ldots e_n}$, and therefore $y \in S_\pi$. Moreover,

$$
\begin{aligned}
\phi_\pi^k(y) &= \phi_\pi^i(\phi_{e_1 \ldots e_n} \phi_{\pi^1}^{i-1}(x)) \\
&= \phi_\pi(\phi_\pi^{i-2}(\phi_{e_1 \ldots e_n} \phi_{\pi^1}^{i-1}(x))) \\
&= \phi_{e_1 \ldots e_n}(\phi_{e_0}(\phi_\pi^{i-1}(\phi_{e_1 \ldots e_n}(\phi_{\pi^1}^{i-1}(x))))) \\
&= \phi_{e_1 \ldots e_n}(\phi_{\pi^1}^i(\phi_{\pi^1}^{i-1}(x))) \\
&= \phi_{e_1 \ldots e_n}(\phi_{\pi^1}^{i-1}(\phi_{\pi^1}^i(x))) \\
&= \phi_{e_1 \ldots e_n}(\phi_{\pi^1}^{i-1}(x)) \\
&= y
\end{aligned}
$$

Thus y is an element in $\operatorname{supp}(\pi^1)$, and more precisely an element in $(\operatorname{supp}(\pi^1))_i$. We therefore showed that $(\operatorname{supp}(\pi^1))_i \subset \phi_{e_0}((\operatorname{supp}(\pi))_i)$.

To show the converse inclusion, we take $x = \phi_{e_0}(y)$ with $y \in (\operatorname{supp}(\pi))_i$. Then $y \in S_{\pi^k}$ and therefore $y \in S_{e_0 e_1 \ldots e_n e_0}$. Finally $\phi_{e_0}(y) \in S_{\pi^1}$. Moreover, we have:

$$
\begin{aligned}
\phi_{\pi^1}^k(x) &= \phi_{\pi^1}^k(\phi_{e_0}(y)) \\
&= \phi_{e_0}(\phi_\pi^k(y)) \\
&= \phi_{e_0}(y) \\
&= x
\end{aligned}
$$

As a consequence, x is an element in $(\operatorname{supp}(\pi))_i$, which shows the converse inclusion.

More generally, if π^k denotes the cycle $e_k e_{k+1} \ldots e_n e_0 \ldots e_{k-1}$, we have

$$
\phi_{e_0 \ldots e_k}(\operatorname{supp}(\pi)_i) = \operatorname{supp}(\pi^k)_i
$$

A similar argument shows that $\phi_{e_n}(\operatorname{supp}(\pi^n)_i) = \operatorname{supp}(\pi)_i$.

We then compute:

$$
\begin{aligned}
q_m(\pi) &= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \int_{\operatorname{supp}(\pi)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_i}(x) \\
&= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \sum_{j \in \mathbf{N}} \int_{\operatorname{supp}(\pi)_j} \frac{m(\omega(\pi)^j)}{j} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x) \\
&= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \sum_{j \in \mathbf{N}} \int_{\phi_{e_n}^{-1}(\operatorname{supp}(\pi)_j)} \frac{m(\omega(\pi)^j)}{j} d\lambda_* \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_i}(x) \\
&= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \sum_{j \in \mathbf{N}} \int_{\operatorname{supp}(\pi^n)_j} \frac{m(\omega(\pi)^j)}{j} d\lambda_* \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_i}(x) \\
&= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \int_{\operatorname{supp}(\pi^n)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_i}(x)
\end{aligned}
$$

One can now notice that $\lambda_* \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_0 = \lambda$. Indeed, ϕ_π is measure-inflating as a composition of measure-inflating maps. But since it has its domain

equal to its codomain, it is necessarily measure-preserving. This implies that for all measurable map $f : X \to \mathbf{R}$:

$$\int_{\mathrm{supp}(\phi_{pi})} f(x)d\lambda(x) = \int_{\mathrm{supp}()\phi_{pi}} f(x)d\lambda_*\phi_{e_{n-1}} \circ \phi_{e_{n-2}} \circ \cdots \circ \phi_{e_0} \circ \phi_{e_n}(x)$$

Using this equality, we compute:

$$
\begin{aligned}
q_m(\pi) &= \frac{1}{\lg(\pi)}\int_{\mathrm{supp}(\pi^n)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda(x) + \ldots \\
&\quad \ldots + \frac{1}{\lg(\pi)}\sum_{i=0}^{\lg(\pi)-2}\int_{\mathrm{supp}(\pi^n)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_*\phi_{e_{n-1}}\circ\cdots\circ\phi_{e_i}(x) \\
&= \frac{1}{\lg(\pi)}\int_{\mathrm{supp}(\pi^n)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_*\phi_{e_{n-1}}\circ\phi_{e_{n-2}}\circ\cdots\circ\phi_{e_0}\circ\phi_{e_n}(x) + \ldots \\
&\quad \ldots + \frac{1}{\lg(\pi)}\sum_{i=0}^{\lg(\pi)-2}\int_{\mathrm{supp}(\pi^n)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_*\phi_{e_{n-1}}\circ\cdots\circ\phi_{e_i}(x) \\
&= q_m(\pi^n)
\end{aligned}
$$

Thus, by induction, $q_m(\pi) = q_m(\pi^j)$ for all integer j. We can now conclude that q_m takes a constant value over the equivalence classes of cycles modulo the action of cyclic permutations. $\square$

Lemma 44. *For all measurable map m, the function q_m is refinement-invariant.*

Proof. Let F, G be weighted graphings, and $F^{(e)}$ a simple refinement of F along $e \in E^F$. We will denote by f, f' the two elements of $F^{(e)}$ which are the decompositions of e. Up to almost everywhere equality, one can suppose that $S_f \cap S_{f'} = \emptyset$. Let us now chose π a representative of a 1-circuit $\bar{\pi}$. Since we are working with 1-circuits, the set π^ω is equal to $\{\pi\}$. We suppose that π contains occurrences of e, and write $\pi = \rho_0 e_{i_0}\rho_1 e_{i_1}\ldots e_{i_{n-1}}\rho_n$ where for all j, $e_{i_j} = e$ and ρ_j is a path (where the paths ρ_0 and ρ_n may be empty). We denote by E_π the set of 1-cycles $\mu = \rho_0\epsilon_{i_0^0}\rho_1\epsilon_{i_1^0}\ldots\epsilon_{i_{n-1}^0}\rho_n\rho_0\epsilon_{i_0^1}\rho_1\ldots\epsilon_{n-1}^1\rho_n\ldots\rho_0\epsilon_{i_0^k}\ldots\epsilon_{i_{n-1}^k}\rho_n$ where $k \in \mathbf{N}$ — which we will denote by $\lg(\mu)$, and where for all values of l, m, $\epsilon_{i_l^m}$ is either equal to f or equal to f'. We will denote by $\bar{E}_\pi$ the set of 1-circuits in E_π, i.e. the set $E_\pi^{(F,\theta)}$ introduced in Theorem 30.

Let us pick $x \in \mathrm{supp}(\pi) - \rho_\pi^{-1}(\infty)$. Then $x \in (\mathrm{supp}(\pi))_k$ for a given value k in $\mathbf{N}$, i.e. $\phi_\pi^k(x) = x$. Since $S_e = S_f \cup S_{f'}$, we have, for each occurrence e_{i_p} of e and each integer l:

$$\phi_\pi^k = \phi_\pi^l \circ \phi_{\rho_{p+1}e_{i_{p+1}}\ldots e_{i_n}\rho_n} \circ \phi_{e_{i_p}} \circ \phi_{\rho_0 e_{i_0}\rho_1\ldots e_{i_{p-1}}\rho_j} \circ \phi_\pi^{k-l-1}$$

Then $\phi_{\rho_0 e\ldots\rho_j} \circ \phi_\pi^{k-l-1}(x)$ is either an element in S_f or an element in $S_{f'}$. For each occurrence e_i of e, we will write $d_{i_{p,l}} = f$ or $d_{i_{p,l}} = f'$ according to wether $\phi_{\rho_0 e\ldots\rho_j} \circ \phi_\pi^{k-l-1}(x)$ is an element in S_f or an element in $S_{f'}$. We then obtain, for

all integer $0 \leqslant l \leqslant k$, paths $v_l = \rho_0 d_{i_{0,l}} \rho_1 d_{i_{1,l}} \ldots d_{i_{n-1,l}} \rho_n$. By concatenation, we can define a cycle $v = v_0 v_1 \ldots v_k$. This cycle is a d-cycle for a given integer d, i.e. $v = \tilde{\pi}^d$ where $\tilde{\pi}$ is a 1-cycle in E_π. It is clear from the definition of $\tilde{\pi}$ that $x \in \mathrm{supp}(\tilde{\pi})$ and that, for all 1-cycle μ in E_π, $x \notin \mathrm{supp}(\mu)$ when $\mu \neq \tilde{\pi}$.

Moreover, it is clear that if $x \in \mathrm{supp}(\mu)$ for a given 1-cycle $\mu \in E_\pi$, then one necessarily has $x \in \mathrm{supp}(\pi)$. We deduce from this that the family $(\mathrm{supp}(\mu))_{\mu \in E_\pi}$ is a partition of the set $\mathrm{supp}(\pi)$. Notice that $\omega_\mu = \omega_\pi^{\lg(\mu)}$. Moreover, for all $x \in \mathrm{supp}(\mu)$, one has $\rho_{\phi_\mu}(x) \times \lg(\mu) = \rho_{\phi_\pi}(x)$, and therefore $\omega_\pi^{\rho_{\phi_\pi}(x)} = \omega_\mu^{\rho_{\phi_\mu}(x)}$.

We now notice that if $\mu = \mu_1 \ldots \mu_{\lg(\mu)} \in E_\pi$, and if σ is the cyclic permutations over $\{1, \ldots, \lg(\mu)\}$ such that $\sigma(i) = i + 1$, then the 1-cycles

$$\mu_{\sigma^k} = \mu_{\sigma^k(1)} \mu_{\sigma^k(2)} \cdots \mu_{\sigma^k(\lg(\mu))}$$

for $0 \leqslant k \leqslant \lg(\mu) - 1$ are pairwise disjoint elements in E_π. Indeed, these are 1-cycles since μ is a 1-cycle, and they are pairwise disjoint because if $\mu_{\sigma^k} = \mu_{\sigma_{k'}}$ (supposing that $k > k'$), we can show that $\mu_{\sigma(k-k')} = \mu$ and that $k - k'$ divides $\lg(\mu)$, which contradicts the fact that μ is a 1-cycle.

We can now deduce that:

$$\int_{\mathrm{supp}(\pi)} \frac{m(\omega_\pi^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x)$$

$$= \sum_{\mu \in E_\pi} \int_{\mathrm{supp}(\mu)} \frac{m(\omega_\pi^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x)$$

$$= \sum_{\mu \in E_\pi} \int_{\mathrm{supp}(\mu)} \frac{m(\omega_\mu^{\rho_{\phi_\mu}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x)$$

$$= \sum_{\bar{\mu} \in E_\pi} \int_{\mathrm{supp}(\bar{\mu})} \frac{lg(\bar{\mu}) m(\omega_{\bar{\mu}}^{\rho_{\phi_{\bar{\mu}}}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x)$$

$$= \sum_{\bar{\mu} \in E_\pi} \int_{\mathrm{supp}(\bar{\mu})} \frac{lg(\bar{\mu}) m(\omega_{\bar{\mu}}^{\rho_{\phi_{\bar{\mu}}}(x)})}{\rho_{\phi_{\bar{\mu}}}(x) \times lg(\bar{\mu})} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x)$$

$$= \sum_{\bar{\mu} \in \bar{E}_\pi} \int_{\mathrm{supp}(\bar{\mu})} \frac{m(\omega_{\bar{\mu}}^{\rho_{\phi_{\bar{\mu}}}(x)})}{\rho_{\phi_{\bar{\mu}}}(x)} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x)$$

We will use in this computation the fact that if $\bar{\mu} \in \bar{E}_\pi$, the associated map $\phi_{\bar{\mu}}$ is equal to ϕ_π^k for k is defined by $\lg(\bar{\mu}) = k \times \lg(\pi)$. We will also need i.e. the computation to name the edges in an element $\bar{\mu} \in \bar{E}_\pi$; we will denote them by $f_0, f_1, \ldots, f_p$ where $p = \lg(\bar{\mu})$. Using this notation and the preceding remark, we have that for all measurable map $f : X \to \mathbf{R}$ and all integer $l \in \{0, \ldots, k-1\}$:

$$\int_{\mathrm{supp}(\bar{\mu})} f(x) d\lambda_* (\phi_\pi)^l \circ \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_i}(x) = \int_{\mathrm{supp}(\bar{\mu})} f(x) d\lambda_* \phi_{f_p} \circ \phi_{f_{p-1}} \circ \cdots \circ \phi_{f_{i+(k-l)\lg(\pi)}}$$

Using what we have proved up to now, we can now compute q_m:

$$
\begin{aligned}
q_m(\pi) &= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \int_{\mathrm{supp}(\pi)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_i}(x) \\[2mm]
&= \frac{1}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \sum_{\bar{\mu} \in \bar{E}_\pi} \int_{\mathrm{supp}(\bar{\mu})} \frac{m(\omega_{\bar{\mu}}^{\rho_{\phi_{\bar{\mu}}}(x)})}{\rho_{\phi_{\bar{\mu}}}(x)} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x) \\[2mm]
&= \sum_{\bar{\mu} \in \bar{E}_\pi} \frac{1}{\lg(\mu)} \left(\frac{\lg(\mu)}{\lg(\pi)} \sum_{i=0}^{\lg(\pi)-1} \int_{\mathrm{supp}(\bar{\mu})} \frac{m(\omega_{\bar{\mu}}^{\rho_{\phi_{\bar{\mu}}}(x)})}{\rho_{\phi_{\bar{\mu}}}(x)} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x) \right) \\[2mm]
&= \sum_{\bar{\mu} \in \bar{E}_\pi} \frac{1}{\lg(\mu)} \left(\sum_{l=0}^{\frac{\lg(\mu)}{\lg(\pi)}-1} \sum_{i=0}^{\lg(\pi)-1} \int_{\mathrm{supp}(\bar{\mu})} \frac{m(\omega_{\bar{\mu}}^{\rho_{\phi_{\bar{\mu}}}(x)})}{\rho_{\phi_{\bar{\mu}}}(x)} d\lambda_* \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x) \right) \\[2mm]
&= \sum_{\bar{\mu} \in \bar{E}_\pi} \frac{1}{\lg(\mu)} \left(\sum_{l=0}^{\frac{\lg(\mu)}{\lg(\pi)}-1} \sum_{i=0}^{\lg(\pi)-1} \int_{\mathrm{supp}(\bar{\mu})} \frac{m(\omega_{\bar{\mu}}^{\rho_{\phi_{\bar{\mu}}}(x)})}{\rho_{\phi_{\bar{\mu}}}(x)} d\lambda_* \phi_\pi^l \circ \phi_{e_n} \circ \cdots \circ \phi_{e_i}(x) \right) \\[2mm]
&= \sum_{\bar{\mu} \in \bar{E}_\pi} \frac{1}{\lg(\mu)} \sum_{i=0}^{\lg(\mu)-1} \int_{\mathrm{supp}(\bar{\mu})} \frac{m(\omega_{\bar{\mu}}^{\rho_{\phi_{\bar{\mu}}}(x)})}{\rho_{\phi_{\bar{\mu}}}(x)} d\lambda_* \phi_{f_p} \circ \phi_{f_{p-1}} \circ \cdots \circ \phi_{f_i}(x) \\[2mm]
&= \sum_{\bar{\mu} \in \bar{E}_\pi} q_m(\bar{\mu})
\end{aligned}
$$

Which shows that q_m is refinement-invariant. $\qquad\square$

The two preceding lemmas have as a direct consequence the following proposition which shows that we defined a family of circuit-quantifying maps.

Proposition 45. *Let $(X, \mathscr{B}, \mu)$ be a trefoil space, $\mathfrak{m}$ the microcosm of measure-inflating maps, and $m : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$ be a measurable map. The function q_m is a $\mathfrak{m}$-circuit-quantifying map.*

5.3. Circuit-Quantifying Map in the General Case

This result can now be extended to the microcosm of all non-singular Borel preserving transformations (the *macrocosm* on X). We first show an easy lemma.

Lemma 46. *If $\rho : X \to \mathbf{N}$ is measurable and for all $i \in \mathbf{N}$ the maps ϕ_i are measurable, then the following map is measurable:*

$$
f(x) = \sum_{i=0}^{\rho(x)} \phi_i(x)
$$

Proof. Indeed, if X_i denotes the measurable set $\rho^{-1}(i)$ for all integer i, then the restriction of f to X_i is equal to the finite sum $\sum_{k=0}^{i} \phi_i(x)$ which is measurable on X_i. $\qquad\square$

This lemma insures us that the following definition makes sense.

Definition 47. Let X be a trefoil space, and $m : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$ be a measurable function. We define the map:

$$\bar{q}_m : \pi = e_0 \dots e_n \mapsto \sum_{j=0}^{n+1} \int_{\mathrm{supp}(\pi)} \sum_{k=0}^{\rho_{\phi_\pi}(x)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(\phi_\pi^k(x))})}{(n+1)^{\rho_\pi(x)} \rho_{\phi_\pi}(\phi_\pi^k(x))} d\lambda_* \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_j}(x)$$

We now have to check that Theorem 43 and Theorem 44 still hold in this general setting. This can easily be seen because of the following computation, where we use the convention that e_k denote $e_{k \mod n+1}$ and $\tilde{\pi}^i$ denotes the restriction of π^i, the i-times concatenation of π, to $\mathrm{supp}(\pi)_i$:

$$
\begin{aligned}
\bar{q}_m(\pi) &= \sum_{j=0}^{\mathrm{lg}(\pi)} \int_{\mathrm{supp}(\pi)} \sum_{k=0}^{\rho_{\phi_\pi}(x)} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(\phi_\pi^k(x))})}{\mathrm{lg}(\pi)^{\rho_\pi(x)} \rho_{\phi_\pi}(\phi_\pi^k(x))} d\lambda_* \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_j}(x) \\
&= \sum_{j=0}^{\mathrm{lg}(\pi)} \sum_{i \in \mathbf{N}} \int_{\mathrm{supp}(\pi)_i} \sum_{k=0}^{i} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(\phi_\pi^k(x))})}{\mathrm{lg}(\pi)^i \rho_{\phi_\pi}(\phi_\pi^k(x))} d\lambda_* \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_j}(x) \\
&= \sum_{j=0}^{\mathrm{lg}(\pi)} \sum_{i \in \mathbf{N}} \sum_{k=0}^{i} \int_{\mathrm{supp}(\pi)_i} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(\phi_\pi^k(x))})}{\mathrm{lg}(\pi)^i \rho_{\phi_\pi}(\phi_\pi^k(x))} d\lambda_* \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_j}(x) \\
&= \sum_{i \in \mathbf{N}} \sum_{j=0}^{\mathrm{lg}(\pi)} \frac{1}{\mathrm{lg}(\pi)^i} \sum_{k=0}^{i} \int_{\mathrm{supp}(\pi)_i} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(\phi_\pi^k(x))})}{\rho_{\phi_\pi}(\phi_\pi^k(x))} d\lambda_* \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_j}(x) \\
&= \sum_{i \in \mathbf{N}} \sum_{j=0}^{\mathrm{lg}(\pi)} \frac{1}{\mathrm{lg}(\pi)^i} \sum_{k=0}^{i} \int_{\mathrm{supp}(\pi)_i} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_\pi^k \circ \phi_{e_n} \circ \phi_{e_{n-1}} \circ \cdots \circ \phi_{e_j}(x) \\
&= \sum_{i \in \mathbf{N}} \sum_{j=0}^{\mathrm{lg}(\pi^i)} \frac{1}{\mathrm{lg}(\pi^i)} \int_{\mathrm{supp}(\pi)_i} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda_* \phi_{e_{n \times i}} \circ \phi_{e_{n \times i-1}} \circ \cdots \circ \phi_{e_j}(x) \\
&= \sum_{i \in \mathbf{N}} q_m(\tilde{\pi}^i)
\end{aligned}
$$

From this result, and the fact that $\phi_{\tilde{\pi}^i}$ is measure-preserving, one can adapt the proofs of Theorem 43 and Theorem 44, and show the following theorem which, together with Theorem 22 and Theorem 35, finishes the proof of Theorem 1.

Theorem 48. *Let $(X, \mathscr{B}, \mu)$ be a trefoil space, $\mathfrak{m}$ the associated macrocosm, and $m : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$ be a measurable map. The map $\bar{q}_m$ is a $\mathfrak{m}$-circuit-quantifying map.*

Example. This setting is a far-reaching extension of our previous work on directed weighted graphs [Sei12a]. Indeed, this previous framework is recovered as the special case of a discrete space endowed with the counting measure. In this case, one can notice that the map ρ_{ϕ_π} is constantly equal to 1 and therefore the family of measurement just defined can be computed with the simpler expression $\bar{q}_m(\pi) = m(\omega(\pi))$. The family of measurements defined from these functions thus turn out to be equal to the family of measurement considered on graphs [Sei12a]. In particular, the measurement defined from the map $\bar{q}_m = -\log(1-x)$ corresponds to Girard's measurement based on the determinant [Sei12b].

Example. Let us consider the trefoil space $X = [0,1]$ endowed with Lebesgue measure, and the microcosm of measure-preserving maps. Since each measure-preserving map on X defines a unitary acting on the Hilbert space $L^2(X)$ by pre-composition, it is easy to associate to any **C**-weighted graphing G a linear combination $[G]$ of partial isometries on $L^2(X)$. This might not define an operator in general since the obtained operator might not be bounded, but we will restrict the discussion to the set of graphings for which $[G]$ is an operator. This set of graphings can be shown to have the following properties:

- it contains, for each integer k, the "$k \times k$-matrices algebra[11]" of graphings constructed from translations between intervals $I_l = [\frac{l}{k}, \frac{l+1}{k}]$, i.e. directed weighted graphs on k vertices;

- it has a trace: for each $f \in \mathfrak{m}$ one can define $tr(f)$ as the measure of the set of fixed points of f; this trace, when restricted to the $k \times k$ matrix algebra defined above yields the usual (normalized, i.e. $tr(1) = 1$) trace of matrices;

Thus, the set of such graphings plays the rôle of the type II_1 hyperfinite factor. The same reasoning shows that the **C**-weighted graphings in the microcosm of measure-preserving maps on $X = \mathbf{R}$ with the Lebesgue measure play the rôle of the type II_∞ hyperfinite factor.

Remark. We did not show here a formal correspondance, but a proof of such a result most surely exists. In particular, in the case where X is the real line, it is known that the type II_∞ factor arises as the von Neumann algebra generated by:

- elements of $L^\infty(\mathbf{R})$ acting on $L^2(\mathbf{R})$ by multiplication;

- the unitaries induced by precomposition by rational translation.

We did not think however that such a result would be of great interest in this paper, as we already know from the discrete case discussed above and previous results [Sei12b, Sei12a] that our setting generalizes Girard's constructions using operator algebras.

5.4. Example: Unification "Algebras"

As already mentioned, it can be shown using previous results [Sei12b, Sei12a] that the framework of graphing generalizes Girard's constructions based on operators [Gir89a, Gir88, Gir11]. We will now explain how the alternative approach he uses, namely using "algebras of clauses" [Gir95b] or "unification algebra" [Gir13a, Gir13b], is also a particular case of our constructions on graphings.

We thus show how Girard's notions of flows and wirings can be understood in terms of graphings. This gives intuitions on what he calls the "unification algebra" which is nothing more than the algebra generated by the set of graphings on the adequate space $B(\Sigma)$.

In the following we fix a countable (infinite) set of variables Var.

---

[11]Of course, this is not the matrix algebra, but one can show that the operator $[G]$ associated to such a graphing G is the image of a $k \times k$ matrix through a well-chosen injective morphism.

Definition 49. A *signature* is a tuple (Const, Fun) where Const contains *symbols of constants* and Fun contains a finite number of *symbols of functions*. We say the signature is *free* if the set Const is empty.

Definition 50. The terms defined by a signature Σ are defined by the grammar:

$$T := x \mid c \mid f(T, \ldots, T) \qquad (x \in \text{Var}, c \in \text{Const}, f \in \text{Fun})$$

A *closed term* is a term that does not contain any variables. A term which is not closed is said to be *open*.

Definition 51. If Σ is a free signature, there are no closed terms. In this particular case, we define the set of closed terms as the trees defined co-inductively as follows:

$$T := f(T, \ldots, T) \quad (f \in \text{Fun})$$

We can understand these closed terms as infinite rooted trees labelled by function symbols.

Definition 52. Let Σ be a free signature. We define the topological space $B(\Sigma)$ as the set of closed terms considered with the topology induced by the set of open terms: $\mathcal{O}(u(x_1, \ldots, x_n)$ is defined as the set of closed terms $u(t_1, \ldots, t_n)$ where the t_i are closed terms. This space can be endowed with a σ-finite radon measure λ defined inductively following the definition of open terms. We define for each enumeration $e : \text{Fun} \to \mathbf{N}^*$:

$$
\begin{aligned}
\lambda_e(x_i) &= 1 && \text{when } x_i \in \text{Var} \\
\lambda_e(f_i(t_1, \ldots, t_{k_i})) &= \tfrac{1}{2^{e(f_i)}} \textstyle\sum_{i=1}^{k_i} \lambda_e(t_i)
\end{aligned}
$$

Remark. One can define other measures (which are more satisfying in some respect) in some specific cases:

- If the set Fun is finite, we write $K = \sum_{f_i \in \text{Fun}} k_i$, where k_i is the arty of f_i, and define:

$$
\begin{aligned}
\lambda(x_i) &= 1 && \text{when } x_i \in \text{Var} \\
\lambda(f_i(t_1, \ldots, t_{k_i})) &= \tfrac{1}{K} \textstyle\sum_{i=1}^{k_i} \lambda(t_i)
\end{aligned}
$$

- If the set Fun is infinite but the number of functions of a given arity k is finite (we denote it by a_k), we can define:

$$
\begin{aligned}
\lambda(x_i) &= 1 && \text{when } x_i \in \text{Var} \\
\lambda(f_i(t_1, \ldots, t_{k_i})) &= \tfrac{1}{a_i \times 2^{k_i+1}} \textstyle\sum_{i=1}^{k_i} \lambda(t_i)
\end{aligned}
$$

Definition 53. Let Σ be a non-free signature. We define the topological space $B(\Sigma)$ as the set of closed terms endowed with the discrete topology. This space can be endowed with the counting measure.

Theorem 54. *For any signature Σ, the space $B(\Sigma)$ is a trefoil space.*

Proof. It is clear in the case of a non-free signature. In the case of a free signature, the space is clearly Hausdorff. It is second-countable since we defined the topology as induced by a countable number of open sets. □

Definition 55. A *flow* is an ordered pair $u \leftharpoonup t$, where u, t are terms with the same variables. A *wiring* is a sum of flows.

Definition 56. A flow $u(x_1, \ldots, x_k) \leftharpoonup t(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)$ represents the following non-singular borel-preserving map from the open $\mathcal{O}(t)$ to the open $\mathcal{O}(u)$:

$$[u \leftharpoonup t] := t(T_1, \ldots, T_k) \mapsto u(T_1, \ldots, T_k)$$

Given a wiring W, we can therefore associate a graphing $[W]$ to it.

Theorem 57. *Let Σ be a signature. The map $W \mapsto [W]$ commutes with execution.*

Proof. This is shown easily by looking at the definition of composition of flows. A global substitution is a map from the set of variables to the set of terms, and we denote by $v\theta$ the result of the substitution of each variable x_i in v by the term $\theta(x_i)$. We say two terms v, v' are *unifiable* when there exists a global substitution θ such that $v\theta = v'\theta$. In this case, there exists a *principal unifier*, i.e. a substitution θ_0 such that any substitution θ satisfying $v\theta = v'\theta$ can be factorized through θ_0, i.e. there exists θ' such that $\theta = \theta_0\theta'$. The composition $(u \leftharpoonup v)(v' \leftharpoonup w)$ is then equal to 0 if v and v' are not unifiable, and to $u\theta_0 \leftharpoonup w\theta_0$ if they are unifiable and θ_0 is the principal unifier.

Now, it is not hard to see that two terms u, v are unifiable if and only if the open sets $\mathcal{O}(u)$ and $\mathcal{O}(v)$ have a non-trivial (of strictly positive measure) intersection. This intersection is then an open set equal to $\mathcal{O}(u\theta_0) = \mathcal{O}(v\theta_0)$ where θ_0 is the principal unifier of u and v. Thus composition of flows corresponds to considering the partial composition of the associated measurable maps.

This implies that the composition of wirings $W \circ W' = (\sum_{i \in I} f_i) \circ (\sum_{j \in J} g_j)$, which is defined as $(\sum_{i \in I, j \in J} f_i \circ g_j$, corresponds to taking the graphing of alternating paths of length 2 between the graphings $[W]$ and $[W']$.

Finally, the execution formula[12] $\mathrm{Ex}(U, \sigma) = (1 - \sigma^2)U(1 - \sigma U)^{-1}(1 - \sigma^2)$, which is computed as:

$$\mathrm{Ex}(U, \sigma) = (1 - \sigma^2)\left(\sum_{i \geq 0} U(\sigma U)^k\right)(1 - \sigma^2)$$

corresponds to the execution of graphings $[U] :\!\mathrm{m}\!: [\sigma]$ because:

- the conjugation by $(1 - \sigma^2)$ is used to restrict the result to wirings living *outside of the cut*, which is dealt with in the execution of graphings by considering the restriction of paths ϕ to their *outside component* $[\phi]_o^o$;

---

[12]We restrict ourselves here to the simple case of the execution formula where σ corresponding to the modus ponens. The case of the more involved formula corresponding to the general cut rule obviously holds, as it can be recovered from the simpler one considered here.

- for each integer k, the terms $U(\sigma U)^k$ correspond to the set of alternating paths of length $2k + 1$ as we already noticed, which are the only possible lengths of alternating paths in this case[13]

Thus the embedding $W \hookrightarrow [W]$ commutes with execution, i.e. the execution of graphings computes the execution formula on wirings. $\square$

This shows that the notion of graphing is a non-trivial generalization of the notion of wirings considered lately by Girard. In particular, the construction of GoI models based on wirings can be expressed in terms of graphings. Moreover, the notion of graphing is much more powerful than Girard's notion of wiring. Indeed, the syntactic definitions of wiring do not allow for the quantitative features of graphings, namely the family of measurement considered above. In particular, the only definable notion of orthogonality one can consider on wiring is defined as the nilpotency of the product of two wirings, which corresponds to the measurement defined above with the dull circuit-quantifying map $m(x) = \infty$.

5.5. *Digression: Topological Graphings*

One generalization of Girard's framework based on unification would be to consider a weakened definition of flow $t \leftharpoonup u$ where the variables of u and t do not match exactly but only the inclusion $\mathrm{Var}(u) \subset \mathrm{Var}(t)$ holds. In this case, the interpretation of flows as non-singular maps is no longer valid as the "weakening" thus allowed makes it possible that the inverse image of a set of measure zero is of strictly positive measure. This is seen by taking the inverse image through $[u(x) \leftharpoonup v(x,y)]$ of a closed term $u(T)$ in the free signature case.

This mismatch is due to the fact that in the process of generalizing the notion of flows, we stepped outside of measure theory. The map interpreting the flows are no longer non-singular (they are still measurable though, but non-singularity is necessary to obtain the associativity of execution), but they are continuous. A topological notion of graphing, corresponding somehow to a notion of *pseudo-monoid* to recall Cartan's notion of pseudo-group [Car04, Car09, KN96], could be applied here instead of our measurable approach.

Indeed, define a *topological graphing* on a topological space X as a countable family $F = \{(\omega_e^F, \phi_e^F : S_e^F \to T_e^F)\}_{e \in E^F}$, where, for all $e \in E^F$ (the set of *edges*):
- ω_e^F is an element of Ω, the *weight* of the edge e;
- S_e^F and T_e^F are open sets, the *source* and *target* of the edge e;
- ϕ_e^F is an open continuous map from S_e^F to T_e^F, the *realization* of the edge e.

Then the notions of paths and cycles can be defined as in the more complex case of measurable graphings considered until now. We can therefore define the execution between topological graphings and show associativity. We note here that the mismatch with associativity in the measurable case which arose from the non-singularity is no longer a problem since we do not quotient by sets of measure 0 anymore. In the same way we defined refinements, one can define refinements

---

[13]This is due to the fact that we restricted to the simpler case of the execution formula, see footnote 12.

in this topological setting and define a corresponding equivalence relation. It is easily shown that execution is compatible with this equivalence relation, and we therefore can mimic almost all results of Section 3 and Section 4, forgetting about almost-everywhere equality. However, the contents of Section 5 depends greatly on the fact that we are dealing with measurable spaces. The only obvious way to obtain the trefoil property in the topological case is therefore to consider the measurement to be ∞ when there exists a cycle and 0 otherwise. This means that the only sensible notion of orthogonality one can define corresponds to nilpotency. Of course, one may be able to define other measurements, but it would be much more difficult than in the measurable case where we can use the radon measure on the space.

This explains why, even though one could do all the constructions we considered in this easier setting, we chose to work with measured spaces. The fact that the topological approach is easier comes with its drawback: the topological setting is much poorer and we would miss the quantitative flavor we obtained here. In particular, we loose the generalization of the determinant measure, as well as any measurement built on circuit-quantifying maps which take values outside $\{0, \infty\}$.

6. The Real Line and Quantification

We now consider any microcosm on the real line endowed with Lebesgue measure which contains the microcosm of affine[14] maps. We fix $\Omega =]0, 1]$ endowed with the usual multiplication and we chose any measurable map $m : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$ such that $m(1) = \infty$. Then, as we showed in the preceding section, the map q_m is a $\mathfrak{m}$-circuit-quantifying map. We can thus define the measurement $\llbracket \cdot, \cdot \rrbracket_m$ corresponding to q_m following Theorem 32. This measurement and the execution of graphings satisfy the trefoil property. We will now show how to interpret in this case multiplicative-additive linear logic with second-order quantification.

As remarked earlier, the set of Ω-weighted graphings in the microcosm of measure-preserving maps on the real line with Lebesgue measure corresponds intuitively to the hyperfinite type II_∞ factor. We are therefore considering an extension of the setting of Girard's hyperfinite geometry of interaction by considering the larger microcosm of affine transformations. The general result we obtained earlier allows us to do so while still disposing of a measurement, which in the particular case of the map $m(x) = -\log(1-x)$ generalizes the measurement based on Fuglede-Kadison determinant [FK52]. We point out that this would correspond in Girard's setting to extend the set of operators considered (i.e. consider an algebra $\mathfrak{A}$ containing strictly the type II_∞ hyperfinite factor), while still disposing of the Fuglede-Kadison determinant. The existence of such an extension is not clear, and should it exists, its definition would be far from trivial!

---

[14]An affine map is a map $x \mapsto \alpha x + \beta$ where α, β are real numbers. These maps are the only ones we will use in order to interpret proofs of MALL^2.

The extension to affine maps gives us the possibility of defining real second-order quantification, which was not the case of Girard in his hyperfinite GoI model. Indeed, the fact that projects — which interpret proofs — have a *location* forces him to consider quantification over a given location, something that we also consider here. However, Girard cannot interpret the right existential introduction (from $\vdash B[A/X], \Gamma$ deduce $\vdash \exists X\ B, \Gamma$) correctly because the location of the formula A and the location of the variable X might not have the same size[15]! We bypass this problem here by using *measure-inflating faxes*, i.e. bijective bimeasurable transformations that multiply the size by a scalar.

6.1. Basic Definitions

The model is based on the same constructions as the one described in previous work [Sei12a]. We recall the basic definitions of projects and behaviors, which will be respectively be used to interpret proofs and formulas, as well as the definition of connectives.

- a *project* of carrier V^A is a triple $\mathfrak{a} = (a, V^A, A)$, where a is a real number, $A = \sum_{i \in I^A} \alpha_i^A A_i$ is a finite formal (real-)weighted sum of graphings of carrier included in V^A; here the projects considered always have a carrier of finite measure;
- two projects $\mathfrak{a}, \mathfrak{b}$ are *orthogonal* when:

$$\ll \mathfrak{a}, \mathfrak{b} \gg_m = a(\sum_{i \in I^A} \alpha_i^B) + b(\sum_{i \in I^B} \alpha_i^B) + \sum_{i \in I^A} \sum_{j \in I^B} \alpha_i^A \alpha_j^B [\![A_i, B_j]\!]_m \neq 0, \infty$$

- the *execution* of two projects $\mathfrak{a}, \mathfrak{b}$ is defined as (Δ denotes the symmetric difference):

$$\mathfrak{a} :: \mathfrak{b} = (\ll \mathfrak{a}, \mathfrak{b} \gg_m, V^A \Delta V^B, \sum_{i \in I^A} \sum_{j \in I^B} \alpha_i^A \alpha_j^B A_i ::_m : B_j)$$

- if $\mathfrak{a}$ is a project and V is a measurable set such that $V^A \subset V$, we define the extension $\mathfrak{a}_{\upharpoonright V}$ as the project (a, V, A);
- a *behavior* $\mathbf{A}$ of carrier V^A is a set of projects of carrier V^A which is equal to its bi-orthogonal $\mathbf{A}^{\perp\perp}$, and such that for all $\lambda \in \mathbf{R}$,

$$\begin{aligned}
\mathfrak{a} \in \mathbf{A} &\quad\Rightarrow\quad \mathfrak{a} + \lambda \mathfrak{o} \in \mathbf{A} \\
\mathfrak{b} \in \mathbf{A}^{\perp} &\quad\Rightarrow\quad \mathfrak{b} + \lambda \mathfrak{o} \in \mathbf{A}^{\perp}
\end{aligned}$$

- we define, for every measurable set the *empty* behavior of carrier V as the empty set $\mathbf{0}_V$, and the *full behavior* of carrier V as its orthogonal $\mathbf{T}_V = \{\mathfrak{a} \mid \mathfrak{a} \text{ of support } V\}$;

---

[15]The restriction to operators in the type II_∞ factor implies that unitaries preserve the sizes.

- if $\mathbf{A}, \mathbf{B}$ are two behaviors of disjoint carriers, we define:

$$
\begin{aligned}
\mathbf{A} \otimes \mathbf{B} &= \{\mathfrak{a} :: \mathfrak{b} \mid \mathfrak{a} \in \mathbf{A}, \mathfrak{b} \in \mathbf{B}\}^{\perp\perp} \\
\mathbf{A} \multimap \mathbf{B} &= \{\mathfrak{f} \mid \forall \mathfrak{a} \in \mathbf{A}, \mathfrak{f} :: \mathfrak{a} \in \mathbf{B}\} \\
\mathbf{A} \oplus \mathbf{B} &= (\{\mathfrak{a}_{\upharpoonright V^A \cup V^B} \mid \mathfrak{a} \in \mathbf{A}\}^{\perp\perp} \cup \{\mathfrak{b}_{\upharpoonright V^A \cup V^B} \mid \mathfrak{b} \in \mathbf{B}\}^{\perp\perp})^{\perp\perp} \\
\mathbf{A} \,\&\, \mathbf{B} &= \{\mathfrak{a}_{\upharpoonright V^A \cup V^B} \mid \mathfrak{a} \in \mathbf{A}^{\perp}\}^{\perp} \cap \{\mathfrak{b}_{\upharpoonright V^A \cup V^B} \mid \mathfrak{b} \in \mathbf{B}^{\perp}\}^{\perp}
\end{aligned}
$$

We now define *localized second order quantification* and show the duality between second order universal quantification and second order existential quantification.

Definition 58. We define the localized second order quantification as, for any measurable set L:

$$
\begin{aligned}
\forall_L \mathbf{X}\, \mathbf{F}(\mathbf{X}) &= \bigcap_{\mathbf{A}, V^A = L} \mathbf{F}(\mathbf{A}) \\
\exists_L \mathbf{X}\, \mathbf{F}(\mathbf{X}) &= \left(\bigcup_{\mathbf{A}, V^A = L} \mathbf{F}(\mathbf{A}) \right)^{\perp\perp}
\end{aligned}
$$

Proposition 59.

$$
(\forall_L \mathbf{X}\, \mathbf{F}(\mathbf{X}))^{\perp} = \exists_L \mathbf{X}\, (\mathbf{F}(\mathbf{X}))^{\perp}
$$

Proof. The proof is straightforward. Using the definitions:

$$
\begin{aligned}
(\forall_L \mathbf{X}\, \mathbf{F}(\mathbf{X}))^{\perp} &= \left(\bigcap_{\mathbf{A}, V^A = L} \mathbf{F}(\mathbf{A}) \right)^{\perp} \\
&= \left(\bigcup_{\mathbf{A}, V^A = L} (\mathbf{F}(\mathbf{A}))^{\perp} \right)^{\perp\perp} \\
&= \exists_L \mathbf{X}\, F^{\perp}(X)
\end{aligned}
$$

Where we used the fact that taking the orthogonal turns an intersection into a union. □

6.2. Truth

We now define a notion of *successful project*, which intuitively correspond to the notion of *winning strategy* in game semantics. This notion should be understood as a tentative characterization of those projects which arise as interpretation of proofs. The notion of success defined here is the natural generalization of the corresponding notion on graphs [Sei12b, Sei12a]. The graphing of a successful project will therefore be a disjoint union of "transpositions". In the following, we say a weighted sum of graphings $\sum_{i \in I^A} \alpha_i^A A_i$ is *balanced* when for all $i, j \in I^A$, we have $\alpha_i^A = \alpha_j^A$.

Definition 60. A project $\mathfrak{a} = (a, A)$ is *successful* when it is balanced, $a = 0$ and A is a disjoint union of transpositions:

- for all $e \in E^A$, $\omega_e^A = 1$;
- for all $e \in E^A$, $\exists e^* \in E^A$ such that $\phi_{e^*}^A = (\phi_e^A)^{-1}$ — in particular $S_e^A = T_{e^*}^A$ and $T_e^A = S_{e^*}^A$;
- for all $e, f \in E^A$ with $f \notin \{e, e^*\}$, $S_e^A \cap S_f^A$ and $T_e^A \cap T_f^A$ are of null measure;

A conduct $\mathbf{A}$ is *true* when it contains a successful project.

Proposition 61 (Consistency). *The conducts $\mathbf{A}$ and $\mathbf{A}^{\perp}$ cannot be simultaneously true.*

Proof. We suppose that $\mathfrak{a} = (0, A)$ and $\mathfrak{b} = (0, B)$ are successful project in the conducts $\mathbf{A}$ and $\mathbf{A}^{\perp}$ respectively. Then:

$$\ll \mathfrak{a}, \mathfrak{b} \gg_m = [\![A, B]\!]_m$$

If there exists a cycle whose support is of strictly positive measure between A and B, then $[\![A, B]\!]_m = \infty$ since we suppose that $m(1) = \infty$. Otherwise, $[\![A, B]\!]_m = 0$. In both cases we obtained a contradiction since $\mathfrak{a}$ and $\mathfrak{b}$ cannot be orthogonal. $\quad\square$

Proposition 62 (Compositionnality). *If $\mathbf{A}$ and $\mathbf{A} \multimap \mathbf{B}$ are true, then $\mathbf{B}$ is true.*

Proof. Let $\mathfrak{a} \in \mathbf{A}$ and $\mathfrak{f} \in \mathbf{A} \multimap \mathbf{B}$ be successful projects. Then:
- If $\ll \mathfrak{a}, \mathfrak{f} \gg_m = \infty$, the conduct $\mathbf{B}$ is equal to $\mathbf{T}_{V^B}$, which is a true conduct since it contains $(0, \emptyset)$;
- Otherwise $\ll \mathfrak{a}, \mathfrak{f} \gg_m = 0$ (this is shown in the same manner as in the preceding proof) and it is sufficient to show that $F :_\mathfrak{m} : A$ is a disjoint union of transpositions. But this is straightforward: to each path there corresponds an opposite path and the weights of the paths are all equal to 1, the conditions on the source and target sets S_π and T_π are then easily checked.

Finally, if $\mathbf{A}$ and $\mathbf{A} \multimap \mathbf{B}$ are true, then $\mathbf{B}$ is true. $\quad\square$

6.3. Interpretation of proofs

Definition 63. We fix an infinite (countable) set of variables $\mathcal{V}$ and w define formulas of MALL^2 inductively by the following grammar:

$$F \quad := \quad \mathbf{T} \mid \mathbf{0} \mid X \mid X^{\perp} \mid F \otimes F \mid F \mathbin{⅋} F \mid F \oplus F \mid F \mathbin{\&} F \mid \forall X\, F \mid \exists X\, F \quad (X \in \mathcal{V})$$

Definition 64 (The Sequent Calculus MALL^2). A proof in the sequent calculus MALL^2 is a derivation tree constructed from the derivation rules shown in Figure 7 page 42.

To prove soundness, we will follow the proof technique used in our previous papers [Sei12b, Sei12a]. We will first define a localized sequent calculus and show a result of full soundness for it. The soundness result for the non-localized calculus is then obtained by noticing that one can always *localize* a derivation. We will consider here that the variables are defined with the carrier equal to an interval in $\mathbf{R}$ of the form $[i, i + 1[$.

$$\frac{}{\vdash C^{\perp}, C} \ \text{ax} \qquad\qquad \frac{\Delta_1 \vdash \Gamma_1, C \qquad \Delta_2 \vdash \Gamma_2, C^{\perp}}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2} \ \text{cut}$$

(a) Identity Group

$$\frac{\Delta_1 \vdash \Gamma_1, C_1 \qquad \Delta_2 \vdash \Gamma_2, C_2}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2, C_1 \otimes C_2} \ \otimes \qquad\qquad \frac{\Delta \vdash \Gamma, C_1, C_2}{\Delta \vdash \Gamma, C_1 \,\mathfrak{P}\, C_2} \ \mathfrak{P}$$

(b) Multiplicative Group

$$\frac{\vdash \Gamma, C_i}{\vdash \Gamma, C_1 \oplus C_2} \ \oplus_i \qquad\qquad \frac{\vdash \Gamma, C_1 \qquad \vdash \Gamma, C_2}{\vdash \Gamma, C_1 \,\&\, C_2} \ \&$$

$$\frac{}{\vdash \Gamma, \top} \ \top \qquad\qquad\qquad \text{No rules for } 0.$$

(c) Additive Group

$$\frac{\vdash \Gamma, C \qquad X \notin FV(\Gamma)}{\vdash \Gamma, \forall X \ C} \ \forall \qquad\qquad \frac{\vdash \Gamma, C[A/X]}{\vdash \Gamma, \exists X \ C} \ \exists$$

(d) Quantifier Group

Figure 7: Rules for the sequent calculus MALL^2

Definition 65. We fix a set $\mathcal{V} = \{X_i(j)\}_{i,j \in \mathbf{N} \times \mathbf{Z}}$ of *localized variables*. For $i \in \mathbf{N}$, the set $X_i = \{X_i(j)\}_{j \in \mathbf{Z}}$ will be called the *variable name X_i*, and an element of X_i will be called a *variable of name X_i*.

For $i, j \in \mathbf{N} \times \mathbf{Z}$ we define the *location* $\sharp X_i(j)$ of the variable $X_i(j)$ as the set

$$\{x \in \mathbf{R} \mid 2^i(2j+1) \le m < 2^i(2j+1)+1\}$$

Definition 66 (Formulas of locMALL^2)**.** We inductively define the formulas of *localized second order multiplicative-additive linear logic* locMALL^2 as well as their *locations* as follows:
- A variable $X_i(j)$ of name X_i is a formula whose location is defined as $\sharp X_i(j)$;
- If $X_i(j)$ is a variable of name X_i, then $(X_i(j))^{\perp}$ is a formula whose location is $\sharp X_i(j)$.
- The constants $\mathbf{T}_{\sharp\Gamma}$ are formulas whose location is defined as $\sharp\Gamma$;
- The constants $\mathbf{0}_{\sharp\Gamma}$ are formulas whose location is defined as $\sharp\Gamma$.
- If A, B are formulas with respective locations X, Y such that $X \cap Y = \emptyset$, then $A \otimes B$ (resp. $A \,\mathfrak{P}\, B$, resp. $A \,\&\, B$, resp. $A \oplus B$) is a formula whose location is $X \cup Y$;
- If X_i is a variable name, and $A(X_i)$ is a formula of location $\sharp A$, then $\forall X_i \ A(X_i)$ and $\exists X_i \ A(X_i)$ are formulas of location $\sharp A$.

Definition 67 (Interpretations)**.** An *interpretation basis* is a function Φ which

associates to each variable name X_i a behavior of carrier[16] $[0,1[\times\{*\}$.

Definition 68 (Interpretation of locMALL^2 formulas)**.** Let Φ be an interpretation basis. We define the interpretation $I_\Phi(F)$ along Φ of a formula F inductively:

- If $F = X_i(j)$, then $I_\Phi(F)$ is the delocation (i.e. a behavior) of $\Phi(X_i)$ defined by the function $x \mapsto 2^i(2j+1)+x$;
- If $F = (X_i(j))^\perp$, we define the behavior $I_\Phi(F) = (I_\Phi(X_i(j)))^\perp$;
- If $F = \mathbf{T}_{\sharp\Gamma}$ (resp. $F = \mathbf{0}_{\sharp\Gamma}$), we define $I_\Phi(F)$ as the behavior $\mathbf{T}_{\sharp\Gamma}$ (resp. $\mathbf{0}_{\sharp\Gamma}$);
- If $F = \mathbf{1}$ (resp. $F = \perp$), we define $I_\Phi(F)$ as the behavior $\mathbf{1}$ (resp. $\perp$);
- If $F = A \otimes B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \otimes I_\Phi(B)$;
- If $F = A \invamp B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \invamp I_\Phi(B)$;
- If $F = A \oplus B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \oplus I_\Phi(B)$;
- If $F = A \mathbin{\&} B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \mathbin{\&} I_\Phi(B)$;
- If $F = \forall X_i A(X_i)$, we define the conduct $I_\Phi(F) = \forall \mathbf{X_i} I_\Phi(A(X_i))$;
- If $F = \exists X_i A(X_i)$, we define the conduct $I_\Phi(F) = \exists \mathbf{X_i} I_\Phi(A(X_i))$.

Moreover, a sequent $\vdash \Gamma$ will be interpreted as the $\invamp$ of formulas in Γ, which will be written $\invamp \Gamma$.

Definition 69. Let F be a formula, A a subformula of F, n the number of occurrences of A in F, and X_i be a variable name that does not appear in F. We define an enumeration $e_{A/F}$ of the occurrences of A in F whose image is $\{1,\ldots,n\}$. For each $j \in \{1,\ldots,n\}$, we define $\psi_j : \sharp e^{-1}(j) \to \sharp X_i(j)$ as the natural (order-preserving) measure-inflating map between $\sharp e^{-1}(j)$, a disjoint union of unit segments, and $\sharp X_i(j)$, a unit segment. We then define the *measure-inflating fax* $[e^{-1}(j) \leftrightarrow X_i(j)]$ as the graphing:

$$\{(1,\psi),(1,\psi^{-1})\}$$

Definition 70 (Interpretation of locMALL^2 proofs)**.** Let Φ be an interpretation basis. We define the interpretation $I_\Phi(\pi)$ — a project — of a proof π inductively:

- if π is a single axiom rule introducing the sequent $\vdash (X_i(j))^\perp, X_i(j')$, we define $I_\Phi(\pi)$ as the project $\mathfrak{Fax}$ defined by the translation $x \mapsto 2^i(2j'-2j)+x$;
- if π is composed of a single rule $\mathbf{T}_{\sharp\Gamma}$, we define $I_\Phi(\pi) = \mathfrak{o}_{\sharp\Gamma}$;
- if π is obtained from π' by using a $\invamp$ rule, then $I_\Phi(\pi) = I_\Phi(\pi')$;
- if π is obtained from π_1 and π_2 by performing a $\otimes$ rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \otimes I_\Phi(\pi')$;
- if π is obtained from π' using a $\oplus_i$ rule introducing a formula of location V, we define $I_\Phi(\pi) = I_\Phi(\pi') \otimes \mathfrak{o}_V$;
- if π of conclusion $\vdash \Gamma, A_0 \mathbin{\&} A_1$ is obtained from π_0 and π_1 using a $\&$ rule, we define the interpretation of π in the same way it was defined in our previous paper [Sei12a];
- If π is obtained from a $\forall$ rule applied to a derivation π', we define $I_\Phi(\pi) = I_\Phi(\pi')$;

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[16]We consider $[0,1[\times\{*\}$ and not simply $[0,1[$ only to insure that the image of Φ is disjoint from the locations of the variables.

- If π is obtained from a $\exists$ rule applied to a derivation π' replacing the formula **A** by the variable name X_i, we define $I_\Phi(\pi) = I_\Phi(\pi'){:}\mathrm{m}{:}(\bigotimes[e^{-1}(j) \leftrightarrow X_i(j)])$;
- if π is obtained from π_1 and π_2 by applying a cut rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \pitchfork I_\Phi(\pi_2)$.

Theorem 71 (locMALL^2 soundness). *Let Φ be an interpretation basis. Let π be a derivation in locMALL^2 of conclusion $\vdash \Gamma$. Then $I_\Phi(\pi)$ is a successful project in $I_\Phi(\vdash \Gamma)$.*

As it was remarked in our previous papers, one can chose an enumeration of the occurrences of variables in order to "localize" any formula A and any proof π of MALL^2: we then obtain formulas A^e and proofs π^e of locMALL^2. The following theorem is therefore a direct consequence of the preceding one.

Theorem 72 (Full MALL^2 Soundness). *Let Φ be an interpretation basis, π an MALL^2 proof of conclusion $\Delta \vdash \Gamma$; and e an enumeration of the occurrences of variables in the axioms in π. Then $I_\Phi(\pi^e)$ is a successful project in $I_\Phi(\Delta^e \vdash \Gamma^e;)$.*

7. Perspectives

We described in this paper a general construction of models of multiplicative-additive linear logic (MALL). This general construction can be performed on any *trefoil space*, that is a measured space subject to a few conditions. Given a trefoil space X, we obtain a hierarchy of models of MALL corresponding to the hierarchy of *microcosms*, i.e. monoids of non-singular transformations from X to itself, and the hierarchy of weight monoids. In particular, all previously considered geometry of interaction constructions can be recovered for particular trefoils spaces X, weight monoids and microcosms.

The perspectives of this work are numerous. First, one can extend the model on the real line described at the end of this paper in order to deal with exponential connectives, following the approach described in the author's PhD thesis [Sei12c]. Following this approach, we will obtain a model of Elementary Linear Logic. But more expressive exponentials can be defined here, and we will also define a model interpreting full linear logic.

The most exciting perspective of this work concerns the field of computational complexity. As described in a short note [Sei14a], we can show a correspondence between a part of the hierarchy of models obtained here and a part of the hierarchy of complexity classes. Indeed, as we consider bigger microcosms, the type of predicates $!\mathbf{Nat_2} \multimap \mathbf{Bool}$ becomes larger. Intuitively, a microcosm describes the computational principles allowed in the system. By adapting earlier results obtained with von Neumann algebras [AS12, AS13] we can define microcosms for which the type of predicates characterizes the class of regular languages on one hand, and the class of logarithmic space predicates on the other.

We can also apply the techniques developed here for quantum computation. Indeed, it is possible to model quantum circuits in a very nice way in some of the models defined in this paper. Once again, one could gain from the possibility

of considering smaller and/or larger microcosms. For instance, one could study restrictions of these models of quantum computation where the available unitary gates are limited to a chosen basis. It would then be possible to understand how the different choices of bases of unitaries affect the model from a computational and/or logical point of view.

Lastly, we believe the theory of dynamical systems and ergodic theory might shed new light on the field of computational complexity. In particular, we will study how mathematical invariants, such as l^2-Betti numbers, are related to computation.

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